Università
degli Studi di
Messina
DIPARTIMENTO DI INGEGNERIA

Dependable computing modeling and simulation

Stochastic processes and Markov chains

Master degree in

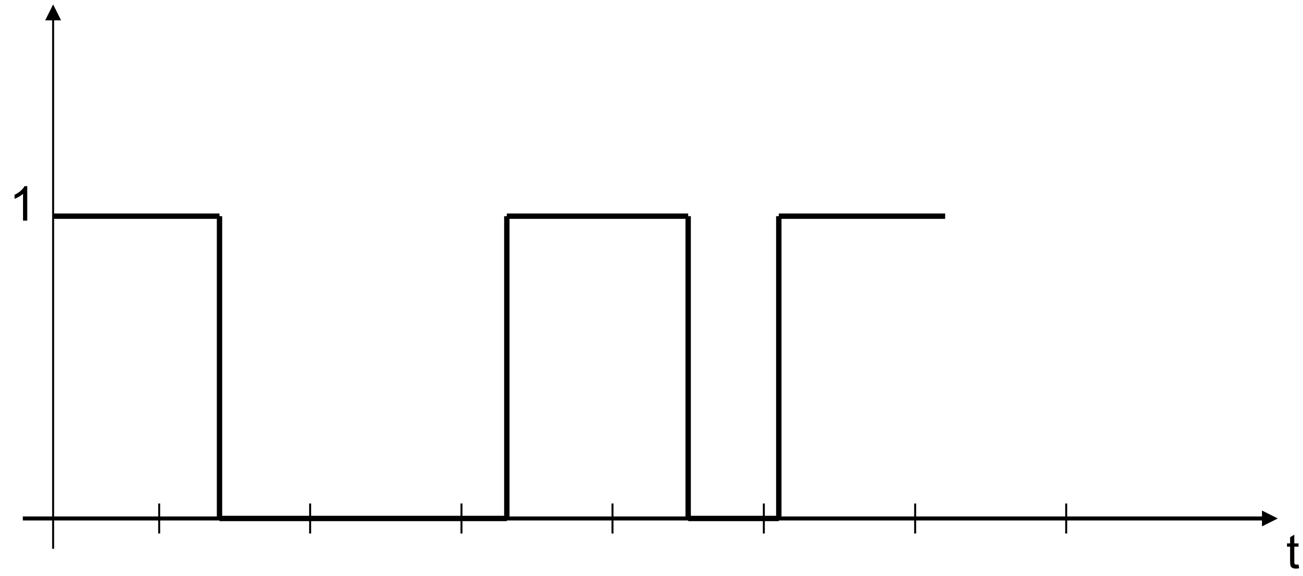
Engineering in Computer Science

Stochastic process

- Family of r.v. defined over a sample space
- The values of *X(t)* are called *states*
- When *X(t)* changes its value we say a *transition state* happens

An example

- A system into two possible states: "UP" or "DOWN"
- $S = \{0, 1\}$



Sample space

- Sample space can be *discrete* or continuous
- When the sample space is discrete the stochastic process is called *chain*
- The parameter can be *discrete* or *continuous*
  - Network protocol based on *time slots*

Markov chain

- Let X(t) be a discrete states stochastic process
- $P[X(t_n)=j]$ -> probability the process is into the state j at time t_n
- X(t) is said a Markov chain when

$$P[X(t_n)=j|X(t_{n-1})=i_{n-1},X(t_{n-2})=i_{n-2},\cdots,X(t_0)=i_0]=$$

$$=P[X(t_n)=j|X(t_{n-1})=i_{n-1}]$$

Homogeneous Time Markov chain

$$P[X(t)=j|X(t_n)=i_n]=$$
$$P[X(t-t_n)=j|X(0)=i_n]$$

- Process dynamic does not depend on time origin
- Sojourn time is exponentially distributed

Computation

$$p_i(t) = P[X(t) = i]$$

$$p(t) = [p_0(t), p_1(t), \cdots]$$

- Transition state probability

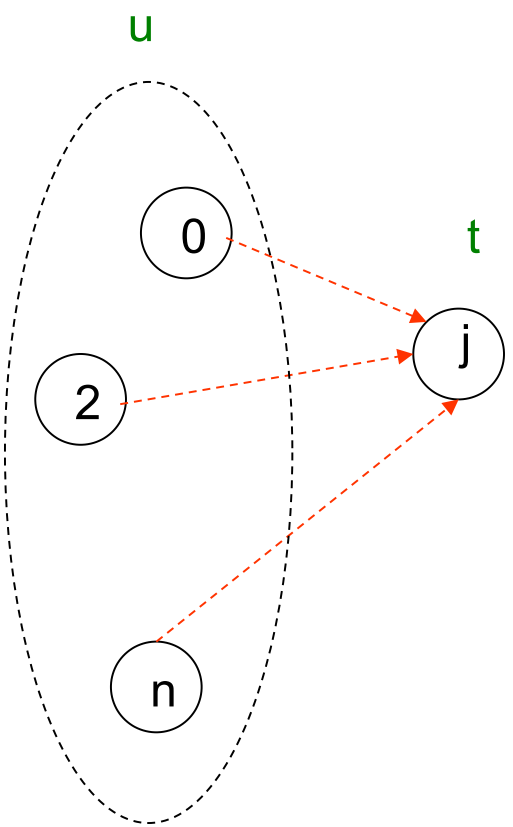
$$P_{ij}(t-u) = P[X(t) = j \mid X(u) = i]$$

$$P(t) = [P_{ij}(t)]$$

Chapman-Kolmogorov equations

$$p_j(t) = p_0(u)P_{0j}(t-u) + p_1(u)P_{1j}(t-u) + \cdots + p_n(u)P_{nj}(t-u)$$

$$p(t) = p(u) \cdot P(t-u)$$



Discrete time Markov chain (DTMC)

- The state changes in discrete time instants $\{0, 1, 2, \ldots\}$
- $\mathbf{P}=\mathbf{P}(1)$, p_{ij} is the (i,j) element of $\mathbf{P}$
- Chapman-Kolmogorov equation can be written as:

$$\mathbf{p}(n+1)=\mathbf{p}(n) \cdot \mathbf{P} \Rightarrow \mathbf{p}(n)=\mathbf{p}(0) \cdot \mathbf{P}^n$$

- If the limit exists when $n \to \infty$

$$\mathbf{p}=\mathbf{p} \cdot \mathbf{P}$$

$$\mathbf{p} \cdot \mathbf{e} = 1$$

Normalization equation

$$e = [1 \cdots 1]^T = \begin{bmatrix} 1 \\ \ldots \\ 1 \end{bmatrix}$$

$$[p_0 \; p_1 \; \cdots \; p_n] \cdot \begin{bmatrix} 1 \\ \ldots \\ 1 \end{bmatrix} = 1$$

Continuous time Markov chain (CTMC)

$$p(t) = p(u) \cdot P \qquad u = t - \Delta t$$

$$p(t) - p(t - \Delta t) = p(t - \Delta t) \cdot [P(\Delta t) - I]$$

$$\frac{p(t) - p(t - \Delta t)}{\Delta t} = p(t - \Delta t) \cdot \frac{[P(\Delta t) - I]}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{p(t) - p(t - \Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \{p(t - \Delta t) \underbrace{\frac{[P(\Delta t) - I]}{\Delta t}}_{= \mathbf{Q}} \}$$

$$\frac{dp(t)}{dt} = p(t)\mathbf{Q}$$

Stable conditions -Steady state

- If the limit exists when t->∞

$$p = \lim_{t \to +\infty} p(t)$$

$$\lim_{t \to +\infty} \frac{dp(t)}{dt} = \lim_{t \to +\infty} p(t) \cdot Q$$

$$\begin{cases} 0 = p \cdot Q \\ p \cdot e = 1 \end{cases}$$

Q Matrix

$$q_{ij} = \lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t) - \delta_{ij}}{\Delta t} \qquad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

- when Δt is small

$$q_{ij} \cdot \Delta t \simeq P_{ij}(\Delta t) - \delta_{ij}$$

Matrix Q - Interpretation

$$i = j \quad \begin{array}{l} q_{ii} \cdot \Delta t \simeq P_{ii}(\Delta t) - 1 \\ -q_{ii} \cdot \Delta t \simeq 1 - P_{ii}(\Delta t) \end{array}$$

Exit rate from state *i*

$$i \neq j \quad q_{ij} \cdot \Delta t \simeq P_{ij}(\Delta t)$$

Transition rate from *i* to *j*

- **Q** is named *transition rate matrix* or *infinitesimal generator matrix*

Some properties

$$\sum_{j,\, j \neq i} P_{ij}(\Delta t) + P_{ii}(\Delta t) = 1 \;\Rightarrow\; \sum_{j} q_{ij} = 0$$

- The off diagonal elements are non negative ($q_{ij}\Delta t$ are probabilities)
- The diagonal elements of Q are non positive

State classification

- Transient state = finite number of visits over an infinite time
- Starting from the state *i*, there is a non-zero probability the process will never return to state *i*

State classification (cont.)

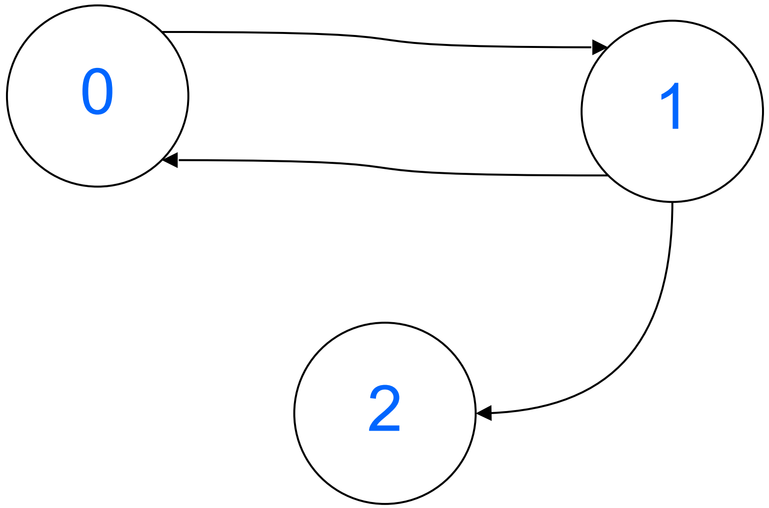
- Recurrent state = the state is not transient
- Starting from state *i* the process returns to *i* with probability 1
- The state is visited infinitely often over an infinite observation time
  - recurrent non null when the first return time is finite
  - recurrent null when the time between two visits is not finite (infinite number of states)

State classification (cont.)

- Absorbing state = a state that, once entered, cannot be left

Characterization

- A Markov chain is said *irreducible* iff each state is reachable from each other



  - An irreducible Markov chain has all recurrent states

Characterization

- Finite state space

  irreducible => all the states are

  recurrent non null

- A Markov chain is said acyclic iff a state exists such that it cannot be visited any more when it has been left
- Definition: a CTMC is *ergodic* if it is irreducible and recurrent non null

Ergodic Markov chain

- Ergodicity is an important characteristic because:
  - Always the steady state probability vector **p** exists
  - **p** does not depend on the initial conditions (**p**(0))
  - linear equation system could be used

Remarks

- Finite state space
  - All the state can be transient states
- Acyclic Markov chain with finite state space
  - either *transient* or *absorbing* states
  - if k is the only absorbing state then $p_k=1$