Section 1.4: The Matrix Equation $Ax = b$

Section 1.5: Solution Sets of Linear Systems

Section 1.7: Linear Independence

Matrix Equation Ax = b

- **Definition:** If A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if **x** is in $\mathbb{R}^n$, then the **product of** A **and x**, denoted by A**x**, **is the linear combination of the columns of** A **using the corresponding entries in x as weights**; that is,

$$A\mathbf{x} = [a_1 \quad a_2 \quad \cdots \quad a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n.$$

- A**x** is defined only if the number of columns of A equals the number of entries in **x**.

Matrix Equation Ax = b

- **Example**: For v_1, v_2, v_3 in $\mathbb{R}^m$, write the linear combination $3v_1 - 5v_2 + 7v_3$ as a matrix times a vector.
- **Solution:** Place v_1, v_2, v_3 into the columns of a matrix *A* and place the weights $3, -5,$ and 7 into a vector ***x.***
- That is,

$$3v_1 - 5v_2 + 7v_3 = [v_1 \quad v_2 \quad v_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}$$

Matrix Equation Ax = b

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \qquad ---(1)$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \qquad ---(2)$$

Matrix Equation Ax = b

- As in the given example (1), the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \qquad ---(3)$$

- Equation (3) has the form $A\mathbf{x} = \mathbf{b}$**.** Such an equation is called a **matrix equation**.

Matrix Equation Ax = b

- **Theorem**: If *A* is an $m \times n$ matrix, with columns $a_1, \dots, a_n$, and if *b* is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = b$$

has the same solution set as the vector equation

$$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = b,$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[a_1 \quad a_2 \quad \cdots \quad a_n \quad b].$$

Existence of Solutions

- The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if **b** is a linear combination of the columns of *A*.
- **Theorem**: Let *A* be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular *A*, either they are all true statements or they are all false.

  a. For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
  b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of *A*.
  c. The columns of *A* span $\mathbb{R}^m$.
  d. A has a pivot position in every row.

Computation of *A*x

- **Example**: Compute $A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

- **Solution**: From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

Computation of *A*x

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \qquad ---(1)$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

Row-vector Rule for Computing Ax

- The matrix with 1s on the diagonal and 0s elsewhere is called an **identity matrix** and is denoted by *I*.

- For example, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

  If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, compute $I\mathbf{x}$.

Properties of the Matrix-vector Product *A*x

- **Theorem**: If *A* is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbb{R}^n$, and c is a scalar, then
  a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$,
  b. $A(c\mathbf{u}) = c(A\mathbf{u})$.
- **Proof**: For simplicity, take $n = 3$, $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ and **u**, **v** $\text{in } \mathbb{R}^3$.
- For $i = 1,2,3,$ $\text{let } u_i \text{ and } v_i$ be the *i*th entries in **u** and **v**, respectively.

Properties of the Matrix-vector Product Ax

- To prove statement (a), compute $A(\mathbf{u}+\mathbf{v})$

  as a linear combination of the

  columns of A using the entries in **u** + **v** as weights.

$$A(\mathbf{u}+\mathbf{v}) = [a_1 \quad a_2 \quad a_3]\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

Entries in $\mathbf{u}+\mathbf{v}$

$$= (u_1+v_1)\mathbf{a}_1 + (u_2+v_2)\mathbf{a}_2 + (u_3+v_3)\mathbf{a}_3$$

Columns of A

$$= (u_1a_1 + u_2a_2 + u_3a_3) + (v_1a_1 + v_2a_2 + v_3a_3)$$

$$= A\mathbf{u} + A\mathbf{v}$$

Properties of the Matrix-vector Product *A*x

- To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of *A* using the entries in *c***u** as weights.

$$A(cu) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)a_1 + (cu_2)a_2 + (cu_3)a_3$$

$$= c(u_1a_1) + c(u_2a_2) + c(u_3a_3)$$

$$= c(u_1a_1 + u_2a_2 + u_3a_3)$$

$$= c(A\mathbf{u})$$

Section 1.4: The Matrix Equation

Section 1.5: Solution Sets of Linear Systems

Section 1.7: Linear Independence

Homogeneous Linear Systems

- A system of linear equations is said to be **homogeneous** if it can be written in the form ***A*x = 0,** where *A* is an $m \times n$ matrix and **0** is the zero vector in $\mathbb{R}^m$.

- Such a system ***A*x = 0** **always** has at least one solution, namely, **x = 0** (the zero vector in $\mathbb{R}^n$).

- This zero solution is usually called the **trivial solution**.

- The homogeneous equation ***A*x = 0** has a nontrivial solution if and only if the equation has at least one free variable.

Homogeneous Linear Systems

- **Example:** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

- **Solution:** Let *A* be the matrix of coefficients of the system and row reduce row the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to echelon form:

Homogeneous Linear Systems

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Since x_3 is a free variable, ***A*x = 0** has nontrivial solutions

  (one for each choice of x_3.)
- Continue the row reduction of $[A \quad \mathbf{0}]$ to **reduced** echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 4/3x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

Homogeneous Linear Systems

- Solve for the basic variables x_1 and x_2 to obtain

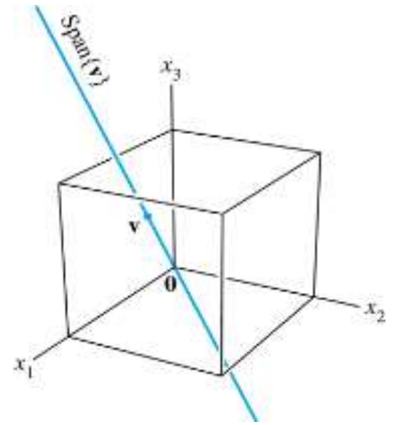
$$x_1 = \frac{4}{3}x_3, \quad x_2 = 0, \quad \text{with } x_3 \text{ free.}$$

  - As a vector, the general solution of ***A*x = 0** has the form given below.

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3 v, \text{ where } v = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous Linear Systems

- Here x_3 is factored out of the expression for the general solution vector.
- This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of **v**.
- The trivial solution is obtained by choosing $x_3 = 0$.
- Geometrically, the solution set is a line through **0** in $\mathbb{R}^3$. See the figure below.



Parametric Vector Form

- The equation of the for $\mathbf{x} = s\mathbf{u} + t\mathbf{v}\,(s, t \text{ in} \mathbb{R})$

  is called a **parametric vector equation** of the plane.

  $\mathbf{x} = x_3\mathbf{v}$ $(\text{with } x_3$ free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$)

- Whenever a solution set is described explicitly with vectors as in Example 1, we say that the solution is in **parametric vector form.**

Solutions of Nonhomogeneous Systems

- When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.
- **Example:** Describe all solutions of ***A*x** = **b**, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathrm{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Solutions of Nonhomogeneous Systems

- **Solution:** Row operations on $[\mathrm{A} \quad \mathbf{b}]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 - \frac{4}{3}x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned} .$$

- Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free.

Solutions of Nonhomogeneous Systems

- As a vector, the general solution of $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ has the form given below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underset{\mathbf{p}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + x_3 \underset{\mathbf{v}}{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}$$

Solutions of Nonhomogeneous Systems

- The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \qquad\qquad ----(1)$$

  describes the solution set of ***A*x** = **b** in parametric vector form.

- The solution set of ***A*x** = **0** has the parametric vector equation

$$\mathbf{x} = t\mathrm{v} \quad (t \text{ in } \mathbb{R}) \qquad\qquad ----(2)$$

  [with the same **v** that appears in (1)].

- Thus the solutions of ***A*x** = **b** are obtained by adding the vector **p** to the solutions of ***A*x** = **0.**

- The vector **p** itself is just one particular solution of ***A*x** = **b** [corresponding to

Solutions of Nonhomogeneous Systems

- **Theorem:** Suppose the equation $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where v_h is any solution of the homogeneous equation $\boldsymbol{A}\mathbf{x} = \mathbf{0}$.

Writing a Solution Set (of a Consistent System) In Parametric Vector Form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Section 1.4: The Matrix Equation

Section 1.5: Solution Sets of Linear Systems

Section 1.7: Linear Independence

Linear Independence

- **Definition:** An indexed set of vectors $\{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots,\ c_p,$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0} \quad ----(1)$$

Linear Independence

- Equation (1) is called a **linear dependence relation** among $\mathrm{v}_1, \ldots, \mathrm{v}_p$ when the weights are not all zero.

- An indexed set is linearly dependent if and only if it is not linearly independent.

- **Example:** Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$

Linear Independence

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1,\ \mathbf{v}_2, \text{and } \mathbf{v}_3.$

- **Solution:** We must determine if there is a nontrivial solution of the following equation.

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Independence

- Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & \\ 2 & 5 & 1 & \\ 3 & 6 & 0 & \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & \\ 0 & -3 & -3 & \\ 0 & 0 & 0 & \end{bmatrix}.$$

- x_1 and x_2 are basic variables, and x_3 is free.
- Each nonzero value of x_3 determines a nontrivial solution of (1).
- Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

Linear Independence

b. To find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$, row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & \\ 0 & 1 & 1 & \\ 0 & 0 & 0 & \end{bmatrix} \qquad \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

- Thus, $x_1 = 2x_3, x_2 = -x_3,$ and x_3 is free.
- Choose any nonzero value for x_3 —say, $x_3 = 5$.
- Then $x_1 = 10$ and $x_2 = -5$.

Linear Independence

- Substitute these values into equation (1) and obtain the equation below.

$$10v_1 - 5v_2 + 5v_3 = 0$$

- This is one (out of infinitely many) possible linear dependence relations among v_1, v_2, and v_3.

Linear Independence of Matrix Columns

- Suppose that we begin with a matrix $A = [a_1 \quad \cdots \quad a_n]$ instead of a set of vectors.
- The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = 0.$$

- Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.
- Thus, the columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Sets of One or Two Vectors

- A set containing only one vector – say, $\mathbf{v}$ – is linearly independent if and only if $\mathbf{v}$ is not the zero vector.

- This is because the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$.

- The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

Sets of One or Two Vectors

- A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Sets of Two or More Vectors

- **Theorem**: Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in *S* is a linear combination of the others.

- In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors,

$\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Sets of Two or More Vectors

- **Proof:** If some v_j in *S* equals a linear combination of the other vectors, then v_j can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (-1) on v_j.
- [For instance, if $v_1 = c_2v_2 + c_3v_3$, then

$$0 = (-1)v_1 + c_2v_2 + c_3v_3 + 0v_4 + \ldots + 0v_p.]$$

- Thus *S* is linearly dependent.
- Conversely, suppose *S* is linearly dependent.
- If v_1 is zero, then it is a (trivial) linear combination of the other vectors in *S*.

Sets of Two or More Vectors

- Otherwise, $v_1 \neq 0$, and there exist weights $c_1 \ldots, , c_p$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0.$$

- Let j be the largest subscript for which $c_j \neq 0$.
- If $j = 1$, then $c_1 v_1 = 0$, which is impossible because $v_1 \neq 0$.

Sets of Two or More Vectors

- So $j > 1$, and

$$c_1\mathbf{v}_1 + \ldots + c_j\mathbf{v}_j + 0\mathbf{v}_j + 0\mathbf{v}_{j+1} + \ldots + 0\mathbf{v}_p = \mathbf{0}$$

$$c_j\mathbf{v}_j = -c_1\mathbf{v}_1 - \ldots - c_{j-1}\mathbf{v}_{j-1}$$

$$\mathbf{v}_j = \left(-\frac{c_1}{c_j}\right)\mathbf{v}_1 + \ldots + \left(-\frac{c_{j-1}}{c_j}\right)\mathbf{v}_{j-1}.$$

Sets of Two or More Vectors

- **Theorem:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
- **Proof:** Let $A = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_p]$.
- Then *A* is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of *n* equations in *p* unknowns.
- If $p > n$, there are more variables than equations, so there must be a free variable.

Sets of Two or More Vectors

- Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and the columns of A are linearly dependent.

- See the figure below for a matrix version of this theorem.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If $p > n$, the columns are linearly dependent.

Sets of Two or More Vectors

- **Theorem:** If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

- **Proof:** By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$.

- Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_p = \mathbf{0}$ shows that *S* in linearly dependent.