Section 1.8: Linear Transformations

Linear Transformations

- A **transformation** (or **function** or **mapping**) *T* from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector **x** in $\mathbb{R}^n$ a vector *T* **(x)** in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called **domain** of *T*, and $\mathbb{R}^m$ is called the **codomain** of *T*.
- The notation $T: \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of *T* is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For **x in** $\mathbb{R}^n$, the vector *T* **(x) in** $\mathbb{R}^m$ is called the **image** of **x** (under the action of *T*).
- The set of all images *T*(**x**) is called the **range** of *T*.

Matrix Transformations

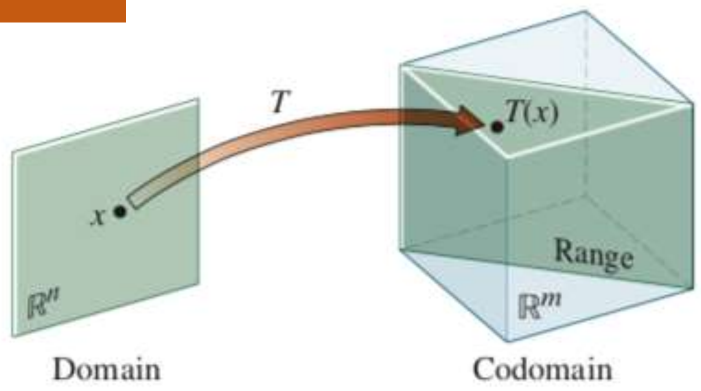

FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

- For each **x** in $\mathbb{R}^n$, *T* **(x)** is computed as *A***x**, where *A* is an $m \times n$ matrix.
- For simplicity, we denote such a **matrix transformation** by $\mathbf{x} \mapsto A\mathbf{x}$.
- The domain of *T* is $\mathbb{R}^n$ when *A* has *n* columns and the codomain of *T* is $\mathbb{R}^m$ when each column of *A* has *m* entries.

Matrix Transformations

- The range of *T* is the set of all linear combinations of the columns of *A*, because each image *T* **(x)** is of the form *A***x**.

  - **Example:** Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix},$$

and define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.$$

Matrix Transformations

a. Find $T(\mathbf{u})$, the image of **u** under the transformation *T*.

b. Find an **x in** $\mathbb{R}^2$ whose image under *T* is **b**.

c. Is there more than one **x** whose image under *T* is **b**?

d. Determine if **c** is in the range of the transformation *T*.

Matrix Transformations

- **Solution:**

a. Compute

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

b. Solve $T(\mathbf{x}) = \mathbf{b}$ for $\mathbf{x}$. That is, solve $A\mathbf{x} = \mathbf{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}. \qquad \text{----(1)}$$

Matrix Transformations

- Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$$

----(2)

- Hence $x_1 = 1.5, x_2 = -.5,$ and $\mathbf{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}.$

- The image of this **x** under *T* is the given vector **b**.

Matrix Transformations

c. Any **x** whose image under *T* is **b** must satisfy equation (1).
   - From (2), it is clear that equation (1) has a unique solution.
   - So there is exactly one **x** whose image is **b**.

d. The vector **c** is in the range of *T* if **c** is the image of some **x** in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some x.
   - This is another way of asking if the system $A\mathrm{x} = \mathbf{c}$ is consistent.

Matrix Transformations

- To find the answer, row reduce the augmented matrix.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

- The third equation, $0 = -35$, shows that the system is inconsistent.
- So **c** is **not** in the range of *T*.

Linear Transformations

- **Definition:** A transformation (or mapping) T is **linear** if:
  i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u, v** in the domain of T;
  ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all **u** in the domain of T.

- **Linear transformations** preserve the operations of vector addition and scalar multiplication.

- These two properties lead to the following useful facts.
- If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$