Chapter 2 Matrix Algebra

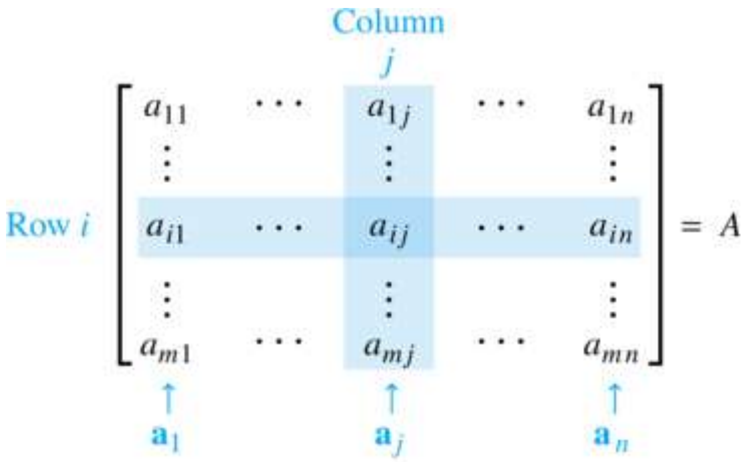
Section 2.1: Matrix Operations

Section 2.2: The Inverse of a Matrix

Section 2.3: Characterizations of Invertible Matrices

Matrix Operations

- If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A.
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.



Matrix notation.

Matrix Operations

- The columns are denoted by $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and the matrix *A* is written as $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$
- The number a_{ij} is the *i*th entry (from the top) of the *j*th column vector $\mathbf{a}_j$.
- The **diagonal entries** in an $m \times n$ matrix $A = \left[a_{ij} \right]$ are $a_{11}, a_{22}, a_{33}, \ldots,$ and they form the **main diagonal** of *A*.
- A **diagonal matrix** is an $n \times n$ matrix whose nondiagonal entries are zero.
- An example is the $n \times n$ identity matrix, I_n.

Sums and Scalar Multiples

- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0.
- Two matrices are **equal** if they have the same size and their corresponding entries are equal.
- If *A* and *B* are $m \times n$ matrices, then the **sum** *A*+*B* is the $m \times n$ matrix whose columns are the sums of the corresponding columns in *A* and *B*.

Sums and Scalar Multiples

The **sum $A + B$** is defined only when A and B are the **same size**.

- **Example**: Let $A=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix},$

  and $C=\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. Find $A+B$ and $A+C$.

- **Solution:** $A+B=\begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$ but $A+C$ is not defined

  because A and C have different sizes.

Sums and Scalar Multiples

- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.

Example: For $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$, compute $3C$.

Sums and Scalar Multiples

- **Theorem**: Let *A*, *B*, and *C* be matrices of the same size, and let *r* and *s* be scalars.
  a. $A + B = B + A$
  b. $(A + B) + C = A + (B + C)$
  c. $A + 0 = A$
  d. $r(A + B) = rA + rB$
  e. $(r + s)A = rA + sA$
  f. $r(sA) = (rs)A$
- Each quantity in Theorem 1 is verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal.

Matrix Multiplication

- **Row—column rule for computing *AB***
- If a product *AB* is defined, then the entry in row *i* and column *j* of *AB* is the sum of the products of corresponding entries from row *i* of *A* and column *j* of *B*.
- If $(AB)_{ij}$ denotes the (*i*, *j*)-entry in *AB*, and if *A* is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + \ldots + a_{in}b_{nj}.$$

Matrix Multiplication

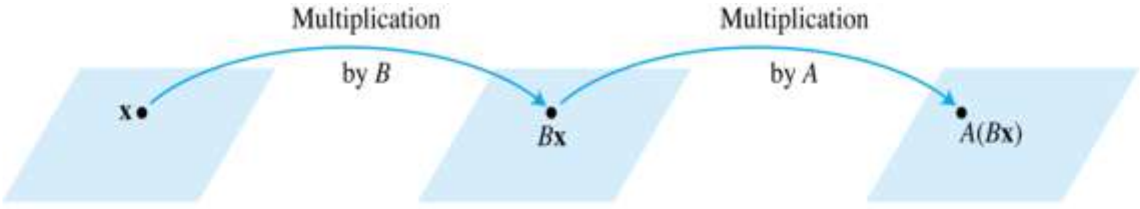
- **Definition**: If *A* is an $m \times n$ matrix, and if *B* is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p,$ then the product *AB* is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p.$
- That is, $AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$

- **Multiplication of matrices corresponds to composition of linear transformations.**

Each column of *AB* is a linear combination of the columns of *A* using weights from the corresponding column of *B*.

Matrix Multiplication

- When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.
- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$.



Multiplication by B and then A.

- Thus $A(B\mathbf{x})$ is produced from $\mathbf{x}$ by a composition of mappings.

Matrix Multiplication

- **Example**: Compute *AB*, where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}.$$

- **Solution**: Write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ and compute:

Matrix Multiplication

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

- Then

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{matrix}$$

Properties of Matrix Multiplication

- **Theorem:** Let *A* be an $m \times n$ matrix, and let *B* and *C* have sizes for which the indicated sums and products are defined.
  a. $A(BC) = (AB)C$ (associative law of multiplication)
  b. $A(B+C) = AB + AC$ (left distributive law)
  c. $(B+C)A = BA + CA$ (right distributive law)
  d. $r(AB) = (rA)B = A(rB)$ for any scalar r
  e. $I_m A = A = A I_n$ (identity for matrix multiplication)

Properties of Matrix Multiplication

- If $AB = BA$, we say that A and B **commute** with one another.
- **Warnings:**
  1. In general, $AB \neq BA$.
  2. The cancellation laws do **not** hold for matrix multiplication. That is, if $AB = AC$, then it is **not** true in general that $B = C$.
  3. If a product AB is the zero matrix, you **cannot** conclude in general that either $A = 0$ or $B = 0$.

Powers of a Matrix

- If *A* is an $n \times n$ matrix and if *k* is a positive integer, then A^k denotes the product of *k* copies of *A*:

$$A^k = \underbrace{A \cdots A}_{k}$$

- If *A* is nonzero and if **x** is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying **x** by *A* repeatedly *k* times.
- If $k = 0$, then $A^0\mathbf{x}$ should be **x** itself.
- Thus A^0 is interpreted as the identity matrix.

The Transpose of a Matrix

- Given an $m \times n$ matrix *A*, the **transpose** of *A* is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of *A*.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

The Transpose of a Matrix

Theorem: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

The transpose of a product of matrices equals the product of their transposes in the **reverse** order.

Section 2.1: Matrix Operations

Section 2.2: The Inverse of a Matrix

Section 2.3: Characterizations of Invertible Matrices

Matrix Operations

- An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I$$

where $I = I_n$, the $n \times n$ identity matrix.

  - In this case, C is an inverse of A.
  - In fact, C is **uniquely determined** by A, because if B were another inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

  - This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \text{ and } AA^{-1} = I$$

Matrix Operations

- **Theorem**: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  If $ad - bc = 0$, then A is not invertible.

- The quantity $ad - bc$ is called the determinant of A, and we write $\det A = ad - bc$
- This theorem says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Matrix Operations

- **Theorem**: If A is an invertible $n \times n$ matrix, then for each **b** in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}.$
  - **Proof**: Take any **b** in $\mathbb{R}^n$.
  - A solution exists because if $A^{-1}\mathbf{b}$ is substituted for **x**, then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}.$
  - So $A^{-1}\mathbf{b}$ is a solution.
  - To prove that the solution is unique, show that if **u** is any solution, then **u** must be $A^{-1}\mathbf{b}$.
  - If $A\mathbf{u} = \mathbf{b}$, we can multiply both sides by A^{-1} and obtain $A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}$, $I\mathbf{u} = A^{-1}\mathbf{b}$, and $\mathbf{u} = A^{-1}\mathbf{b}$.

Matrix Operations

- **Theorem:**

a. If *A* is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If *A* and *B* are $n \times n$ invertible matrices, then so is *AB*, and the inverse of *AB* is the product of the inverses of *A* and *B* in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If *A* is an invertible matrix, then so is A^T, and the inverse of A^T, is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

- **Proof**: To verify statement (a), find a matrix *C* such that

$$A^{-1}C = I \text{ and } CA^{-1} = I$$

- These equations are satisfied with *A* in place of *C*. Hence A^{-1} is invertible, and *A* is its inverse.
- Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

- A similar calculation shows that $(B^{-1}A^{-1})(AB) = I.$
- For statement (c), use Theorem 3(d), read from right to left, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$
- Similarly, $A^T(A^{-1})^T = I^T = I.$

Matrix Operations

- Hence A^T is invertible, and its inverse is $(A^{-1})^T$.
- The generalization of Theorem 6(b) is as follows:

  The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.
- An invertible matrix *A* is row equivalent to an identity matrix, and we can find A^{-1} by watching the row reduction of *A* to *I*.

Elementary Matrices

- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

Elementary Matrices

- **Example**: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on *A*.

Elementary Matrices

- **Solution**: Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

- Addition of -4 times row 1 of A to row 3 produces E_1A.

Elementary Matrices

- An interchange of rows 1 and 2 of *A* produces E_2A, and multiplication of row 3 of *A* by 5 produces E_3A.
- Left-multiplication by E_1 in Example 1 has the same effect on any $3 \times n$ matrix.
- Since $E_1 I = E_1$, we see that E_1 itself is produced by this same row operation on the identity.

Elementary Matrices

- If an elementary row operation is performed on an $m \times n$ matrix *A*, the resulting matrix can be written as *EA*, where the $m \times m$ matrix *E* is created by performing the same row operation on I_m.
- Each elementary matrix *E* is invertible. The inverse of *E* is the elementary matrix of the same type that transforms *E* back into *I*.

Elementary Matrices

- **Theorem 7**: An $n \times n$ matrix *A* is invertible if and only if *A* is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces *A* to I_n also transforms I_n into A^{-1}.
- **Proof**: Suppose that *A* is invertible.
- Then, since the equation *A***x** = **b** has a solution for each **b** (Theorem 5), *A* has a pivot position in every row.
- Because *A* is square, the *n* pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n. That is, $A \sim I_n$.

Elementary Matrices

- Now suppose, conversely, that $A \sim I_n$.
- Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $E_1, \ldots, E_p$ such that

$$A \sim E_1 A \sim E_2(E_1 A) \sim \ldots \sim E_p(E_{p-1} \ldots E_1 A) = I_n.$$

- That is, $$E_p \ldots E_1 A = I_n \qquad ---(1)$$
- Since the product $E_p \ldots E_1$ of invertible matrices is invertible, (1) leads to $(E_p \ldots E_1)^{-1}(E_p \ldots E_1)A = (E_p \ldots E_1)^{-1} I_n$

$$A = (E_p \ldots E_1)^{-1}$$

Algorithm for Finding Inverse of A

- Thus *A* is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also,

$$A^{-1} = \left[(E_p \ldots E_1)^{-1}\right]^{-1} = E_p \ldots E_1.$$

- Then $A^{-1} = E_p \ldots E_1 \cdot I_n$, which says that A^{-1} results from applying $E_1, \ldots, E_p$ successively to I_n.
- This is the same sequence in (1) that reduced *A* to I_n.
- Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If *A* is row equivalent to *I*, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, *A* does not have an inverse.

Algorithm for Finding Inverse of A

- **Example**: Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}, \text{ if it exists.}$$

- **Solution:**

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

Algorithm for Finding Inverse of A

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Reasonable Answers

- Theorem 7 shows, since $A \sim I,$ that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

- Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 2.1: Matrix Operations

Section 2.2: The Inverse of a Matrix

Section 2.3: Characterizations of Invertible Matrices

The Invertible Matrix Theorem

- Let A be an $n \times n$ matrix. The following are equivalent:
  a. The matrix A is an invertible matrix.
  b. The matrix A is row equivalent to the $n \times n$ identity matrix.
  c. A has n pivot positions.
  d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  e. The columns of A form a linearly independent set.

The Invertible Matrix Theorem

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

The Invertible Matrix Theorem

- Additionally, whenever *A* is invertible:

  A^{-1} is invertible and $(A^{-1})^{-1} = A$

- The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes:

the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

The Invertible Matrix Theorem

- **Example:** Use the Invertible Matrix Theorem to decide if A is invertible:

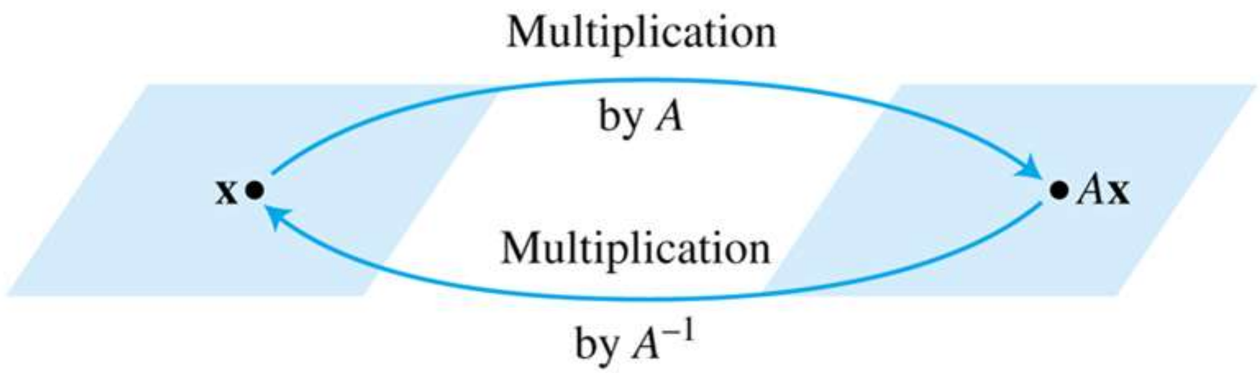
$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

  - **Solution:**

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Invertible Linear Transformations

- Matrix multiplication corresponds to composition of linear transformations.
- When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. See the following figure.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

Invertible Linear Transformations

- A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathrm{x})) = \mathrm{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad \text{----(1)}$$

$$T(S(\mathrm{x})) = \mathrm{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad \text{----(2)}$$

- **Theorem:** Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let *A* be the standard matrix for *T*. Then *T* is invertible if and only if *A* is an invertible matrix. In that case, the linear transformation *S* given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equation (1) and (2).