Chapter 4 Vector Spaces

Section 4.4: Coordinate Systems

Section 4.5: The Dimension of a Vector Space

Section 4.6: Change of Basis

The Unique Representation Theorem

- **Theorem:** Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space *V*. Then for each **x** in *V*, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n \qquad (1)$$

- **Proof:** Since B spans *V*, there exist scalars such that (1) holds.

  Suppose **x** also has the representation for scalars $d_1, \ldots, d_n$.

$$\mathbf{x} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$$

The Unique Representation Theorem

- Then, subtracting, we have

$$0 = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \ldots + (c_n - d_n)\mathbf{b}_n \quad (2)$$

- Since B is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \le j \le n$.
- **Definition:** Suppose B $= \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for *V* and **x** is in *V*. The coordinates of x relative to the basis B (or the B-coordinate of x) are the weights $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

The Unique Representation Theorem

- If $c_1, \ldots, c_n$ are the **B**-coordinates of **x**, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathrm{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

  is the **coordinate vector of x** (relative to B), or the **B-coordinate vector of x**.

- The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathrm{B}}$ is the coordinate mapping (determined by B).

Coordinates in $\mathbb{R}^n$

- When a basis B for $\mathbb{R}^n$ is fixed, the B-coordinate vector of a specified **x** is easily found, as in the example below.
- **Example:** Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and

  $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_B$ of **x** relative to B.
- **Solution:** The B-coordinate c_1, c_2 of **x** satisfy

$$\underbrace{c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{b}_1} + c_2 \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2} = \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\mathbf{x}}$$

Coordinates in $\mathbb{R}^n$

or

$$\begin{array}{cc} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ \mathbf{b}_1 \quad \mathbf{b}_2 \end{array} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ \mathbf{x} \end{array} \qquad (3)$$

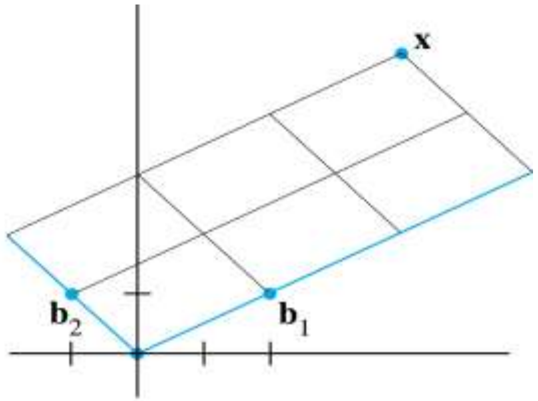
- This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left.
- In any case, the solution is $c_1 = 3$, $c_2 = 2$.
- Thus $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$ and

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Coordinates in $\mathbb{R}^n$

$$\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$$

- See the following figure.



The $\mathcal{B}$-coordinate vector of **x** is (3, 2).

- The matrix in (3) changes the B-coordinates of a vector **x** into the standard coordinates for **x**.
- An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $\text{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.
- Let $P_{\text{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$

Coordinates in $\mathbb{R}^n$

- Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n$$

is equivalent to

$$\mathrm{x} = P_{\mathcal{B}}[\mathrm{x}]_{\mathcal{B}} \qquad (4)$$

- P_B is called the **change-of-coordinates matrix** from B to the standard basis in $\mathbb{R}^n$.

- Left-multiplication by P_B transforms the coordinate vector $[\mathbf{x}]_B$ into $\mathbf{x}$.

- Since the columns of P_B form a basis for $\mathbb{R}^n$, P_B is invertible (by the Invertible Matrix Theorem).

Coordinates in $\mathbb{R}^n$

- Left-multiplication by P_{B}^{-1} converts **x** into its B-coordinate vector:

$$P_{\mathrm{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathrm{B}}$$

- The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathrm{B}}$, produced by P_{B}^{-1}, is the coordinate mapping.

- Since P_{B}^{-1} is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$, by the Invertible Matrix Theorem.

The Coordinate Mapping

- **Theorem:** Let B $=\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ be a basis for a vector space *V*. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from *V* onto $\mathbb{R}^n$.

- **Proof:** Take two typical vectors in *V*, say,

$$\mathbf{u} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

$$\mathbf{w} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$$

- Then, using vector operations,

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{b}_1 + \ldots + (c_n + d_n)\mathbf{b}_n$$

The Coordinate Mapping

- It follows that

$$[\mathbf{u}+\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{w}]_{\mathcal{B}}$$

- So the coordinate mapping preserves addition.
- If *r* is any scalar, then

$$r\mathbf{u} = r(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n) = (rc_1)\mathbf{b}_1 + \ldots + (rc_n)\mathbf{b}_n$$

The Coordinate Mapping

- So

$$[r\mathbf{u}]_B = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\mathbf{u}]_B$$

- Thus the coordinate mapping also preserves scalar multiplication and hence is a linear transformation.
- The linearity of the coordinate mapping extends to linear combinations.
- If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, c_p$ are scalars, then

$$[c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p]_B = c_1[\mathbf{u}_1]_B + \ldots + c_p[\mathbf{u}_p]_B$$

The Coordinate Mapping

- The coordinate mapping in the above theorem is an important example of an *isomorphism* from *V* onto $\mathbb{R}^n$.
- In general, a one-to-one linear transformation from a vector space *V* onto a vector space *W* is called an **isomorphism** from *V* onto *W*.
- *Every vector space calculation in V is accurately reproduced in W, and vice versa.*
- In particular, any real vector space with a basis of *n* vectors is indistinguishable from $\mathbb{R}^n$.

The Coordinate Mapping

- **Example:** Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$,

  and B$=\{\mathbf{v}_1, \mathbf{v}_2\}$. Then B is a basis for $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if **x** is in *H*, and if it is, find the coordinate vector of **x** relative to B.

The Coordinate Mapping

- **Solution:** If **x** is in *H*, then the following vector equation is consistent:

$$c_1\begin{bmatrix}3\\6\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}3\\12\\7\end{bmatrix}$$

- The scalars c_1 and c_2, if they exist, are the B-coordinates of **x**.

The Coordinate Mapping

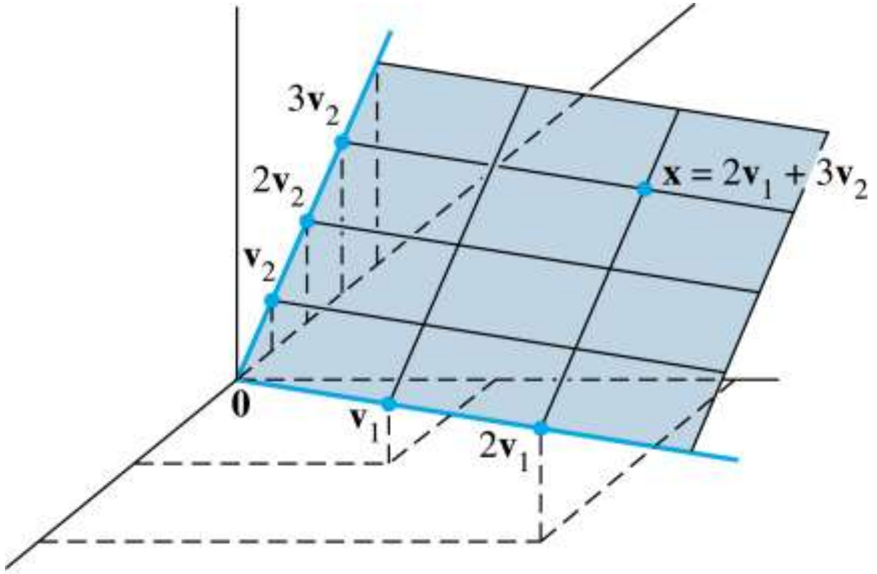
- Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

The Coordinate Mapping

- The coordinate system on *H* determined by B is shown in the following figure.



A coordinate system on a plane H in $\mathbb{R}^3$.

$$B = \{v_1, v_2\}$$

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Section 4.4: Coordinate Systems

Section 4.5: The Dimension of a Vector Space

Section 4.6: Change of Basis

Dimension of a Vector Space

- **Theorem:** If a vector space V has a basis B $= \{\mathbf{b}_1, ..., \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

- **Proof:** Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a set in V with more than n vectors.

The coordinate vectors $[\mathbf{u}_1]_B, \ldots, [\mathbf{u}_p]_B$ form a linearly dependent set in $\mathbb{R}^n$, because there are more vectors (p) than entries (n) in each vector.

Dimension of a Vector Space

- So there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1[\mathrm{u}_1]_{\mathcal{B}} + \ldots + c_p[\mathrm{u}_p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The zero vector in $\mathbb{R}^n$

Since the coordinate mapping is a linear transformation,

$$[c_1\mathrm{u}_1 + \ldots + c_p\mathrm{u}_p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Dimension of a Vector Space

- The zero vector on the right displays the *n* weights needed to build the vector $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ from the basis vectors in B.

- That is, $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p = 0\mathbf{b}_1 + \ldots + 0\mathbf{b}_n = 0.$

- Since the c_i are not all zero, $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly dependent.
- The following theorem implies that if a vector space V has a basis

$\mathrm{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then each linearly independent set in V has no more than n vectors.

Dimension of a Vector Space

- **Theorem:** If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

- **Proof:** Let B_1 be a basis of n vectors and B_2 be any other basis (of V).
- Since B_1 is a basis and B_2 is linearly independent, B_2 has no more than n vectors. Also, since B_2 is a basis and B_1 is linearly independent, B_2 has at least n vectors.
- Thus B_2 consists of exactly n vectors.

Dimension of a Vector Space

- **Definition:** If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space **{0}** is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.
- **Example 1:** Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

Dimension of a Vector Space

- *H* is the set of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

- Clearly, $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, but $\mathbf{v}_3$ is a multiple of $\mathbf{v}_2$.
- By the Spanning Set Theorem, we may discard $\mathbf{v}_3$ and still have a set that spans *H*.

Subspaces of a Finite-Dimensional Space

- Finally, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent and hence is a basis for H.
- Thus dim

$$H = 3$$

- **Theorem:** Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Subspaces of a Finite-Dimensional Space

- **Proof:** If $H = \{\mathbf{0}\}$, then certainly $\dim H = 0 \le \dim V$.
- Otherwise, let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be any linearly independent set in *H*.
- If *S* spans *H*, then *S* is a basis for *H*.
- Otherwise, there is some $\mathbf{u}_{k+1}$ in *H* that is not in Span *S*.

Subspaces of a Finite-Dimensional Space

- But then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ will be linearly independent, because no vector in the set can be a linear combination of vectors that precede it.
- So long as the new set does not span *H*, we can continue this process of expanding *S* to a larger linearly independent set in *H*.
- But the number of vectors in a linearly independent expansion of *S* can never exceed the dimension of *V*.

So eventually the expansion of S will span H and hence will be a basis for H, and $\dim H \le \dim V$.

The Basis Theorem

- **Theorem:** Let V be a p-dimensional vector space, $p \geq 1$.

a. Any linearly independent set of exactly p elements in V is automatically a basis for V.

b. Any set of exactly p elements that spans V is automatically a basis for V.

The Basis Theorem

- **Proof:** By Theorem, a linearly independent set S of p elements can be extended to a basis for V.
- But that basis must contain exactly p elements, since dim $V = p$.
- So S must already be a basis for V.
- Now suppose that S has p elements and spans V.

The Basis Theorem

- Since V is nonzero, the Spanning Set Theorem implies that a subset S' of S is a basis of V.

- Since $\dim V = p$, S' must contain p vectors.
- Hence $S = S'$.

The Rank Theorem

- **Definition:** The **rank** of A is the dimension of the column space of A.
- Since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T.
- The dimension of the null space is called the **nullity** of A.

The Rank Theorem

- **Theorem:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal.
- This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank}\, A + \text{nullity}\, A = n$$

The Rank Theorem

- **Example:**
  a. If *A* is a 7×9 matrix with a two-dimensional null space, what is the rank of *A*?
  b. Could a 6×9 matrix have a two-dimensional null space?
- **Solution:**
  a. Since *A* has 9 columns, $(\text{rank } A) + 2 = 9$, and hence rank $A = 7$.

The Rank Theorem

b. No. If a 6×9 matrix, call it *B*, has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem.

But the columns of *B* are vectors in $\mathbb{R}^6$, and so the dimension of Col *B* cannot exceed 6; that is, rank *B* cannot exceed 6.

Rank and Nullity

- **Example:** Find the rank and nullity of *A*.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Rank and Nullity

- **Solution:** Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are three free variable—x_2, x_4 and x_5.
- Hence the nullity $A = 3$.
- Also rank $A = 2$ because A has two pivot columns.

The Invertible Matrix Theorem

- **Theorem:** Let *A* be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that *A* is an invertible matrix.
  - m. The columns of *A* form a basis of $\mathbb{R}^n$.
  - n. Col $A = \mathbb{R}^n$
  - o. Rank $A = n$
  - p. Nullity *A* = 0
  - q. Nul *A* = {**0**}

Section 4.4: Coordinate Systems

Section 4.5: The Dimension of a Vector Space

Section 4.6: Change of Basis

Change of Basis

- **Example:** Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2 \tag{1}$$

- Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 \tag{2}$$

- That is, suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

Change of Basis

- **Solution** Apply the coordinate mapping determined by $\mathcal{C}$ to **x** in (2). Since the coordinate mapping is a linear transformation,

$$[\mathbf{x}]_{\mathcal{C}} = [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}}$$
$$= [3\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}}$$

- We can write the vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad (3)$$

Change of Basis

- This formula gives $[\mathbf{x}]_{\mathcal{C}}$, once we know the columns of the matrix. From (1),

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

- Thus, (3) provides the solution:

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Change of Basis

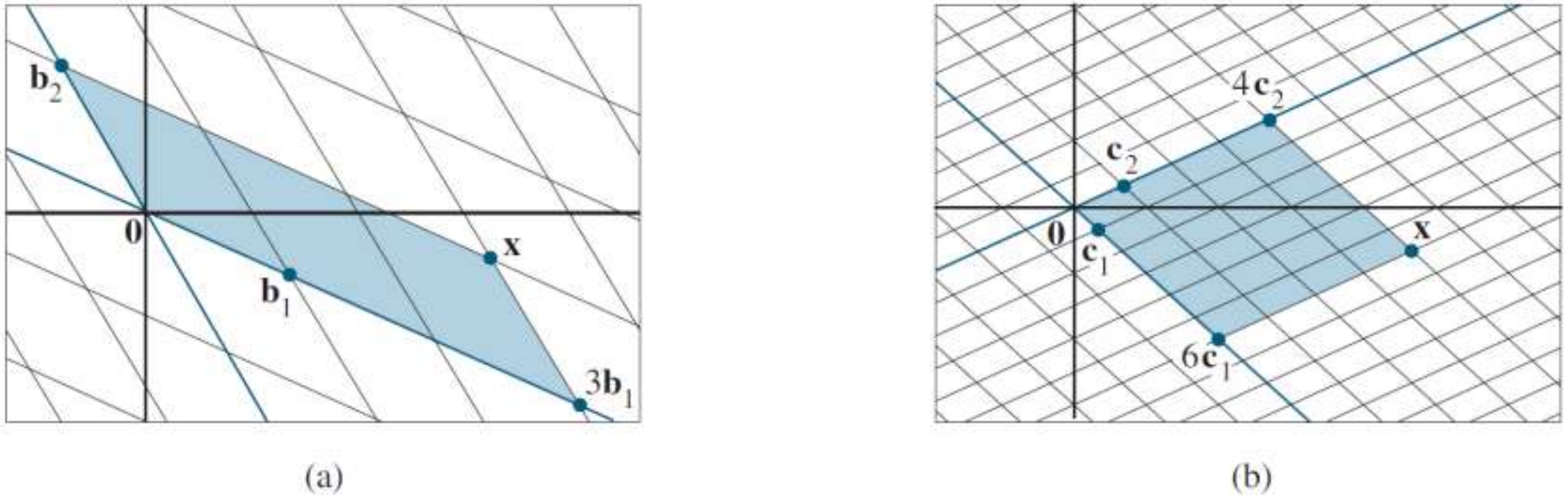


FIGURE 1 Two coordinate systems for the same vector space.

Change of Basis

- **Theorem**: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ for a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}} \qquad (4)$$

- The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis . That is,

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[\, [\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}} \,\right] \qquad (5)$$

Change of Basis

- The matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ in is called the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. Multiplication by $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.
- Figure 2 below illustrates the change-of-coordinates equation (4).

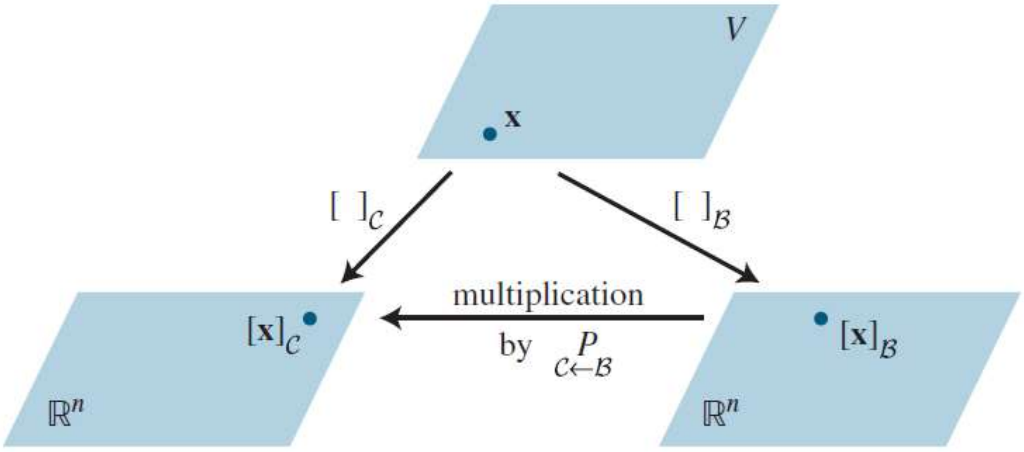


FIGURE 2 Two coordinate systems for V.

Change of Basis

- The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$.
- Since $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of equation (4) by $\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1}$ yields

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}$$

Change of Basis

- Thus $\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates into $\mathcal{B}$–coordinates. That is,

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} \qquad (6)$$

Change of Basis in $\mathbb{R}^n$

- If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_{\mathcal{E}} = \mathbf{b}_1$, and likewise for the other vectors in $\mathcal{B}$. In this case, $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4, namely,

$$P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \ \ldots \ \mathbf{b}_n\,]$$

- To change coordinates between two nonstandard bases in $\mathbb{R}^n$, we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

Change of Basis in $\mathbb{R}^n$

- **Example:** Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for $\mathbb{R}^n$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

- **Solution** The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ involves the $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[\mathbf{c}_1 \quad \mathbf{c}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad [\mathbf{c}_1 \quad \mathbf{c}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2$$

Change of Basis in $\mathbb{R}^n$

- To solve both systems simultaneously, augment the coefficient matrix with $\mathbf{b}_1$ and $\mathbf{b}_2$, and row reduce:

$$[\mathbf{c}_1 \;\; \mathbf{c}_2 \;\; \mathbf{b}_1 \;\; \mathbf{b}_2] = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix} \tag{7}$$

- Thus

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{ and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

- The desired change-of-coordinates matrix is therefore

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[\mathbf{b}_1]_{\mathcal{C}} \;\; [\mathbf{b}_2]_{\mathcal{C}} \right] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$