Università degli Studi di Messina
DIPARTIMENTO DI INGEGNERIA

Dependable computing modelling and simulation

Reliability

Master degree in

Engineering in Computer Science

Reliability

- Probability a system is continuously working over a time interval
- Probabilistic approach
- The time to failure is a r.v.
- Probability that a component fails within a fixed time

Definition

Reliability of a component is its ability to perform the work which it has been designed for, over the time interval (0, t) given the behavioral conditions and the stress subject to

- Time t is said *mission time*

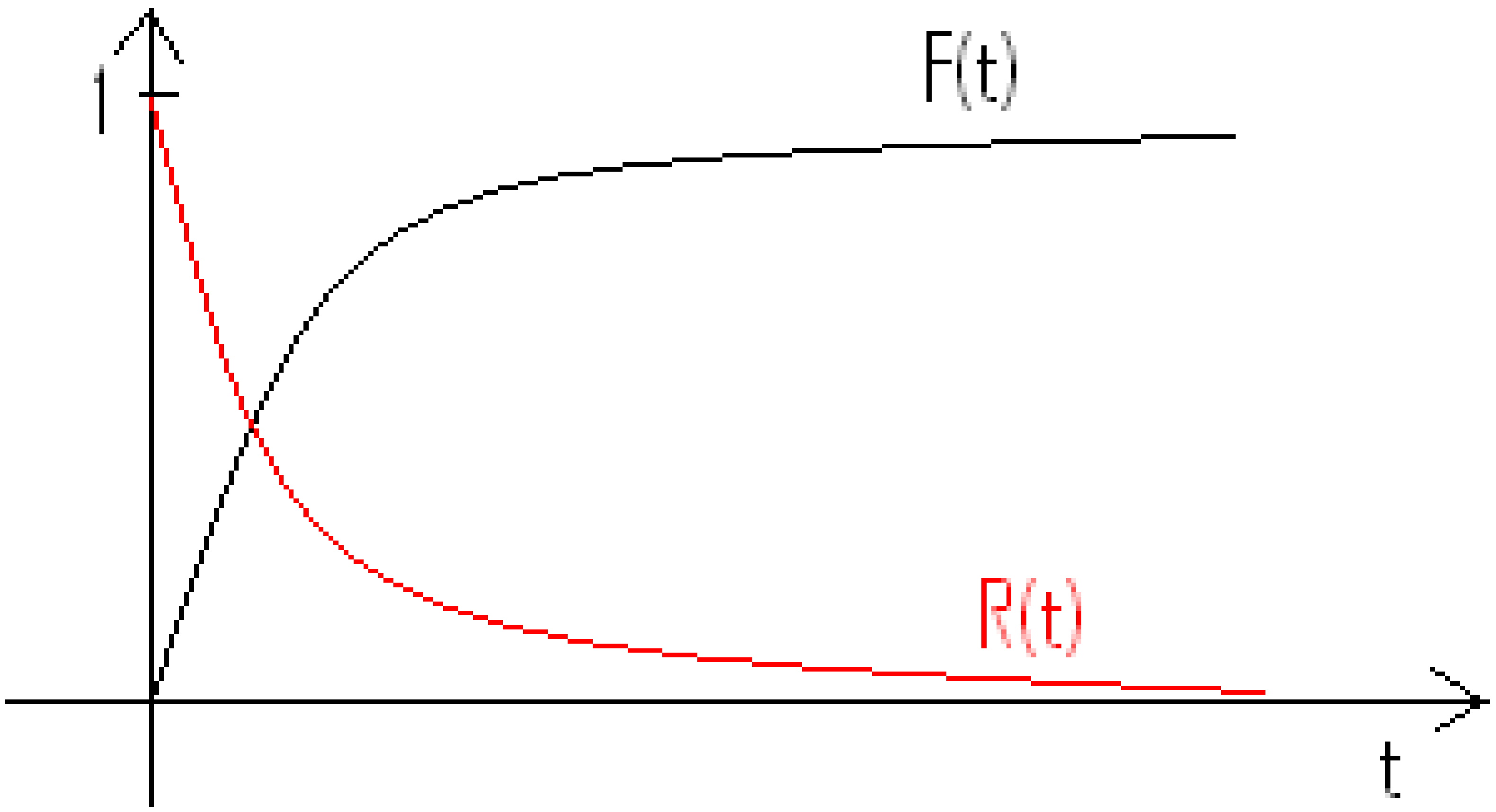
Reliability: a quantity

- Let X be the r.v. representing the *system life time* (or its *time to failure*)
- $F_X(t) = Pr[X \leq t]$
- $R(t) = Pr[X > t] = 1 - Pr[X \leq t] = 1 - F_X(t)$
- The function $R(t)$ is said *reliability*

Properties

- $R(t)$ is not a CDF (it is **not** a non decreasing function)
- $lim_{t \to +\infty} R(t) = 0$
- $R(t)$ is a non increasing function
- $0 \leq R(t) \leq 1 \quad in\ [0,+\infty)$

Properties (cont.)

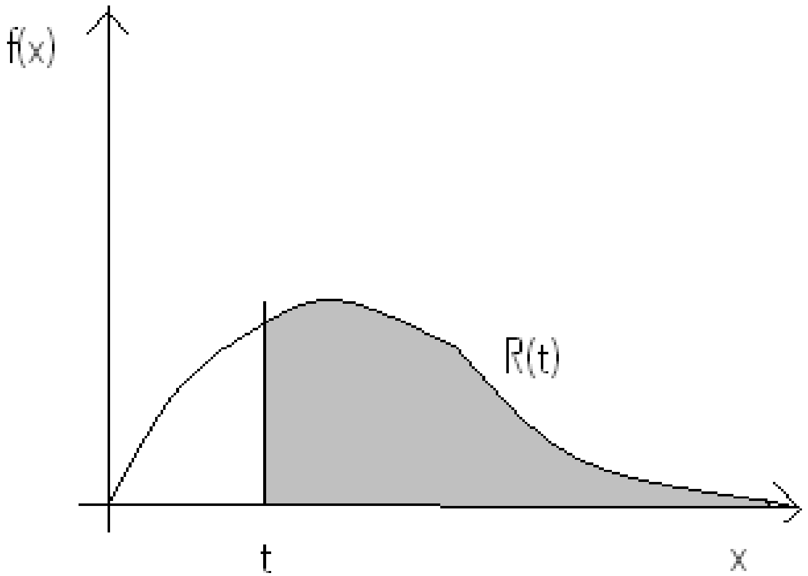


Properties (cont.)

- Let $f_X(t)$ be the pdf of X

$$R(t) = \int_{t}^{+\infty} f_x(\tau)\, d\tau$$

$$f_X(t) = \frac{dF_X(t)}{dt} = -\frac{dR(t)}{dt}$$



Failure time r.v.: moments

- Mean time to failure (MTTF)

$$E[\tau] = MTTF = \int_0^{\infty} t\, f(t)\, dt = \int_0^{\infty} R(t)\, dt$$

- Second order moment

$$m_2 = E[\tau^2] = \int_0^{\infty} t^2 f(t)\, dt$$

- Variance show how the possible values scatter around the mean time

$$V[\tau] = E[(\tau - m_1)^2] = \int_0^{\infty} (t - m_1)^2 f(t)\, dt = m_2 - m_1^2$$

Failure rate

- Unconditioned probability that a fault occurs in [t, t+Δt]

$$f(t)\cdot\Delta t$$

- Non conditioned probability
- Let us suppose the system correctly operated till time t
  - We can argue the failre probability Δt is different

Faiulre time (cont.)

- The conditioned probability is

$$P[t<X<t+\Delta t | X>t] = \frac{P[t<X<t+\Delta t]}{P[X>t]} = \frac{F(t+\Delta t)-F(t)}{R(t)}$$

- The istantaneous failure rate (hazard rate) is defined as:

$$h(t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t)-F(t)}{R(t)} \frac{1}{\Delta t}$$

Failure rate - property

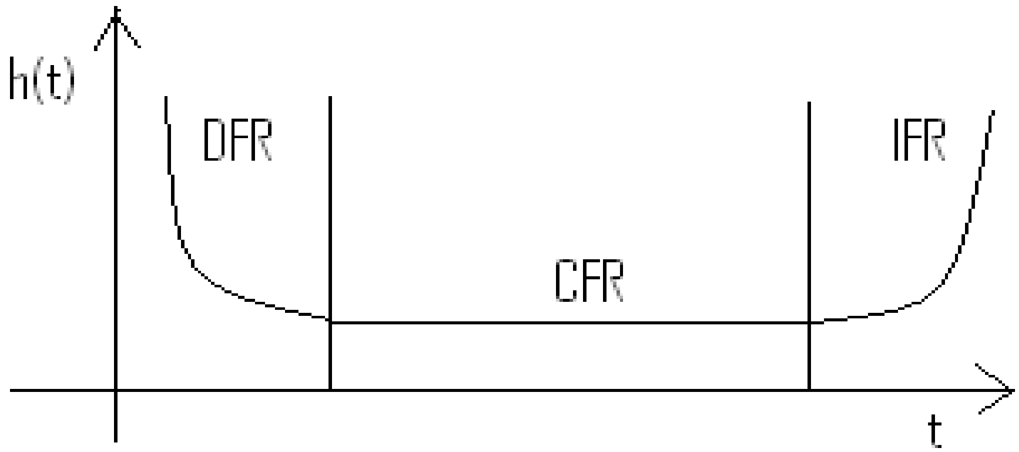
$$h(t)=\lim_{\Delta t\to 0}\frac{F(t+\Delta t)-F(t)}{\Delta t}\frac{1}{R(t)}=$$

$$=\frac{f(t)}{R(t)}=-\frac{1}{R(t)}\frac{dR(t)}{dt}$$

- Solving the equation above

$$R(t)=e^{-\int_0^t h(x)\,dx}$$

Failure rate over the time



- DFR = decreasing failure rate
- CFR = constant failure rate
- IFR = increasing failure rate
- When a failure time is exponentially distributed h(t) is constant

Exponential distribution

$$F(t) = 1 - e^{-\lambda t}$$

$$f(t) = \lambda e^{-\lambda t}$$

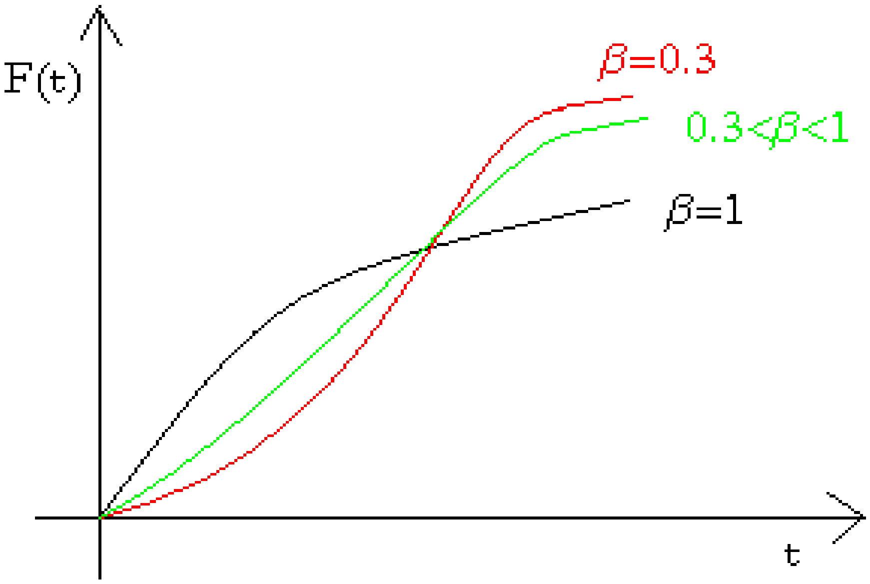
$$R(t) = e^{-\lambda t}$$

$$h(t) = \lambda$$

Weibull distribution

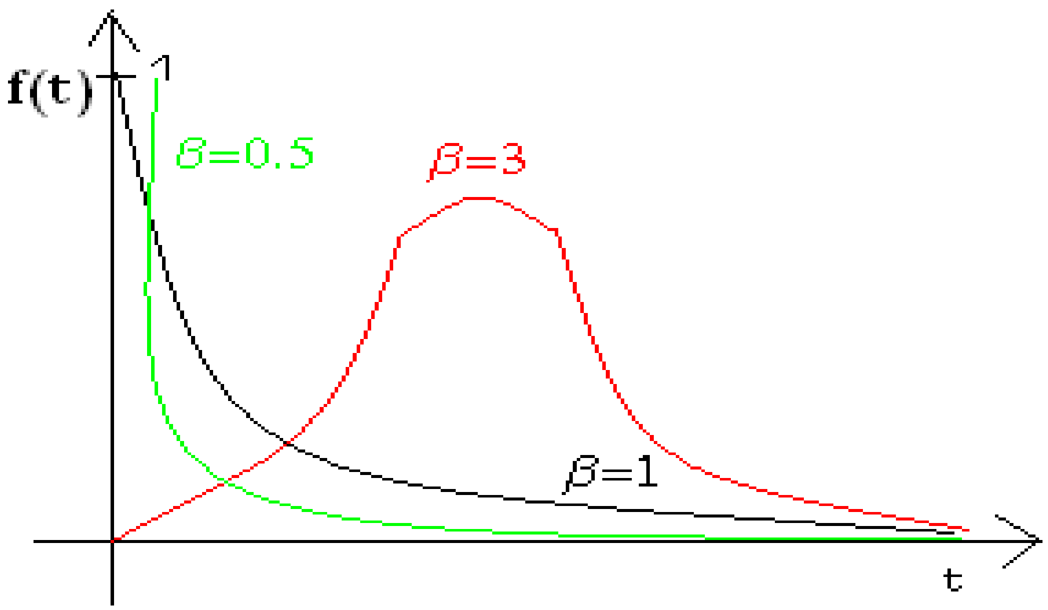
- *α = scale factor*
- *β= form factor*

$$F(t) = 1 - e^{-\left(\frac{t}{\alpha}\right)^{\beta}}$$



Weibull distribution - pdf

$$f(t) = \frac{\beta}{\alpha^\beta} t^{\beta-1} e^{-(\frac{t}{\alpha})^\beta}$$

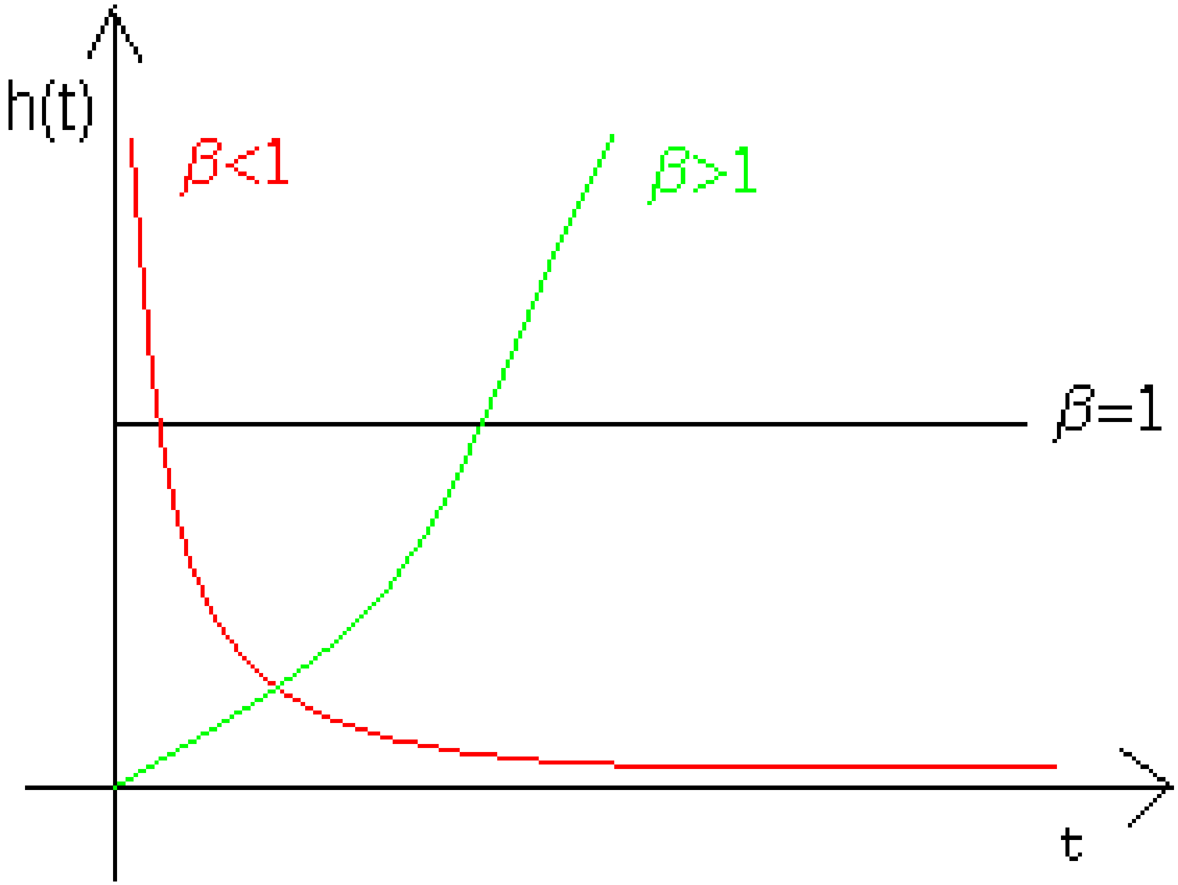


Weibull distribution – h(t)

$$h(t)=\frac{\beta}{\alpha^{\beta}}t^{\beta-1} \qquad\qquad R(t)=e^{-(\frac{t}{\alpha})^{\beta}}$$

- $\beta<1$ => h(t) decreasing
- $\beta=1$ => h(t) constant
- $\beta>1$ => h(t) increasing

Weibull distribution – h(t)



Summary

$$F(t) = 1 - e^{-(\frac{t}{\alpha})^{\beta}}$$

$$f(t) = \frac{\beta}{\alpha^{\beta}} t^{\beta-1} e^{-(\frac{t}{\alpha})^{\beta}}$$

$$R(t) = e^{-(\frac{t}{\alpha})^{\beta}}$$

$$h(t) = \frac{\beta}{\alpha^{\beta}} t^{\beta-1}$$

- Weibull distribution is a generalization of the exponential distribution
- When β = 1, it is an exponential distribution;
- When β = 1, h(t) is constant and equal to $1/\alpha$

Mean Time To Failure (MTTF)

- If *X* is a failure time, its *expected value* is a notable measure
- *E[X]* is said *MTTF*

$$MTTF = E[X] = \int_0^{+\infty} t \cdot f(t)\,dt$$

- E.g.: exponential distribution

$$MTTF = \frac{1}{\lambda}$$

Remark

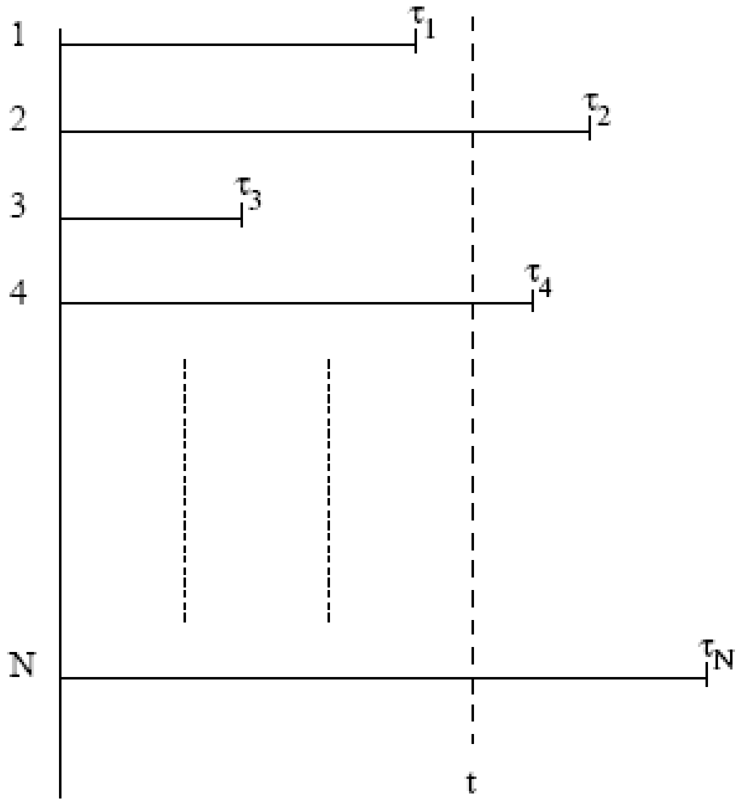
- λ is dimentionally equally to t^{-1}
- λ t is a pure number
- Often, $R(t)$ is expressed in terms of λt

Remark (cont.)

$-\lambda t$	$R(t)$		$-\lambda t$	$R(t)$
0.00	0.000000		1.00	0.367879
0.05	0.951229		1.10	0.332871
0.15	0.860708		1.15	0.316637
0.20	0.818731		1.20	0.301194
0.25	0.778801		1.25	0.286505
0.30	0.740818		1.30	0.272532
0.35	0.704688		1.35	0.259240
0.40	0.670320		1.40	0.246597
0.45	0.637628		1.45	0.234570
0.50	0.606531		1.50	0.223130
0.55	0.576950		1.55	0.212248
0.60	0.548812		1.60	0.201897
0.65	0.522046		1.65	0.192050
0.70	0.496585		1.70	0.182684
0.75	0.472367		1.75	0.173774
0.80	0.449329		1.80	0.165299
0.85	0.427415		1.85	0.157237
0.90	0.406570		1.90	0.149569
0.95	0.386741		1.95	0.142274

An experimentation

- N equal components
- Started to operate at the same time ($t=0$)
- τ_i is the failure time of component i



Experimental evaluation

- $N_v(t)$ = number of working components at time t
- $N_r(t)$ = number of fault components at time t

$$N = N_v(t)+N_r(t)$$

Experimental evaluation (cont.)

$$\hat{F}(t) = \frac{N_r(t)}{N} \qquad \hat{R}(t) = \frac{N_v(t)}{N} = 1 - \hat{F}(t)$$

$$\hat{f}(t) = \frac{d\hat{F}(t)}{dt} = \frac{1}{N}\frac{dN_r(t)}{dt} = -\frac{d\hat{R}(t)}{dt}$$

pdf represents the speed at which the components fail with respect the total number of components

Experimental evaluation (cont.)

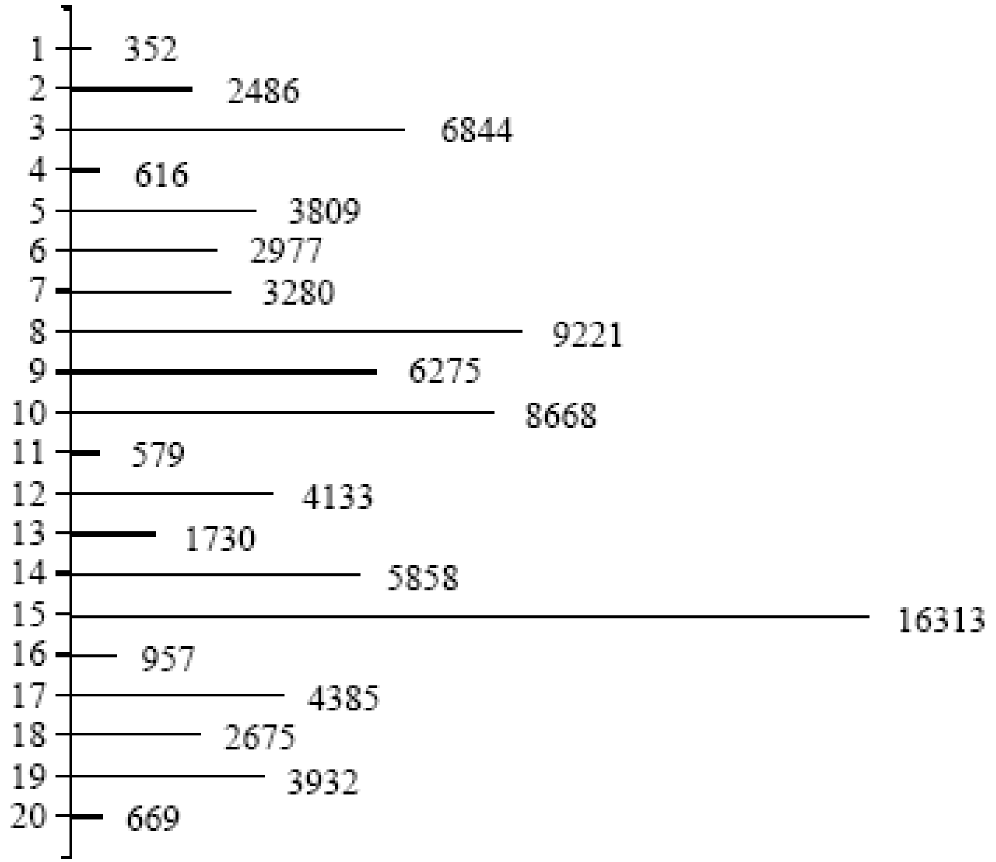
$$\hat{h}(t) = -\frac{1}{\hat{R}(t)} \frac{d\hat{R}(t)}{dt} = \frac{\hat{f}(t)}{\hat{R}(t)} = \frac{1}{N_v(t)} \frac{dN_r(t)}{dt}$$

Failre rate represents the speed at which the components fail, with respect the operational ones

$$\overline{MTTF} = \frac{\tau_1 + \tau_2 + \cdots + \tau_N}{N} = \frac{1}{N} \sum_{i=0}^{N} \tau_i$$

An example

- N=20

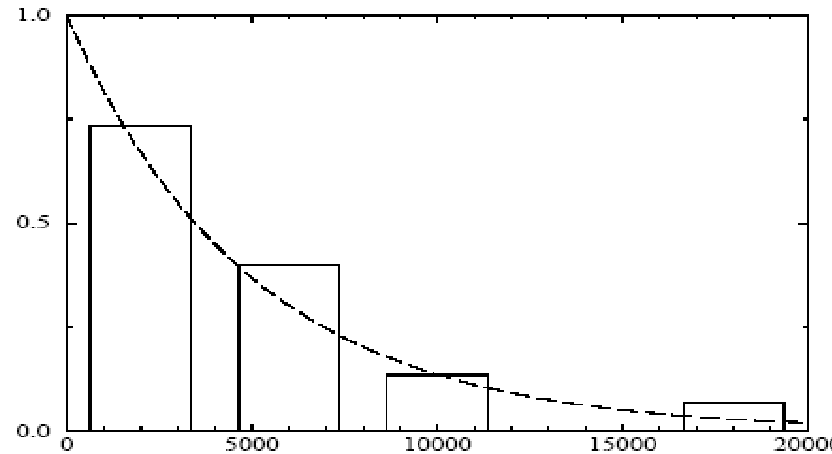


An example (cont.)

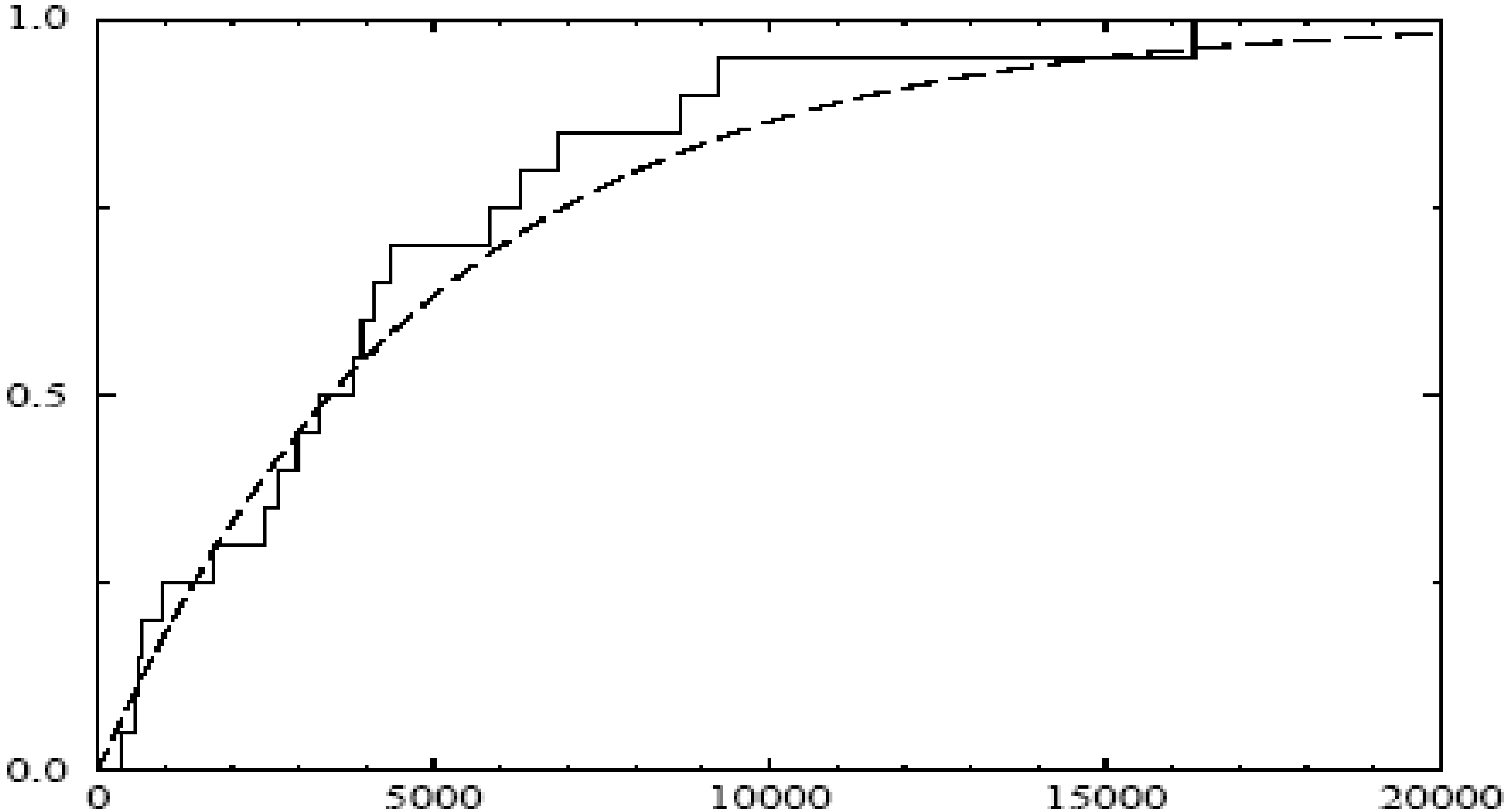
Num. Ordine Campione	Tempo al guasto τ (h)
1	352
2	579
3	616
4	669
5	957
6	1730
7	2486
8	2675
9	2977
10	3280
11	3809
12	3932
13	4133
14	4385
15	5858
16	6275
17	6844
18	8668
19	9221
20	16312

An example (cont.)

Intervallo	Occorrenze
0-4000	11
4000-8000	6
8000-12000	2
12000-16000	0
16000-20000	1



An example (cont.)



An example (cont.)

$$\overline{MTTF} = 4288\,h$$

$$\bar{\lambda} = \frac{1}{\overline{MTTF}} \approx 2.332 \times 10^{-4}\,h^{-1}$$

Remark: the same procedure is useful to manage experimental data in general, not only failure times