Formal Modelling and Verification
Master CORO – M2 ERTS

Didier Lime

École Centrale de Nantes – LS2N

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Plan I

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Concurrent Programming

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demo: concurrency.c

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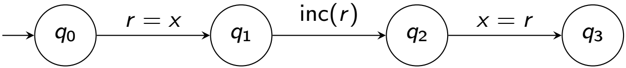
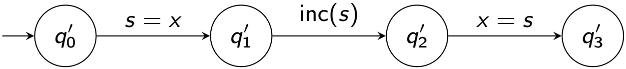
Two wide classes of properties:

- Qualitative : safety and liveness;
- Quantitative: Deadlines, duration of execution, number of function calls...(Formalisms modelling time, probabilities, energy, ...)

A First Model





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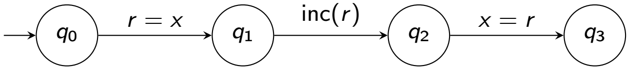

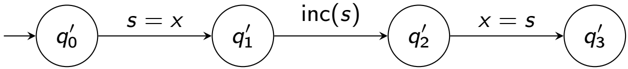

Informal property

We never have both $(q_1 \vee q_2)$ and $(q'_1 \vee q'_2)$ at the same time.

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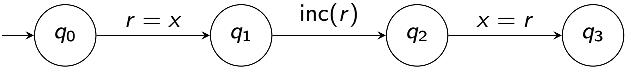

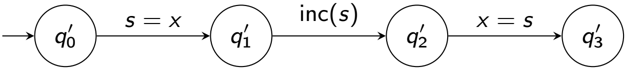

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Abstraction with respect to the property?

Peterson's Mutual Exclusion Algorithm [Ray84]

```
P :
    d ← false
loop
    <non critical section>
    d ← true
    turn ← 1
    attendre(¬d′∨ turn= 0)
    <critical section>
    d ← false
end_loop
```

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P′ :
    d′ ← false
loop
    <non critical section>
    d′ ← true
    turn ← 0
    attendre(¬d∨ turn= 1)
    <critical section>
    d′ ← false
end_loop
```

Peterson's Mutual Exclusion Algorithm [Ray84]

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i0:  d ← false
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i1:  d ← true
i2:  turn ← 1
i3:  attendre(¬d'∨ turn= 0)
    <critical section>
i4:  d ← false
end_loop
```

```
P':
i'0:  d' ← false
loop
    <non critical section>
i'1:  d' ← true
i'2:  turn ← 0
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    <critical section>
i'4:  d' ← false
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i_4: $d \leftarrow$ `false`
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P':

i'_0: $d' \leftarrow$ `false`
`loop`
    `<non critical section>`
i'_1: $d' \leftarrow$ `true`
i'_2: `turn` $\leftarrow$ 0
i'_3: `attendre`($\neg d \vee$ `turn`$= 1$)
    `<critical section>`
i'_4: $d' \leftarrow$ `false`
`end_loop`

How can we prove that this program works as intended?

Peterson's Mutual Exclusion Algorithm

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- The state of the system is $(d, d', \text{turn}, n, n') \in \mathbb{B} \times \mathbb{B} \times \{0, 1\} \times [0..4] \times [0..4]$, where n and n' represent the number of the next instruction to be executed for resp. P and P'

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- The number of states is finite: we can explore exhaustively all possibilities.

Peterson's Mutual Exclusion Algorithm

$$
\begin{tikzcd}
{} \arrow[dr] & {} \arrow[d, dashed] & {} \arrow[d, dashed] & {} \arrow[d, dashed] & & \\
{} \arrow[r, dashed] & (F, F, ?, 1, 1) \arrow[r, "i_1"] \arrow[d, "i'_1"'] & (T, F, ?, 2, 1) \arrow[r, "i_2"] \arrow[d, "i'_1"'] & (T, F, 1, 3, 1) \arrow[r, "i_3"] \arrow[d, "i'_1"'] & (T, F, 1, 4, 1) \arrow[r, dashed, "i_4"] \arrow[d, "i'_1"'] & {} \\
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 & & & & \textcolor{red}{(T, T, ?, 4, 4)} & \\
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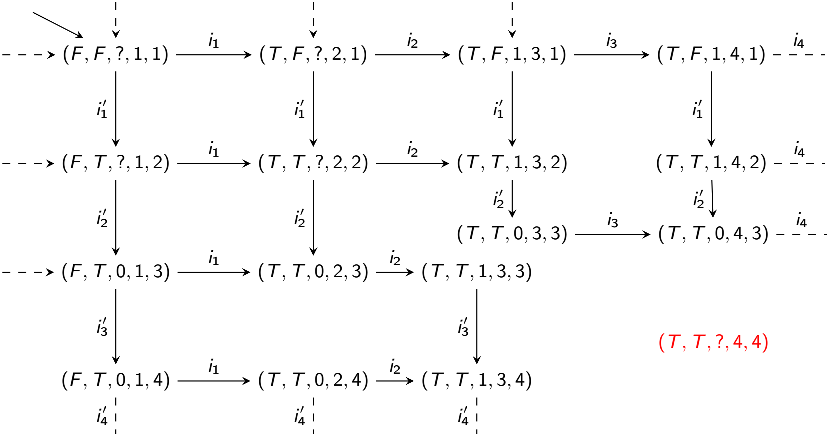

Mutual Exclusion? Deadlocks? Access to the CS for P and P'?

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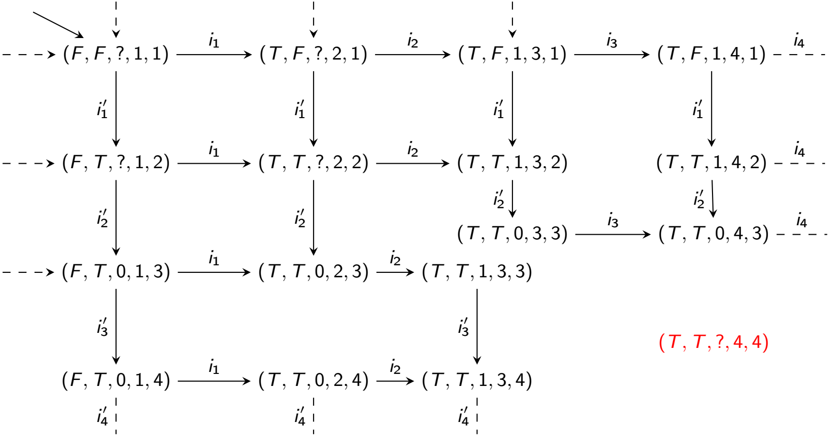


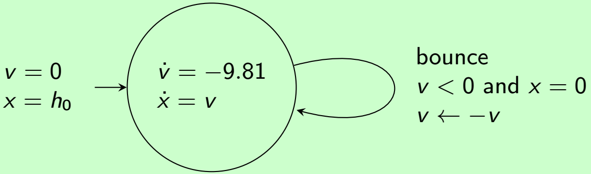
Mutual Exclusion? Deadlocks? Access to the CS for P and P'?
What about starvation?

Models for Complex Systems

- Many complex systems can be described by hybrid systems:
  - A continuous evolution governed by differential equations:
  - Discrete events, that change these these differential equations, or their initial conditions.

Example

Consider a ball dropped from some initial height h_0 and bouncing on the floor:



- The general setting is hard / impossible to analyze automatically
- We need to find an abstraction of the system relevant preserving the properties of interest.

Some Intuition About Decidability and Complexity

Some problems (e.g. termination of an arbitrary program) cannot be solved by an automatic procedure.

- Assume we have a function $H(f, x)$ that for any function f and any of its input x, returns whether the computation of $f(x)$ terminates;
- This might not be the case if f contains a `while (true)` loop for instance;
- Now consider the following function $H'(g)$, where g is a function with a function as input:

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function H'(g):
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- What happens for $H'(H')$?
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  - if the computation does not terminate, then $H(H',H')$ is false and the computation should terminate (and return true).
- So function H cannot exist.

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- Even within decidable problems, we can make differences in:
  For a given model of computation, and considering a worst-case instance (or best-case, average-case, etc.)
  - the temporal complexity: number of instructions for the best algorithm to decide;
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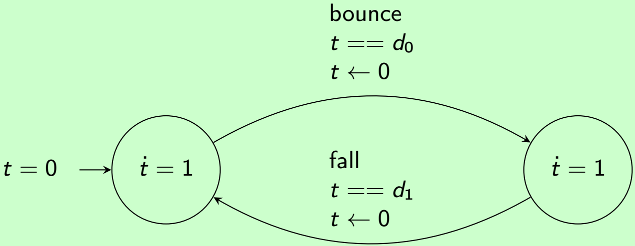
  - the temporal complexity: number of instructions for the best algorithm to decide;
  - the spatial complexity: amount of memory for the best algorithm to decide.
- For Turing machines and worst-case instances, we have, e.g.:
  - PTIME (aka P): CTL model-checking on finite automata (incl. reachability, liveness, etc.)
  - PSPACE: LTL model-checking on finite automata / TCTL model-checking on timed automata;
  - EXPTIME: Timed control for timed automata;
  - EXPSPACE: Timed language inclusion for strongly non-zeno timed automata;
  - Undecidable: Reachability in hybrid automata / language inclusion in timed automata.

Models for Complex Systems

- In the previous example, solve the equations to find the time to bouncing d_0, and the duration between bouncing and falling down again d_1:

Example

Consider a ball dropped from some initial height h_0, bouncing on the floor after d_0 seconds, and falling again d_1 seconds after bouncing:



- The time between the *bounce* events is preserved by this abstraction;
- It much easier to analyze automatically.

Models for Complex Systems

- Suppose now there is some dampening when bouncing: $v \leftarrow -v\epsilon$, with $0 < \epsilon < 1$;
- Now only an upper bound on the time between two *bounce* events is preserved in timed model;
- Still the number of bounces in the full model is infinite:

Exercise

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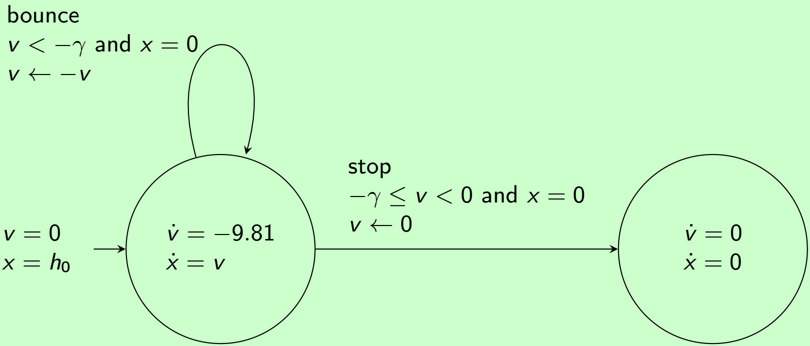
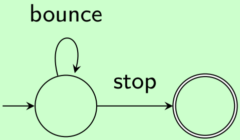


Models for Complex Systems

- Using no quantitative information, we can have a very coarse abstraction:

Example



- It is very easy to analyze and still preserves some properties, like:

  After bouncing a finite number of times, the ball eventually stops.

Plan I

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Definitions I

Definition (Labeled Transition Systems [BK08])

A Labeled Transition System (LTS) is a tuple $(S, S_0, \Sigma, \rightarrow)$, with:

- S is a set of elements called states ;
- S_0 is a non-empty subset of S containing the initial states ;
- Σ is a non-empty set of elements called actions;
- $\rightarrow \subset S \times \Sigma \times S$ is a relation called transition relation. We note $s \xrightarrow{a} s'$ when $(s, a, s') \in \rightarrow$.

Definition (Run)

A run of $\mathcal{S} = (S, S_0, \Sigma, \rightarrow)$ is possibly infinite sequence $s_0, a_0, s_1, a_1 \ldots, a_{n-1} s_n, \ldots$ such that $\forall i, s_i \xrightarrow{a_i} s_{i+1}$. We note $[\![\mathcal{S}]\!]$ the set of the runs of $\mathcal{S}$.

Definition (Reachable State)

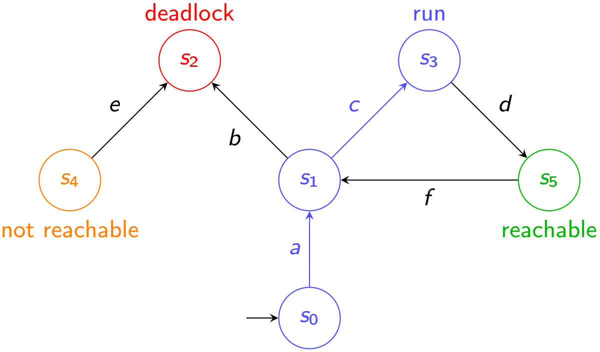
A state s' is reachable from a state s if there exists a run starting in s end ending in s'.

Definitions II

Definition (Deadlock State)

A state s is a deadlock if $\forall s' \in S, \forall a \in \Sigma, (s, a, s') \not\in \rightarrow$.

Example


deadlock
run
s_2
s_3
e
b
c
d
s_4
s_1
s_5
f
not reachable
reachable
a
s_0

Complete LTS

Definition

Complete LTS An LTS $(S, S_0, \Sigma, \rightarrow)$ is complete if from all states, every action is possible:

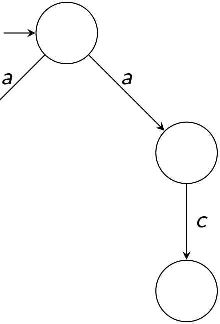
$$\forall s \in S, \forall a \in \sigma, \exists s' \in S \text{ s.t. } s \xrightarrow{a} s'$$

- A complete LTS means that all (sequential) behaviors are accounted for in the modeled system;
- It is particularly important for a specification;
- An LTS can always be made complete (preserving sequences of actions):
  - Add to S an *error* state s_{err};
  - For all $s \in S$ and $a \in \Sigma$ s.t. there is no s' s.t. $s \xrightarrow{a} s'$, add a transition (s, a, s_{err}) to $\rightarrow$.

Determinism

Definition (Deterministic LTS)

An LTS $\mathcal{S} = (S, S_0, \Sigma, \rightarrow)$ is deterministic if $|S_0| = 1$ and $s \xrightarrow{a} s' \land s \xrightarrow{a} s'' \Rightarrow s' = s''$.



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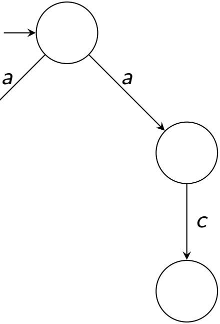

Usefulness ?

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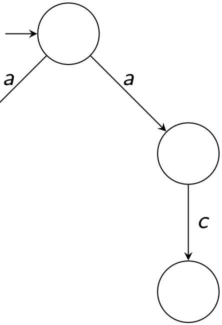
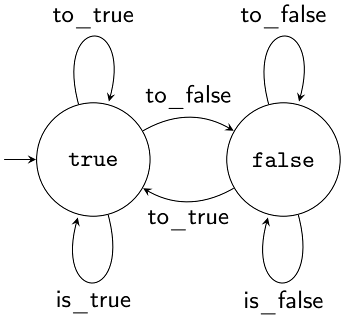


Usefulness ? Abstraction.

LTS Example: Boolean Variable



Building Systems from Components

A complex system $\mathcal{S}$ is often described as the assembly of different components $(\mathcal{S}_i)_i$.

To analyze $\mathcal{S}$ we can:

- reason on the individual components $\mathcal{S}_i$ (for local properties);
- build $\mathcal{S}$ (possibly on-the-fly);
- compose local analyses (compositional approaches).

Synchronized Product of LTSs [BK08]

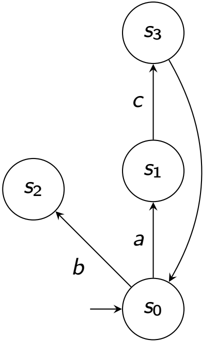
Consider n sets of actions Σ_i and n LTS $\mathcal{S}_i = (S^i, S^i_0, \Sigma_i, \rightarrow_i)_{n \in \mathbb{N}}$. We note $\Sigma_i^\bullet = \Sigma_i \cup \{\bullet\}$ with $\forall i, \bullet \notin \Sigma_i$.

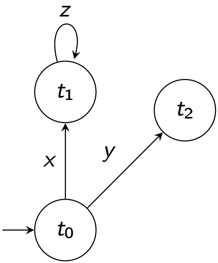
Definition (Synchronized Product)

The synchronised product $(\mathcal{S}_1, \ldots, \mathcal{S}_n)_f$ of the $\mathcal{S}_i$'s by the synchronization function $f : \Sigma_1^\bullet \times \cdots \times \Sigma_n^\bullet \rightarrow B$ is the LTS $(S, S_0, B, \rightarrow)$ such that:

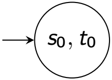
- $S = S^1 \times \ldots \times S^n$;
- $S_0 = S_0^1 \times \ldots \times S_0^n$;
- $\rightarrow \subset S \times B \times S$ is such that $(s_1, \ldots, s_n) \xrightarrow{b} (s'_1, \ldots, s'_n)$ iff $\forall i, \exists a_i \in \Sigma_i^\bullet$ s.t. $s_i \xrightarrow{a_i} s'_i$ and $f(a_1, \ldots, a_n) = b$.

Example





$$
\begin{aligned}
f(a, x) &= a \\
f(a, z) &= f(c, z) = a \\
f(b, \bullet) &= b \\
f(c, y) &= k
\end{aligned}
$$



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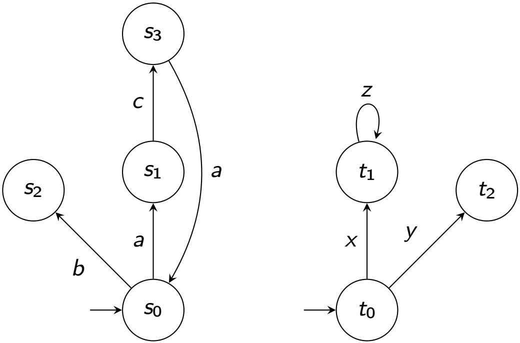

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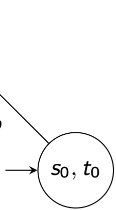

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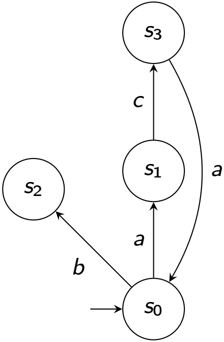

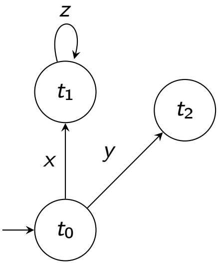

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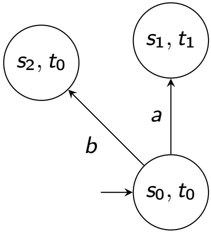
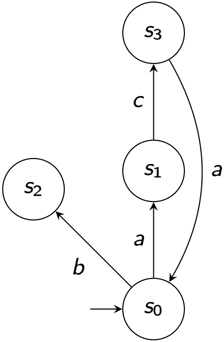
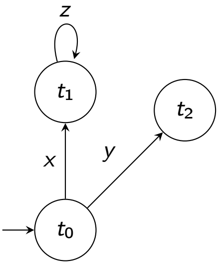


Example





$$
\begin{aligned}
f(a,x) &= a\\
f(a,z) &= f(c,z) = a\\
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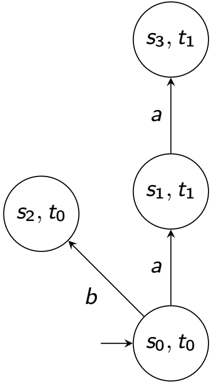
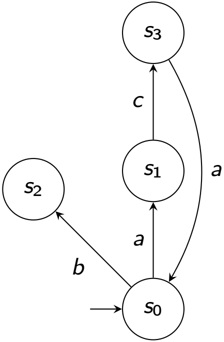
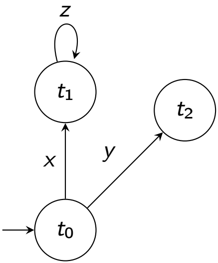
\end{aligned}
$$



Example





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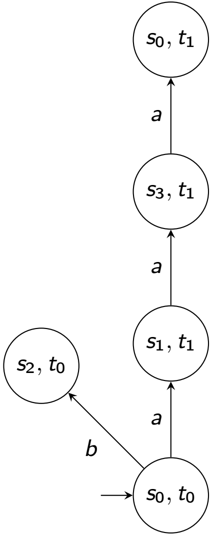

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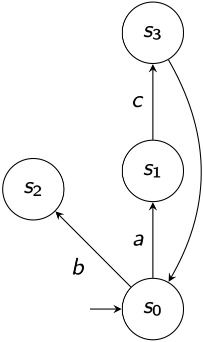

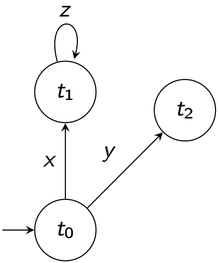

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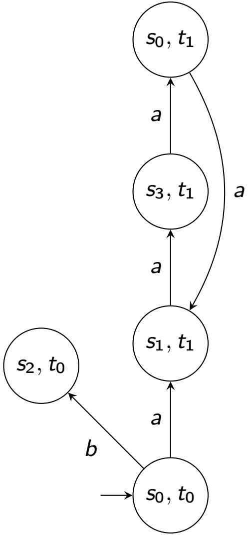

Common Synchronization Schemes: No Synchronization

- We might have no synchronization at all between the components;
- This can be modeled by the asynchronous product;
- In f exactly one component is not $\bullet$; define all combinations of one action from some Σ_i with $\bullet$'s.

Common Synchronization Schemes: Complete Synchronization

- In some systems, components always progress all at the same time: e.g. hardware circuits;
- This can be modeled by the synchronous product;
- f never uses $\bullet$ and should be defined on all possible combinations of actions.

Common Synchronization Schemes: Rendez-vous

- When two processes need to wait for each other;

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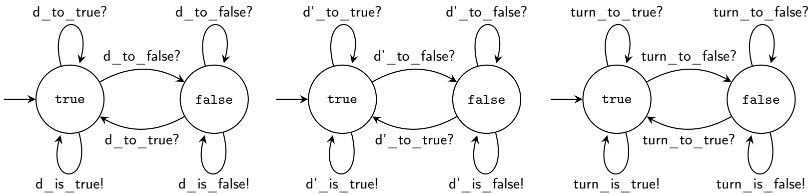
- In that context, actions without any synchronization mark are assumed independent:

$$f(b, \bullet) = b \text{ or } f(\bullet, b) = b$$

Example


d_to_true?
d_to_false?
d_to_false?
true
false
d_to_true?
d_is_true!
d_is_false!
d'_to_true?
d'_to_false?
d'_to_false?
true
false
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turn_to_true?
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true
false
turn_to_true?
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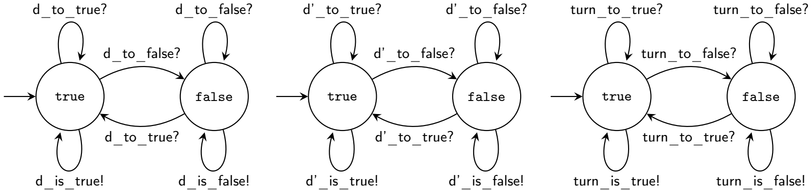
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true
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true
false
turn_to_true?
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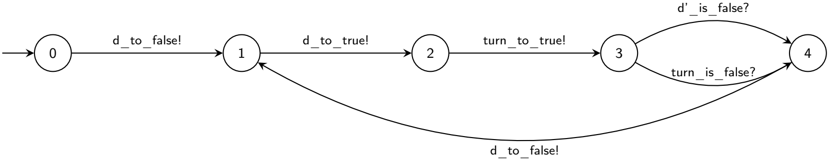
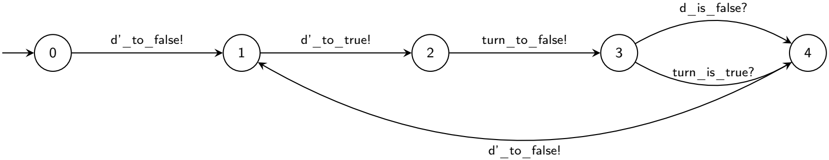

0
d_to_false!
1
d_to_true!
2
turn_to_true!
3
d'_is_false?
turn_is_false?
4
d_to_false!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- The notation with ! and ? (or !! and ??) is often used also;
- The non-blocking sending is often modeled by making sure that all processes can always recieve the message (possibly through a self-loop).

Exercise

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The philosophers' dinner. n philosopher are gathered around a big table with a spaghetti dish in the middle. There is one fork between each pair of adjacent philosophers. Each philospher is always either eating or thinking but never both at the same time. In order to eat, a philosopher needs the two forks, situated on its left and right.

1. Model a fork with a finite LTS;
2. Model a philosopher with a finite LTS (assuming that when it has taken a fork it will not release it before having eaten);
3. Build a substantial part of the product of two philosophers and two forks, including a deadlock.

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  - A tree of executions, with branching points.

Plan I

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  Let $(S, S_0, A, \rightarrow)$ be an LTS and $G \subseteq S$, can we stay in G forever?
- The safety property can also be written as:
  Let $(S, S_0, A, \rightarrow)$ be an LTS and $G \subseteq S$, are all states in $S \setminus G$ not reachable?

Erroneous Version of Peterson's Algorithm

P:

```
i0:  d ← false
loop
     <non critical section>
i1:  d ← true
i2:  turn ← 1
i3:  attendre(¬d' ∨ turn = 0)
     <critical section>
i4:  d ← false
end_loop
```

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```
i'0:  d' ← false
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i_0: $d \leftarrow$ `false`
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    `<critical section>`
i'_4: $d' \leftarrow$ `false`
`end_loop`

We have removed `turn`.

Erroneous Version of Peterson's Algorithm

$$
\begin{tikzcd}
 & & {} \arrow[d, dashed] & & {} \arrow[d, dashed] & & & \\
{} \arrow[r, dashed] & {} \arrow[r, phantom] & (F,F,1,1) \arrow[rr, "i_1"] \arrow[d, "i_1'"'] & & (T,F,3,1) \arrow[rr, "i_3"] \arrow[d, "i_1'"'] & & (T,F,4,1) \arrow[r, dashed, dash, "i_4"] \arrow[d, "i_1'"'] & {} \\
{} \arrow[rr, dashed] & & (F,T,1,2) \arrow[rr, "i_1"] \arrow[d, "i_3'"'] & & \color{blue}{(T,T,3,3)} & & (T,T,4,3) \arrow[r, dashed, dash, "i_4"] & {} \\
 & & (F,T,1,4) \arrow[rr, "i_1"] \arrow[d, dashed, dash, "i_4'"] & & (T,T,3,4) \arrow[d, dashed, dash, "i_4'"] & & \color{red}{(T,T,4,4)} & \\
 & & {} & & {} & & &
\end{tikzcd}
$$

- **mutual exclusion**: is state $\color{red}{(T,T,4,4)}$ reachable?
- **non blocking**: is state $\color{blue}{(T,T,3,3)}$ reachable?

A Reachability Algorithm

- For a finite LTS $(S, s_0, A, \rightarrow)$, reachability of $G \subseteq S$ can be checked by a graph walk:

```
W ← S0 ; P ← ∅; r ← false
while W ≠ ∅ and not r
    s ← next(W)
    if s ∈ G then
        r ← true
    else
        if s ∉ P then
            add s to P
            for all s′ ∈ S, a ∈ A s.t. s –a→ s′
                add s′ to W
            endfor
        endif
    endif
endwhile
```

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- We compute a least fix point: $(\mu X . S_0 \cup \mathsf{Succ}(X))$

$X \leftarrow \emptyset$
$Y \leftarrow S$
while $X \neq Y$
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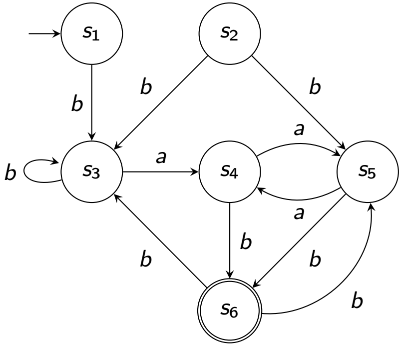
```
X ← ∅
Y ← S
while X ≠ Y
    Y ← X
    X ← S0 ∪ Succ(Y)
endwhile
```

Why does the algorithm terminate?

Forward Reachability: Example



We compute: $X_0 = \emptyset$ et $X_{n+1} = \{s_1\} \cup \mathrm{Succ}(X_n)$

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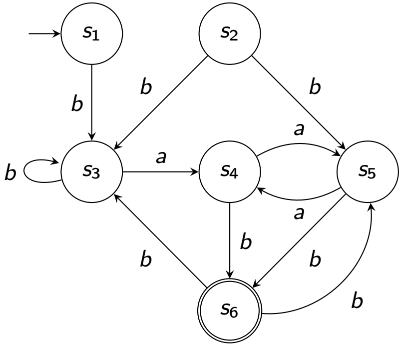

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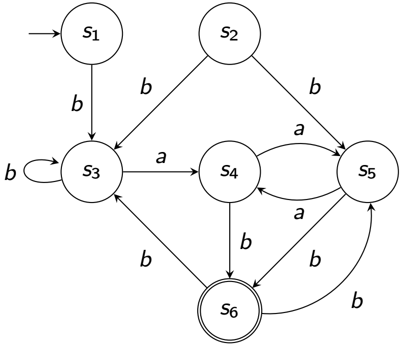

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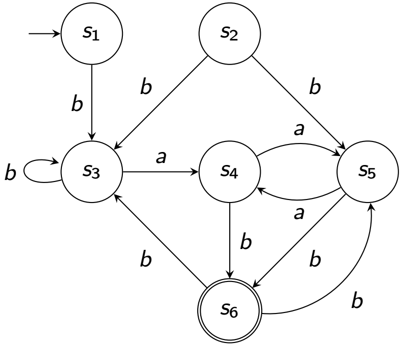

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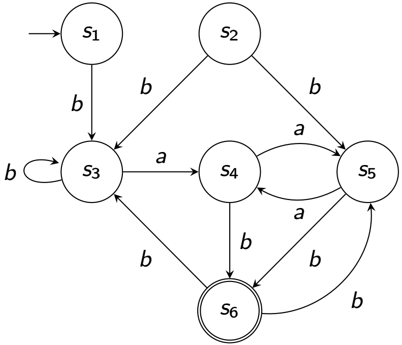

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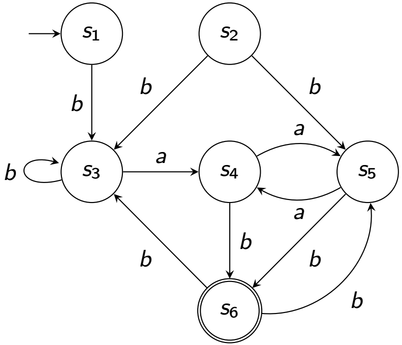

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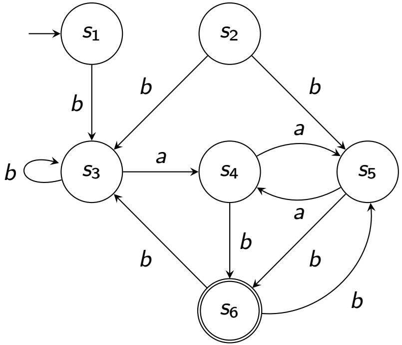

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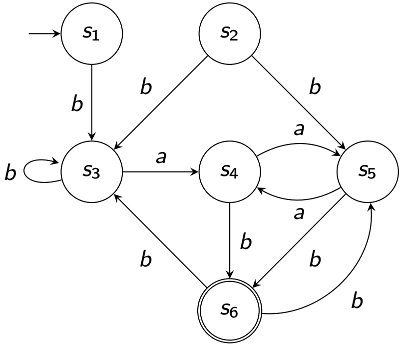

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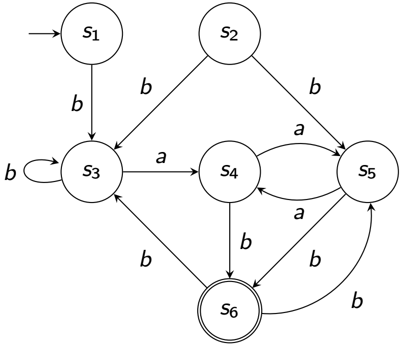

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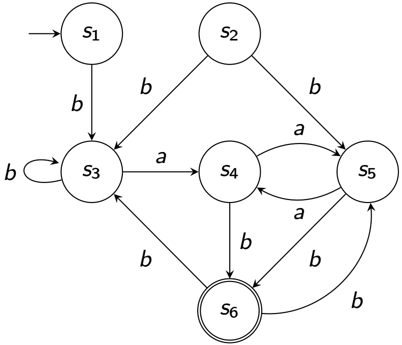

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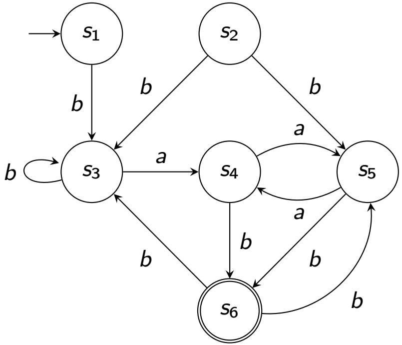
Backward Reachability: Example



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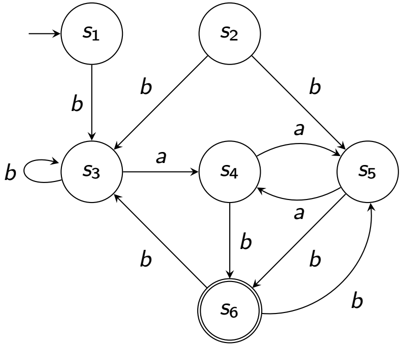

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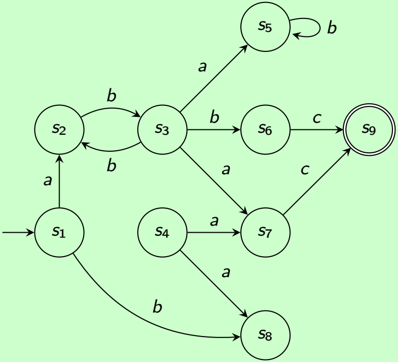
- $X_5 = X_4$.

Reachability: Exercises

Exercise

1. Compute the set of states that are reachable from s_1;
2. Compute the set of states that are co-reachable for s_9;
3. Compute the set of states that cannot reach s_9 (safety).



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Exercise

Compute the set of states that are safe for $\overline{\{s_9\}}$ for the previous exemple.

Reachability: Exercise

Exercise

1. Prove that $s \in X_n$ iff there exists a run of length less than n that starts in s and ends in G;
2. Prove that the co-reachability algorithm is correct (gives states that are indeed coreachable) and complete (gives all of them).

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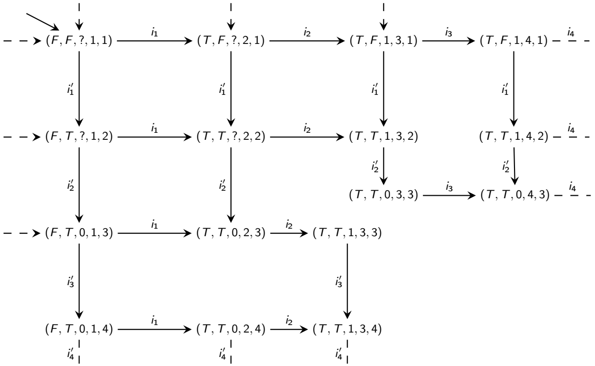
Observers Example: Peterson's Algorithm



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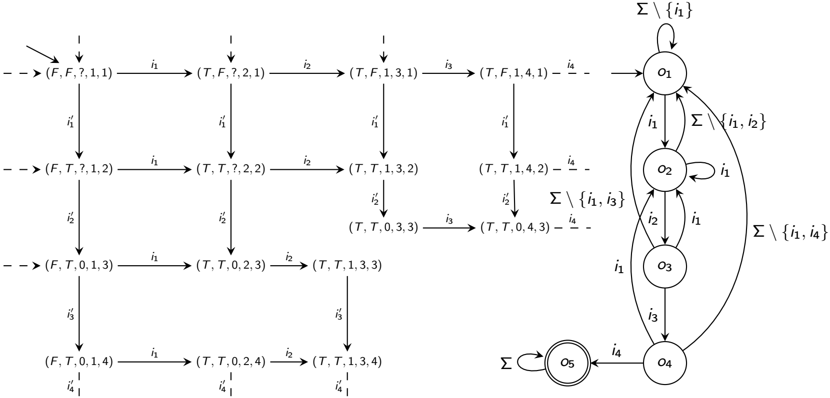

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  Is a state of the form $(*, *, *, *, *, o_5)$ reachable in the product?

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Let $(S, S_0, \Sigma, \rightarrow)$ be an LTS and $s \in S$. $a \in \Sigma$ is quasi-live in s if there exists a state s' reachable from s, and from which a is possible ($\exists s''$ s.t. $s' \xrightarrow{a} s''$).

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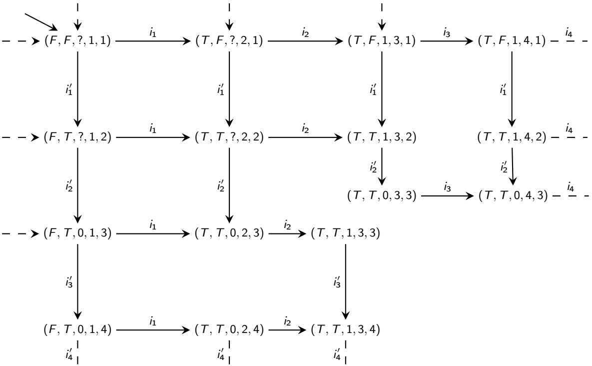

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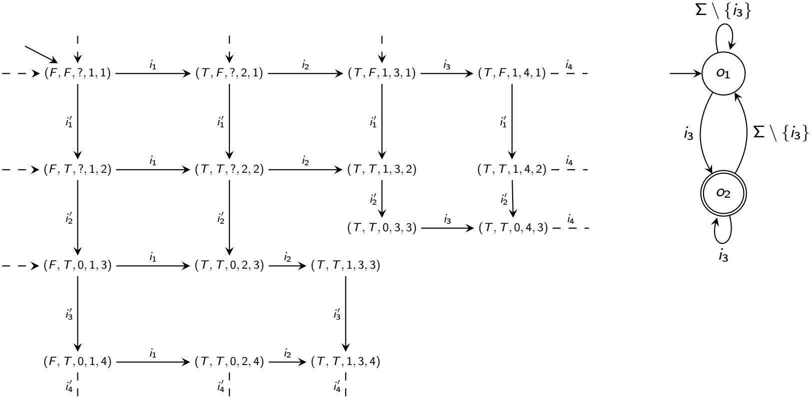

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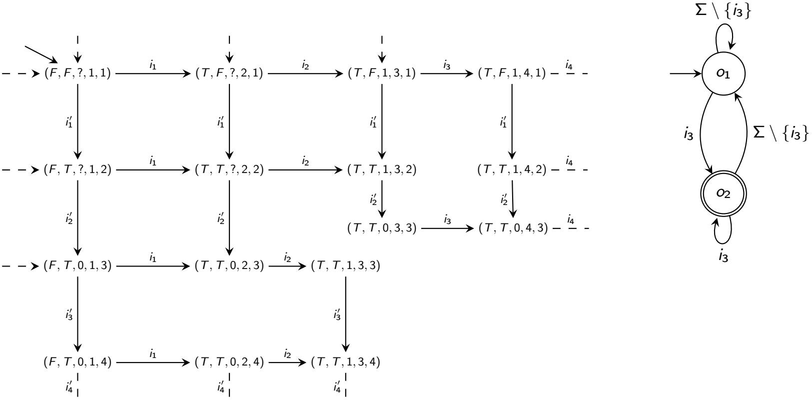

- **possible continuous access to the resource**: is i_3 live?
  Reachability of $(*, *, *, *, *, o_2)$ is not enough
- We need (at least) the repeated reachability of $(*, *, *, *, *, o_2)$.

Repeated Reachability Algorithm

- Let $(S, S_0, \Sigma, \rightarrow)$ be an LTS and a set of states to repeat infinitely often $G \subseteq S$;
- We give a symbolic algorithm;
- We compute nested fix points: $\nu V.\mu X.\mathrm{Pred}(G \cap V) \cup \mathrm{Pred}(X)$

```
V ← S
W ← ∅
while  V ≠ W
      W ← V
      X ← ∅
      Y ← S
      while  X ≠ Y
            Y ← X
            X ← Pred(V ∩ G) ∪ Pred(Y)
      endwhile
      V ← X
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- After iteration $n \geq 1$, V contains the set of states that can reach G, in at least one step, and at least n times.

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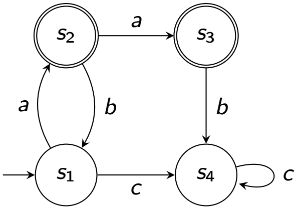

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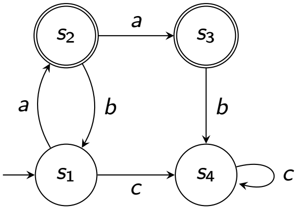

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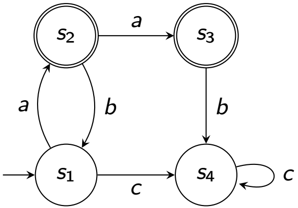

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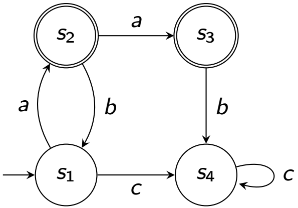

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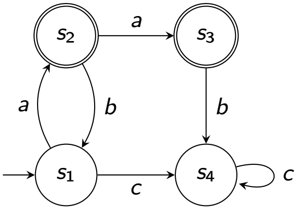

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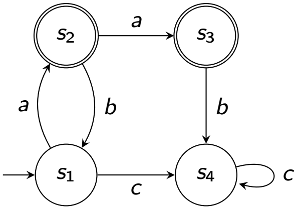

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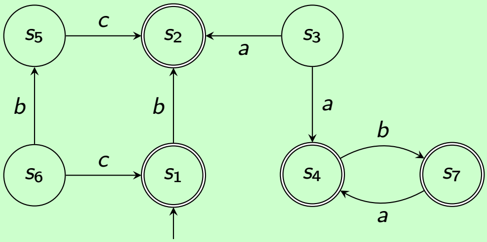
- $V_2 = \{s_1, s_2\} = V_1$;

Repeated Reachability: Exercise

Exercise

Compute the set of states from which $\{s_1, s_2, s_4, s_7\}$ can be repeated infinitely often.



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Plan I

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- Sequences of actions can be characterized with the notion of language of a finite automaton.

Formal Languages

Letters and words:

- Consider a finite set Σ, called alphabet;
- Elements of Σ are called letters;
- Words are sequences of letters;
- We note Σ^* the set of all words on Σ;
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Languages :

- A language on Σ is a subset of Σ^*;
- Let U and V be two languages on Σ, $UV = \{uv | u \in U, v \in V\}$;
- $U^* = \{u_0 \dots u_n | n \geq 0, \forall i, u_i \in U\}$.

Language Recognized by an NFA

Definition (Trace)

The trace of a run $\rho = s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} s_n \cdots$ is the possibly infinite word $\text{trace}(\rho) = a_0 a_1 \ldots a_{n-1} \ldots$.

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Definition (Word Recognized by an NFA)

A finite $w \in \Sigma^*$ is recognized by an NFA $(S, S_0, \Sigma, \rightarrow, F)$ if there exists a runs ρ of $\mathcal{S}$ starting from an initial state and ending in a state of F such that $\mathrm{trace}(\rho) = w$.

Language Recognized by an NFA

Definition (Trace)

The trace of a run $\rho = s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} s_n \cdots$ is the possibly infinite word $\text{trace}(\rho) = a_0 a_1 \ldots a_{n-1} \ldots$.

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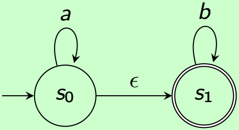
Definition (Language Recognized by an NFA)

The language $\mathcal{L}(\mathcal{S})$ recognized by an NFA $\mathcal{S}$ is the set of words it recognizes.

Finite Automata: Examples

Example



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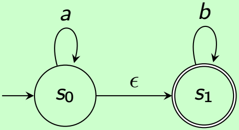

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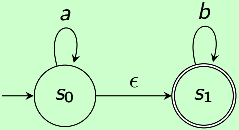

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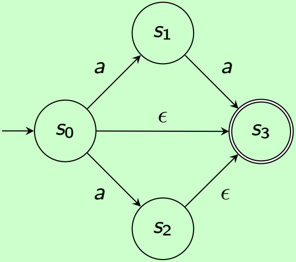

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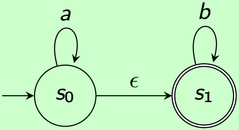

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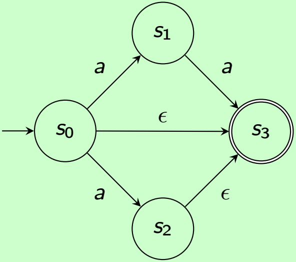

$$\mathcal{L}(\mathcal{A}) = \{\epsilon, a, aa\}$$

Exercises

Exercise

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two finite automata. Show how to build a finite automaton the language of which is:

- The concatenation $\mathcal{L}(\mathcal{A}_1).\mathcal{L}(\mathcal{A}_2)$;
- The union $\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$;
- The Kleene Star $\mathcal{L}(\mathcal{A}_1)^*$;
- The intersection $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$.

Exercise

Prove that the two following problems are equivalent:

- Given an LTS and s one of its states, is s reachable?
- Given a finite automaton, is its language empty?

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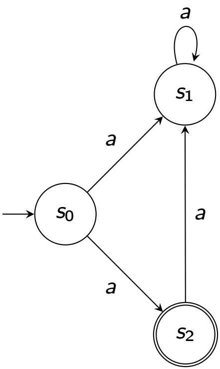
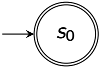
- Complement: invert accepting states and non-accepting states but B should be complete et deterministic.

Complementation: Non-determinism



- Language:
$$\mathcal{L} = \{a\}$$
- Complement:
$$\overline{\mathcal{L}} = \{a^n | n \neq 1\}$$
- Inverting accepting and non-accepting states:
$$\mathcal{L}' = \{a^n | n \geq 0\}$$

Complementation: Incompleteness



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Determinization of an NFA

Theorem

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Computing $\Delta(\mathcal{A})$ is called *determinization* of $\mathcal{A}$.

Determinization of an NFA

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NFA. We define $\Delta(\mathcal{A}) = (Q', \Sigma, \delta', q_0', F')$

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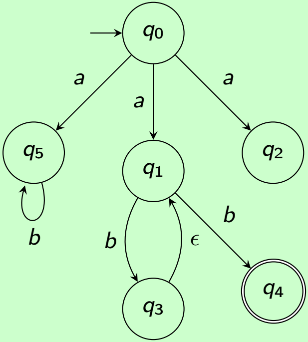
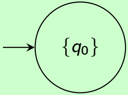
- $\forall q' \in Q', \forall a \in \Sigma, \delta'(q') = \mathcal{F}_\epsilon(\bigcup_{q \in q'} \delta(q, a))$ where $\mathcal{F}_\epsilon$ is the fix point of the function: $S \mapsto S \cup \{q | \exists q', q \in \delta(q', \epsilon)\}$.

Determinization of an NFA: Example

Example





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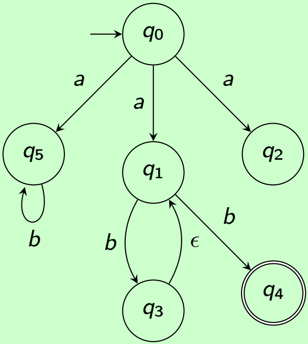

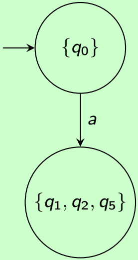

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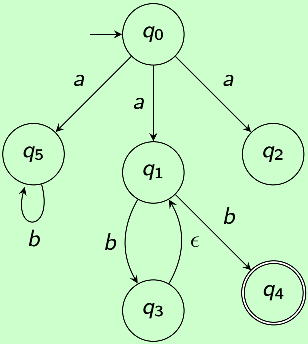

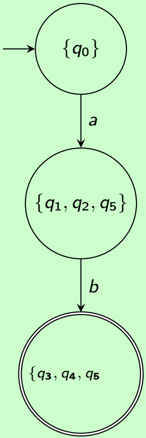

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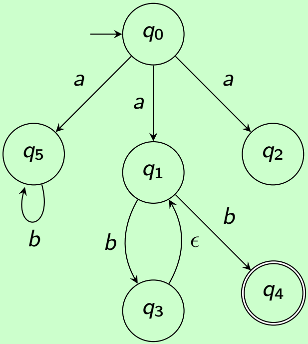

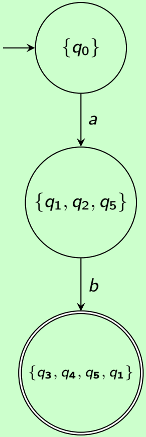

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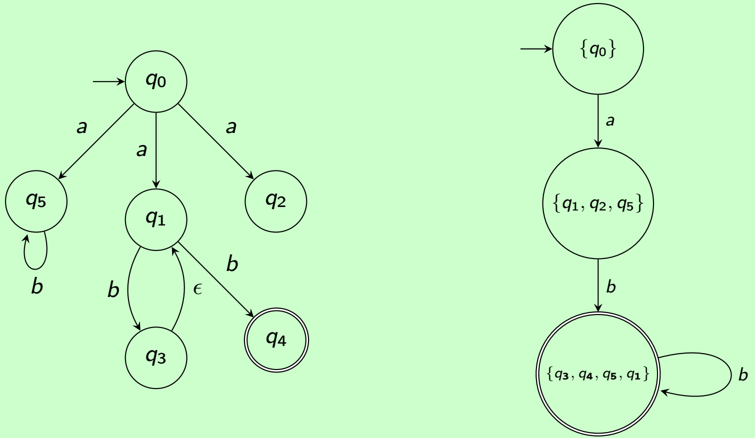
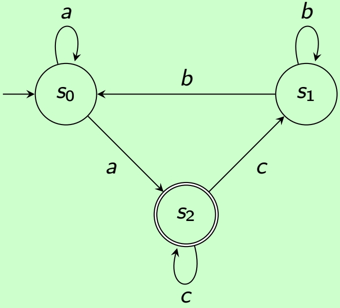
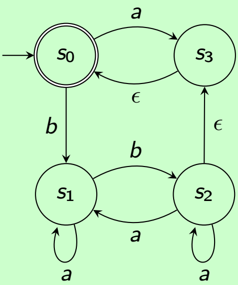


Determinization of an NFA: Exercise

Exercise

Determinize and complete the following NFAs:





Infinite Words and Büchi Automata

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A (non-deterministic) Büchi automaton (BA) is a tuple $(S, S_0, A, \rightarrow, R)$ where:

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- The language of a Büchi automaton can be defined similarly as in the finite case.

Language recognized by a BA

Definition (Word recognized by a BA)

A infinite word $w \in \Sigma^\omega$ is recognized by a BA $(S, S_0, \Sigma, \rightarrow, R)$ if there exists an infinite run ρ of $\mathcal{S}$ starting from an initial state and ending in a state of R.

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The language $\mathcal{L}(\mathcal{S})$ recognized by a BA $\mathcal{S}$ is the set of its recognized words.

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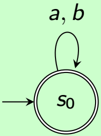

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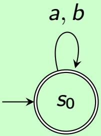

$$\mathcal{L}(\mathcal{A}) = (a|b)^\omega$$

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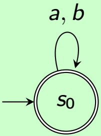

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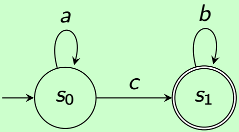

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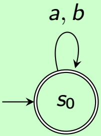

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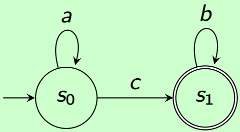

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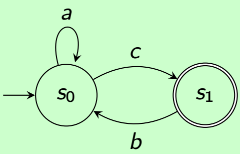

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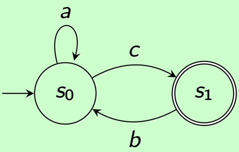

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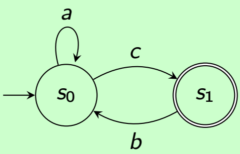

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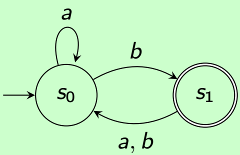

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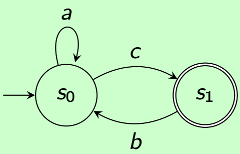

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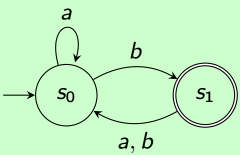

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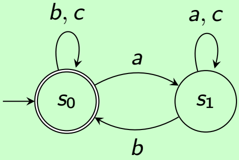

1. Give an intuitive expression for the language of this BA;
2. Give a BA that recognizes the complement of that language.

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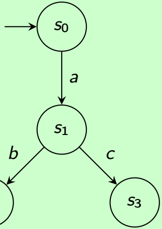
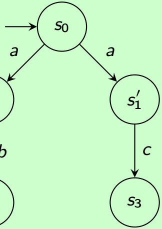
  - Building the complement is *hard* (but possible; we will skip it).

Limits of Trace Equivalence

Example





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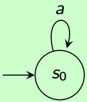

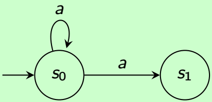

Simulation

Definition (Simulation)

Consider $\mathcal{A} = (S, S_0, A, \rightarrow)$. Let $\mathcal{R} \subseteq (S \times S)$ be a preorder (reflexive and transitive) relation. $\mathcal{R}$ is a simulation if: $\forall (s_1, s_2) \in \mathcal{R}, \forall s_2' \in S$ s.t. $s_2 \xrightarrow{a} s_2', \exists s_1' \in S$ s.t. $s_1 \xrightarrow{a} s_1'$ and $(s_1', s_2') \in \mathcal{R}$.

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This can be straightforwardly extended ot the simulation of an LTS by another.

Theorem

$\mathcal{A}_1$ *simulate* $\mathcal{A}_2 \Rightarrow \mathcal{L}(\mathcal{A}_2) \subseteq \mathcal{L}(\mathcal{A}_1)$.

Simulation

Definition (Simulation)

Consider $\mathcal{A} = (S, S_0, A, \rightarrow)$. Let $\mathcal{R} \subseteq (S \times S)$ be a preorder (reflexive and transitive) relation. $\mathcal{R}$ is a simulation if: $\forall (s_1, s_2) \in \mathcal{R}, \forall s_2' \in S$ s.t. $s_2 \xrightarrow{a} s_2', \exists s_1' \in S$ s.t. $s_1 \xrightarrow{a} s_1'$ and $(s_1', s_2') \in \mathcal{R}$.

We often write $s\mathcal{R}s'$ for $(s, s') \in \mathcal{R}$.
This can be straightforwardly extended ot the simulation of an LTS by another.

Theorem

$\mathcal{A}_1$ *simulate* $\mathcal{A}_2 \Rightarrow \mathcal{L}(\mathcal{A}_2) \subseteq \mathcal{L}(\mathcal{A}_1)$.

what about the other direction?

Abstraction

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Definition (Simulation with respect to an abstraction criterion)

A preorder on $S \times S$, $\mathcal{R}$ is a simulation w.r.t. the abstraction criterion $\alpha \subseteq A^* \times A^*$ if: $\forall (s_1, s_2) \in \mathcal{R}, \forall s_2' \in S$ s.t. $s_2 \stackrel{u}{\Rightarrow} s_2', \exists v \in A^*$ s.t. $(u, v) \in \alpha, \exists s_1' \in S$ s.t. $s_1 \stackrel{v}{\Rightarrow} s_1'$ and $(s_1', s_2') \in \mathcal{R}$.

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- (Strong) Simulation : $\alpha = \{(a, a), a \in A\}$;

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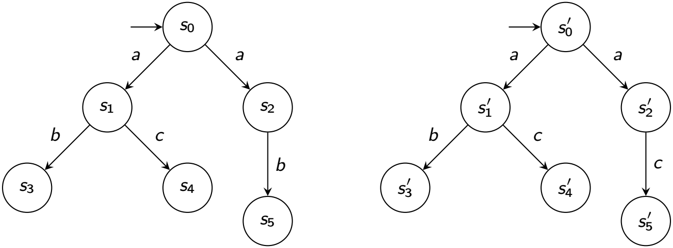
Computing the greatest bisimulation can be done in polynomial time for finite transition systems.

- We can have $\mathcal{S}$ simulates $\mathcal{S}'$ by some relation $\mathcal{R}$, $\mathcal{S}'$ simulates $\mathcal{S}$ by some relation $\mathcal{R}'$ and $\mathcal{R}' \neq \mathcal{R}^{-1}$ (We call this a co-simulation).
  Example ?

Co-simulation



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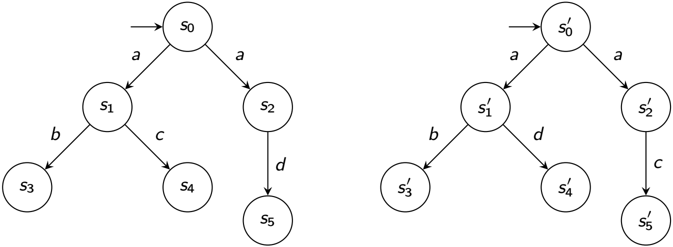

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  7. Termination?

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- Let $F : S_1 \times S_2 \to S_1 \times S_2$ be defined by:

$$F(E) = \{(s_1, s_2) \in E | s_1 \xrightarrow{a} \Leftrightarrow s_2 \xrightarrow{a} \text{ et } s_1 \xrightarrow{a} s_1' \text{ et } s_2 \xrightarrow{a} s_2' \Rightarrow (s_1', s_2') \in E\}$$

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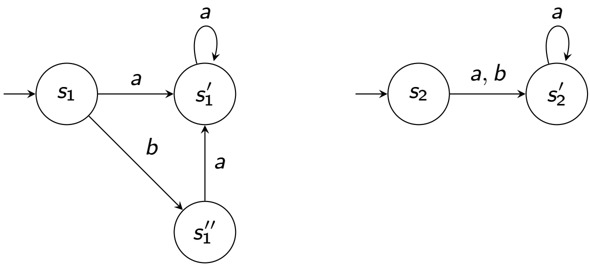
Computing the Greatest Bisimulation: Example



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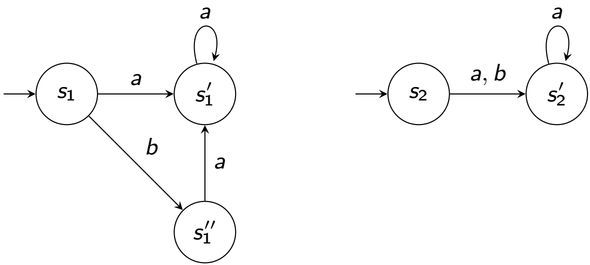

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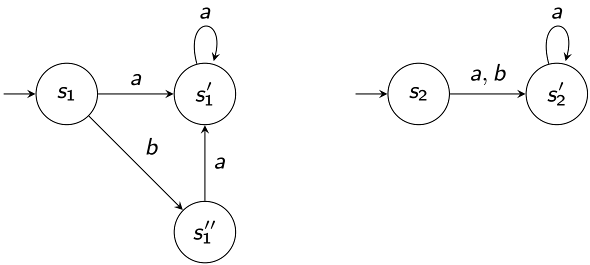

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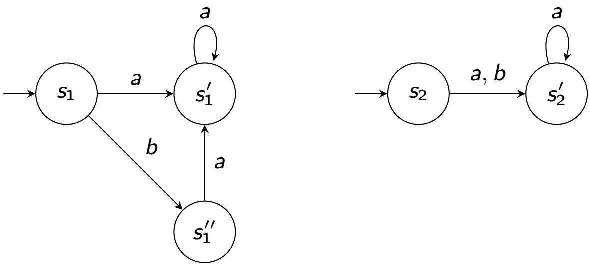

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Quotient of an LTS by an equivalence relation

Definition (Quotient System)

Let $\mathcal{S} = (S, S_0, A, \rightarrow)$ be an LTS and $\approx$ be an equivalence relation on $S \times S$. The quotient of $\mathcal{S}$ by $\approx$ is the LTS $\mathcal{S}/\approx = (\hat{S}, \hat{S}_0, \hat{\rightarrow})$ defined by:

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- Minimization by bisimulation, observation equivalence, etc.

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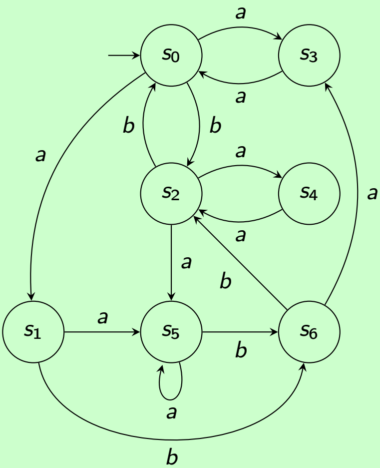
Example



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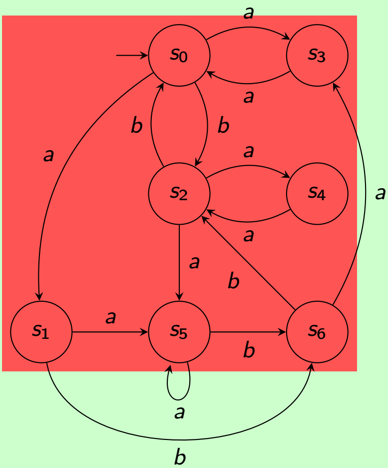
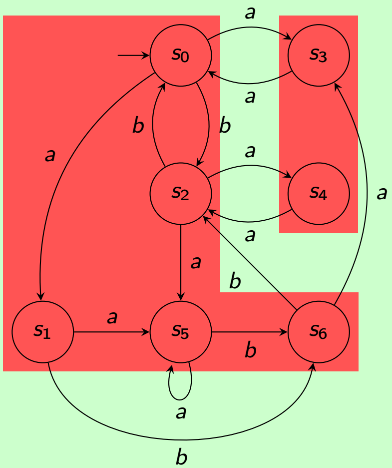


Equivalence classes of $\approx_0$:
$A = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$

Minimization by Bisimulation

Example



$\approx_1$	a	b
s_0	A	A
s_1	A	A
s_2	A	A
s_3	A	$\emptyset$
s_4	A	$\emptyset$
s_5	A	A
s_6	A	A

Equivalence classes of $\approx_1$:
$B = \{s_0, s_1, s_2, s_5, s_6\}$ and
$C = \{s_3, s_4\}$

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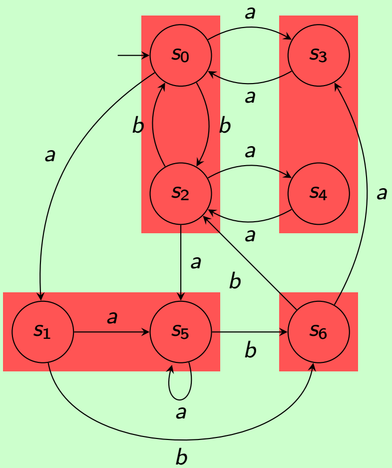
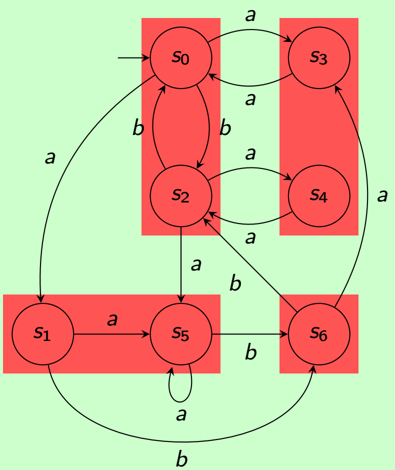


$\approx_2$	a	b
s_0	B, C	B
s_1	B	B
s_2	B, C	B
s_5	B	B
s_6	C	B
s_3	B	$\emptyset$
s_4	B	$\emptyset$

Equivalence classes of $\approx_2$:
$D = \{s_0, s_2\}$, $E = \{s_1, s_5\}$,
$F = \{s_6\}$, $G = \{s_3, s_4\}$

Minimization by Bisimulation

Example

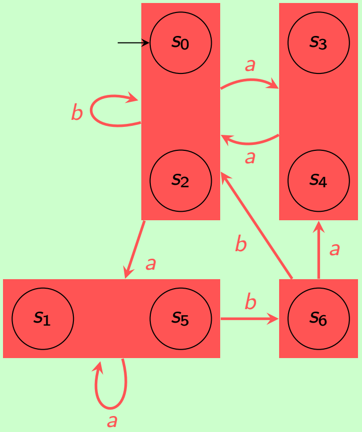


$\approx_3$	a	b
s_0	E, G	D
s_2	E, G	D
s_1	E	F
s_5	E	F
s_6	G	D
s_3	D	$\emptyset$
s_4	D	$\emptyset$

We have $\approx_3 = \approx_2$

Minimization by Bisimulation

Example



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- In the fix point, the initial equivalence classes of $\approx_0$ are F and $S \setminus F$.

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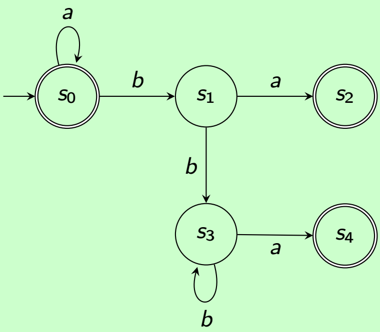

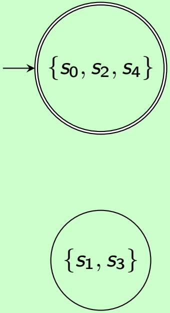

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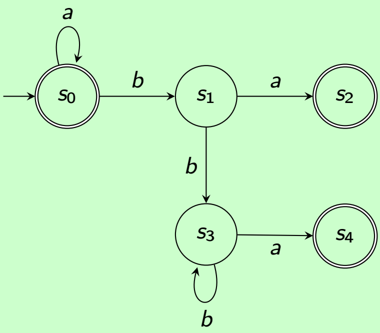

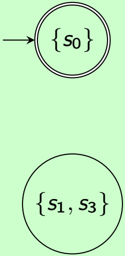

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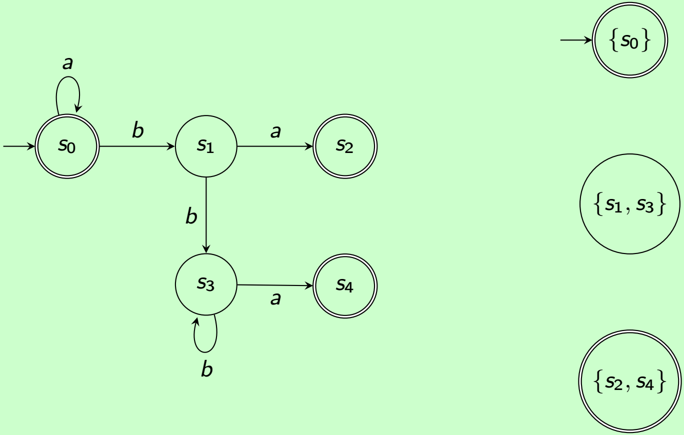

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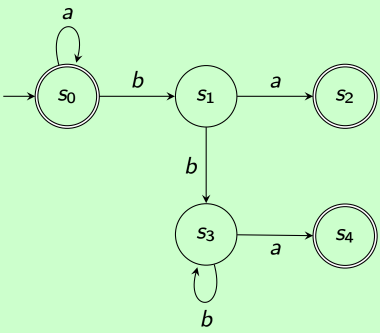

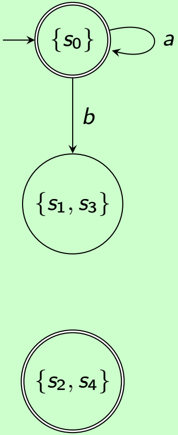

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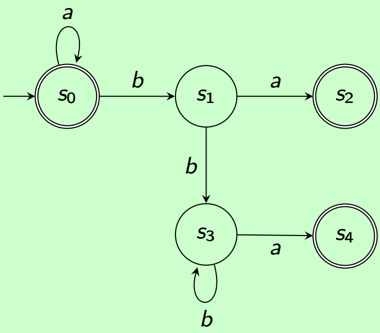

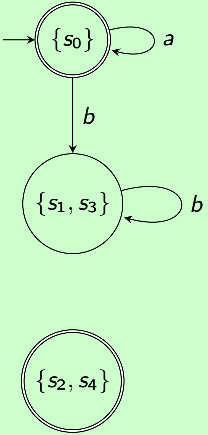

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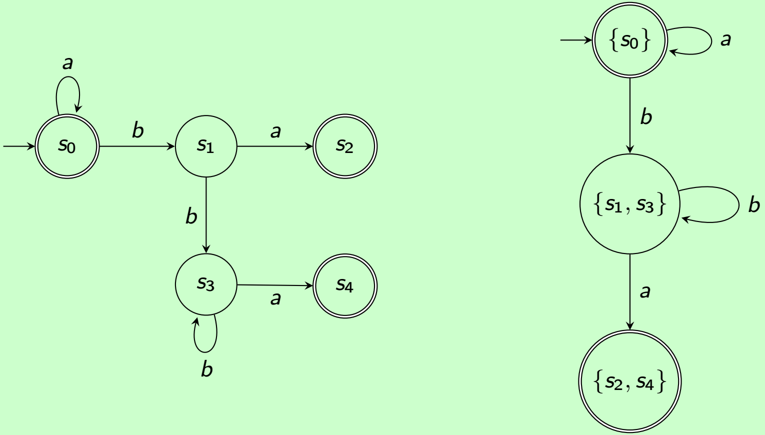
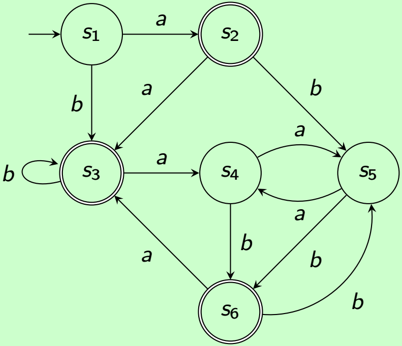


Minimization of an NFA: Exercise

Exercise

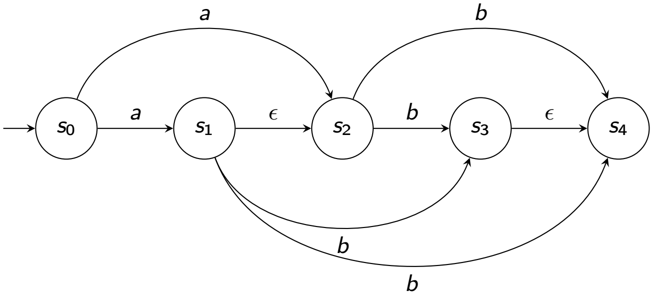
Minimize the following NFA, preserving bisimulation:



Minimizing using Observational Equivalence

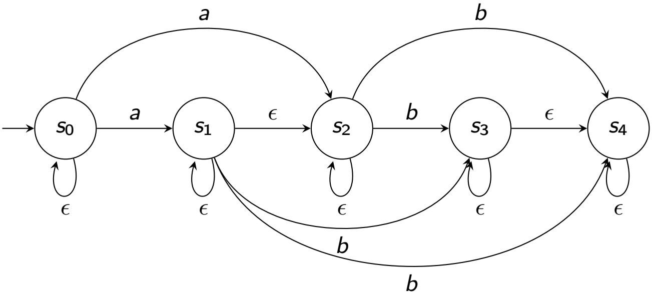
$$\rightarrow s_0 \xrightarrow{a} s_1 \xrightarrow{\epsilon} s_2 \xrightarrow{b} s_3 \xrightarrow{\epsilon} s_4$$

Minimizing using Observational Equivalence



1. ϵ-saturation (transitive closure);

Minimizing using Observational Equivalence



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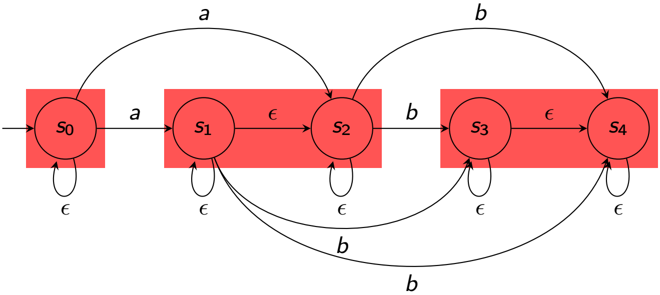

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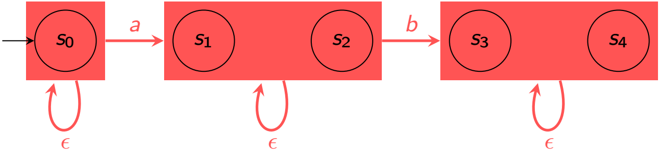

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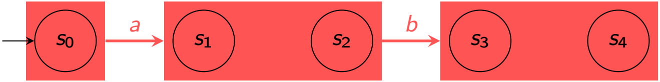

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Minimizing using Observational Equivalence: Exercise

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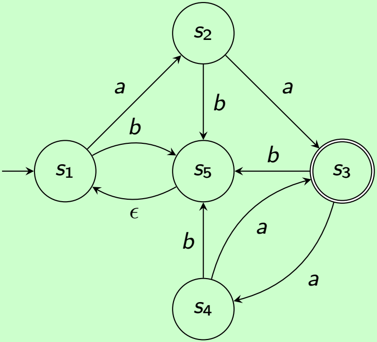
Minimize the following NFA using observational equivalence (weak bisimulation):



Plan I

Introduction

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- Each requirement can then be check on the model of the system;
- For an automatic and sound procedure we need fomalized requirements:

Model-checking

Let $\mathcal{S}$ be a model of the system and let φ be a (temporal) logic formula

$$\mathcal{S} \models \varphi?$$

Classic Logics

- Propositionnal logic, under Backus-Naur Form (BNF)):

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- Second-order logic, monadic second-order logic, etc.

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  - LTL: properties on runs;
  - CTL: properties on execution trees.

Kripke Structures

Definition (Kripke Structure)

A Kripke Structure is a tuple $(AP, W, \rightarrow, \ell)$ where:

- AP is a set of atomic propositions;
- W is a non-empty set of states;
- $\rightarrow \subseteq W \times W$ is a (left-total) relation called transition relation;
- $\ell : W \rightarrow 2^{\mathrm{AP}}$ is a labeling (or interpretation) function.

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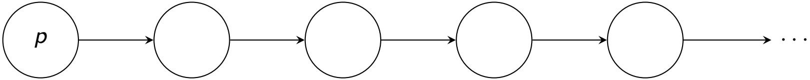
- Time modalities: Until **U** and neXt **X**.

Intuitive Semantics of LTL

p :



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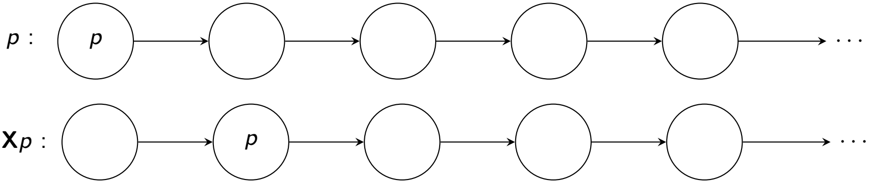

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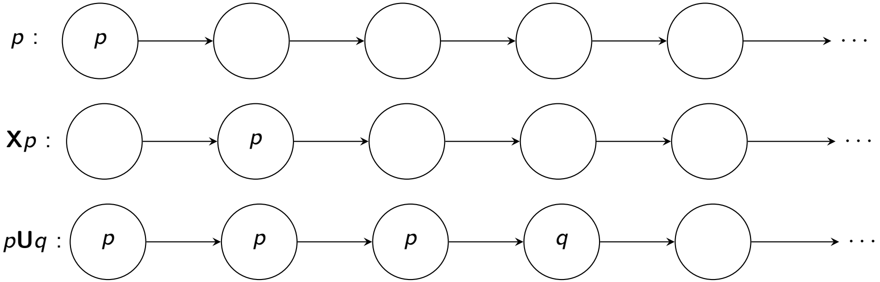

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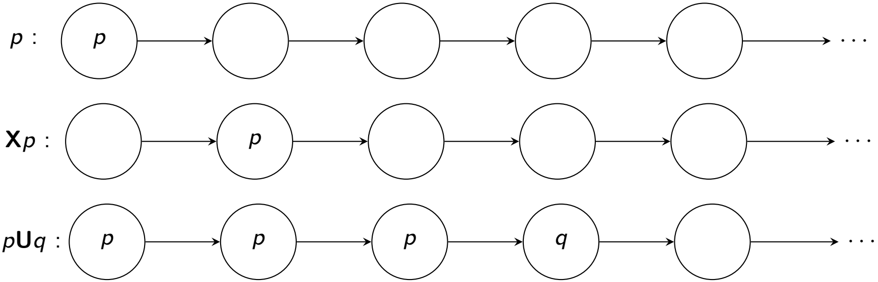

Exercise

Give an example run statisfying $p\mathbf{U}\neg(\mathtt{true}\mathbf{U}\neg q)$.

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We consider the infinite traces (wrt. atomic propositions) of a Kripke structure:

$$a_0 a_1 a_2 \cdots \text{ with } \forall i, a_i \in 2^{\mathsf{AP}}$$

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- $a_0 a_1 a_2 \cdots \models \varphi \mathbf{U} \psi$ iff $\exists u \geq 0$ s.t. $a_u a_{u+1} \cdots \models \psi$ and $\forall v$ s.t. $0 \leq v < u, a_v a_{v+1} \cdots \models \varphi$.

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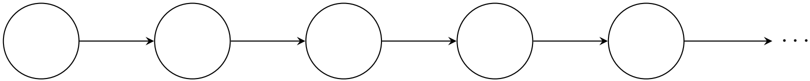
Exercise

Release is the dual of Until: $\varphi\mathbf{R}\psi = \neg(\neg\varphi\mathbf{U}\neg\psi)$. Express $\mathbf{R}$ in function of $\mathbf{U}$ and $\mathbf{G}$ without any negation.

Examples

$\mathbf{F}p$:



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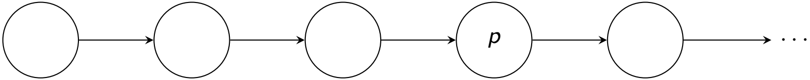

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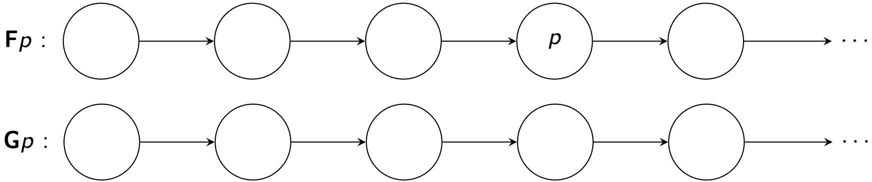

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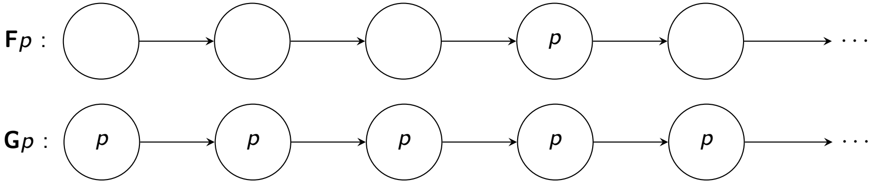

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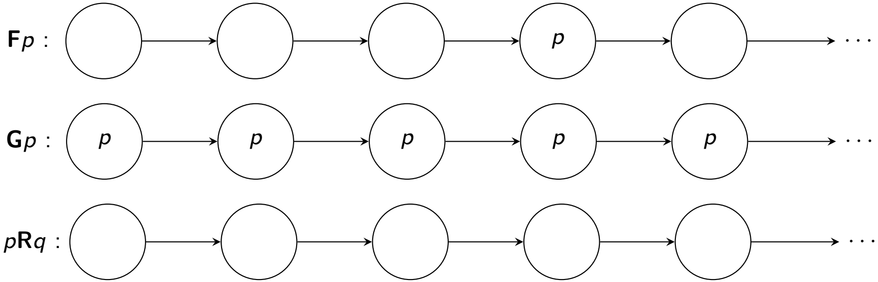

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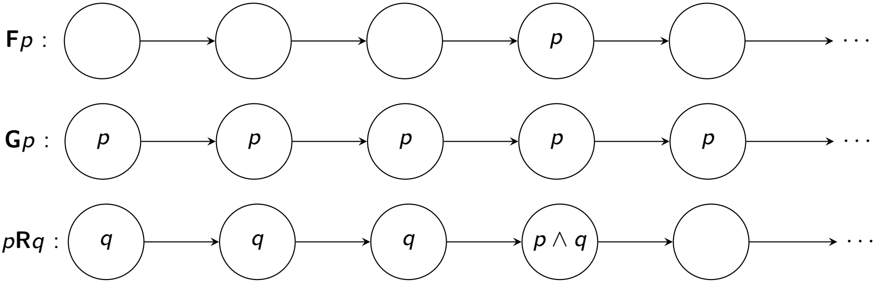

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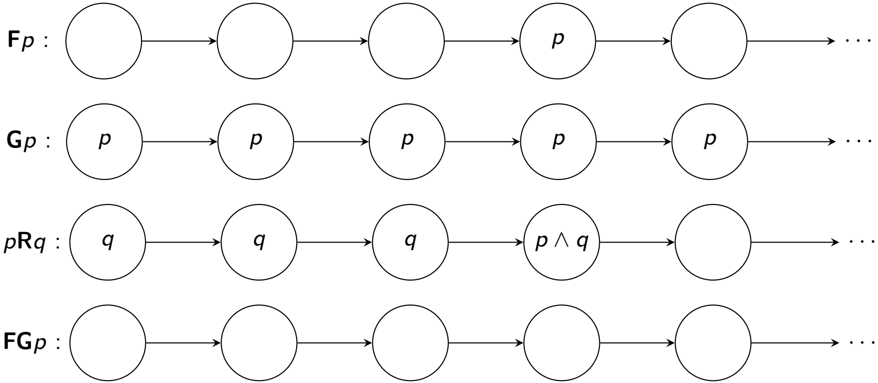

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p
· · ·
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p
p
p
p
p
· · ·
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q
q
$p \wedge q$
· · ·
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· · ·

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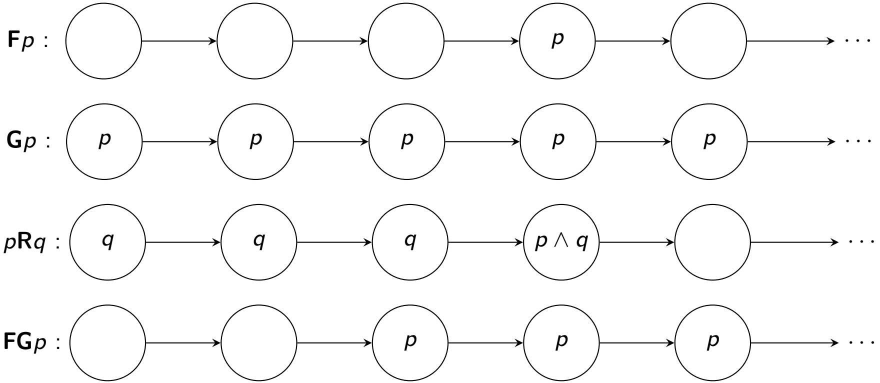

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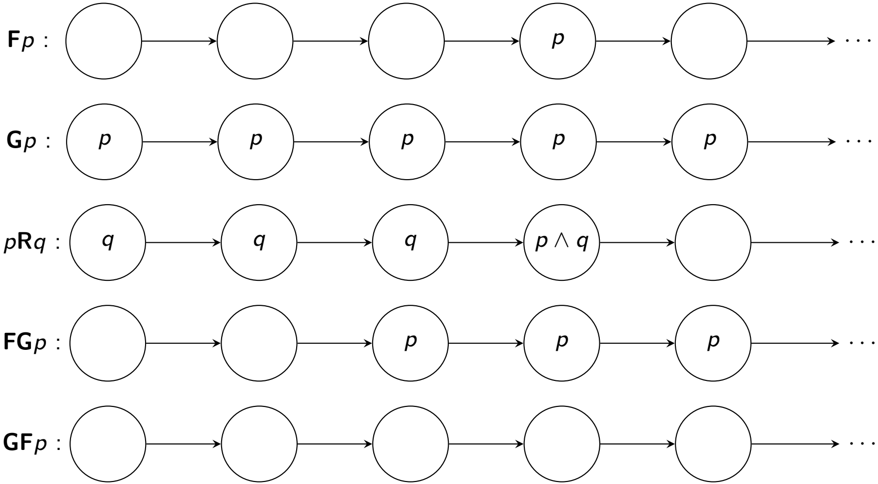

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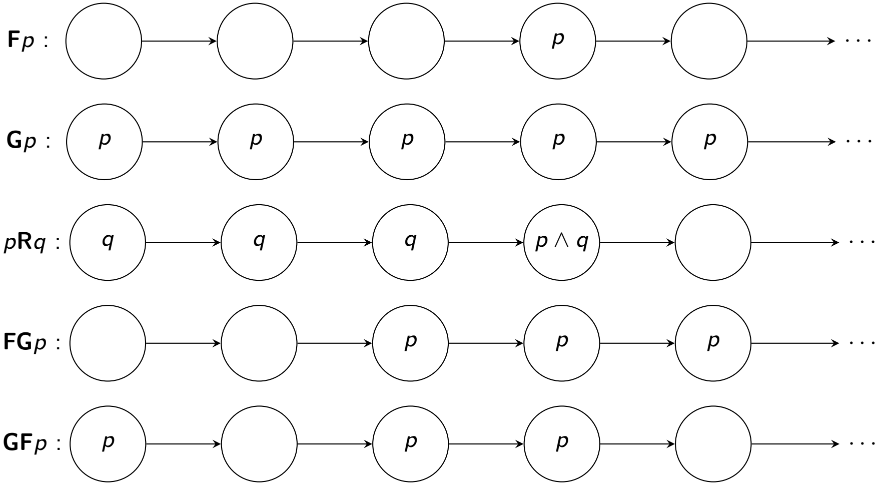

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LTL Properties: Exercise

Exercise

Consider a plane transporting passengers between Nantes and Amsterdam. It continuously executes the following cycle: the plane is empty in Nantes; wait 1h for next departure (might be skipped or repeated); passengers embark; the plane flies to Amsterdam; passengers disembark; wait 1h for next departure (might be skipped or repeated); new passengers embark; the plane flies back to Nantes; passenger disembark.

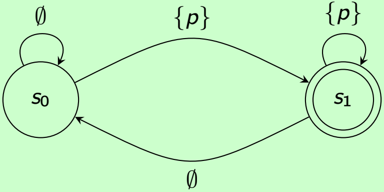
Model this problem as a Kripke Structure and write LTL formulas for the following properties and assess their truth on the model:

1. The plane will eventually be empty in Amsterdam;
2. The plane is always infinitely often in Nantes;
3. If it never waits indefinitely, the plane is always infinitely often in Nantes;
4. Whenever the plane is in Amsterdam, it is next full in Nantes;
5. Whenever the plane is full, it will be empty and then full again (assuming fairness);
6. Whenever the plane is full in Nantes and ready to fly out, it cannot be empty in Nantes before being first empty in Amsterdam.

LTL and Büchi Automata

Example

$\mathbf{GF}p$



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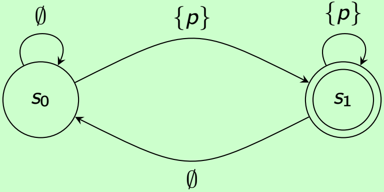

Example

G$(p \Rightarrow$ **F**$q)$

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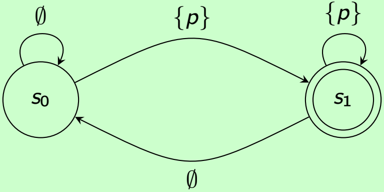

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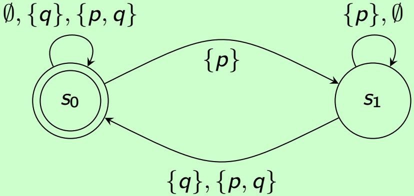

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- And finally:

$$\mathcal{S} \models \varphi \Leftrightarrow \mathcal{L}(\mathcal{S} \times A_{\neg\varphi}) = \emptyset$$

Büchi Automaton of an LTL formula

Definition (Fermeture)

The closure $Cl(\varphi)$ of a formula φ is the smallest set of LTL formulas that is closed under the following rules:

- $\varphi \in Cl(\varphi)$;
- if $\psi \in Cl(\varphi)$ then $\neg\psi \in Cl(\varphi)$ (recall that $\neg\neg\psi = \psi$);
- if $\psi_1 \wedge \psi_2 \in Cl(\varphi)$ then $\psi_1 \in Cl(\varphi)$ and $\psi_2 \in Cl(\varphi)$;
- if $\psi_1 \vee \psi_2 \in Cl(\varphi)$ then $\psi_1 \in Cl(\varphi)$ and $\psi_2 \in Cl(\varphi)$;
- if $\mathbf{X}\psi \in Cl(\varphi)$ then $\psi \in Cl(\varphi)$;
- if $\psi_1 \mathbf{U} \psi_2 \in Cl(\varphi)$ then $\psi_1 \in Cl(\varphi)$, $\psi_2 \in Cl(\varphi)$ and $\mathbf{X}(\psi_1 \mathbf{U} \psi_2) \in Cl(\varphi)$.

Example

The closure of $\varphi = (p \vee q)\mathbf{U}(\mathbf{X}\mathit{true})$ is:

$$\{\varphi, \neg\varphi, (p \vee q), \neg(p \vee q), \mathbf{X}\mathtt{true}, \neg(\mathbf{X}\mathtt{true}), p, \neg p, q, \neg q, \mathtt{true}, \mathtt{false}\}$$

Büchi Automaton of an LTL formula

Definition (Maximally Consistent Subset)

A subset C of $Cl(\varphi)$ is maximally consistent if:

1. C is consistent:
   - $\texttt{true} \in Cl(\varphi) \Rightarrow \texttt{true} \in C$;
   - $\forall \psi \in Cl(\varphi), \psi \in C$ iff $\neg\psi \notin C$;
   - $\forall \psi = \psi_1 \wedge \psi_2 \in Cl(\varphi), \psi \in C$ iff $\psi_1 \in C$ $\psi_2 \in C$;
   - $\forall \psi = \psi_1 \vee \psi_2 \in Cl(\varphi), \psi \in C$ iff $\psi_1 \in C$ or $\psi_2 \in C$;
   - $\forall \psi = \psi_1 \mathbf{U} \psi_2 \in Cl(\varphi)$:
     - if $\psi_2 \in C$ then $\psi \in C$;
     - if $\psi \in C$ and $\psi_2 \notin C$ then $\psi_1 \in C$.
2. C is maximal: $\forall \psi \in Cl(\varphi)$, either ψ or $\neg\psi$ belongs to C.

Example

The following sets are maximally consistent for $\varphi = (p \vee q)\mathbf{U}(\mathbf{X} true)$:

1. $\{\varphi, p \vee q, \neg(\mathbf{X}\texttt{true}), \neg p, q, \texttt{true}\}$,
2. $\{\varphi, \neg(p \vee q), \mathbf{X}\texttt{true}, \neg p, \neg q, \texttt{true}\}$,
3. $\{\neg\varphi, p \vee q, \neg(\mathbf{X}\texttt{true}), p, q, \texttt{true}\}$,
4. ...

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Theorem

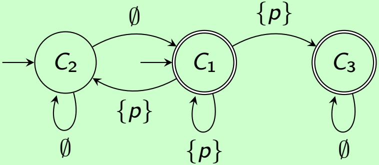
- *$\mathcal{A}_\varphi$ recognizes exactly the sequences satisfying φ ;*
- *$\mathcal{A}_\varphi$ has at most $2^{4|\varphi|}$ states.*

Exercises

Example

We build the automaton for $\varphi = \mathbf{Fp} = \texttt{true}\mathbf{U}p$, with $\mathsf{AP} = \{p\}$.

- $Cl(\varphi) = \{\varphi, \neg\varphi, p, \neg p, \texttt{true}, \texttt{false}\}$;
- The maximally consistent subsets are:
  - $C_1 = \{\varphi, p, \texttt{true}\}$;
  - $C_2 = \{\varphi, \neg p, \texttt{true}\}$;
  - $C_3 = \{\neg\varphi, \neg p, \texttt{true}\}$;
- The automaton:



Exercises

Exercise

Build a Büchi automaton for each of the following formula:

1. $\mathbf{Gp}$;
2. $\mathbf{pUXq}$;
3. $\mathbf{G}(\mathbf{p} \Rightarrow \mathbf{Xq})$.

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Let $\mathcal{S} = (W, \rightarrow, \ell)$ be a Kripke structure, $t \in W$ and $\mathcal{S}$ (so $c_0 = t$):

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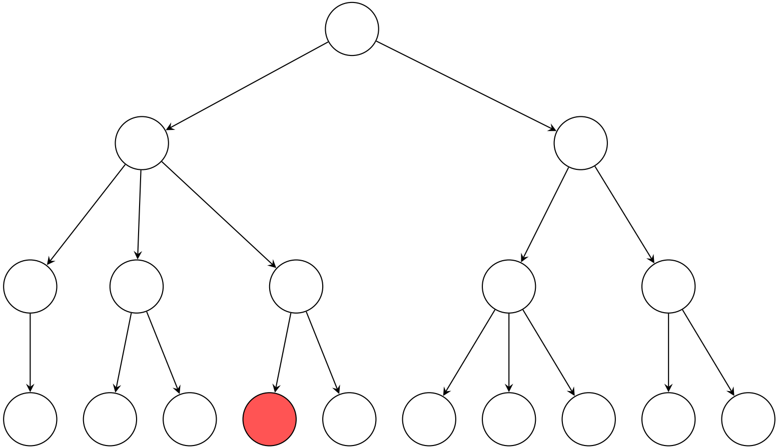

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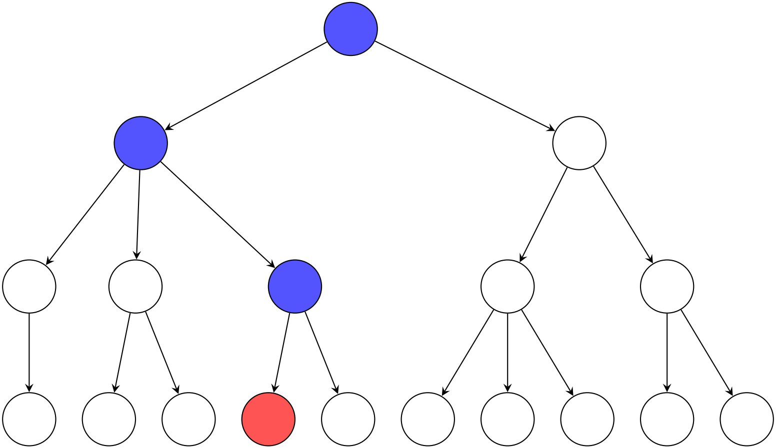

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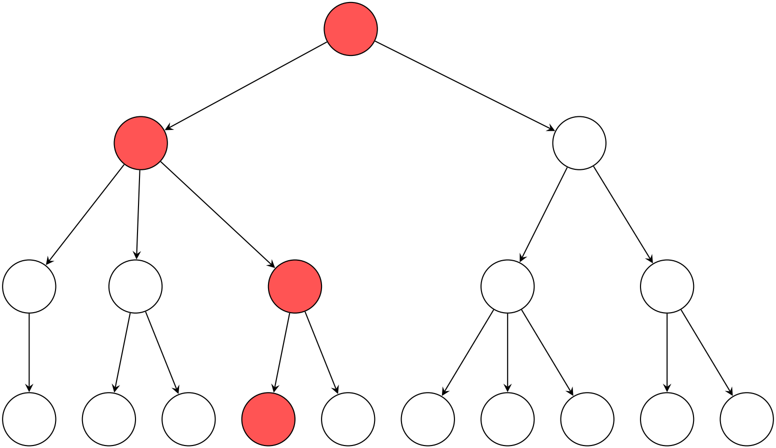

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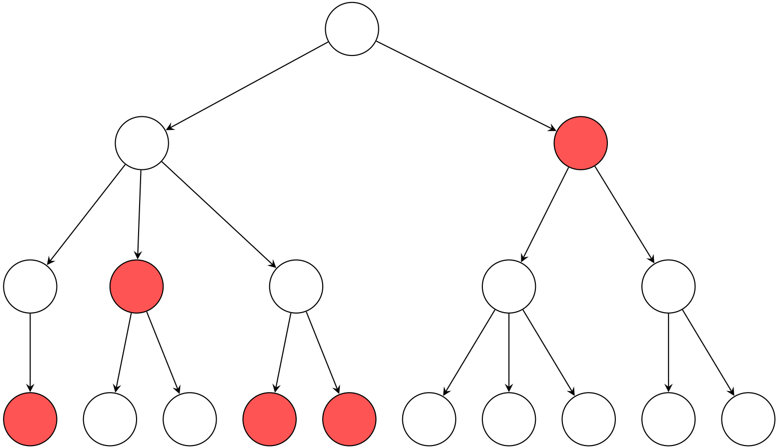

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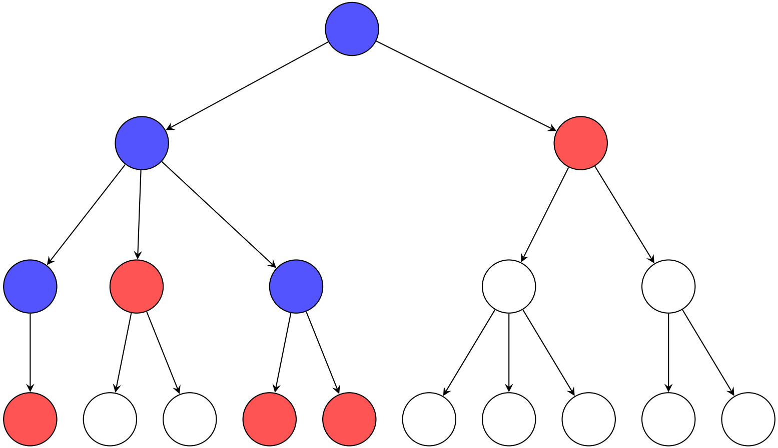

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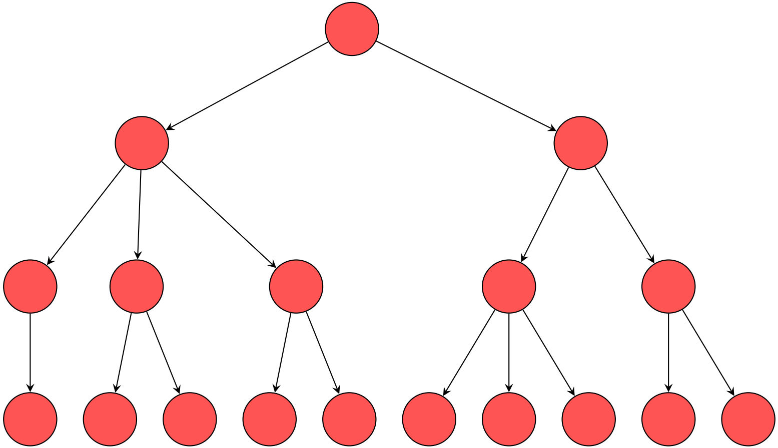
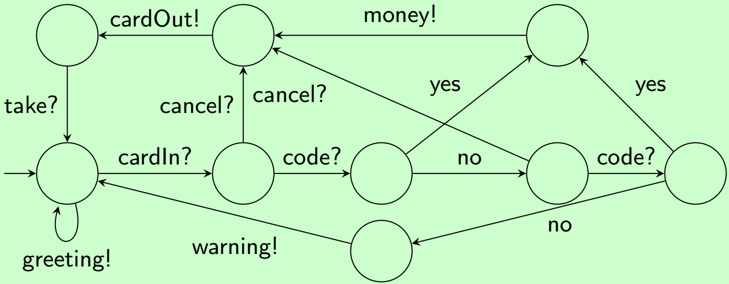


AGφ

CTL: Exercise

Exercise

Consider the following (labeled) Kripke Structure representing a simplified ATM:



1. Label the states with the following properties:
   - *i* the ATM is available for a new transaction;
   - *c* the card has been inserted in the ATM (and not withdrawn);
   - *b* the card has been ejected and is ready to take;
   - *e* exactly one error with the code has occurred;
   - *x* two errors with the code have occurred.

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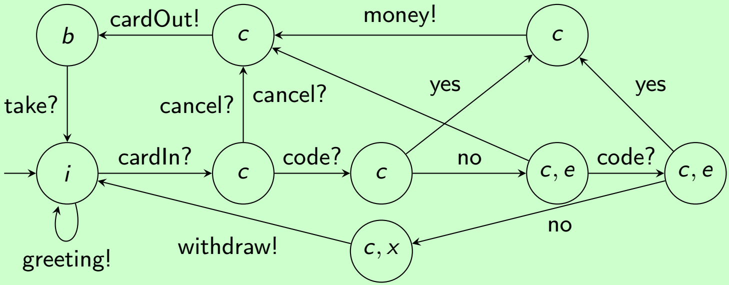

2. Write formulas for the following properties and assess their truth value:
   - There is no deadlock;
   - A new session is always eventually started;
   - While less than two errors have been, the card can still be gotten back;
   - Whenever a session is started, if less than two errors are made, we can always take the card back;
   - When two errors have been made, no card can be taken back until a new one is inserted.

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- It is based on symbolic set operations, using fix points.

Symbolic Set Evaluation: Propositional Logic

- States satisfying a given atomic proposition p: $[\![p]\!] = \{s | p \in \ell(s)\}$;
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- States satisfying $\neg\varphi$: $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$.

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  - $[\![\mathbf{AX}\varphi]\!] =$;

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  - $\widetilde{\mathrm{Pred}}(X) = \{s \in W | \forall s' \in W, s \longrightarrow s' \Rightarrow s' \in X'\}$;
  - Duality: $\widetilde{\mathrm{Pred}}(X) = W \setminus \mathrm{Pred}(W \setminus X)$;
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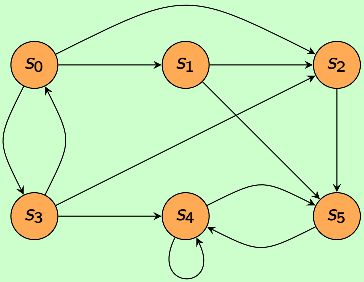
Symbolic Set Evaluation: Example

Consider $\varphi = \mathbf{AF}s_5$. Does $(\mathcal{S}, s_0) \models \varphi$?

Example



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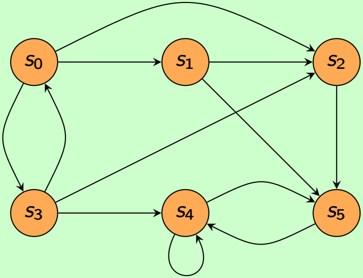

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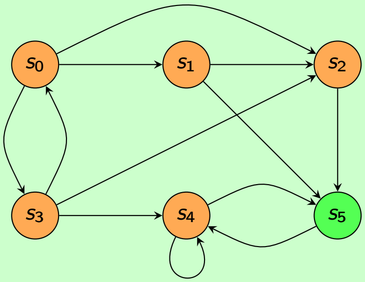

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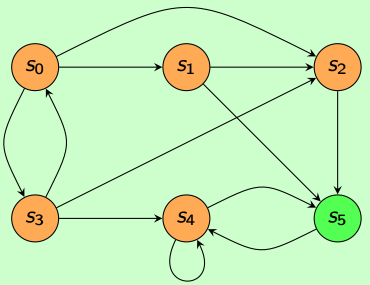

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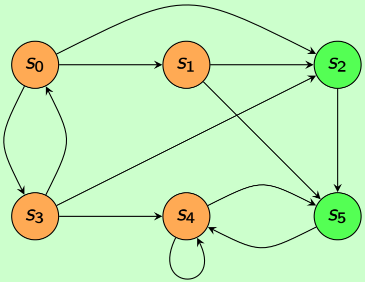

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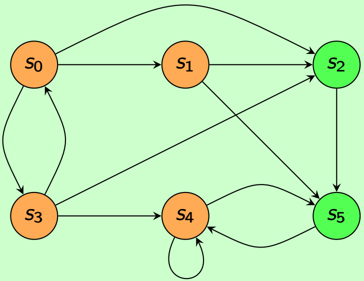

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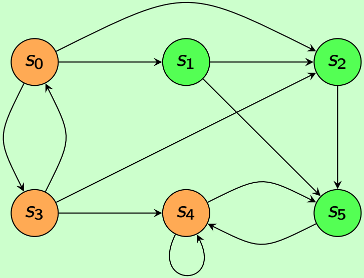

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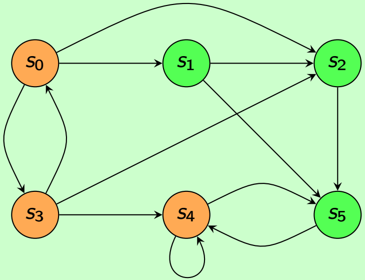

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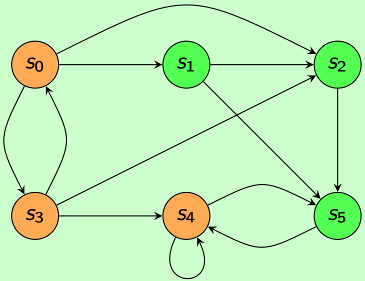

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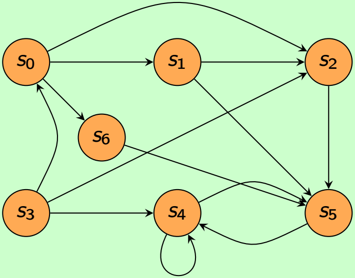

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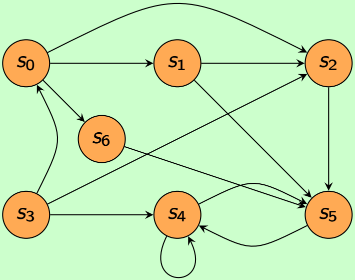

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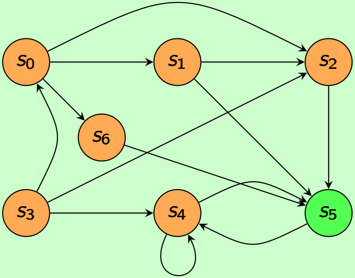

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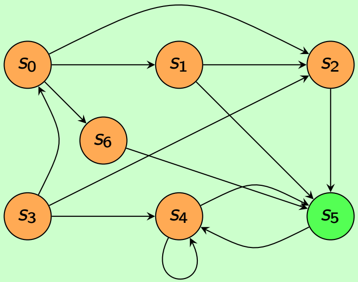

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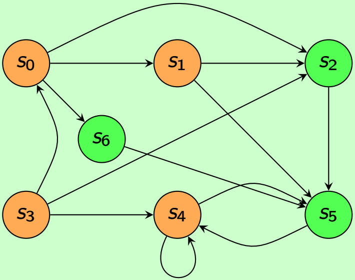

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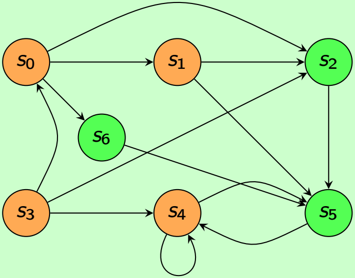

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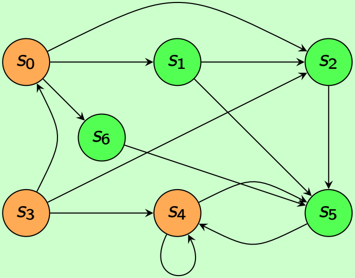

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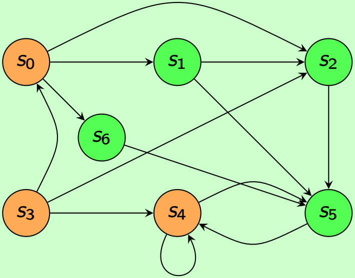

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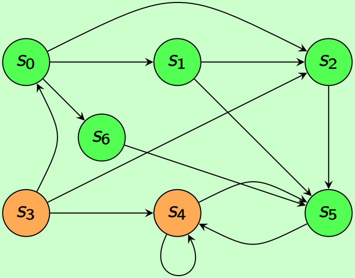

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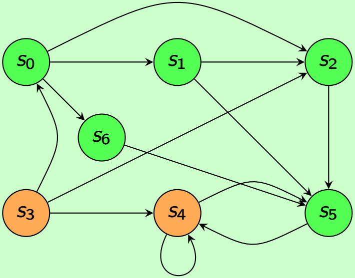

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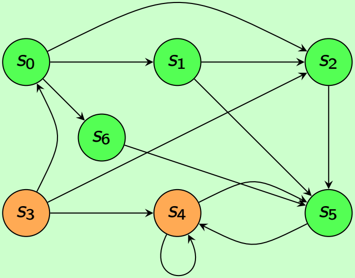

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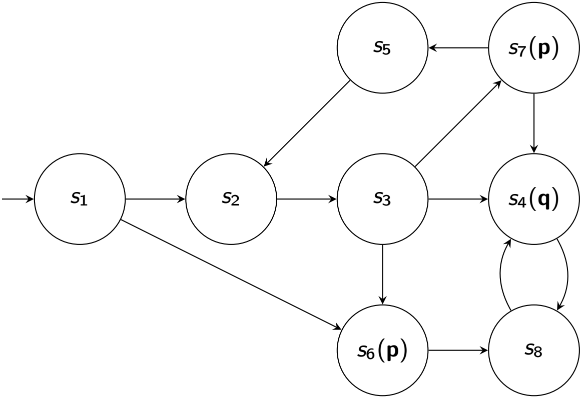
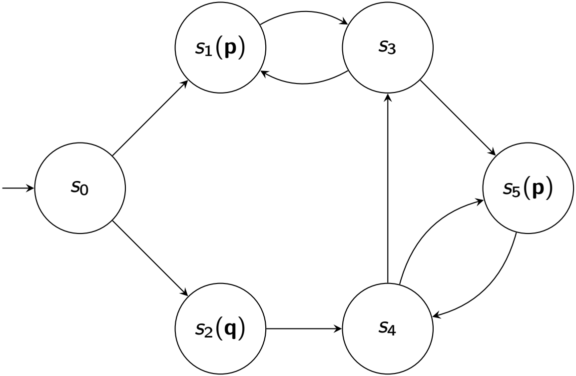
Symbolic Set Evaluation: Exercise



1. Does the Kripke Structure satisfy the CTL property **AF q**? Give details of the computation.
2. Does the Kripke Structure satisfy the CTL property **EF (p** $\wedge$ **AF q)**? Give details of the computation.

Symbolic Set Evaluation: Exercise

Check the CTL property $\phi = \mathbf{AF\ AG}(\mathbf{p} \vee \neg\mathbf{q})$ on the following Kripke structure:



LTL and CTL

Exercise

1. Exhibit two Kripke structures (KS) such that:
   - there exists a CTL formula that distinguishes them (true for one KS and false for the other one);
   - there is no LTL formula that distinguishes them.
2. Exhibit a CTL formula with no equivalent in LTL;
   Two formulas are equivalent if they are true exactly for the same KS
3. Exhibit an LTL formula with no equivalent in CTL.

Plan I

Control

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- We model the control problem in the framework of the theory of games on graphs.

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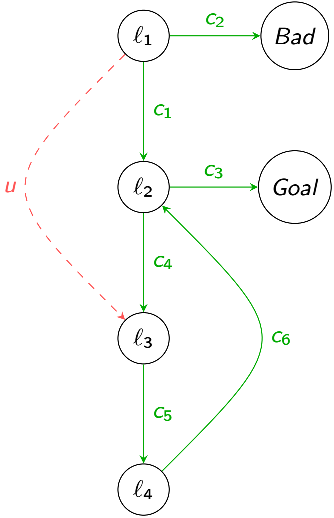
  - others, e.g. ω-regular objectives on Büchi game automata, etc.

Example



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Let $\mathcal{A} = (Q, A, \delta, Q_0)$ be a game automaton. A strategy f on $\mathcal{A}$ is a partial function of $[\![\mathcal{A}]\!]$ in A_c. If $f : Q \to A_c$ then f is said to be memoryless/positional.

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- An infinite run belongs to $Outcome(q, f)$ if all its finite prefixes do.

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Definition

A run is maximal within some set of runs R if it is infinite or cannot be extended within R by a controllable action. We simply say that it is maximal if $R = [\![\mathcal{A}]\!]$.

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- The reachability objective for *Goal* contains exactly the maximal runs ρ s.t.: $States(\rho) \cap Goal \neq \emptyset$ and there is always a possible controllable transition in all states before *Goal*.

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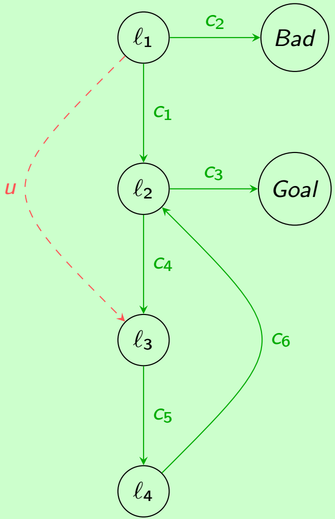

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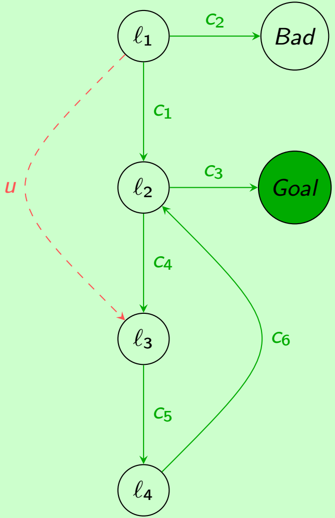

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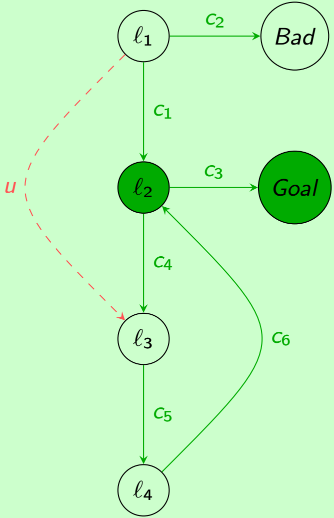

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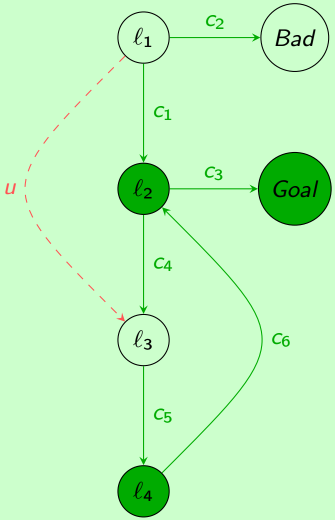

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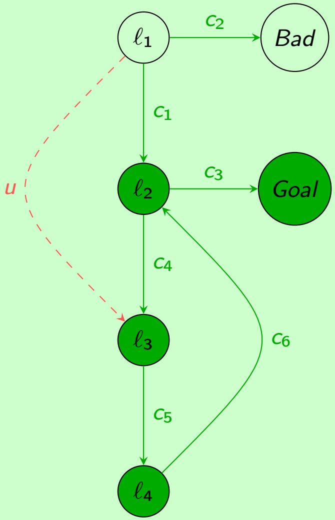

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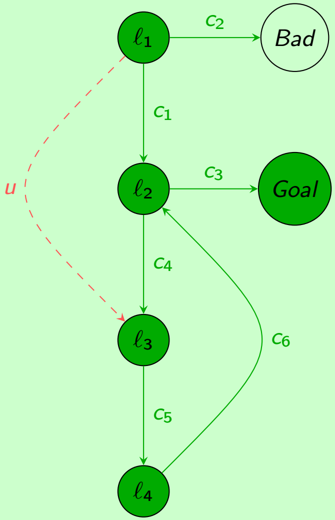

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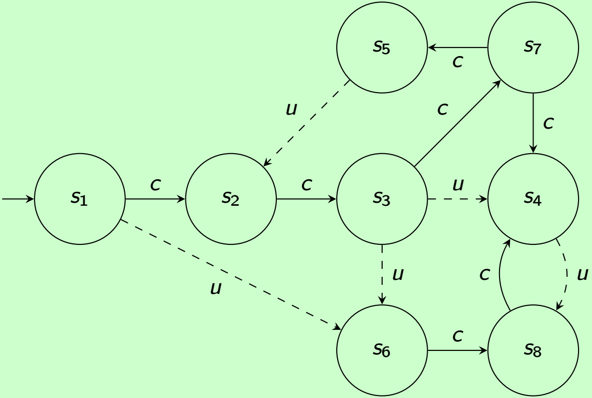

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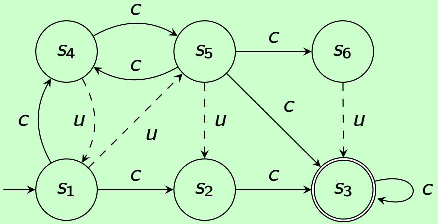

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1. Prove that checking the CTL property EF can be done by solving a reachability game;
2. Prove the same for AF.

Exercise

Given a set of safe states G, the safety game consists in forcing the system to stay in G.

1. define the corresponding objective in terms of set of runs;
2. show that the winning states are not the complement of those for which the environment has a strategy to enforce the reachability of $\overline{G}$;
3. propose an algorithm to compute the winning states (and a winning strategy) in a safety game;
4. how can we define a notion of "most permissive strategy"?

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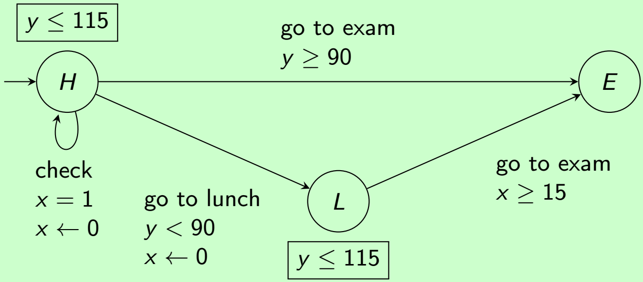
Plan I

Timed Automata: Example

Example

It is noon, you are at home (H), and you want to go out for a lunch (L) with a friend before the FMOV exam (E) at 2pm. You have to wait for your friend to be ready and however, so you check every minute.



Do all paths lead to E in less than 120 minutes?

Timed Automata

Definition (Timed Automata [AD94, HNSY94])

A timed automaton is a tuple (L, l_0, X, A, E, Inv) where:

- L is a finite set of locations;
- l_0 is the initial location;
- X is a finite set of clocks with non-negative real values;
- A is a finite set of actions;
- $E \subset L \times \mathcal{C}(X) \times A \times 2^X \times L$ is a finite set of edges. Let $e = (l, \delta, \alpha, R, l') \in E$. Edge e links location l to location l', with guard δ, action α, set of clocks to reset to zero R.
- $Inv \in \mathcal{C}(X)^L$ assigns an *invariant* to each location.

$\mathcal{C}(X)$ is the set of conjunctions of simple constraints on X: $x \sim c$ with $x \in X, c \in \mathbb{Q}$ and $\{\sim \in <, \leq, \geq, >\}$

Semantics of Timed Automata

- The possible actions in a TA are defined by the current location and the value of all clocks;
- We call this a concrete state of a TA;
- The behavior of a TA is defined as a timed transition system, called its behavioral semantics.

Timed Transition Systems

Definition (Timed Transition System)

A Timed Transition System (TTS) is a tuple $(S, A, s_0, \rightarrow)$ where:

- S is a set of states;
- A is a set of actions;
- $s_0 \in S$ is the initial state;
- $\rightarrow$ is the transition relation, decomposed into:
  - continuous/time transitions: $\xrightarrow{d \in \mathbb{R}^+}$,
  - discrete transitions: $\xrightarrow{a \in A}$.

For $d \in \mathbb{R}^+$, $\xrightarrow{d}$ is the action consisting in letting d time units elapse.

Semantics of Timed Automata

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The (concrete/behavioral) semantics of a timed automaton $\mathcal{A}$ is the TTS $\mathcal{S}_{\mathcal{A}} = (Q, A \cup \{d\}_{d \in \mathbb{R}^+}, Q_0, \rightarrow)$ where:

- $Q = \{(\ell, \nu) \in L \times (\mathbb{R}^+)^X | \nu \models Inv(\ell)\}$,
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- $\rightarrow \in Q \times (A \cup \{d\}_{d \in \mathbb{R}^+}) \times Q$ is defined for $a \in A$ and $d \in \mathbb{R}^+$ by:
  - $(l, \nu) \xrightarrow{a} (\ell', \nu')$ iff $\exists(\ell, \delta, a, R, \ell') \in E$ s.t.:

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$$\begin{cases} \nu' = \nu + d, \\ \forall d' \in [0, d], Inv(l)(\nu + d') = \mathtt{true} \end{cases}$$

Runs in Timed Transition Systems

Definition (Run)

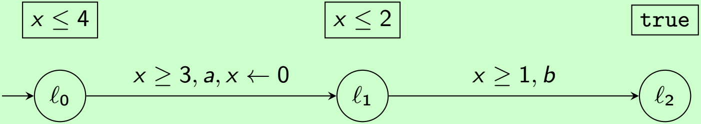
A run σ from s in a TTS $(S, A, s_0, \rightarrow)$ is a sequence $(s_i)_{i \geq 1}$ s.t.:
$s_1 \xrightarrow{d_1} s_2 \xrightarrow{a_1} s_3 \xrightarrow{d_2} s_4 \xrightarrow{a_2} \cdots$ with $\forall i, a_i \in A, d_i \in \mathbb{R}^+$ and $s_1 = s$. We note
$\sigma = s_1 \xrightarrow{(d_1, a_1)} s_3 \xrightarrow{(d_2, a_2)} \cdots$.

Definition (Duration, divergence)

Let $\sigma = s_1 \xrightarrow{(d_1, a_1)} s_3 \xrightarrow{(d_2, a_2)} \cdots$ be a run in a TTS. La duration of σ is $d(\sigma) = \sum_i d_i$. If $d(\sigma) = \infty$, we say that σ is divergent.

Timed Automata: Another Example

Example



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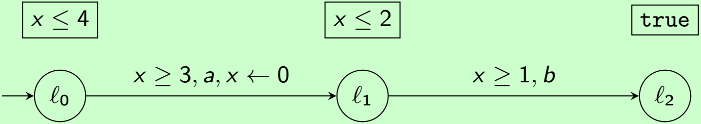


$$\rho = (\ell_0, 0) \xrightarrow{(3.2, a)} (\ell_1, 0) \xrightarrow{(1.6, b)} (\ell_2, 1.6)$$

Timed Automata: Exercise

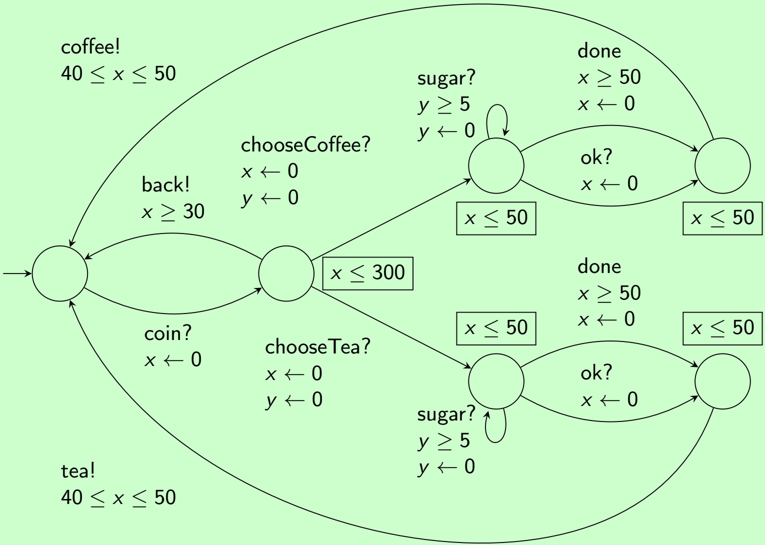
Exercise

Consider a beverage vending machine that serves coffee and tea. First you insert one coin, then you get to choose either coffee or tea. If no choice is made within 30 seconds, the session is aborted and the coin is given back. When a beverage has been selected the number of sugar doses must be chosen by repeatedly pressing a button. To avoid unwanted presses of the button, consecutive presses are only taken into account if they are separated by at least 0.5s. After 5s, or whenever the OK button is pressed, the machine delivers the beverage, which takes between 4s and 5s, and gets back to the initial state.

Model this system using a timed automaton.

Timed Automata: Exercise

Exercise



Plan I

Languages of Timed Automata

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The trace of a run $\rho = (\ell_0, \nu_0) \xrightarrow{(a_0, d_0)} \ell_1, \nu_1) \xrightarrow{(a_1, d_1)} \cdots$ is the (timed) word $(a_0, d_0)(a_1, d_0 + d_1)(a_2, d_0 + d_1 + d_2) \cdots$

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  Just check:

$$\mathcal{A}_{\text{universel}} \subseteq \mathcal{L}$$

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Corollary (Reachability)

Reachability of a concrete state is PSPACE-complete in TA.

- This is proved (and done in practice) by building finite abstractions of the state-space: region and zone graphs.

Other Properties

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Theorem (Complement)

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Theorem (Determinisation)

- *There might be no deterministic TA that accepts the timed language of a given TA.*
- *Knowing whether there exists a deterministic TA that accepts the same timed language as a TA is undecidable.*

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Regions

- We need an equivalence relation $\equiv$ that respects the timing constraints:

$$\nu \equiv \nu' \Rightarrow \begin{cases} \nu \models g \Leftrightarrow \nu' \models g \\ \forall t \in \mathbb{R}^+, \exists t' \in \mathbb{R}^+ \text{ s.t. } \nu + t \equiv \nu' + t' \end{cases}$$

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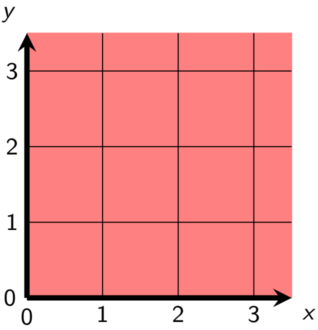

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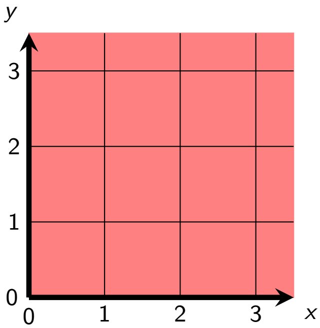
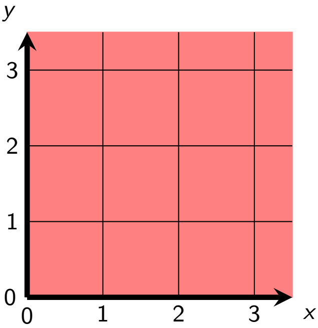


What about strict constraints?

Regions



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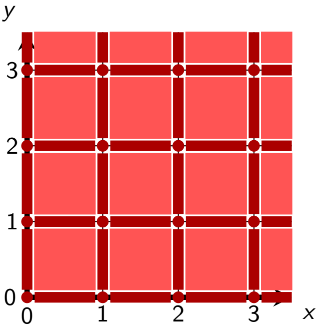

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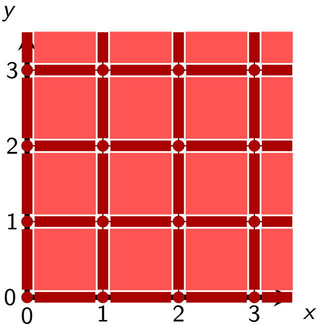

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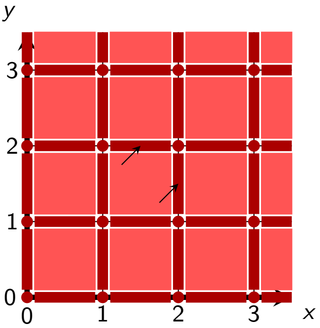

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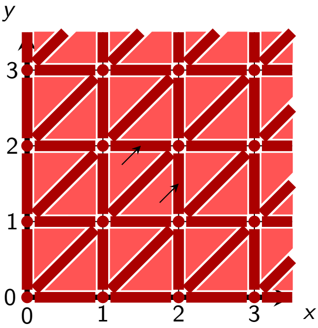

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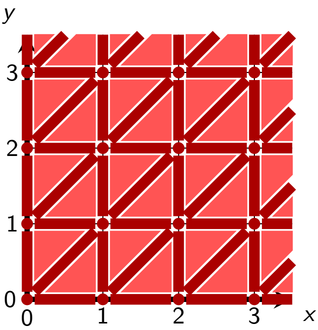

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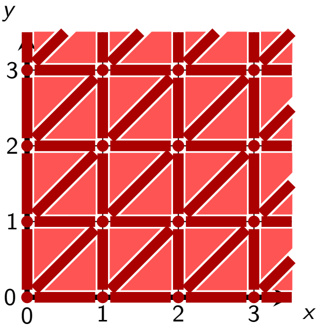

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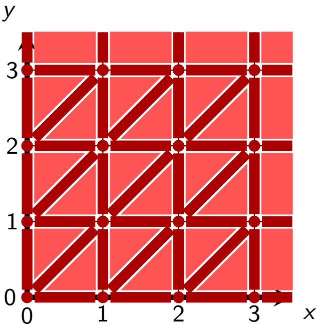

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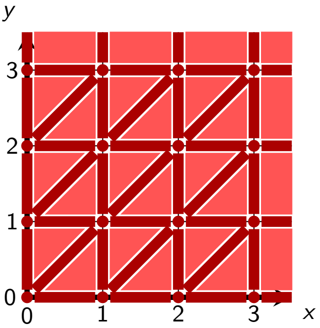

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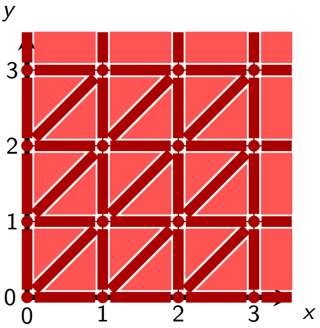

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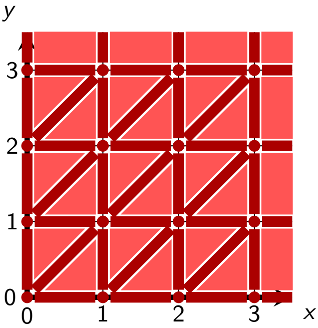

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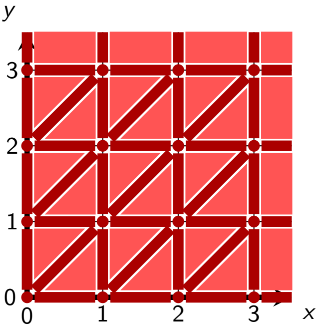
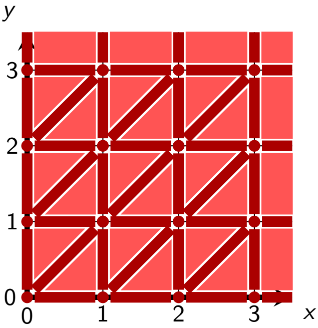


$$\nu \equiv \nu' \Leftrightarrow \begin{cases} E(\nu(x)) = E(\nu'(x)) \\ \quad \text{or } (\nu(x) > max \text{ and } \nu'(x) > max) \\ \text{if } (\nu(x) \leq max \text{ and } \nu'(y) \leq max) \text{ then} \\ \quad \text{frac}(\nu(x)) < \text{frac}(\nu(y)) \Leftrightarrow \\ \quad \text{frac}(\nu'(x)) < \text{frac}(\nu'(y)) \end{cases}$$

Region Automaton



- Let $Succ(R)$ be the set of successor regions of R by time elapsing:
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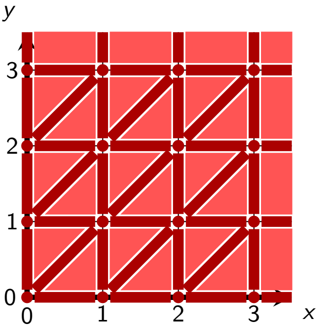

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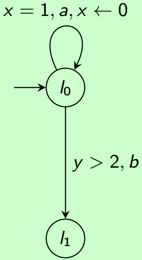
- The region automaton $\mathcal{A} = (Q, A, q_0, \rightarrow)$ is defined by:
  - $Q = L \times \mathbb{R}^+ / \equiv$;
  - $q_0 = (l_0, \mathbf{0})$;
  - $\rightarrow = \{(q, R) \xrightarrow{a} (q', R') | \exists q \xrightarrow{\delta,\alpha,r} q' \in E$ and $\exists R'' \in Succ(R)$ s.t. $R'' \subseteq \delta$ and $R' = R''[r \leftarrow 0]\}$.

Region Automaton

Example



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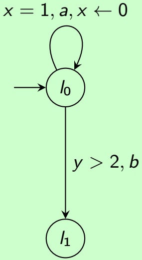

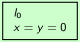

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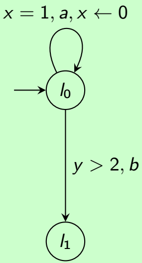

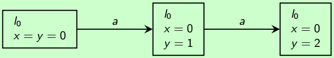

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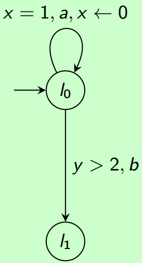

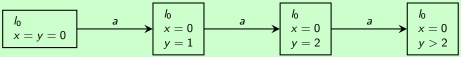

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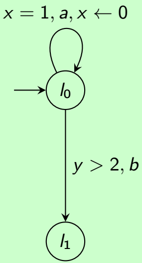

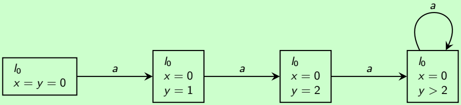

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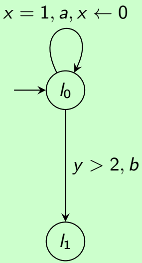

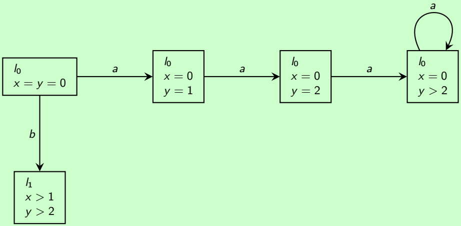

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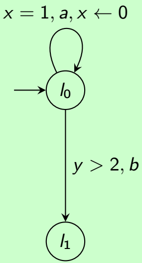

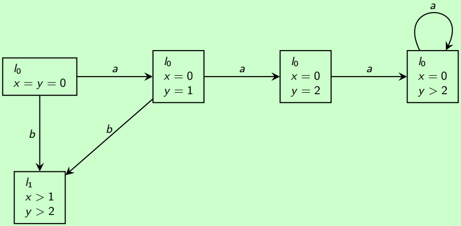

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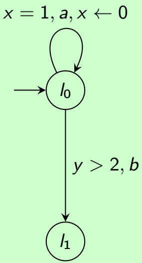

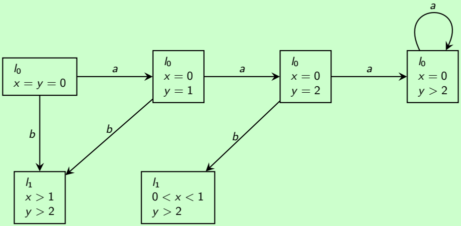

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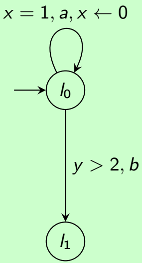

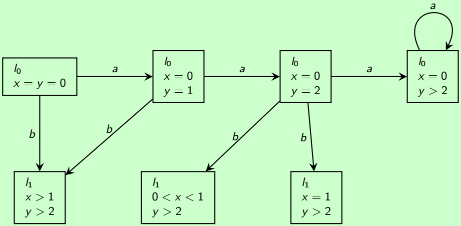

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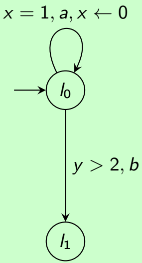

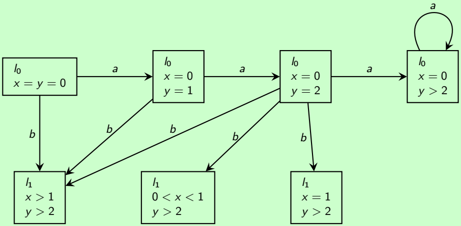

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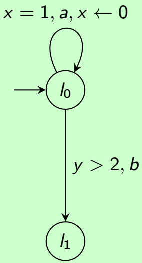

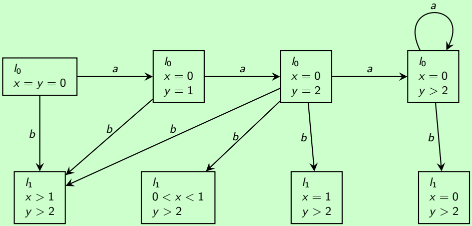

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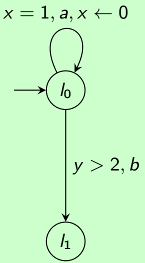

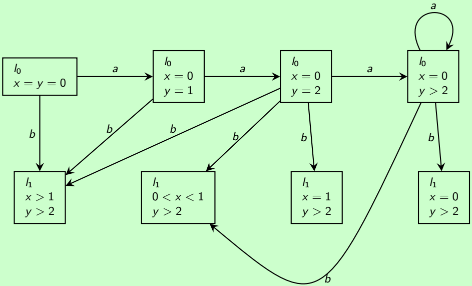

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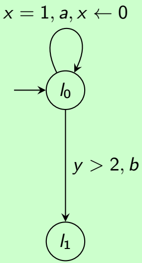

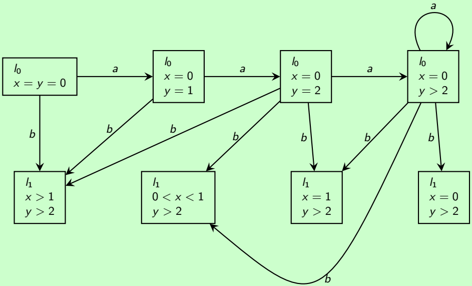

Region Automaton

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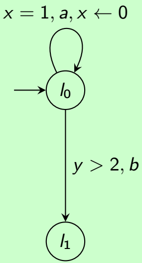

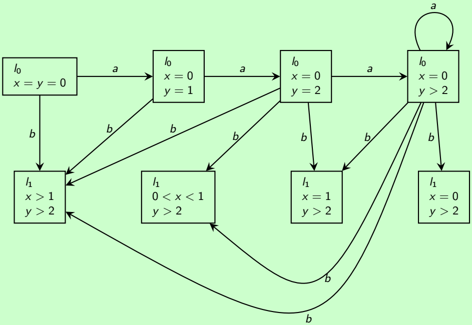

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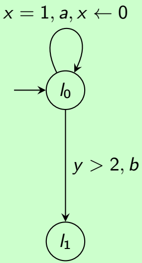

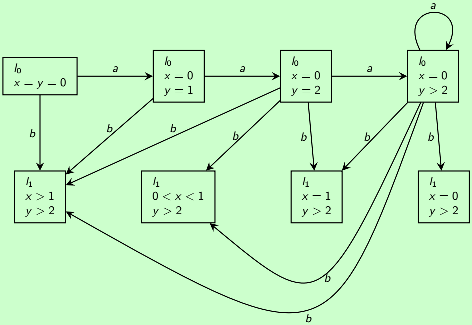

Exercise

Draw the transitions in the clock space.

Region Automaton

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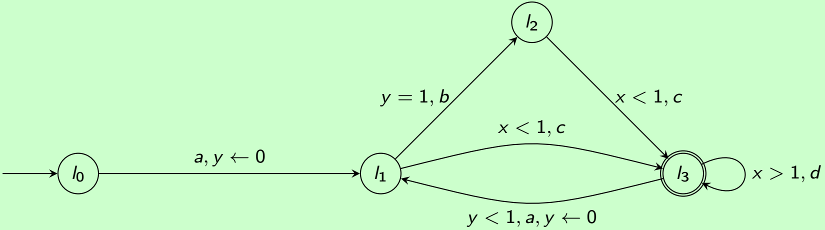
We assume **only for this exercise** that we can take a transition only after having waited for a positive duration (> 0).

Compute the region graph of the following TA from [AD94]:



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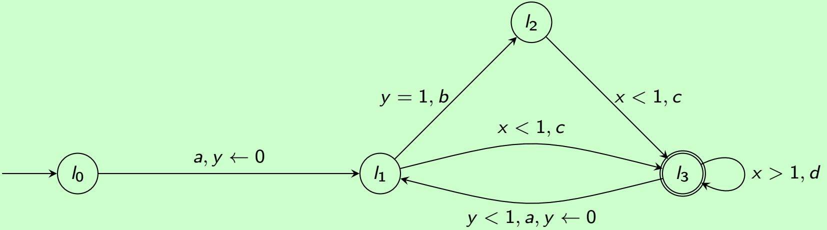


Remark: This automaton can do cycle without letting time elapse.

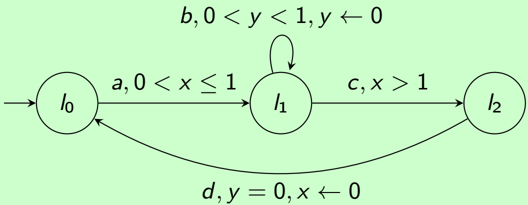
Definition (Zeno Run)

An infinite run is Zeno if its duration is finite.

Region Automaton

Exercise

Compute the region automaton of the following TA:

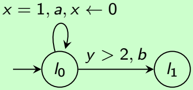


Model-checking

- The region automaton preserves time-abstract bisimulation
  bisimulation were dad' is equivalent to a whenever d and d' are delays
- We can then use it to verify CTL properties on TA:
  Ignoring sequences of delays without any action

Exercise

1. Does the following TA (from the example for the region automaton) verify $\mathbf{AF} l_1$?



2. What about $\mathbf{EF}(l_0 \text{ and } x = 1)$?

Plan I

Timed Computation Tree Logic

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Syntax of TCTL

$$\varphi ::= \mathbf{p} \,|\, x \sim k \,|\, \neg\varphi \,|\, \varphi \vee \varphi \,|\, z \,\mathbf{in}\, \varphi \,|\, \mathbf{A}\varphi\mathbf{U}_I\varphi \,|\, \mathbf{E}\varphi\mathbf{U}\varphi$$

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- $x \in X$, $k \in \mathbb{N}$, and $\sim \in \{<, \leq, =, \geq, >\}$ is a constraint on the clocks of the system.

Semantics of TCTL

Sémantics of TCTL

Let $\mathcal{S} = (W, \rightarrow, \ell)$ be a timed Kripke structure, $s = (l, \nu) \in W$ and $c = (c_i)_{i \in \mathbb{N}} \in Path(t)A$, where $Path(t)$ is the set of paths starting at s in $\mathcal{S}$ (thus $c_0 = t$). Let $\Delta(c, i)$ be the duration of $c_0 \longrightarrow \cdots \longrightarrow c_i$

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Timed Until

We often use shorthands for intervals in the Until operator:

- $\mathbf{U}_{<2}$ is $\mathbf{U}_{[0,2]}$;
- $\mathbf{U}_{\geq 2}$ is $\mathbf{U}_{[2,+\infty)}$;
- $\mathbf{U}_{=2}$ is $\mathbf{U}_{[2,2]}$;
- etc.

Timed Properties: Examples

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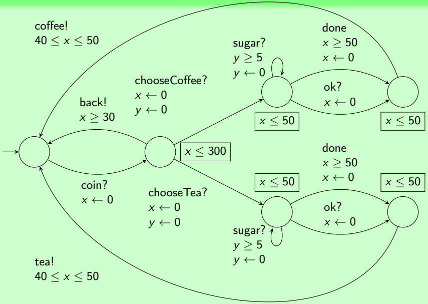
Timed Properties: Exercise

Exercise



1. Add atomic propositions: i no session started, c coffee choosed, t tea choosed, C coffee served, T tea served (add some locations for the last two);

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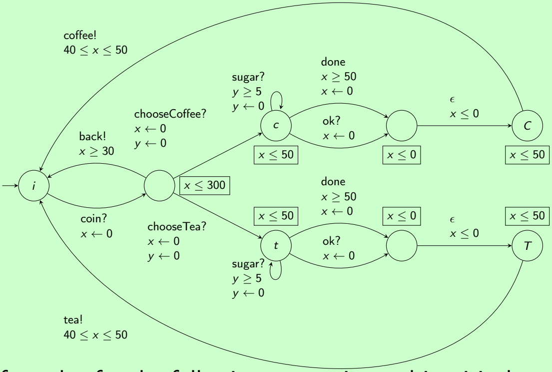
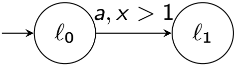
Exercise

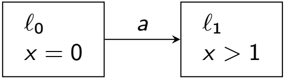


2. Write TCTL formulas for the following properties and intuitively assess their truth:
   - When a vending session has started, a beverage is always obtained in less than 40s;
   - When coffee has been chosen, tea cannot be obtained before the session is ended;
   - When coffee has been chosen, tea cannot be obtained before at least 5s;
   - It is possible to never have any coffee or tea;
   - It is possible to never have any coffe or tea but still start sessions infinitely often.

TCTL Model-checking with Untimed Until

- If all until operators have interval $[0, inf)$ we can reduce the verification to CTL;
- We cannot use the previous region construction though, because it does not distinguish the values of clocks when time elapses:





Is $\mathbf{EF}(x = 1)$ satisfied?

- We build a variant: a region automaton with delays.

Region Automaton With Delays

- We make explicit the passing from one region to another by delaying [ACD93]:

Region Automaton With Delays

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  - actions: $(l, r) \xrightarrow{a \in \Sigma} (l', r')$ iff $\exists (l, \gamma, a, R, l')$ s.t.:

$$\left\{ \begin{array}{l} r' = r[R \leftarrow 0] \\ r \models \gamma \\ r' \models Inv(l') \end{array} \right.$$

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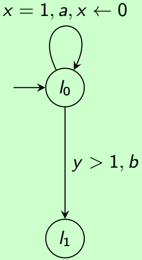
  - time: $(l, r) \xrightarrow{\delta} (l, Succ_1(r))$ with $Succ_1(r)$ the first region that is reachable from r by delaying or r if none exist (beyond the maximal constant).

Region Automaton With Delays

Example



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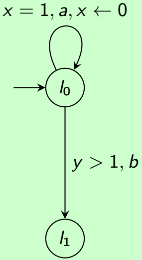

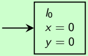

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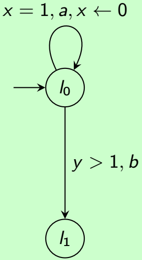

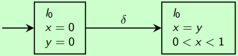

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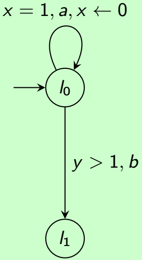

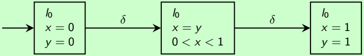

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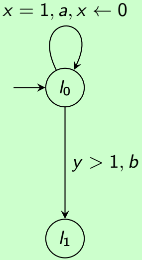

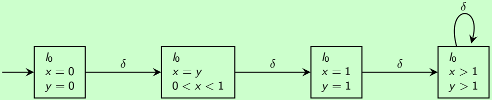

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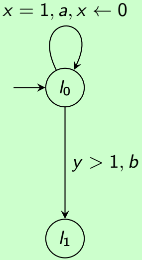

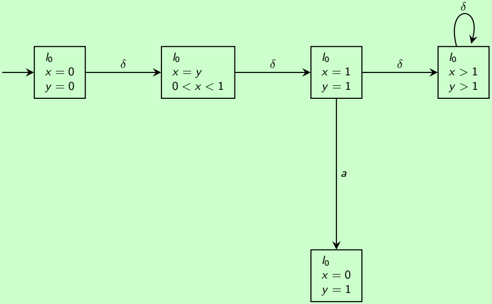

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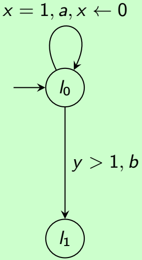

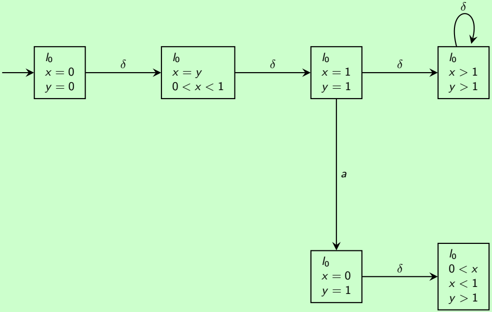

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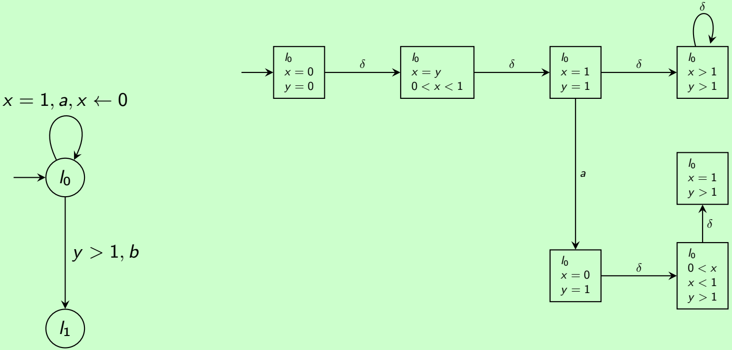

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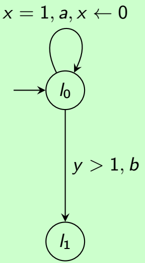

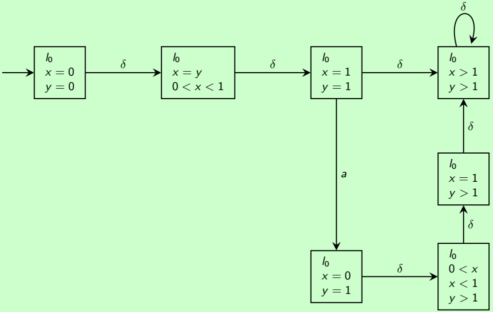

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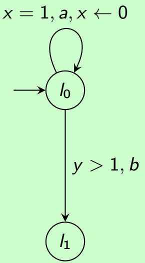

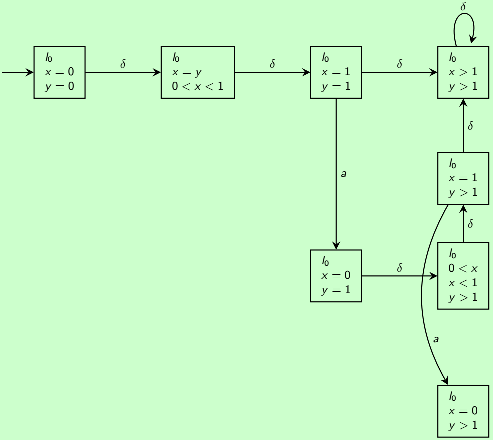

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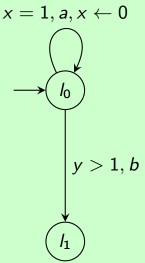
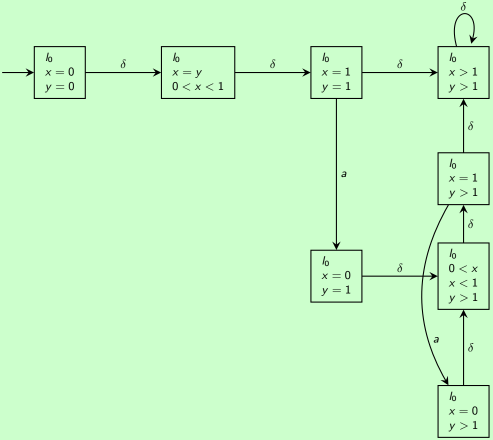
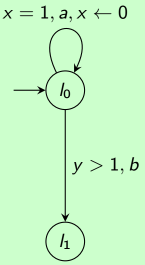
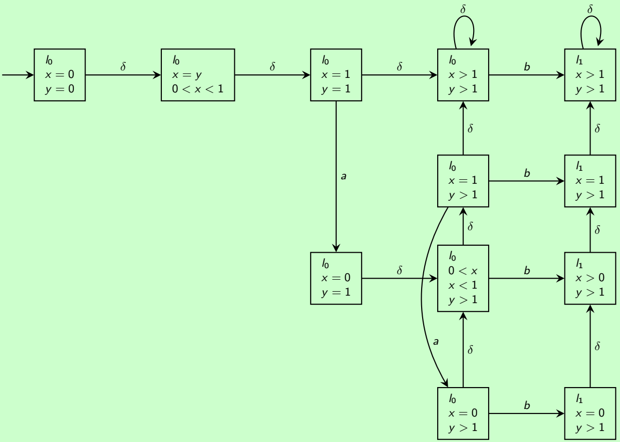




Region Automaton With Delays

Example





TCTL Model-checking with Timed Untils

- We want to check $\varphi_1 U_I \varphi_2$ in state $s = (\ell, \nu)$;
- This is equivalent to $\varphi_1 U(z \in I \wedge \varphi_2)$ in s with an additional fresh clock z such that $\nu(z) = 0$;
- In case of a non-nested formula, just add z and proceed as before;
- In case of a nested formula, we need to test the nested timed until subformulas for each state of the region automaton;

Example

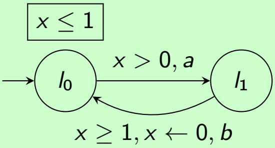
Consider formula $\varphi = \mathbf{EF}\psi$ with $\psi = \varphi_1 U_{[1,2]} \varphi_2$

1. build the region automaton (with delays);
2. for each of its states (ℓ, r), check $\psi' = \varphi_2 U(z \in [1,2] \wedge \varphi_2$ in the region automaton starting from $(\ell, r[z \leftarrow 0])$, where $r[z \leftarrow 0]$ is r plus an additional clock z with constraint $z = 0$;
3. label each of the (ℓ, r) that satisfy ψ by a new atomic proposition p_ψ;
4. check $\mathbf{EF} p_\psi$.

TCTL Model-checking

Exercise

Is property $\varphi = \mathbf{AG}(l_1 \Rightarrow \mathbf{AF}_{\leq 1} l_0)$ satisfied by the following TA:



Plan I

Zones

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- We group regions into zones (convex unions of regions) s.t.:

$$(l,\nu) \equiv (l',\nu') \Leftrightarrow \exists \rho,\rho' \in [\![\mathcal{S}]\!], \left\{ \begin{array}{l} \mathit{last}(\rho) = (l,\nu), \mathit{last}(\rho') = (l',\nu') \\ \text{and } \mathit{Untimed}(\rho) = \mathit{Untimed}(\rho') \end{array} \right.$$

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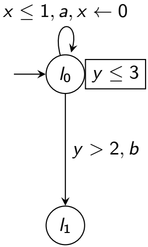
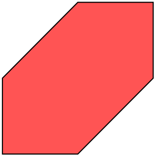
- Zones (like regions) are particular convex polyhedra.

Zones: Example



Zones

- Zones can be encoded by Difference Bound Matrix (DBM):
  $x \leq 1 \Leftrightarrow x - x_0 \leq 1, x_0 = 0$



$$\begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ -1 \leq x - y \leq 1 \end{cases}$$

$$\begin{bmatrix} (0, \leq) & (0, \leq) & (0, \leq) \\ (2, \leq) & (0, \leq) & (1, \leq) \\ (2, \leq) & (1, \leq) & (0, \leq) \end{bmatrix}$$

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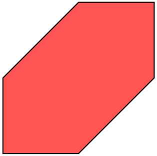

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Exercise

Write the system of inequalities encoded by the following DBM and draw the corresponding polyhedron:

$$\begin{bmatrix} (0, \leq) & (-1, <) & (-2, \leq) \\ (2, \leq) & (0, \leq) & (0, \leq) \\ (4, \leq) & (2, \leq) & (0, \leq) \end{bmatrix}$$

Canonical Form of DBMs

Exercise

Write the systems of inequalities encoded by the following DBMs and draw the corresponding polyhedra:

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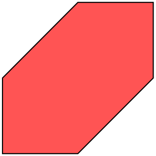
- A DBM is in canonical form if all its coefficients are minimal;
- Allows for efficient equality and inclusion tests;
- Any DBM can be put into canonical form using the Floyd-Warshall algorithm:

Floyd-Warshall Algorithm

for k **from** `1` **to** `N`:
  for i **from** `1` **to** `N`:
    for j **from** `1` **to** `N`:
      $D(i,j) \leftarrow min(D(i,j), D(i,k) + D(k,j))$

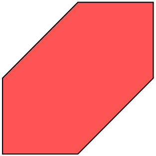
Forward Reachability Using Zones

- We define two specific operations on zones:



Reset of y (to zero):

$$Z[y \leftarrow 0]$$



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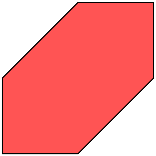

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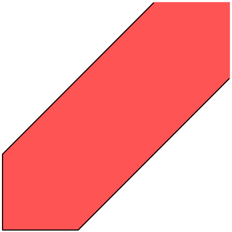

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Reset, future, intersection using DBMs

Exercise

1. if D is a canonical form DBM representing zone Z, compute canonical form DBM $D^{\nearrow}$ representing $Z^{\nearrow}$;
2. if D_1 and D_2 are two DBMs, compute the DBM D' representing the intersection of the two corresponding zones; if D_1 and D_2 are in canonical form, is your D' always in canonical form?
3. if D is the canonical form DBM representing zone Z, compute the canonical form DBM $D[x \leftarrow 0]$ representing $Z[x \leftarrow 0]$.

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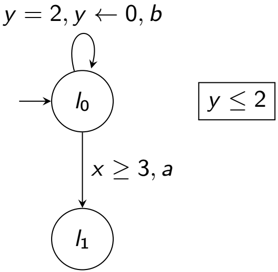
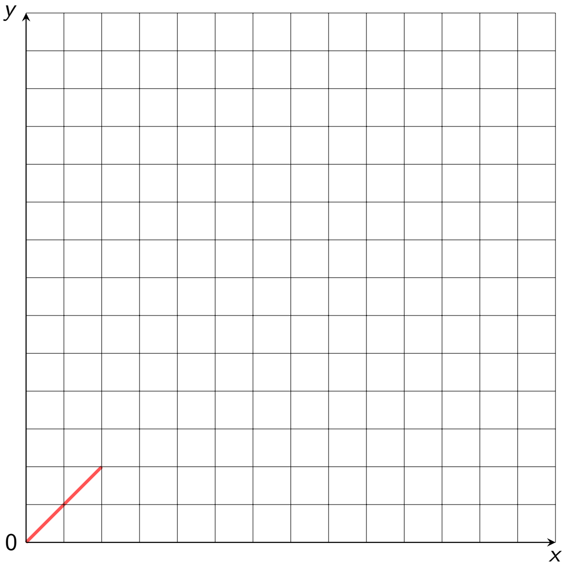
  These tests benefit from the canonical form of DBMs

Extrapolation





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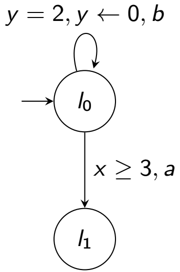

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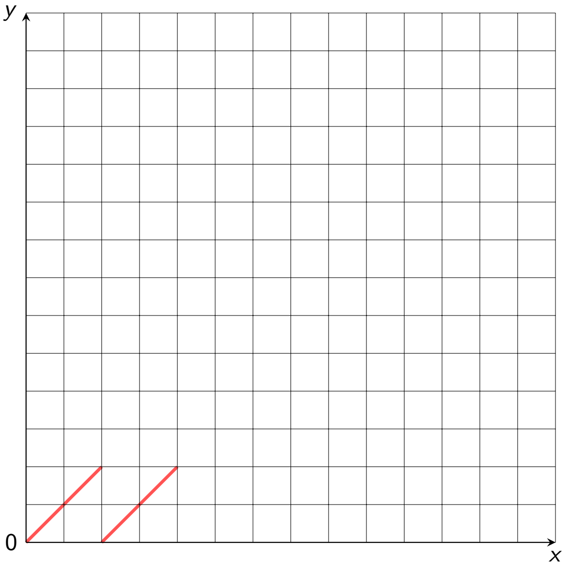

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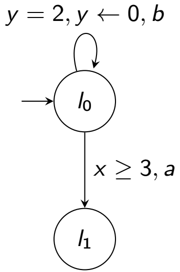

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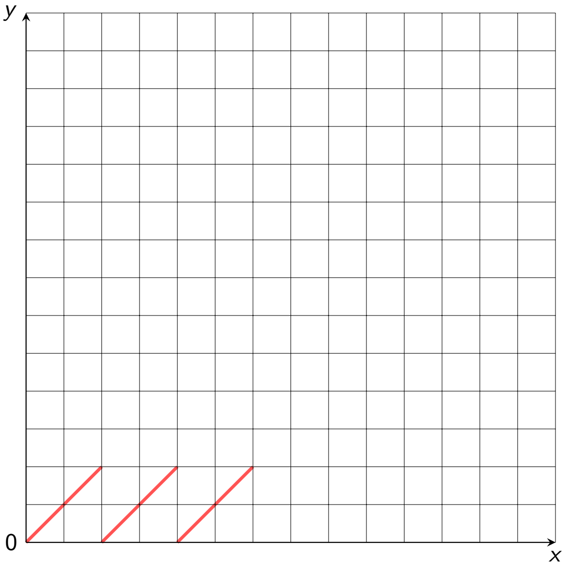

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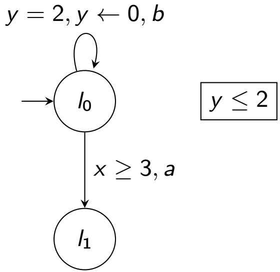

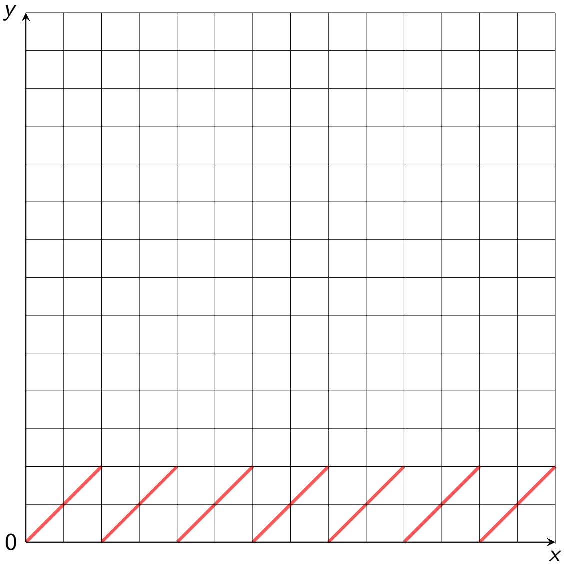

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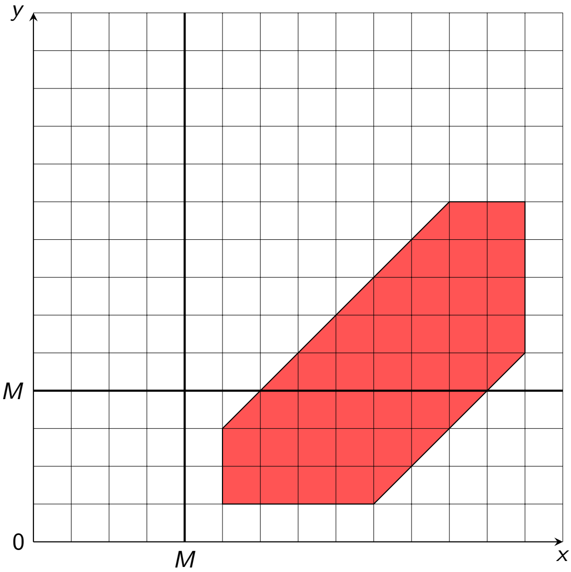

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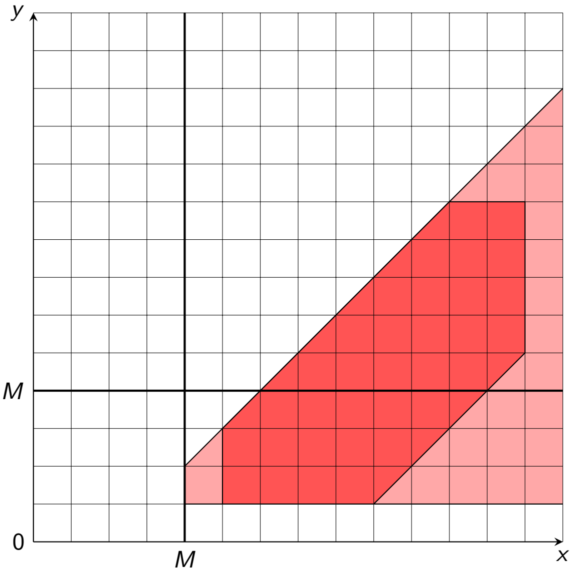

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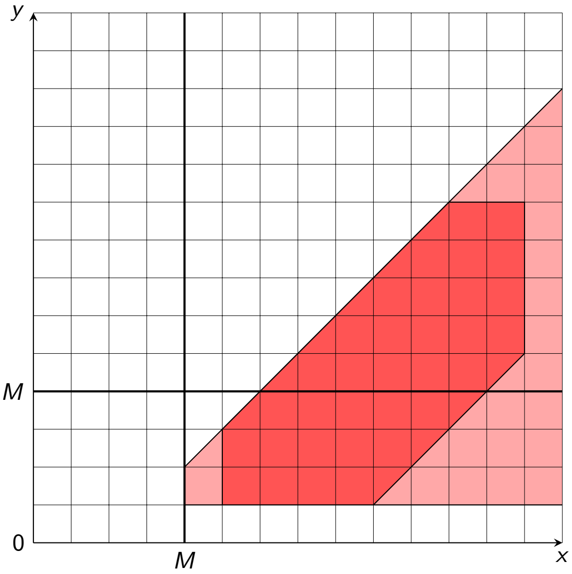
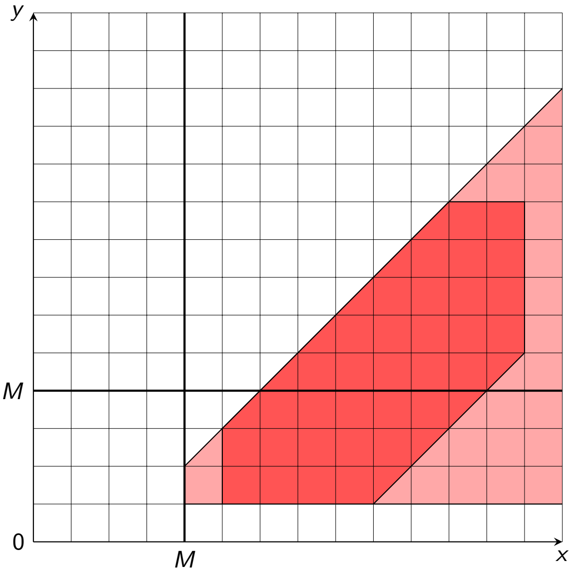


If $D(i,j) > M$ then
$D(i,j) \leftarrow$

Extrapolation



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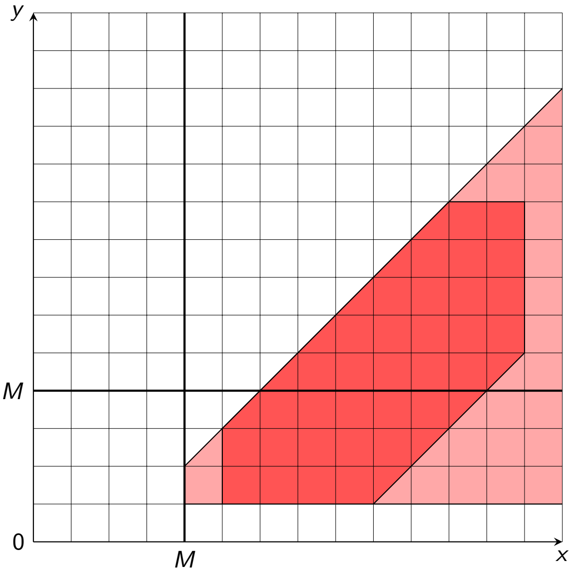

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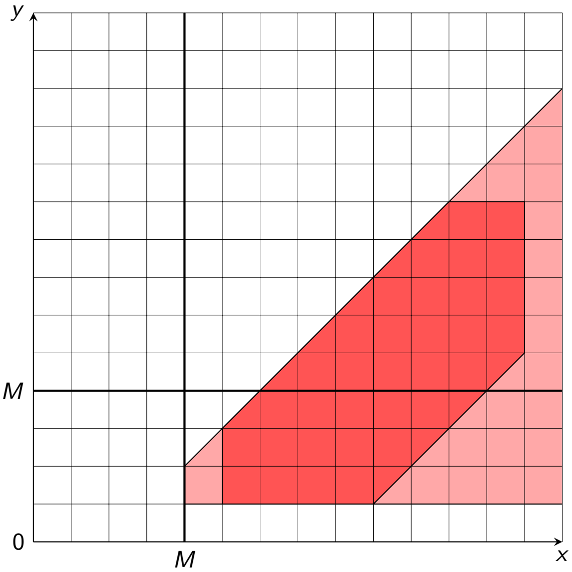
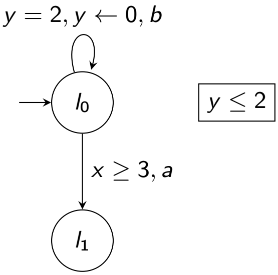


If $D(i,j) > M$ then $D(i,j) \leftarrow +\infty$

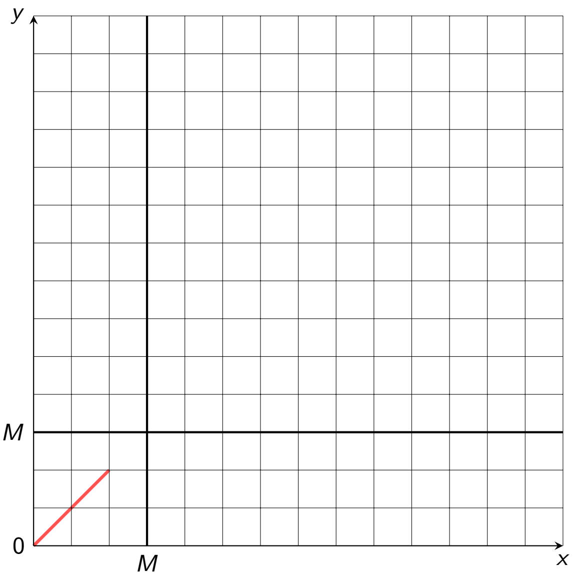
If $D(i,j) < -M$ then $D(i,j) \leftarrow -M$

Not necessarily "optimal": here both constraints could be removed.

Extrapolation



We have M = 3



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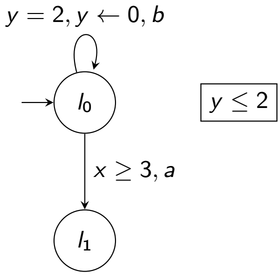

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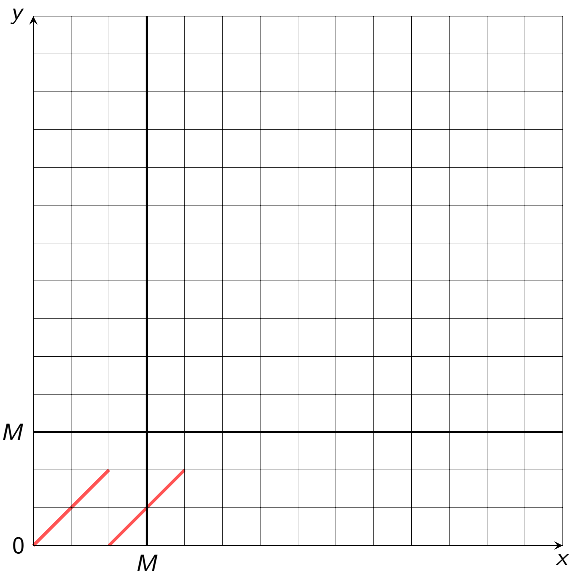

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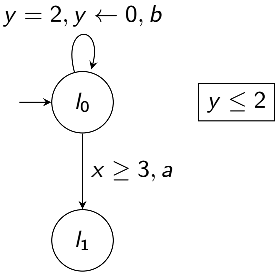

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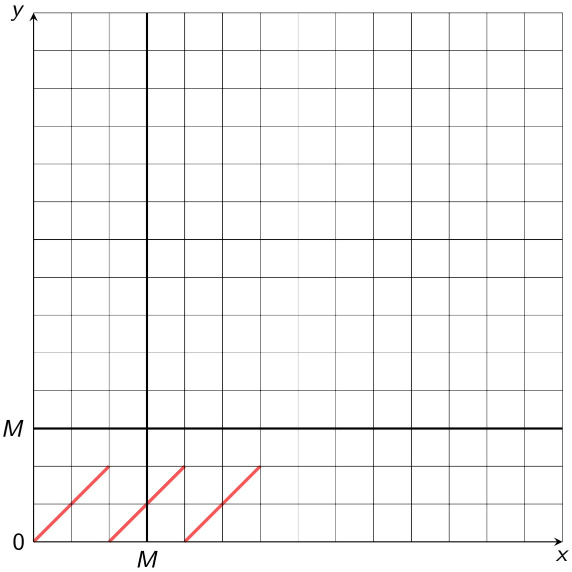

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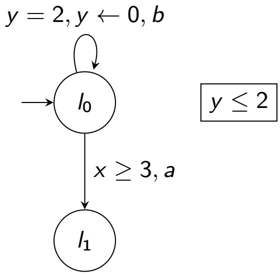

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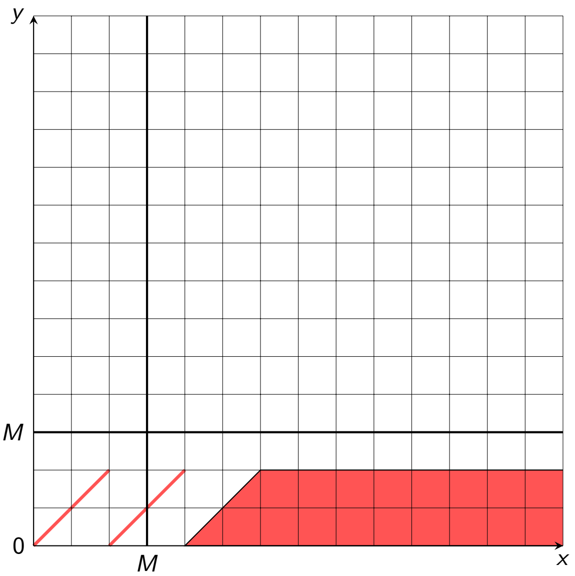
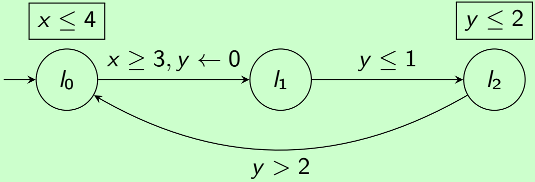


Forward Reachability Using Zones

Exercise

Compute the simulation graph of the following TA:



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- For a subset of TCTL we can write more efficient algorithms;
- E.g, for EU and AU, we can devise simple on-the-fly algorithms:
- We need only compute the simulation graph with an additional clock z for the timed Until:

Algorithm for $\mathbf{Ep U_{[a,b]} q}$

bool checkEU(l,Z):
  $passed \leftarrow passed \cup \{(l, Z)\}$
  if $(\min(Z_{|z}) > b)$
    return false
  else
    return $(q \in \ell(l)$ and $\max(Z_{|z}) \geq a)$
      or $(p \in \ell(l)$ and $\bigvee_{e=(l,\alpha,\delta,R,l')\in E}((l', next(Z, e)) \notin passed$
        ?checkEU$(l', next(Z, e))$
        :false$))$

Rajeev Alur, Costas Courcoubetis, and David Dill.
Model-checking in dense real-time.
Information and Computation, 104(1):2–34, 1993.

R. Alur and D. Dill.
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