Section 1.9: The Matrix of Linear Transformation

The Matrix of Linear Transformation

- **Theorem**: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then there exists a unique matrix *A* such that

$$T(\mathrm{x}) = A\mathrm{x} \quad \text{for all } \mathrm{x} \text{ in } \mathbb{R}^n$$

- In fact, *A* is the $m \times n$ matrix whose *j* th column is the vector

$T(\mathrm{e}_j)$, where e_j is the *j* th column of the identity matrix in $\mathbb{R}^n$

$$A = [T(\mathrm{e}_1) \ \cdots \ T(\mathrm{e}_1)] \qquad (3)$$

The Matrix of Linear Transformation

- **Proof:** Write

$$\mathrm{x} = \mathrm{I}_n\mathrm{x} = [\mathrm{e}_1 \ldots \mathrm{e}_n]\mathrm{x} = x_1\mathrm{e}_1 + \ldots + \mathrm{x}_n\mathrm{e}_n$$

and use the linearity of *T* to compute

$$\begin{aligned} T(\mathrm{x}) &= T(x_1\mathrm{e}_1 + \cdots + x_n\mathrm{e}_n) = x_1T(\mathrm{e}_1) + \cdots + x_nT(\mathrm{e}_n) \\ &= [T(\mathrm{e}_1) \quad \cdots \quad T(\mathrm{e}_1)]\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = A\mathrm{x} \end{aligned}$$

- The matrix *A* in (3) is called the **standard matrix for the linear transformation *T*.**

The Matrix of Linear Transformation

- **Example:** Find the standard matrix *A* for the dilation transformation $T(\mathrm{x}) = 3\mathrm{x}$, for x in $\mathbb{R}^2$.

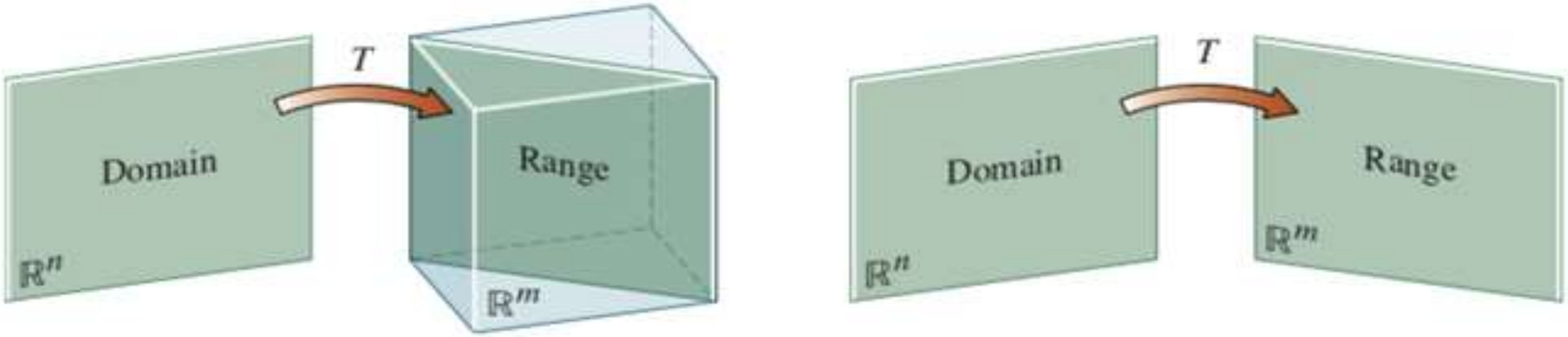
- **Solution:** Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Existence And Uniqueness Questions

- **Definition:** A mapping $T\colon \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^n$ if each **b** in $\mathbb{R}^m$ is the image of at least one **x** in $\mathbb{R}^n$.



T is *not* onto $\mathbb{R}^m$ | *T* is onto $\mathbb{R}^m$

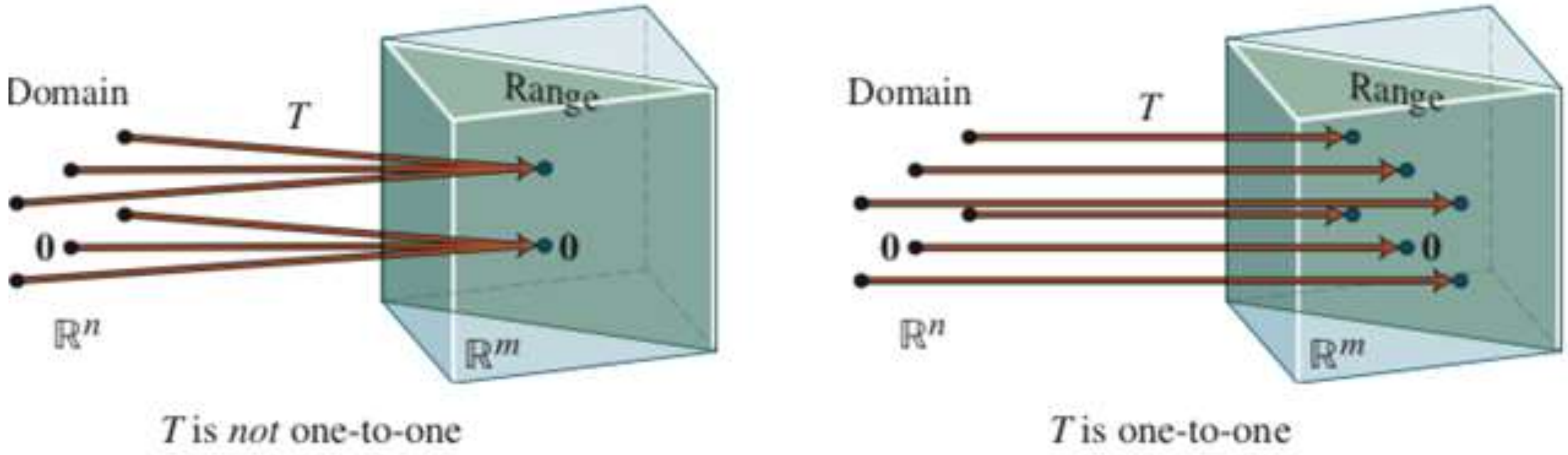
Existence And Uniqueness Questions

-- *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each **b** in the codomain $\mathbb{R}^m$, there exists at least one solution of $\mathrm{T(x) = b}$.

-- The mapping *T* is not onto when there is some **b** in $\mathbb{R}^m$ for which the equation $\mathrm{T(x) = b}$ has no solution.

Existence And Uniqueness Questions

- **Definition**: A mapping $T\colon\ \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.



Existence And Uniqueness Questions

- **Example**: Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

  - Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Existence And Uniqueness Questions

- **Solution**: Since *A* happens to be in echelon form, we can see at once that *A* has a pivot position in each row.

  By Theorem 4 in Section 1.4, for each **b** in $\mathbb{R}^3$, the equation *A***x** = **b** is consistent. In other words, the linear transformation *T* maps $\mathbb{R}^4$ (its domain) onto $\mathbb{R}^3$.

- However, since the equation *A***x** = **b** has a free variable (because there are four variables and only three basic variables), each **b** is the image of more than one **x**. This is, *T* is not one-to-one.

Existence And Uniqueness Questions

- **Theorem**: Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then *T* is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

- **Proof**: -- Since *T* is linear, $\mathrm{T}(0) = 0$. If *T* is one-to-one, then the equation $\mathrm{T}(\mathrm{x}) = 0$ has at most one solution and hence only the trivial solution.

  -- If *T* is not one-to-one, then there is a **b** that is the image of at least two different vectors in $\mathbb{R}^n$ -- say, **u** and **v**. That is $\mathrm{T}(\mathrm{u}) = \mathrm{b}$ and $\mathrm{T}(\mathrm{v}) = \mathrm{b}$.

  But then, since *T* is linear, $T(\mathrm{u} - \mathrm{v}) = T(\mathrm{u}) - T(\mathrm{v}) = \mathrm{b} - \mathrm{b} = 0$.

  The vector $\mathrm{u} - \mathrm{v}$ is not zero, since $\mathrm{u} \neq \mathrm{v}$. Hence the equation $T(\mathrm{x}) = 0$ has more than one solution. So, either the two conditions in the theorem are both true or they are both false.

Existence And Uniqueness Questions

- **Theorem**: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let *A* be the standard matrix for *T*. Then:

a) *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of *A* span $\mathbb{R}^m$,
b) *T* is one-to-one if and only if the columns of *A* are linearly independent.

- **Proof:** a) The columns of *A* span $\mathbb{R}^m$ if and only if for each **b** in $\mathbb{R}^m$ the equation *A***x** = **b** is consistent.

  In other words, if and only if for every **b**, the equation $\mathrm{T(x)} = \mathrm{b}$ has at least one solution. This is true if and only if *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

b) The equations $\mathrm{T(x)} = 0$ and *A***x** = **0** are the same except for notation.
So, by Theorem, *T* is one-to-one if and only if *A***x** = **0** has only the trivial solution.
This happens if and only if the columns of *A* are linearly independent.