Chapter 2 Matrix Algebra

Section 2.4: Partitioned Matrices

Section 2.5: The LU Factorization

Section 2.8: Subspaces of $\mathbb{R}^n$

Section 2.9: Dimension and Rank

Partitioned Matrices

- A key feature of our work with matrices has been the ability to regard matrix A as a list of column vectors rather than just a rectangular array of numbers.
- This point of view has been so useful that we wish to consider other **partitions** of A, indicated by horizontal and vertical dividing rules, as in Example 1 on the next slide.

Partitioned Matrices

- **Example:** The matrix

$$A = \begin{bmatrix} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ -8 & -6 & 3 & 1 & 7 & -4 \end{bmatrix}$$

- Can also be written 2×3 **partitioned (**or **block) matrix** as the

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

- Whose entries are the **blocks (**or **submatrices)**

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

Addition and Scalar Multiplication

- If matrices A and B are the same size and are partitioned in **exactly the same way**, then it is natural to make the same partition of the ordinary matrix sum $A + B$.
- In this case, each block of $A + B$ is the (matrix) sum of the corresponding blocks of A and B.
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

Addition and Scalar Multiplication

- Partitioned matrices can be multiplied by the usual row—column rule as if the block entries were scalars, provided that for a product *AB*, the column partition of *A* matches the row partition of *B*.
- **Example:** Let

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- The 5 columns of *A* are partitioned into a set of 3 columns and then a set of 2 columns. The 5 rows of *B* are partitioned in the same way—into a set of 3 rows and then a set of 2 rows.

Multiplication of Partitioned Matrices

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \\ 2 & 1 \end{bmatrix}$$

- It is important for each smaller product in the expression for *AB* to be written with the submatrix from *A* on the left, since **matrix multiplication is not commutative**.

Multiplication of Partitioned Matrices

- For instance,

$$A_{11}B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}$$

$$A_{12}B_2 = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

- Hence the top block in *AB* is

- $$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ -6 & 2 \end{bmatrix}$$

Multiplication of Partitioned Matrices

- **Theorem**: Column—Row Expansion of AB

If A is $m \times n$ And B is $n \times p,$ then

$$AB = \begin{bmatrix} \text{col}_1(A) & \text{col}_2(A) & \dots & \text{col}_n(A) \end{bmatrix} \begin{bmatrix} \text{row}_1(B) \\ \text{row}_2(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix} \qquad (1)$$

$$= \text{col}_1(A)\text{row}_1(B) + \cdots + \text{col}_n(A)\text{row}_n(B)$$

- **Proof:** For each row index i and column index j, the (i, j)-entry in $\text{col}_k(A)$ and b_{kj} from $\text{row}_k(B)$ is the product of a_{ik} from $\text{col}_k(A)$ and b_{kj} from $\text{row}_k(B)$.

Multiplication of Partitioned Matrices

- Hence the (*i*, *j*)-entry in the sum shown in equation (1) is

$$\underset{(k=1)}{a_{i1}b_{1j}} \quad + \quad \underset{(k=2)}{a_{i2}b_{2j}} \quad + \quad \cdots \quad + \quad \underset{(k=n)}{a_{in}b_{nj}}$$

- This sum is also the (*i*, *j*)-entry in *AB*, by the row—column rule.

Section 2.4: Partitioned Matrices

Section 2.5: The LU Factorization

Section 2.8: Subspaces of $\mathbb{R}^n$

Section 2.9: Dimension and Rank

Matrix factorization

- A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.

The LU Factorization

- The LU factorization is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \ldots, \quad A\mathbf{x} = \mathbf{b}_p \qquad (1)$$

- When A is invertible, one could compute A^{-1} and then compute $A^{-1}\mathbf{b}_1$, $A^{-1}\mathbf{b}_2$, and so on.
- However, it is more efficient to solve the first equation in the sequence (1) by row reduction and obtain the LU factorization of A at the same time. Thereafter, the remaining equations in sequence (1) are solved with the LU factorization.

The LU Factorization

- At first, assume that A is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*.
- Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.
- Such a factorization is called an **LU factorization** of A. The matrix L is invertible and is called a unit lower triangular matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \# & * & * & * & * \\ 0 & \# & * & * & * \\ 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The LU Factorization

- Before studying how to construct L and U, we should look at why they are so useful. When $A = LU$, the equation $A\mathbf{x} = \mathbf{b}$ can be written as $L(U\mathbf{x}) = \mathbf{b}$.
- Writing y for Ux, we can find x by solving the pair of equations

$$\boxed{\begin{aligned} L\mathbf{y} &= \mathbf{b} \\ U\mathbf{x} &= \mathbf{y} \end{aligned}}$$

- First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$, and then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Each equation is easy to solve because L and U are triangular.

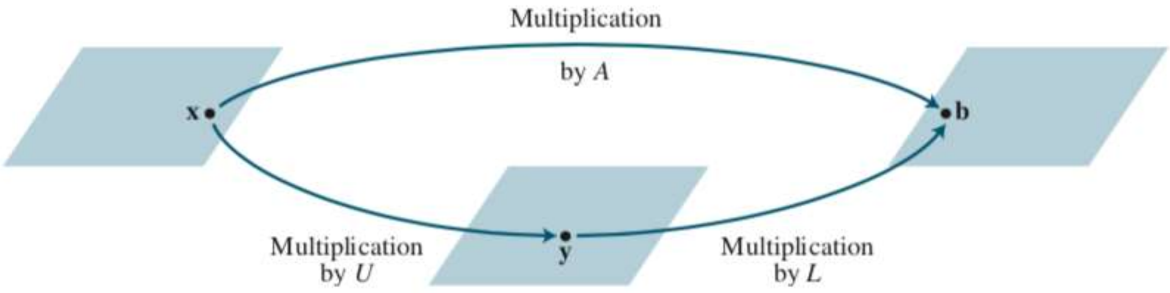

FIGURE 2 Factorization of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

The LU Factorization

- **Example** It can be verified that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

- Use this factorization of *A* to solve *A***x**=**b**, where $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

The LU Factorization

- **Solution**

$$[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I \quad \mathbf{y}]$$

$$[U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

An LU Factorization Algorithm

- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it.
- In this case, there exist unit lower triangular elementary matrices E_1 ,…, E_p such that

$$E_p \ldots E_1 A = U \tag{3}$$

- Then

$$A = (Ep \ldots E_1)^{-1} U = LU$$

where

$$L = (Ep \ldots E_1)^{-1} \tag{4}$$

- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

An LU Factorization Algorithm

- Note that row operations in equation (3), which reduce *A* to *U*, also reduce the *L* in equation (4) to *I*, because $E_p \ldots E_1 L = (E_p \ldots E_1)(E_p \ldots E_1)^{-1} = I$. This observation is the key to *constructing L*.

Algorithm for an LU Factorization

1. Reduce *A* to an echelon form *U* by a sequence of row replacement operations, if possible.
2. Place entries in *L* such that the *same sequence of row operations* reduces *L* to *I*.

An LU Factorization Algorithm

- **Example** Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

- **Solution** Since *A* has four rows, *L* should be 4×4. The first column of *L* is the first column of *A* divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

An LU Factorization Algorithm

$$A=\begin{bmatrix}2 & 4 & -1 & 5 & -2\\ -4 & -5 & 3 & -8 & 1\\ 2 & -5 & -4 & 1 & 8\\ -6 & 0 & 7 & -3 & 1\end{bmatrix}\sim\begin{bmatrix}2 & 4 & -1 & 5 & -2\\ 0 & 3 & 1 & 2 & -3\\ 0 & -9 & -3 & -4 & 10\\ 0 & 12 & 4 & 12 & -5\end{bmatrix}=A_1$$

$$\sim A_2=\begin{bmatrix}2 & 4 & -1 & 5 & -2\\ 0 & 3 & 1 & 2 & -3\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 4 & 7\end{bmatrix}\sim\begin{bmatrix}2 & 4 & -1 & 5 & -2\\ 0 & 3 & 1 & 2 & -3\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 5\end{bmatrix}=U$$

Since *A* has four rows, *L* should be 4×4. The first column of *L* is the first column of *A* divided by the top pivot entry:

$$L=\begin{bmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 1 & & 1 & 0\\ -3 & & & 1\end{bmatrix} \tag{5}$$

An LU Factorization Algorithm

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad [5]$$

$$\div 2 \quad \div 3 \quad \div 2 \quad \div 5$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \qquad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

- An easy calculation verifies that this L and U satisfy $LU = A$.

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Section 2.8: Subspaces of $\mathbb{R}^n$

Section 2.9: Dimension and Rank

Subspaces of $\mathbb{R}^n$

- **Definition**: A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:
  a) The zero vector is in H.
  b) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Subspaces of $\mathbb{R}^n$

- A plane through the origin is the standard way to visualize the subspace.

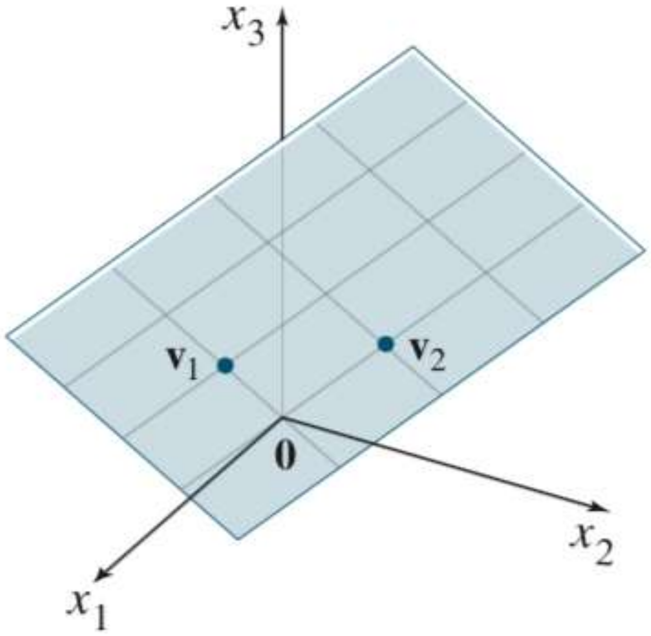

FIGURE 1

Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

Subspaces of $\mathbb{R}^n$

- **Example:** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^n$ and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace of $\mathbb{R}^n$.

Proof:

a. Note that the zero vector is in H because $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$.

b. Take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

Then $\mathbf{u} + \mathbf{v} = (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2$, which is in H.

c. Also, for any scalar c, the vector $c\mathbf{u}$ is in H, because $c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = cs_1(\mathbf{v}_1) + cs_2(\mathbf{v}_2)$.

Column Space and Null Space of A Matrix

- **Definition:** The **column space** of an $m \times n$ matrix A is the set Col A of all linear combinations of the columns of A.

- If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, with columns in $\mathbb{R}^m$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

- The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

Column Space and Null Space of A Matrix

- **Example:** Let $A=\begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.

  Determine whether **b** is in the column space of *A*.

- **Solution:** Row reducing the augmented matrix [*A* **b**],

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can conclude that $A\mathbf{x} = \mathbf{b}$ is consistent and **b** is in Col *A*.

Column Space and Null Space of A Matrix

- **Definition:** The **null space** of a matrix A is the set Nul A of all solutions of the homogenous equation $A\mathbf{x} = 0$.
- **Theorem**: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.
- **Proof:**

a. The zero vector is in Nul A (because $A\mathbf{0} = \mathbf{0}$).

b. Take any $\mathbf{u}$ and $\mathbf{v}$ in Nul A, then

$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{u} + \mathbf{v}$ is in Nul A.

c. For any scalar c, $A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0}$, so $c\mathbf{u}$ is in Nul A

Basis for A Subspace

- **Definition**: A **basis** for a subspace *H* of $\mathbb{R}^n$ is a linearly independent set in *H* that spans *H*.

- **Example:** The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

Basis for A Subspace

- One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$.

Basis for A Subspace

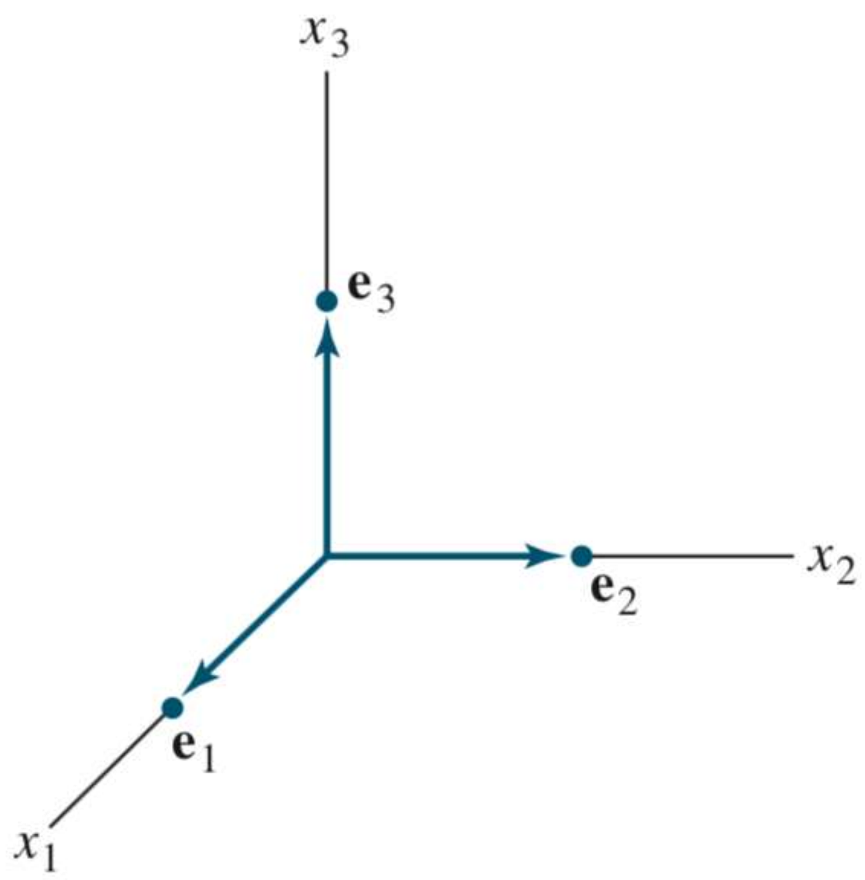

FIGURE 3

The standard basis for $\mathbb{R}^3$.

Basis for A Subspace

- **Theorem**: The pivot columns of a matrix *A* form a basis for the column space of *A*.
- **Example**

$$A = \begin{bmatrix} 2 & 4 & 2 & 6 \\ 1 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Basis for Col *A* = $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 2 \\ 1 \end{bmatrix} \right\}$ Basis for Nul *A* = $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

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Coordinate Systems

- Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for *H*, and suppose a vector **x** in *H* can be generated in two ways, say,

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p \quad \text{and } \mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_p\mathbf{b}_p \qquad (1)$$

- Then, subtracting gives

$$0 = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_p - d_p)\mathbf{b}_p \qquad (2)$$

- Since $\mathcal{B}$ is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \leq j \leq p$, which shows that the two representations in (1) are actually the same.

Coordinate Systems

- **Definition**: Suppose the set B $= \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace *H*. For each **x** in *H*, the coordinates of x relative to the basis B are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

$$[\mathbf{x}]_{\mathrm{B}} = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_p \end{bmatrix}.$$

- is called **the coordinate vector of x (relative to B)** or the **B-coordinate vector of x.**

Coordinate Systems

- **Example:** Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and B = $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- Then B is a basis for *H* = Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
- Determine if **x** is in *H*, and if it is, find the coordinate vector of **x** relative to B.
- **Solution** If **x** is in *H*, then the following vector equation is consistent: $c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$

Coordinate Systems

- The scalars c_1 and c_2, if they exist, are the B-coordinates of **x**. Row operations show that

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basis determines a "coordinate system" on H, which can be visualized by the grid shown on the next slide.

Coordinate Systems

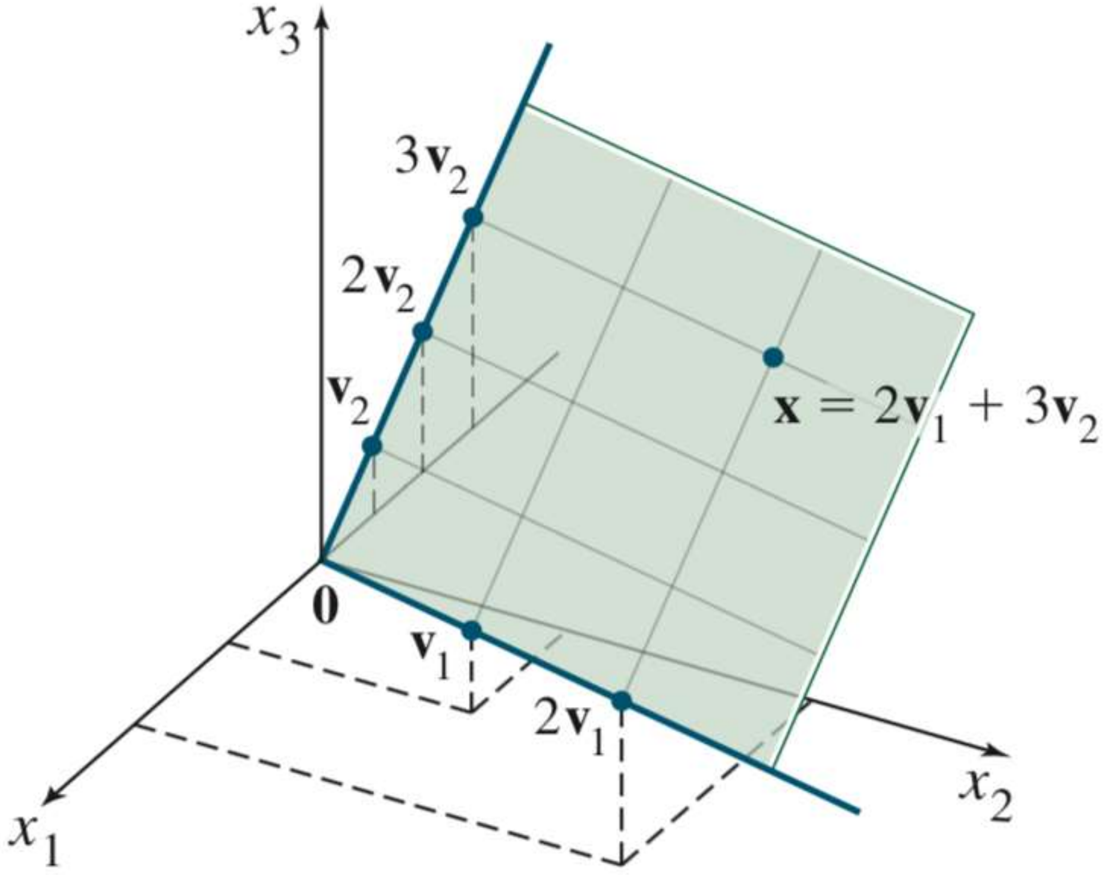

FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

The Dimension of A Subspace

- **Definition**: The **dimension** of a nonzero subspace H, denoted by **dim** $\boldsymbol{H}$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

- **Definition:** The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

The Dimension of A Subspace

- **Example:** Determine the rank of the matrix

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

- **Solution** Reduce *A* to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

- The matrix *A* has 3 pivot columns, so rank *A* = 3.

The Dimension of A Subspace

- **Theorem:** If a matrix A has n columns, then

$$\text{rank } A + \dim \text{Nul } A = n.$$

- **Theorem:** Let H be a p-dimensional subspace of $\mathbb{R}^n$.

a. Any linearly independent set of exactly p elements in H is automatically a basis for H.

b. Also, any set of p elements of H that spans H is automatically a basis for H.

Rank and the Invertible Matrix Theorem

- **The Invertible Theorem (continued)** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. Col $A = \mathbb{R}^n$

  o. rank $A = n$

  p. dim Nul $A = 0$

  q. Nul $A = \{\mathbf{0}\}$

Rank and the Invertible Matrix Theorem

- **Proof** Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning. The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Longrightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (q) \Rightarrow (d)$$

- Statement (g), which says that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, implies statement (n), because Col A is precisely the set of all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Rank and the Invertible Matrix Theorem

- The implications $(n) \Rightarrow (o)$ follow from the definitions of *dimension* and *rank*.
- If the rank of *A* is *n*, the number of columns of *A*, then dim Nul *A* = 0, by the Rank Theorem, and so Nul *A* = {**0**}. Thus $(o) \Rightarrow (p) \Rightarrow (q)$.
- Also, statement (q) implies that the equation *A***x** = **0** has only the trivial solution, which is statement (d).
- Since statements (d) and (g) are already known to be equivalent to the statement that *A* is invertible, the proof is complete.