

# Induction and Inductance

## 30-1 FARADAY'S LAW AND LENZ'S LAW

### Learning Objectives

After reading this module, you should be able to . . .

- 30.01** Identify that the amount of magnetic field piercing a surface (not skimming along the surface) is the magnetic flux  $\Phi_B$  through the surface.
- 30.02** Identify that an area vector for a flat surface is a vector that is perpendicular to the surface and that has a magnitude equal to the area of the surface.
- 30.03** Identify that any surface can be divided into area elements (patch elements) that are each small enough and flat enough for an area vector  $d\vec{A}$  to be assigned to it, with the vector perpendicular to the element and having a magnitude equal to the area of the element.
- 30.04** Calculate the magnetic flux  $\Phi_B$  through a surface by integrating the dot product of the magnetic field vector  $\vec{B}$  and the area vector  $d\vec{A}$  (for patch elements) over the surface, in magnitude-angle notation and unit-vector notation.
- 30.05** Identify that a current is induced in a conducting loop while the number of magnetic field lines intercepted by the loop is changing.
- 30.06** Identify that an induced current in a conducting loop is driven by an induced emf.
- 30.07** Apply Faraday's law, which is the relationship between an induced emf in a conducting loop and the rate at which magnetic flux through the loop changes.
- 30.08** Extend Faraday's law from a loop to a coil with multiple loops.
- 30.09** Identify the three general ways in which the magnetic flux through a coil can change.
- 30.10** Use a right-hand rule for Lenz's law to determine the direction of induced emf and induced current in a conducting loop.
- 30.11** Identify that when a magnetic flux through a loop changes, the induced current in the loop sets up a magnetic field to oppose that change.
- 30.12** If an emf is induced in a conducting loop containing a battery, determine the net emf and calculate the corresponding current in the loop.

### Key Ideas

- The magnetic flux  $\Phi_B$  through an area  $A$  in a magnetic field  $\vec{B}$  is defined as

$$\Phi_B = \int \vec{B} \cdot d\vec{A},$$

where the integral is taken over the area. The SI unit of magnetic flux is the weber, where  $1 \text{ Wb} = 1 \text{ T} \cdot \text{m}^2$ .

- If  $\vec{B}$  is perpendicular to the area and uniform over it, the flux is

$$\Phi_B = BA \quad (\vec{B} \perp A, \vec{B} \text{ uniform}).$$

- If the magnetic flux  $\Phi_B$  through an area bounded by a closed conducting loop changes with time, a current and

an emf are produced in the loop; this process is called induction. The induced emf is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad (\text{Faraday's law}).$$

- If the loop is replaced by a closely packed coil of  $N$  turns, the induced emf is

$$\mathcal{E} = -N \frac{d\Phi_B}{dt}.$$

- An induced current has a direction such that the magnetic field *due to the current* opposes the change in the magnetic flux that induces the current. The induced emf has the same direction as the induced current.

## What Is Physics?

In Chapter 29 we discussed the fact that a current produces a magnetic field. That fact came as a surprise to the scientists who discovered the effect. Perhaps even more surprising was the discovery of the reverse effect: A magnetic field can produce an electric field that can drive a current. This link between a magnetic field and the electric field it produces (*induces*) is now called *Faraday's law of induction*.

The observations by Michael Faraday and other scientists that led to this law were at first just basic science. Today, however, applications of that basic science are almost everywhere. For example, induction is the basis of the electric guitars that revolutionized early rock and still drive heavy metal and punk today. It is also the basis of the electric generators that power cities and transportation lines and of the huge induction furnaces that are commonplace in foundries where large amounts of metal must be melted rapidly.

Before we get to applications like the electric guitar, we must examine two simple experiments about Faraday's law of induction.

## Two Experiments

Let us examine two simple experiments to prepare for our discussion of Faraday's law of induction.

**First Experiment.** Figure 30-1 shows a conducting loop connected to a sensitive ammeter. Because there is no battery or other source of emf included, there is no current in the circuit. However, if we move a bar magnet toward the loop, a current suddenly appears in the circuit. The current disappears when the magnet stops. If we then move the magnet away, a current again suddenly appears, but now in the opposite direction. If we experimented for a while, we would discover the following:

1. A current appears only if there is relative motion between the loop and the magnet (one must move relative to the other); the current disappears when the relative motion between them ceases.
2. Faster motion produces a greater current.
3. If moving the magnet's north pole toward the loop causes, say, clockwise current, then moving the north pole away causes counterclockwise current. Moving the south pole toward or away from the loop also causes currents, but in the reversed directions.

The current produced in the loop is called an **induced current**; the work done per unit charge to produce that current (to move the conduction electrons that constitute the current) is called an **induced emf**; and the process of producing the current and emf is called **induction**.

**Second Experiment.** For this experiment we use the apparatus of Fig. 30-2, with the two conducting loops close to each other but not touching. If we close switch S, to turn on a current in the right-hand loop, the meter suddenly and briefly registers a current — an induced current — in the left-hand loop. If we then open the switch, another sudden and brief induced current appears in the left-hand loop, but in the opposite direction. We get an induced current (and thus an induced emf) only when the current in the right-hand loop is changing (either turning on or turning off) and not when it is constant (even if it is large).

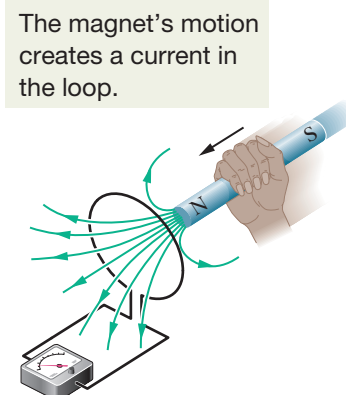
The induced emf and induced current in these experiments are apparently caused when something changes — but what is that “something”? Faraday knew.

## Faraday's Law of Induction

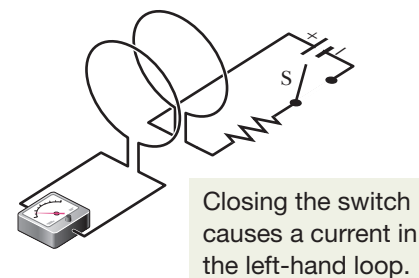
Faraday realized that an emf and a current can be induced in a loop, as in our two experiments, by changing the *amount of magnetic field* passing through the loop. He further realized that the “amount of magnetic field” can be visualized in terms of the magnetic field lines passing through the loop. **Faraday's law of induction**, stated in terms of our experiments, is this:



An emf is induced in the loop at the left in Figs. 30-1 and 30-2 when the number of magnetic field lines that pass through the loop is changing.



**Figure 30-1** An ammeter registers a current in the wire loop when the magnet is moving with respect to the loop.



**Figure 30-2** An ammeter registers a current in the left-hand wire loop just as switch S is closed (to turn on the current in the right-hand wire loop) or opened (to turn off the current in the right-hand loop). No motion of the coils is involved.

The actual number of field lines passing through the loop does not matter; the values of the induced emf and induced current are determined by the *rate* at which that number changes.

In our first experiment (Fig. 30-1), the magnetic field lines spread out from the north pole of the magnet. Thus, as we move the north pole closer to the loop, the number of field lines passing through the loop increases. That increase apparently causes conduction electrons in the loop to move (the induced current) and provides energy (the induced emf) for their motion. When the magnet stops moving, the number of field lines through the loop no longer changes and the induced current and induced emf disappear.

In our second experiment (Fig. 30-2), when the switch is open (no current), there are no field lines. However, when we turn on the current in the right-hand loop, the increasing current builds up a magnetic field around that loop and at the left-hand loop. While the field builds, the number of magnetic field lines through the left-hand loop increases. As in the first experiment, the increase in field lines through that loop apparently induces a current and an emf there. When the current in the right-hand loop reaches a final, steady value, the number of field lines through the left-hand loop no longer changes, and the induced current and induced emf disappear.

### A Quantitative Treatment

To put Faraday's law to work, we need a way to calculate the *amount of magnetic field* that passes through a loop. In Chapter 23, in a similar situation, we needed to calculate the amount of electric field that passes through a surface. There we defined an electric flux  $\Phi_E = \int \vec{E} \cdot d\vec{A}$ . Here we define a *magnetic flux*: Suppose a loop enclosing an area  $A$  is placed in a magnetic field  $\vec{B}$ . Then the **magnetic flux** through the loop is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \quad (\text{magnetic flux through area } A). \quad (30-1)$$

As in Chapter 23,  $d\vec{A}$  is a vector of magnitude  $dA$  that is perpendicular to a differential area  $dA$ . As with electric flux, we want the component of the field that *pierces* the surface (not skims along it). The dot product of the field and the area vector automatically gives us that piercing component.

**Special Case.** As a special case of Eq. 30-1, suppose that the loop lies in a plane and that the magnetic field is perpendicular to the plane of the loop. Then we can write the dot product in Eq. 30-1 as  $B \, dA \cos 0^\circ = B \, dA$ . If the magnetic field is also uniform, then  $B$  can be brought out in front of the integral sign. The remaining  $\int dA$  then gives just the area  $A$  of the loop. Thus, Eq. 30-1 reduces to

$$\Phi_B = BA \quad (\vec{B} \perp \text{area } A, \vec{B} \text{ uniform}). \quad (30-2)$$

**Unit.** From Eqs. 30-1 and 30-2, we see that the SI unit for magnetic flux is the tesla–square meter, which is called the *weber* (abbreviated Wb):

$$1 \text{ weber} = 1 \text{ Wb} = 1 \text{ T} \cdot \text{m}^2. \quad (30-3)$$

**Faraday's Law.** With the notion of magnetic flux, we can state Faraday's law in a more quantitative and useful way:



The magnitude of the emf  $\mathcal{E}$  induced in a conducting loop is equal to the rate at which the magnetic flux  $\Phi_B$  through that loop changes with time.

As you will see below, the induced emf  $\mathcal{E}$  tends to oppose the flux change, so Faraday's law is formally written as

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad (\text{Faraday's law}), \quad (30-4)$$

with the minus sign indicating that opposition. We often neglect the minus sign in Eq. 30-4, seeking only the magnitude of the induced emf.

If we change the magnetic flux through a coil of  $N$  turns, an induced emf appears in every turn and the total emf induced in the coil is the sum of these individual induced emfs. If the coil is tightly wound (*closely packed*), so that the same magnetic flux  $\Phi_B$  passes through all the turns, the total emf induced in the coil is

$$\mathcal{E} = -N \frac{d\Phi_B}{dt} \quad (\text{coil of } N \text{ turns}). \quad (30-5)$$

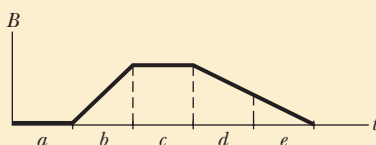
Here are the general means by which we can change the magnetic flux through a coil:

1. Change the magnitude  $B$  of the magnetic field within the coil.
2. Change either the total area of the coil or the portion of that area that lies within the magnetic field (for example, by expanding the coil or sliding it into or out of the field).
3. Change the angle between the direction of the magnetic field  $\vec{B}$  and the plane of the coil (for example, by rotating the coil so that field  $\vec{B}$  is first perpendicular to the plane of the coil and then is along that plane).



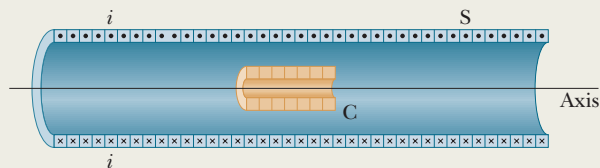
### Checkpoint 1

The graph gives the magnitude  $B(t)$  of a uniform magnetic field that exists throughout a conducting loop, with the direction of the field perpendicular to the plane of the loop. Rank the five regions of the graph according to the magnitude of the emf induced in the loop, greatest first.



### Sample Problem 30.01 Induced emf in coil due to a solenoid

The long solenoid  $S$  shown (in cross section) in Fig. 30-3 has 220 turns/cm and carries a current  $i = 1.5$  A; its diameter  $D$  is 3.2 cm. At its center we place a 130-turn closely packed coil  $C$  of diameter  $d = 2.1$  cm. The current in the solenoid is reduced to zero at a steady rate in 25 ms. What is the magnitude of the emf that is induced in coil  $C$  while the current in the solenoid is changing?



**Figure 30-3** A coil  $C$  is located inside a solenoid  $S$ , which carries current  $i$ .

### KEY IDEAS

1. Because it is located in the interior of the solenoid, coil  $C$  lies within the magnetic field produced by current  $i$  in the solenoid; thus, there is a magnetic flux  $\Phi_B$  through coil  $C$ .
2. Because current  $i$  decreases, flux  $\Phi_B$  also decreases.
3. As  $\Phi_B$  decreases, emf  $\mathcal{E}$  is induced in coil  $C$ .
4. The flux through each turn of coil  $C$  depends on the area  $A$  and orientation of that turn in the solenoid's magnetic field  $\vec{B}$ . Because  $\vec{B}$  is uniform and directed perpendicular to area  $A$ , the flux is given by Eq. 30-2 ( $\Phi_B = BA$ ).
5. The magnitude  $B$  of the magnetic field in the interior of a solenoid depends on the solenoid's current  $i$  and its number  $n$  of turns per unit length, according to Eq. 29-23 ( $B = \mu_0 i n$ ).



**Calculations:** Because coil C consists of more than one turn, we apply Faraday's law in the form of Eq. 30-5 ( $\mathcal{E} = -N d\Phi_B/dt$ ), where the number of turns  $N$  is 130 and  $d\Phi_B/dt$  is the rate at which the flux changes.

Because the current in the solenoid decreases at a steady rate, flux  $\Phi_B$  also decreases at a steady rate, and so we can write  $d\Phi_B/dt$  as  $\Delta\Phi_B/\Delta t$ . Then, to evaluate  $\Delta\Phi_B$ , we need the final and initial flux values. The final flux  $\Phi_{B,f}$  is zero because the final current in the solenoid is zero. To find the initial flux  $\Phi_{B,i}$ , we note that area  $A$  is  $\frac{1}{4}\pi d^2$  ( $= 3.464 \times 10^{-4} \text{ m}^2$ ) and the number  $n$  is 220 turns/cm, or 22 000 turns/m. Substituting Eq. 29-23 into Eq. 30-2 then leads to

$$\begin{aligned}\Phi_{B,i} &= BA = (\mu_0 in)A \\ &= (4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(1.5 \text{ A})(22\,000 \text{ turns/m}) \\ &\quad \times (3.464 \times 10^{-4} \text{ m}^2) \\ &= 1.44 \times 10^{-5} \text{ Wb}.\end{aligned}$$

Now we can write

$$\begin{aligned}\frac{d\Phi_B}{dt} &= \frac{\Delta\Phi_B}{\Delta t} = \frac{\Phi_{B,f} - \Phi_{B,i}}{\Delta t} \\ &= \frac{(0 - 1.44 \times 10^{-5} \text{ Wb})}{25 \times 10^{-3} \text{ s}} \\ &= -5.76 \times 10^{-4} \text{ Wb/s} \\ &= -5.76 \times 10^{-4} \text{ V}.\end{aligned}$$

We are interested only in magnitudes; so we ignore the minus signs here and in Eq. 30-5, writing

$$\begin{aligned}\mathcal{E} &= N \frac{d\Phi_B}{dt} = (130 \text{ turns})(5.76 \times 10^{-4} \text{ V}) \\ &= 7.5 \times 10^{-2} \text{ V} \\ &= 75 \text{ mV}.\end{aligned}\quad (\text{Answer})$$



Additional examples, video, and practice available at WileyPLUS

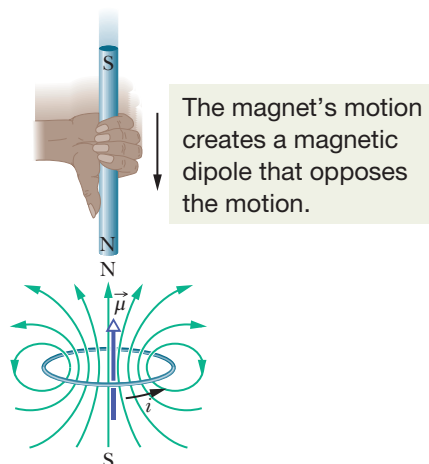
## Lenz's Law

Soon after Faraday proposed his law of induction, Heinrich Friedrich Lenz devised a rule for determining the direction of an induced current in a loop:



An induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current.

Furthermore, the direction of an induced emf is that of the induced current. The key word in Lenz's law is "opposition." Let's apply the law to the motion of the north pole toward the conducting loop in Fig. 30-4.



**Figure 30-4** Lenz's law at work. As the magnet is moved toward the loop, a current is induced in the loop. The current produces its own magnetic field, with magnetic dipole moment  $\vec{\mu}$  oriented so as to oppose the motion of the magnet. Thus, the induced current must be counterclockwise as shown.

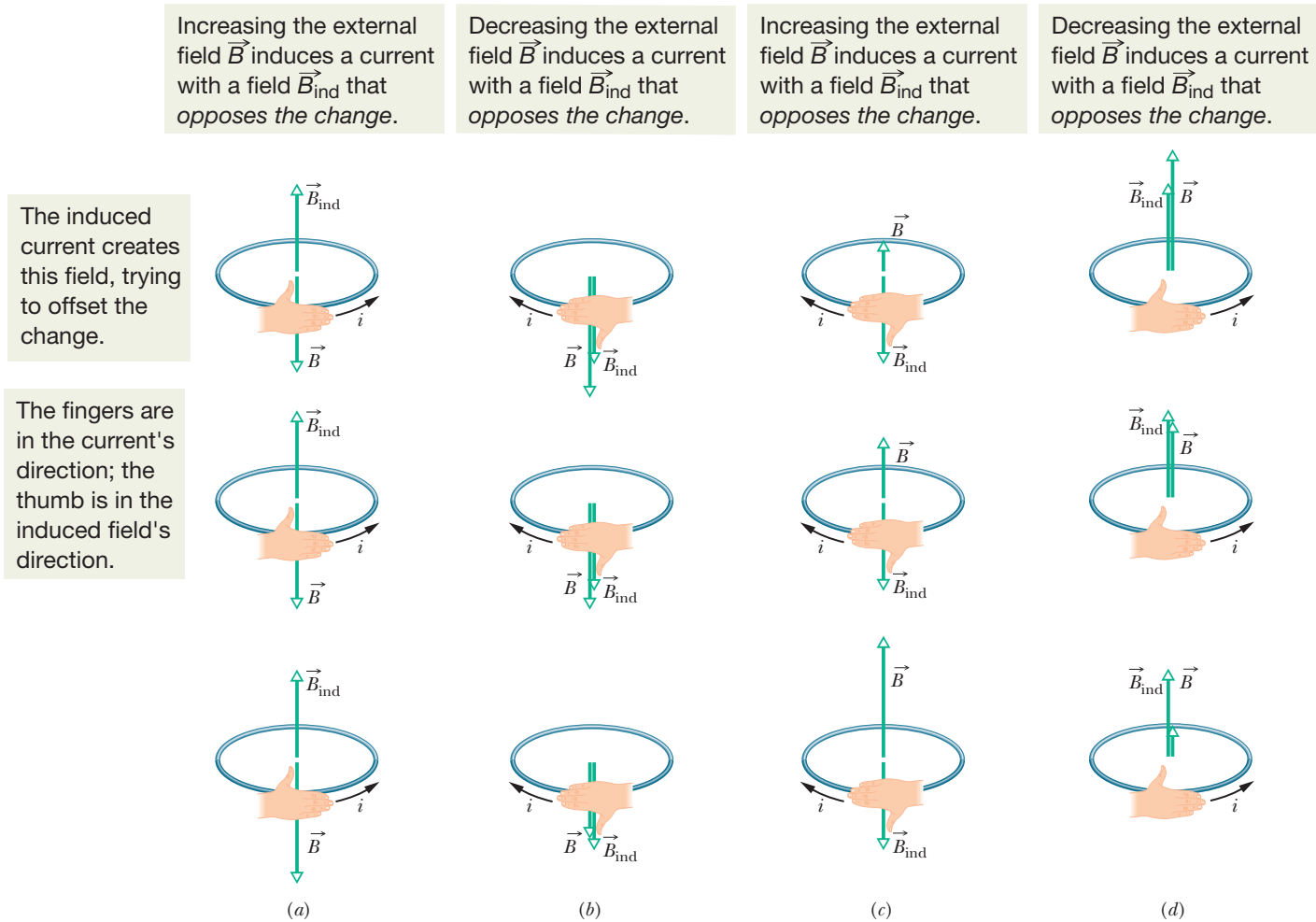
**1. Opposition to Pole Movement.** The approach of the magnet's north pole in Fig. 30-4 increases the magnetic flux through the loop and thereby induces a current in the loop. From Fig. 29-22, we know that the loop then acts as a magnetic dipole with a south pole and a north pole, and that its magnetic dipole moment  $\vec{\mu}$  is directed from south to north. To *oppose* the magnetic flux increase being caused by the approaching magnet, the loop's north pole (and thus  $\vec{\mu}$ ) must face *toward* the approaching north pole so as to repel it (Fig. 30-4). Then the curled-straight right-hand rule for  $\vec{\mu}$  (Fig. 29-22) tells us that the current induced in the loop must be counterclockwise in Fig. 30-4.

If we next pull the magnet away from the loop, a current will again be induced in the loop. Now, however, the loop will have a south pole facing the retreating north pole of the magnet, so as to oppose the retreat. Thus, the induced current will be clockwise.

**2. Opposition to Flux Change.** In Fig. 30-4, with the magnet initially distant, no magnetic flux passes through the loop. As the north pole of the magnet then nears the loop with its magnetic field  $\vec{B}$  directed *downward*, the flux through the loop increases. To oppose this increase in flux, the induced current  $i$  must set up its own field  $\vec{B}_{\text{ind}}$  directed *upward* inside the loop, as shown in Fig. 30-5a; then the upward flux of field  $\vec{B}_{\text{ind}}$  opposes the increasing downward flux of field  $\vec{B}$ . The curled-straight right-hand rule of Fig. 29-22 then tells us that  $i$  must be counterclockwise in Fig. 30-5a.



**Heads Up.** The flux of  $\vec{B}_{\text{ind}}$  always opposes the *change* in the flux of  $\vec{B}$ , but  $\vec{B}_{\text{ind}}$  is not always opposite  $\vec{B}$ . For example, if we next pull the magnet away from the loop in Fig. 30-4, the magnet's flux  $\Phi_B$  is still downward through the loop, but it is now decreasing. The flux of  $\vec{B}_{\text{ind}}$  must now be downward inside the loop, to oppose that *decrease* (Fig. 30-5b). Thus,  $\vec{B}_{\text{ind}}$  and  $\vec{B}$  are now in the same direction. In Figs. 30-5c and d, the south pole of the magnet approaches and retreats from the loop, again with opposition to change.

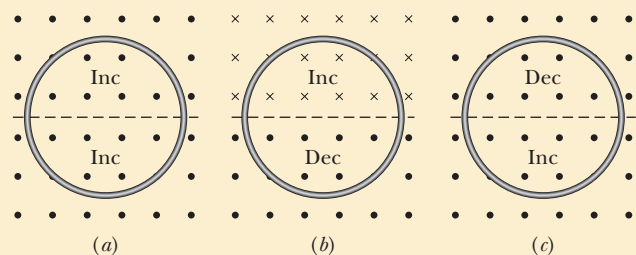


**Figure 30-5** The direction of the current  $i$  induced in a loop is such that the current's magnetic field  $\vec{B}_{\text{ind}}$  opposes the *change* in the magnetic field  $\vec{B}$  inducing  $i$ . The field  $\vec{B}_{\text{ind}}$  is always directed opposite an increasing field  $\vec{B}$  (a, c) and in the same direction as a decreasing field  $\vec{B}$  (b, d). The curled-straight right-hand rule gives the direction of the induced current based on the direction of the induced field.



### Checkpoint 2

The figure shows three situations in which identical circular conducting loops are in uniform magnetic fields that are either increasing (Inc) or decreasing (Dec) in magnitude at identical rates. In each, the dashed line coincides with a diameter. Rank the situations according to the magnitude of the current induced in the loops, greatest first.





### Sample Problem 30.02 Induced emf and current due to a changing uniform $B$ field

Figure 30-6 shows a conducting loop consisting of a half-circle of radius  $r = 0.20$  m and three straight sections. The half-circle lies in a uniform magnetic field  $\vec{B}$  that is directed out of the page; the field magnitude is given by  $B = 4.0t^2 + 2.0t + 3.0$ , with  $B$  in teslas and  $t$  in seconds. An ideal battery with emf  $\mathcal{E}_{\text{bat}} = 2.0$  V is connected to the loop. The resistance of the loop is  $2.0 \Omega$ .

(a) What are the magnitude and direction of the emf  $\mathcal{E}_{\text{ind}}$  induced around the loop by field  $\vec{B}$  at  $t = 10$  s?

#### KEY IDEAS

1. According to Faraday's law, the magnitude of  $\mathcal{E}_{\text{ind}}$  is equal to the rate  $d\Phi_B/dt$  at which the magnetic flux through the loop changes.
2. The flux through the loop depends on how much of the loop's area lies within the flux and how the area is oriented in the magnetic field  $\vec{B}$ .
3. Because  $\vec{B}$  is uniform and is perpendicular to the plane of the loop, the flux is given by Eq. 30-2 ( $\Phi_B = BA$ ). (We don't need to integrate  $B$  over the area to get the flux.)
4. The induced field  $B_{\text{ind}}$  (due to the induced current) must always oppose the *change* in the magnetic flux.

**Magnitude:** Using Eq. 30-2 and realizing that only the field magnitude  $B$  changes in time (not the area  $A$ ), we rewrite Faraday's law, Eq. 30-4, as

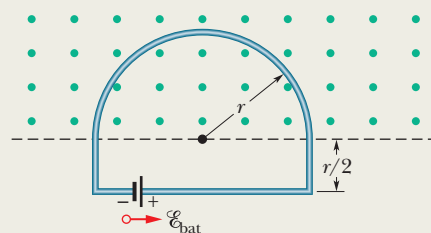
$$\mathcal{E}_{\text{ind}} = \frac{d\Phi_B}{dt} = \frac{d(BA)}{dt} = A \frac{dB}{dt}.$$

Because the flux penetrates the loop only within the half-circle, the area  $A$  in this equation is  $\frac{1}{2}\pi r^2$ . Substituting this and the given expression for  $B$  yields

$$\begin{aligned}\mathcal{E}_{\text{ind}} &= A \frac{dB}{dt} = \frac{\pi r^2}{2} \frac{d}{dt} (4.0t^2 + 2.0t + 3.0) \\ &= \frac{\pi r^2}{2} (8.0t + 2.0).\end{aligned}$$

### Sample Problem 30.03 Induced emf due to a changing nonuniform $B$ field

Figure 30-7 shows a rectangular loop of wire immersed in a nonuniform and varying magnetic field  $\vec{B}$  that is perpendicular to and directed into the page. The field's magnitude is given by  $B = 4t^2x^2$ , with  $B$  in teslas,  $t$  in seconds, and  $x$  in meters. (Note that the function depends on *both* time and position.) The loop has width  $W = 3.0$  m and height  $H = 2.0$  m. What are the magnitude and direction of the induced emf  $\mathcal{E}$  around the loop at  $t = 0.10$  s?



**Figure 30-6** A battery is connected to a conducting loop that includes a half-circle of radius  $r$  lying in a uniform magnetic field. The field is directed out of the page; its magnitude is changing.

At  $t = 10$  s, then,

$$\begin{aligned}\mathcal{E}_{\text{ind}} &= \frac{\pi (0.20 \text{ m})^2}{2} [8.0(10) + 2.0] \\ &= 5.152 \text{ V} \approx 5.2 \text{ V}.\end{aligned}\quad (\text{Answer})$$

**Direction:** To find the direction of  $\mathcal{E}_{\text{ind}}$ , we first note that in Fig. 30-6 the flux through the loop is out of the page and increasing. Because the induced field  $B_{\text{ind}}$  (due to the induced current) must oppose that increase, it must be *into* the page. Using the curled-straight right-hand rule (Fig. 30-5c), we find that the induced current is clockwise around the loop, and thus so is the induced emf  $\mathcal{E}_{\text{ind}}$ .

(b) What is the current in the loop at  $t = 10$  s?

#### KEY IDEA

The point here is that *two* emfs tend to move charges around the loop.

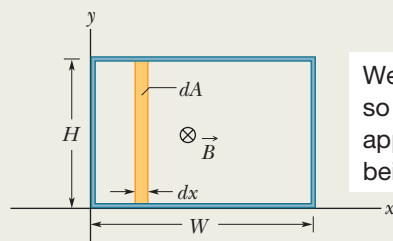
**Calculation:** The induced emf  $\mathcal{E}_{\text{ind}}$  tends to drive a current clockwise around the loop; the battery's emf  $\mathcal{E}_{\text{bat}}$  tends to drive a current counterclockwise. Because  $\mathcal{E}_{\text{ind}}$  is greater than  $\mathcal{E}_{\text{bat}}$ , the net emf  $\mathcal{E}_{\text{net}}$  is clockwise, and thus so is the current. To find the current at  $t = 10$  s, we use Eq. 27-2 ( $i = \mathcal{E}/R$ ):

$$\begin{aligned}i &= \frac{\mathcal{E}_{\text{net}}}{R} = \frac{\mathcal{E}_{\text{ind}} - \mathcal{E}_{\text{bat}}}{R} \\ &= \frac{5.152 \text{ V} - 2.0 \text{ V}}{2.0 \Omega} = 1.58 \text{ A} \approx 1.6 \text{ A}.\end{aligned}\quad (\text{Answer})$$

#### KEY IDEAS

1. Because the magnitude of the magnetic field  $\vec{B}$  is changing with time, the magnetic flux  $\Phi_B$  through the loop is also changing.
2. The changing flux induces an emf  $\mathcal{E}$  in the loop according to Faraday's law, which we can write as  $\mathcal{E} = d\Phi_B/dt$ .

If the field varies with position, we must integrate to get the flux through the loop.



We start with a strip so thin that we can approximate the field as being uniform within it.

**Figure 30-7** A closed conducting loop, of width  $W$  and height  $H$ , lies in a nonuniform, varying magnetic field that points directly into the page. To apply Faraday's law, we use the vertical strip of height  $H$ , width  $dx$ , and area  $dA$ .

3. To use that law, we need an expression for the flux  $\Phi_B$  at any time  $t$ . However, because  $B$  is *not* uniform over the area enclosed by the loop, we *cannot* use Eq. 30-2 ( $\Phi_B = BA$ ) to find that expression; instead we must use Eq. 30-1 ( $\Phi_B = \int \vec{B} \cdot d\vec{A}$ ).

**Calculations:** In Fig. 30-7,  $\vec{B}$  is perpendicular to the plane of the loop (and hence parallel to the differential area vector  $d\vec{A}$ ); so the dot product in Eq. 30-1 gives  $B dA$ . Because the magnetic field varies with the coordinate  $x$  but not with the

coordinate  $y$ , we can take the differential area  $dA$  to be the area of a vertical strip of height  $H$  and width  $dx$  (as shown in Fig. 30-7). Then  $dA = H dx$ , and the flux through the loop is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int B dA = \int BH dx = \int 4t^2 x^2 H dx.$$

Treating  $t$  as a constant for this integration and inserting the integration limits  $x = 0$  and  $x = 3.0$  m, we obtain

$$\Phi_B = 4t^2 H \int_0^{3.0} x^2 dx = 4t^2 H \left[ \frac{x^3}{3} \right]_0^{3.0} = 72t^2,$$

where we have substituted  $H = 2.0$  m and  $\Phi_B$  is in webers. Now we can use Faraday's law to find the magnitude of  $\mathcal{E}$  at any time  $t$ :

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \frac{d(72t^2)}{dt} = 144t,$$

in which  $\mathcal{E}$  is in volts. At  $t = 0.10$  s,

$$\mathcal{E} = (144 \text{ V/s})(0.10 \text{ s}) \approx 14 \text{ V.} \quad (\text{Answer})$$

The flux of  $\vec{B}$  through the loop is into the page in Fig. 30-7 and is increasing in magnitude because  $B$  is increasing in magnitude with time. By Lenz's law, the field  $B_{\text{ind}}$  of the induced current opposes this increase and so is directed out of the page. The curled-straight right-hand rule in Fig. 30-5a then tells us that the induced current is counter-clockwise around the loop, and thus so is the induced emf  $\mathcal{E}$ .



Additional examples, video, and practice available at WileyPLUS



## 30-2 INDUCTION AND ENERGY TRANSFERS

### Learning Objectives

After reading this module, you should be able to . . .

**30.13** For a conducting loop pulled into or out of a magnetic field, calculate the rate at which energy is transferred to thermal energy.

**30.14** Apply the relationship between an induced current and the rate at which it produces thermal energy.

**30.15** Describe eddy currents.

### Key Idea

- The induction of a current by a changing flux means that energy is being transferred to that current. The energy can then be transferred to other forms, such as thermal energy.

## Induction and Energy Transfers

By Lenz's law, whether you move the magnet toward or away from the loop in Fig. 30-1, a magnetic force resists the motion, requiring your applied force to do positive work. At the same time, thermal energy is produced in the material of the loop because of the material's electrical resistance to the current that is induced by the motion. The energy you transfer to the closed *loop + magnet* system via your applied force ends up in this thermal energy. (For now, we neglect energy that is radiated away from the loop as electromagnetic waves during the



induction.) The faster you move the magnet, the more rapidly your applied force does work and the greater the rate at which your energy is transferred to thermal energy in the loop; that is, the power of the transfer is greater.

Regardless of how current is induced in a loop, energy is always transferred to thermal energy during the process because of the electrical resistance of the loop (unless the loop is superconducting). For example, in Fig. 30-2, when switch  $S$  is closed and a current is briefly induced in the left-hand loop, energy is transferred from the battery to thermal energy in that loop.

Figure 30-8 shows another situation involving induced current. A rectangular loop of wire of width  $L$  has one end in a uniform external magnetic field that is directed perpendicularly into the plane of the loop. This field may be produced, for example, by a large electromagnet. The dashed lines in Fig. 30-8 show the assumed limits of the magnetic field; the fringing of the field at its edges is neglected. You are to pull this loop to the right at a constant velocity  $\vec{v}$ .

**Flux Change.** The situation of Fig. 30-8 does not differ in any essential way from that of Fig. 30-1. In each case a magnetic field and a conducting loop are in relative motion; in each case the flux of the field through the loop is changing with time. It is true that in Fig. 30-1 the flux is changing because  $\vec{B}$  is changing and in Fig. 30-8 the flux is changing because the area of the loop still in the magnetic field is changing, but that difference is not important. The important difference between the two arrangements is that the arrangement of Fig. 30-8 makes calculations easier. Let us now calculate the rate at which you do mechanical work as you pull steadily on the loop in Fig. 30-8.

**Rate of Work.** As you will see, to pull the loop at a constant velocity  $\vec{v}$ , you must apply a constant force  $\vec{F}$  to the loop because a magnetic force of equal magnitude but opposite direction acts on the loop to oppose you. From Eq. 7-48, the rate at which you do work — that is, the power — is then

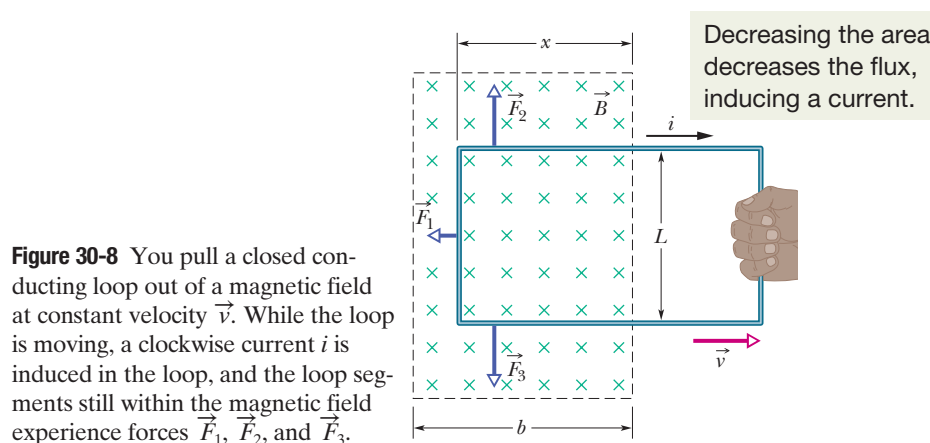
$$P = Fv, \quad (30-6)$$

where  $F$  is the magnitude of your force. We wish to find an expression for  $P$  in terms of the magnitude  $B$  of the magnetic field and the characteristics of the loop — namely, its resistance  $R$  to current and its dimension  $L$ .

As you move the loop to the right in Fig. 30-8, the portion of its area within the magnetic field decreases. Thus, the flux through the loop also decreases and, according to Faraday's law, a current is produced in the loop. It is the presence of this current that causes the force that opposes your pull.

**Induced emf.** To find the current, we first apply Faraday's law. When  $x$  is the length of the loop still in the magnetic field, the area of the loop still in the field is  $Lx$ . Then from Eq. 30-2, the magnitude of the flux through the loop is

$$\Phi_B = BA = BLx. \quad (30-7)$$



**Figure 30-8** You pull a closed conducting loop out of a magnetic field at constant velocity  $\vec{v}$ . While the loop is moving, a clockwise current  $i$  is induced in the loop, and the loop segments still within the magnetic field experience forces  $\vec{F}_1$ ,  $\vec{F}_2$ , and  $\vec{F}_3$ .

As  $x$  decreases, the flux decreases. Faraday's law tells us that with this flux decrease, an emf is induced in the loop. Dropping the minus sign in Eq. 30-4 and using Eq. 30-7, we can write the magnitude of this emf as

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \frac{d}{dt} BLx = BL \frac{dx}{dt} = BLv, \quad (30-8)$$

in which we have replaced  $dx/dt$  with  $v$ , the speed at which the loop moves.

Figure 30-9 shows the loop as a circuit: induced emf  $\mathcal{E}$  is represented on the left, and the collective resistance  $R$  of the loop is represented on the right. The direction of the induced current  $i$  is obtained with a right-hand rule as in Fig. 30-5b for decreasing flux; applying the rule tells us that the current must be clockwise, and  $\mathcal{E}$  must have the same direction.

**Induced Current.** To find the magnitude of the induced current, we cannot apply the loop rule for potential differences in a circuit because, as you will see in Module 30-3, we cannot define a potential difference for an induced emf. However, we can apply the equation  $i = \mathcal{E}/R$ . With Eq. 30-8, this becomes

$$i = \frac{BLv}{R}. \quad (30-9)$$

Because three segments of the loop in Fig. 30-8 carry this current through the magnetic field, sideways deflecting forces act on those segments. From Eq. 28-26 we know that such a deflecting force is, in general notation,

$$\vec{F}_d = i \vec{L} \times \vec{B}. \quad (30-10)$$

In Fig. 30-8, the deflecting forces acting on the three segments of the loop are marked  $\vec{F}_1$ ,  $\vec{F}_2$ , and  $\vec{F}_3$ . Note, however, that from the symmetry, forces  $\vec{F}_2$  and  $\vec{F}_3$  are equal in magnitude and cancel. This leaves only force  $\vec{F}_1$ , which is directed opposite your force  $\vec{F}$  on the loop and thus is the force opposing you. So,  $\vec{F} = -\vec{F}_1$ .

Using Eq. 30-10 to obtain the magnitude of  $\vec{F}_1$  and noting that the angle between  $\vec{B}$  and the length vector  $\vec{L}$  for the left segment is  $90^\circ$ , we write

$$F = F_1 = iLB \sin 90^\circ = iLB. \quad (30-11)$$

Substituting Eq. 30-9 for  $i$  in Eq. 30-11 then gives us

$$F = \frac{B^2 L^2 v}{R}. \quad (30-12)$$

Because  $B$ ,  $L$ , and  $R$  are constants, the speed  $v$  at which you move the loop is constant if the magnitude  $F$  of the force you apply to the loop is also constant.

**Rate of Work.** By substituting Eq. 30-12 into Eq. 30-6, we find the rate at which you do work on the loop as you pull it from the magnetic field:

$$P = Fv = \frac{B^2 L^2 v^2}{R} \quad (\text{rate of doing work}). \quad (30-13)$$

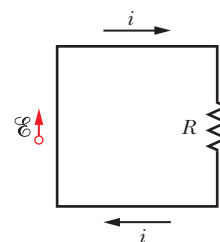
**Thermal Energy.** To complete our analysis, let us find the rate at which thermal energy appears in the loop as you pull it along at constant speed. We calculate it from Eq. 26-27,

$$P = i^2 R. \quad (30-14)$$

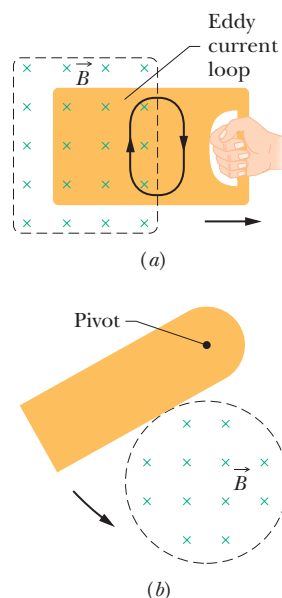
Substituting for  $i$  from Eq. 30-9, we find

$$P = \left( \frac{BLv}{R} \right)^2 R = \frac{B^2 L^2 v^2}{R} \quad (\text{thermal energy rate}), \quad (30-15)$$

which is exactly equal to the rate at which you are doing work on the loop (Eq. 30-13). Thus, the work that you do in pulling the loop through the magnetic field appears as thermal energy in the loop.



**Figure 30-9** A circuit diagram for the loop of Fig. 30-8 while the loop is moving.



**Figure 30-10** (a) As you pull a solid conducting plate out of a magnetic field, *eddy currents* are induced in the plate. A typical loop of eddy current is shown. (b) A conducting plate is allowed to swing like a pendulum and into a region of magnetic field. As it enters and leaves the field, eddy currents are induced in the plate.

### Eddy Currents

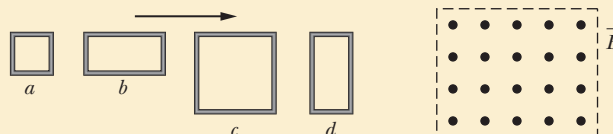
Suppose we replace the conducting loop of Fig. 30-8 with a solid conducting plate. If we then move the plate out of the magnetic field as we did the loop (Fig. 30-10a), the relative motion of the field and the conductor again induces a current in the conductor. Thus, we again encounter an opposing force and must do work because of the induced current. With the plate, however, the conduction electrons making up the induced current do not follow one path as they do with the loop. Instead, the electrons swirl about within the plate as if they were caught in an eddy (whirlpool) of water. Such a current is called an *eddy current* and can be represented, as it is in Fig. 30-10a, as if it followed a single path.

As with the conducting loop of Fig. 30-8, the current induced in the plate results in mechanical energy being dissipated as thermal energy. The dissipation is more apparent in the arrangement of Fig. 30-10b; a conducting plate, free to rotate about a pivot, is allowed to swing down through a magnetic field like a pendulum. Each time the plate enters and leaves the field, a portion of its mechanical energy is transferred to its thermal energy. After several swings, no mechanical energy remains and the warmed-up plate just hangs from its pivot.



### Checkpoint 3

The figure shows four wire loops, with edge lengths of either  $L$  or  $2L$ . All four loops will move through a region of uniform magnetic field  $\vec{B}$  (directed out of the page) at the same constant velocity. Rank the four loops according to the maximum magnitude of the emf induced as they move through the field, greatest first.



## 30-3 INDUCED ELECTRIC FIELDS

### Learning Objectives

After reading this module, you should be able to . . .

**30.16** Identify that a changing magnetic field induces an electric field, regardless of whether there is a conducting loop.

**30.17** Apply Faraday's law to relate the electric field  $\vec{E}$  induced along a closed path (whether it has

conducting material or not) to the rate of change  $d\Phi/dt$  of the magnetic flux encircled by the path.

**30.18** Identify that an electric potential cannot be associated with an induced electric field.

### Key Ideas

● An emf is induced by a changing magnetic flux even if the loop through which the flux is changing is not a physical conductor but an imaginary line. The changing magnetic field induces an electric field  $\vec{E}$  at every point of such a loop; the induced emf is related to  $\vec{E}$  by

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s}.$$

● Using the induced electric field, we can write Faraday's law in its most general form as

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad (\text{Faraday's law}).$$

A changing magnetic field induces an electric field  $\vec{E}$ .

## Induced Electric Fields

Let us place a copper ring of radius  $r$  in a uniform external magnetic field, as in Fig. 30-11*a*. The field — neglecting fringing — fills a cylindrical volume of radius  $R$ . Suppose that we increase the strength of this field at a steady rate, perhaps by increasing — in an appropriate way — the current in the windings of the electromagnet that produces the field. The magnetic flux through the ring will then change at a steady rate and — by Faraday's law — an induced emf and thus an induced current will appear in the ring. From Lenz's law we can deduce that the direction of the induced current is counterclockwise in Fig. 30-11*a*.

If there is a current in the copper ring, an electric field must be present along the ring because an electric field is needed to do the work of moving the conduction electrons. Moreover, the electric field must have been produced by the changing magnetic flux. This **induced electric field**  $\vec{E}$  is just as real as an electric field produced by static charges; either field will exert a force  $q_0\vec{E}$  on a particle of charge  $q_0$ .

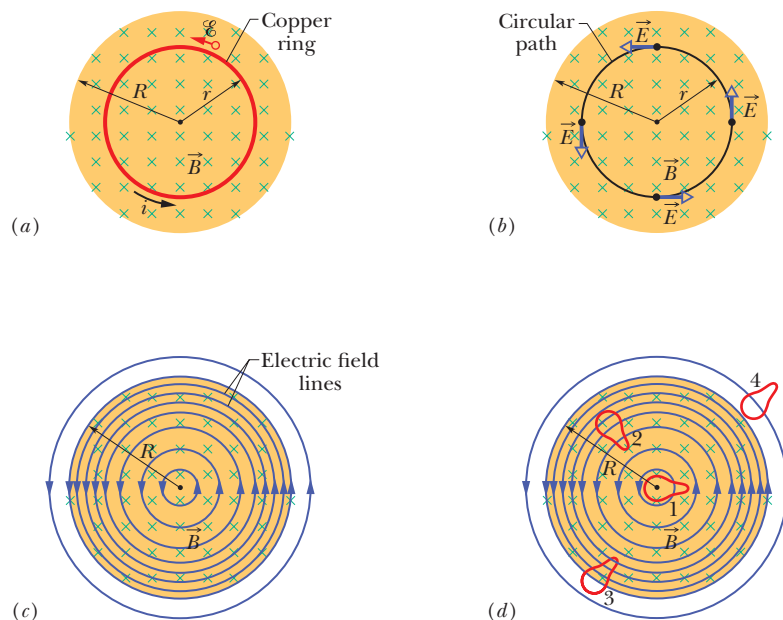
By this line of reasoning, we are led to a useful and informative restatement of Faraday's law of induction:



A changing magnetic field produces an electric field.

The striking feature of this statement is that the electric field is induced even if there is no copper ring. Thus, the electric field would appear even if the changing magnetic field were in a vacuum.

To fix these ideas, consider Fig. 30-11*b*, which is just like Fig. 30-11*a* except the copper ring has been replaced by a hypothetical circular path of radius  $r$ . We assume, as previously, that the magnetic field  $\vec{B}$  is increasing in magnitude at a constant rate  $dB/dt$ . The electric field induced at various points around the



**Figure 30-11** (a) If the magnetic field increases at a steady rate, a constant induced current appears, as shown, in the copper ring of radius  $r$ . (b) An induced electric field exists even when the ring is removed; the electric field is shown at four points. (c) The complete picture of the induced electric field, displayed as field lines. (d) Four similar closed paths that enclose identical areas. Equal emfs are induced around paths 1 and 2, which lie entirely within the region of changing magnetic field. A smaller emf is induced around path 3, which only partially lies in that region. No net emf is induced around path 4, which lies entirely outside the magnetic field.

circular path must — from the symmetry — be tangent to the circle, as Fig. 30-11*b* shows.\* Hence, the circular path is an electric field line. There is nothing special about the circle of radius  $r$ , so the electric field lines produced by the changing magnetic field must be a set of concentric circles, as in Fig. 30-11*c*.

As long as the magnetic field is *increasing* with time, the electric field represented by the circular field lines in Fig. 30-11*c* will be present. If the magnetic field remains *constant* with time, there will be no induced electric field and thus no electric field lines. If the magnetic field is *decreasing* with time (at a constant rate), the electric field lines will still be concentric circles as in Fig. 30-11*c*, but they will now have the opposite direction. All this is what we have in mind when we say “A changing magnetic field produces an electric field.”

### A Reformulation of Faraday's Law

Consider a particle of charge  $q_0$  moving around the circular path of Fig. 30-11*b*. The work  $W$  done on it in one revolution by the induced electric field is  $W = \mathcal{E}q_0$ , where  $\mathcal{E}$  is the induced emf — that is, the work done per unit charge in moving the test charge around the path. From another point of view, the work is

$$W = \int \vec{F} \cdot d\vec{s} = (q_0 E)(2\pi r), \quad (30-16)$$

where  $q_0 E$  is the magnitude of the force acting on the test charge and  $2\pi r$  is the distance over which that force acts. Setting these two expressions for  $W$  equal to each other and canceling  $q_0$ , we find that

$$\mathcal{E} = 2\pi r E. \quad (30-17)$$

Next we rewrite Eq. 30-16 to give a more general expression for the work done on a particle of charge  $q_0$  moving along any closed path:

$$W = \oint \vec{F} \cdot d\vec{s} = q_0 \oint \vec{E} \cdot d\vec{s}. \quad (30-18)$$

(The loop on each integral sign indicates that the integral is to be taken around the closed path.) Substituting  $\mathcal{E}q_0$  for  $W$ , we find that

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s}. \quad (30-19)$$

This integral reduces at once to Eq. 30-17 if we evaluate it for the special case of Fig. 30-11*b*.

**Meaning of emf.** With Eq. 30-19, we can expand the meaning of induced emf. Up to this point, induced emf has meant the work per unit charge done in maintaining current due to a changing magnetic flux, or it has meant the work done per unit charge on a charged particle that moves around a closed path in a changing magnetic flux. However, with Fig. 30-11*b* and Eq. 30-19, an induced emf can exist without the need of a current or particle: An induced emf is the sum — via integration — of quantities  $\vec{E} \cdot d\vec{s}$  around a closed path, where  $\vec{E}$  is the electric field induced by a changing magnetic flux and  $d\vec{s}$  is a differential length vector along the path.

If we combine Eq. 30-19 with Faraday's law in Eq. 30-4 ( $\mathcal{E} = -d\Phi_B/dt$ ), we can rewrite Faraday's law as

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad (\text{Faraday's law}). \quad (30-20)$$

\*Arguments of symmetry would also permit the lines of  $\vec{E}$  around the circular path to be *radial*, rather than tangential. However, such radial lines would imply that there are free charges, distributed symmetrically about the axis of symmetry, on which the electric field lines could begin or end; there are no such charges.



This equation says simply that a changing magnetic field induces an electric field. The changing magnetic field appears on the right side of this equation, the electric field on the left.

Faraday's law in the form of Eq. 30-20 can be applied to *any* closed path that can be drawn in a changing magnetic field. Figure 30-11*d*, for example, shows four such paths, all having the same shape and area but located in different positions in the changing field. The induced emfs  $\mathcal{E}(=\oint \vec{E} \cdot d\vec{s})$  for paths 1 and 2 are equal because these paths lie entirely in the magnetic field and thus have the same value of  $d\Phi_B/dt$ . This is true even though the electric field vectors at points along these paths are different, as indicated by the patterns of electric field lines in the figure. For path 3 the induced emf is smaller because the enclosed flux  $\Phi_B$  (hence  $d\Phi_B/dt$ ) is smaller, and for path 4 the induced emf is zero even though the electric field is not zero at any point on the path.

### A New Look at Electric Potential

Induced electric fields are produced not by static charges but by a changing magnetic flux. Although electric fields produced in either way exert forces on charged particles, there is an important difference between them. The simplest evidence of this difference is that the field lines of induced electric fields form closed loops, as in Fig. 30-11*c*. Field lines produced by static charges never do so but must start on positive charges and end on negative charges. Thus, a field line from a charge can never loop around and back onto itself as we see for each of the field lines in Fig. 30-11*c*.

In a more formal sense, we can state the difference between electric fields produced by induction and those produced by static charges in these words:



Electric potential has meaning only for electric fields that are produced by static charges; it has no meaning for electric fields that are produced by induction.

You can understand this statement qualitatively by considering what happens to a charged particle that makes a single journey around the circular path in Fig. 30-11*b*. It starts at a certain point and, on its return to that same point, has experienced an emf  $\mathcal{E}$  of, let us say, 5 V; that is, work of 5 J/C has been done on the particle by the electric field, and thus the particle should then be at a point that is 5 V greater in potential. However, that is impossible because the particle is back at the same point, which cannot have two different values of potential. Thus, potential has no meaning for electric fields that are set up by changing magnetic fields.

We can take a more formal look by recalling Eq. 24-18, which defines the potential difference between two points  $i$  and  $f$  in an electric field  $\vec{E}$  in terms of an integration between those points:

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}. \quad (30-21)$$

In Chapter 24 we had not yet encountered Faraday's law of induction; so the electric fields involved in the derivation of Eq. 24-18 were those due to static charges. If  $i$  and  $f$  in Eq. 30-21 are the same point, the path connecting them is a closed loop,  $V_i$  and  $V_f$  are identical, and Eq. 30-21 reduces to

$$\oint \vec{E} \cdot d\vec{s} = 0. \quad (30-22)$$

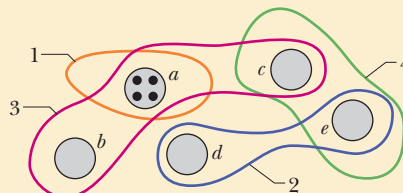
However, when a changing magnetic flux is present, this integral is *not* zero but is  $-d\Phi_B/dt$ , as Eq. 30-20 asserts. Thus, assigning electric potential to an induced electric field leads us to a contradiction. We must conclude that electric potential has no meaning for electric fields associated with induction.



### Checkpoint 4

The figure shows five lettered regions in which a uniform magnetic field extends either directly out of the page or into the page, with the direction indicated only for region *a*. The field is increasing in magnitude at the same steady rate in all five regions; the regions are identical in area. Also shown are four numbered paths along which  $\oint \vec{E} \cdot d\vec{s}$  has the magnitudes given below in terms of a quantity “mag.” Determine whether the magnetic field is directed into or out of the page for regions *b* through *e*.

Path	1	2	3	4
$\oint \vec{E} \cdot d\vec{s}$	mag	2(mag)	3(mag)	0



### Sample Problem 30.04 Induced electric field due to changing *B* field, inside and outside

In Fig. 30-11*b*, take  $R = 8.5$  cm and  $dB/dt = 0.13$  T/s.

(a) Find an expression for the magnitude  $E$  of the induced electric field at points within the magnetic field, at radius  $r$  from the center of the magnetic field. Evaluate the expression for  $r = 5.2$  cm.

#### KEY IDEA

An electric field is induced by the changing magnetic field, according to Faraday’s law.

**Calculations:** To calculate the field magnitude  $E$ , we apply Faraday’s law in the form of Eq. 30-20. We use a circular path of integration with radius  $r \leq R$  because we want  $E$  for points within the magnetic field. We assume from the symmetry that  $\vec{E}$  in Fig. 30-11*b* is tangent to the circular path at all points. The path vector  $d\vec{s}$  is also always tangent to the circular path; so the dot product  $\vec{E} \cdot d\vec{s}$  in Eq. 30-20 must have the magnitude  $E ds$  at all points on the path. We can also assume from the symmetry that  $E$  has the same value at all points along the circular path. Then the left side of Eq. 30-20 becomes

$$\oint \vec{E} \cdot d\vec{s} = \oint E ds = E \oint ds = E(2\pi r). \quad (30-23)$$

(The integral  $\oint ds$  is the circumference  $2\pi r$  of the circular path.)

Next, we need to evaluate the right side of Eq. 30-20. Because  $\vec{B}$  is uniform over the area  $A$  encircled by the path of integration and is directed perpendicular to that area, the magnetic flux is given by Eq. 30-2:

$$\Phi_B = BA = B(\pi r^2). \quad (30-24)$$

Substituting this and Eq. 30-23 into Eq. 30-20 and dropping

the minus sign, we find that

$$E(2\pi r) = (\pi r^2) \frac{dB}{dt}$$

or

$$E = \frac{r}{2} \frac{dB}{dt}. \quad (\text{Answer}) \quad (30-25)$$

Equation 30-25 gives the magnitude of the electric field at any point for which  $r \leq R$  (that is, within the magnetic field). Substituting given values yields, for the magnitude of  $\vec{E}$  at  $r = 5.2$  cm,

$$\begin{aligned} E &= \frac{(5.2 \times 10^{-2} \text{ m})}{2} (0.13 \text{ T/s}) \\ &= 0.0034 \text{ V/m} = 3.4 \text{ mV/m}. \quad (\text{Answer}) \end{aligned}$$

(b) Find an expression for the magnitude  $E$  of the induced electric field at points that are outside the magnetic field, at radius  $r$  from the center of the magnetic field. Evaluate the expression for  $r = 12.5$  cm.

#### KEY IDEAS

Here again an electric field is induced by the changing magnetic field, according to Faraday’s law, except that now we use a circular path of integration with radius  $r \geq R$  because we want to evaluate  $E$  for points outside the magnetic field. Proceeding as in (a), we again obtain Eq. 30-23. However, we do not then obtain Eq. 30-24 because the new path of integration is now outside the magnetic field, and so the magnetic flux encircled by the new path is only that in the area  $\pi R^2$  of the magnetic field region.

**Calculations:** We can now write

$$\Phi_B = BA = B(\pi R^2). \quad (30-26)$$

Substituting this and Eq. 30-23 into Eq. 30-20 (without the minus sign) and solving for  $E$  yield

$$E = \frac{R^2}{2r} \frac{dB}{dt}. \quad (\text{Answer}) \quad (30-27)$$

Because  $E$  is not zero here, we know that an electric field is induced even at points that are outside the changing magnetic field, an important result that (as you will see in Module 31-6) makes transformers possible.

With the given data, Eq. 30-27 yields the magnitude of  $\vec{E}$  at  $r = 12.5$  cm:

$$\begin{aligned} E &= \frac{(8.5 \times 10^{-2} \text{ m})^2}{(2)(12.5 \times 10^{-2} \text{ m})} (0.13 \text{ T/s}) \\ &= 3.8 \times 10^{-3} \text{ V/m} = 3.8 \text{ mV/m}. \quad (\text{Answer}) \end{aligned}$$

Equations 30-25 and 30-27 give the same result for  $r = R$ . Figure 30-12 shows a plot of  $E(r)$ . Note that the inside and outside plots meet at  $r = R$ .

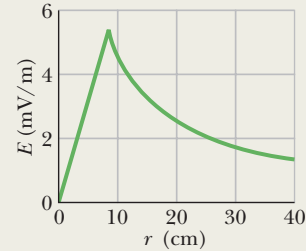


Figure 30-12 A plot of the induced electric field  $E(r)$ .



Additional examples, video, and practice available at WileyPLUS



## 30-4 INDUCTORS AND INDUCTANCE

### Learning Objectives

After reading this module, you should be able to . . .

**30.19** Identify an inductor.

**30.20** For an inductor, apply the relationship between inductance  $L$ , total flux  $N\Phi$ , and current  $i$ .

**30.21** For a solenoid, apply the relationship between the inductance per unit length  $L/l$ , the area  $A$  of each turn, and the number of turns per unit length  $n$ .

### Key Ideas

● An inductor is a device that can be used to produce a known magnetic field in a specified region. If a current  $i$  is established through each of the  $N$  windings of an inductor, a magnetic flux  $\Phi_B$  links those windings. The inductance  $L$  of the inductor is

$$L = \frac{N\Phi_B}{i} \quad (\text{inductance defined}).$$

● The SI unit of inductance is the henry (H), where  $1 \text{ henry} = 1 \text{ H} = 1 \text{ T} \cdot \text{m}^2/\text{A}$ .

● The inductance per unit length near the middle of a long solenoid of cross-sectional area  $A$  and  $n$  turns per unit length is

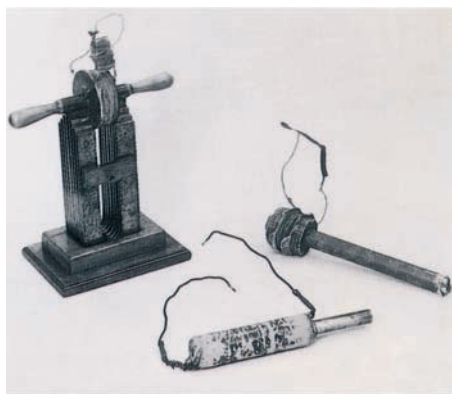
$$\frac{L}{l} = \mu_0 n^2 A \quad (\text{solenoid}).$$

## Inductors and Inductance

We found in Chapter 25 that a capacitor can be used to produce a desired electric field. We considered the parallel-plate arrangement as a basic type of capacitor. Similarly, an **inductor** (symbol  $\text{⏏}$ ) can be used to produce a desired magnetic field. We shall consider a long solenoid (more specifically, a short length near the middle of a long solenoid, to avoid any fringing effects) as our basic type of inductor.

If we establish a current  $i$  in the windings (turns) of the solenoid we are taking as our inductor, the current produces a magnetic flux  $\Phi_B$  through the central region of the inductor. The **inductance** of the inductor is then defined in terms of that flux as

$$L = \frac{N\Phi_B}{i} \quad (\text{inductance defined}), \quad (30-28)$$



The Royal Institution/Bridgeman Art Library/NY

The crude inductors with which Michael Faraday discovered the law of induction. In those days amenities such as insulated wire were not commercially available. It is said that Faraday insulated his wires by wrapping them with strips cut from one of his wife's petticoats.

in which  $N$  is the number of turns. The windings of the inductor are said to be *linked* by the shared flux, and the product  $N\Phi_B$  is called the *magnetic flux linkage*. The inductance  $L$  is thus a measure of the flux linkage produced by the inductor per unit of current.

Because the SI unit of magnetic flux is the tesla-square meter, the SI unit of inductance is the tesla-square meter per ampere ( $\text{T} \cdot \text{m}^2/\text{A}$ ). We call this the **henry** (H), after American physicist Joseph Henry, the codiscoverer of the law of induction and a contemporary of Faraday. Thus,

$$1 \text{ henry} = 1 \text{ H} = 1 \text{ T} \cdot \text{m}^2/\text{A}. \quad (30-29)$$

Through the rest of this chapter we assume that all inductors, no matter what their geometric arrangement, have no magnetic materials such as iron in their vicinity. Such materials would distort the magnetic field of an inductor.

### Inductance of a Solenoid

Consider a long solenoid of cross-sectional area  $A$ . What is the inductance per unit length near its middle? To use the defining equation for inductance (Eq. 30-28), we must calculate the flux linkage set up by a given current in the solenoid windings. Consider a length  $l$  near the middle of this solenoid. The flux linkage there is

$$N\Phi_B = (nl)(BA),$$

in which  $n$  is the number of turns per unit length of the solenoid and  $B$  is the magnitude of the magnetic field within the solenoid.

The magnitude  $B$  is given by Eq. 29-23,

$$B = \mu_0 in,$$

and so from Eq. 30-28,

$$\begin{aligned} L &= \frac{N\Phi_B}{i} = \frac{(nl)(BA)}{i} = \frac{(nl)(\mu_0 in)(A)}{i} \\ &= \mu_0 n^2 l A. \end{aligned} \quad (30-30)$$

Thus, the inductance per unit length near the center of a long solenoid is

$$\frac{L}{l} = \mu_0 n^2 A \quad (\text{solenoid}). \quad (30-31)$$

Inductance — like capacitance — depends only on the geometry of the device. The dependence on the square of the number of turns per unit length is to be expected. If you, say, triple  $n$ , you not only triple the number of turns ( $N$ ) but you also triple the flux ( $\Phi_B = BA = \mu_0 inA$ ) through each turn, multiplying the flux linkage  $N\Phi_B$  and thus the inductance  $L$  by a factor of 9.

If the solenoid is very much longer than its radius, then Eq. 30-30 gives its inductance to a good approximation. This approximation neglects the spreading of the magnetic field lines near the ends of the solenoid, just as the parallel-plate capacitor formula ( $C = \epsilon_0 A/d$ ) neglects the fringing of the electric field lines near the edges of the capacitor plates.

From Eq. 30-30, and recalling that  $n$  is a number per unit length, we can see that an inductance can be written as a product of the permeability constant  $\mu_0$  and a quantity with the dimensions of a length. This means that  $\mu_0$  can be expressed in the unit henry per meter:

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7} \text{ T} \cdot \text{m}/\text{A} \\ &= 4\pi \times 10^{-7} \text{ H}/\text{m}. \end{aligned} \quad (30-32)$$

The latter is the more common unit for the permeability constant.

## 30-5 SELF-INDUCTION

### Learning Objectives

After reading this module, you should be able to . . .

**30.22** Identify that an induced emf appears in a coil when the current through the coil is changing.

**30.23** Apply the relationship between the induced emf in a coil, the coil's inductance  $L$ , and the rate  $di/dt$  at which the current is changing.

**30.24** When an emf is induced in a coil because the current in the coil is changing, determine the direction of the emf by using Lenz's law to show that the emf always opposes the change in the current, attempting to maintain the initial current.

### Key Ideas

- If a current  $i$  in a coil changes with time, an emf is induced in the coil. This self-induced emf is

$$\mathcal{E}_L = -L \frac{di}{dt}.$$

- The direction of  $\mathcal{E}_L$  is found from Lenz's law: The self-induced emf acts to oppose the change that produces it.

## Self-Induction

If two coils — which we can now call inductors — are near each other, a current  $i$  in one coil produces a magnetic flux  $\Phi_B$  through the second coil. We have seen that if we change this flux by changing the current, an induced emf appears in the second coil according to Faraday's law. An induced emf appears in the first coil as well.



An induced emf  $\mathcal{E}_L$  appears in any coil in which the current is changing.

This process (see Fig. 30-13) is called **self-induction**, and the emf that appears is called a **self-induced emf**. It obeys Faraday's law of induction just as other induced emfs do.

For any inductor, Eq. 30-28 tells us that

$$N\Phi_B = Li. \quad (30-33)$$

Faraday's law tells us that

$$\mathcal{E}_L = - \frac{d(N\Phi_B)}{dt}. \quad (30-34)$$

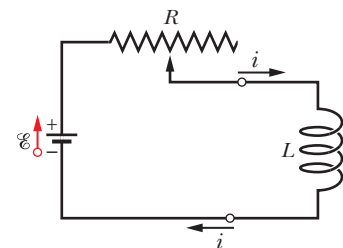
By combining Eqs. 30-33 and 30-34 we can write

$$\mathcal{E}_L = -L \frac{di}{dt} \quad (\text{self-induced emf}). \quad (30-35)$$

Thus, in any inductor (such as a coil, a solenoid, or a toroid) a self-induced emf appears whenever the current changes with time. The magnitude of the current has no influence on the magnitude of the induced emf; only the rate of change of the current counts.

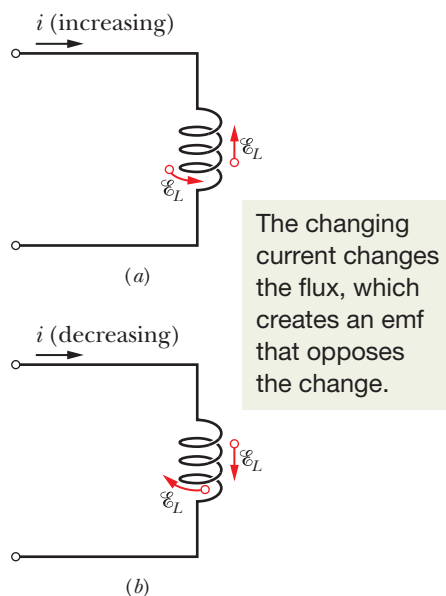
**Direction.** You can find the *direction* of a self-induced emf from Lenz's law. The minus sign in Eq. 30-35 indicates that — as the law states — the self-induced emf  $\mathcal{E}_L$  has the orientation such that it opposes the change in current  $i$ . We can drop the minus sign when we want only the magnitude of  $\mathcal{E}_L$ .

Suppose that you set up a current  $i$  in a coil and arrange to have the current increase with time at a rate  $di/dt$ . In the language of Lenz's law, this increase in the current in the coil is the “change” that the self-induction must oppose. Thus, a self-induced emf must appear in the coil, pointing so as to oppose the increase in the current, trying (but failing) to maintain the initial condition, as



**Figure 30-13** If the current in a coil is changed by varying the contact position on a variable resistor, a self-induced emf  $\mathcal{E}_L$  will appear in the coil *while the current is changing*.





**Figure 30-14** (a) The current  $i$  is increasing, and the self-induced emf  $\mathcal{E}_L$  appears along the coil in a direction such that it opposes the increase. The arrow representing  $\mathcal{E}_L$  can be drawn along a turn of the coil or alongside the coil. Both are shown. (b) The current  $i$  is decreasing, and the self-induced emf appears in a direction such that it opposes the decrease.

shown in Fig. 30-14a. If, instead, the current decreases with time, the self-induced emf must point in a direction that tends to oppose the decrease (Fig. 30-14b), again trying to maintain the initial condition.

**Electric Potential.** In Module 30-3 we saw that we cannot define an electric potential for an electric field (and thus for an emf) that is induced by a changing magnetic flux. This means that when a self-induced emf is produced in the inductor of Fig. 30-13, we cannot define an electric potential within the inductor itself, where the flux is changing. However, potentials can still be defined at points of the circuit that are not within the inductor — points where the electric fields are due to charge distributions and their associated electric potentials.

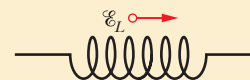
Moreover, we can define a self-induced potential difference  $V_L$  across an inductor (between its terminals, which we assume to be outside the region of changing flux). For an *ideal inductor* (its wire has negligible resistance), the magnitude of  $V_L$  is equal to the magnitude of the self-induced emf  $\mathcal{E}_L$ .

If, instead, the wire in the inductor has resistance  $r$ , we mentally separate the inductor into a resistance  $r$  (which we take to be outside the region of changing flux) and an ideal inductor of self-induced emf  $\mathcal{E}_L$ . As with a real battery of emf  $\mathcal{E}$  and internal resistance  $r$ , the potential difference across the terminals of a real inductor then differs from the emf. Unless otherwise indicated, we assume here that inductors are ideal.



### Checkpoint 5

The figure shows an emf  $\mathcal{E}_L$  induced in a coil. Which of the following can describe the current through the coil: (a) constant and rightward, (b) constant and leftward, (c) increasing and rightward, (d) decreasing and rightward, (e) increasing and leftward, (f) decreasing and leftward?



## 30-6 RL CIRCUITS

### Learning Objectives

After reading this module, you should be able to . . .

- 30.25** Sketch a schematic diagram of an  $RL$  circuit in which the current is rising.
- 30.26** Write a loop equation (a differential equation) for an  $RL$  circuit in which the current is rising.
- 30.27** For an  $RL$  circuit in which the current is rising, apply the equation  $i(t)$  for the current as a function of time.
- 30.28** For an  $RL$  circuit in which the current is rising, find equations for the potential difference  $V$  across the resistor, the rate  $di/dt$  at which the current changes, and the emf of the inductor, as functions of time.
- 30.29** Calculate an inductive time constant  $\tau_L$ .
- 30.30** Sketch a schematic diagram of an  $RL$  circuit in which the current is decaying.

- 30.31** Write a loop equation (a differential equation) for an  $RL$  circuit in which the current is decaying.
- 30.32** For an  $RL$  circuit in which the current is decaying, apply the equation  $i(t)$  for the current as a function of time.
- 30.33** From an equation for decaying current in an  $RL$  circuit, find equations for the potential difference  $V$  across the resistor, the rate  $di/dt$  at which current is changing, and the emf of the inductor, as functions of time.
- 30.34** For an  $RL$  circuit, identify the current through the inductor and the emf across it just as current in the circuit begins to change (the initial condition) and a long time later when equilibrium is reached (the final condition).

### Key Ideas

- If a constant emf  $\mathcal{E}$  is introduced into a single-loop circuit containing a resistance  $R$  and an inductance  $L$ , the current rises to an equilibrium value of  $\mathcal{E}/R$  according to

$$i = \frac{\mathcal{E}}{R} (1 - e^{-t/\tau_L}) \quad (\text{rise of current}).$$

Here  $\tau_L (= L/R)$  governs the rate of rise of the current and is called the inductive time constant of the circuit.

- When the source of constant emf is removed, the current decays from a value  $i_0$  according to

$$i = i_0 e^{-t/\tau_L} \quad (\text{decay of current}).$$

## RL Circuits

In Module 27-4 we saw that if we suddenly introduce an emf  $\mathcal{E}$  into a single-loop circuit containing a resistor  $R$  and a capacitor  $C$ , the charge on the capacitor does not build up immediately to its final equilibrium value  $C\mathcal{E}$  but approaches it in an exponential fashion:

$$q = C\mathcal{E}(1 - e^{-t/\tau_C}). \quad (30-36)$$

The rate at which the charge builds up is determined by the capacitive time constant  $\tau_C$ , defined in Eq. 27-36 as

$$\tau_C = RC. \quad (30-37)$$

If we suddenly remove the emf from this same circuit, the charge does not immediately fall to zero but approaches zero in an exponential fashion:

$$q = q_0 e^{-t/\tau_C}. \quad (30-38)$$

The time constant  $\tau_C$  describes the fall of the charge as well as its rise.

An analogous slowing of the rise (or fall) of the current occurs if we introduce an emf  $\mathcal{E}$  into (or remove it from) a single-loop circuit containing a resistor  $R$  and an inductor  $L$ . When the switch  $S$  in Fig. 30-15 is closed on  $a$ , for example, the current in the resistor starts to rise. If the inductor were not present, the current would rise rapidly to a steady value  $\mathcal{E}/R$ . Because of the inductor, however, a self-induced emf  $\mathcal{E}_L$  appears in the circuit; from Lenz's law, this emf opposes the rise of the current, which means that it opposes the battery emf  $\mathcal{E}$  in polarity. Thus, the current in the resistor responds to the difference between two emfs, a constant  $\mathcal{E}$  due to the battery and a variable  $\mathcal{E}_L (= -L di/dt)$  due to self-induction. As long as this  $\mathcal{E}_L$  is present, the current will be less than  $\mathcal{E}/R$ .

As time goes on, the rate at which the current increases becomes less rapid and the magnitude of the self-induced emf, which is proportional to  $di/dt$ , becomes smaller. Thus, the current in the circuit approaches  $\mathcal{E}/R$  asymptotically.

We can generalize these results as follows:



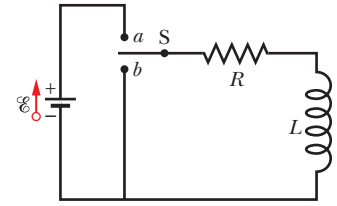
Initially, an inductor acts to oppose changes in the current through it. A long time later, it acts like ordinary connecting wire.

Now let us analyze the situation quantitatively. With the switch  $S$  in Fig. 30-15 thrown to  $a$ , the circuit is equivalent to that of Fig. 30-16. Let us apply the loop rule, starting at point  $x$  in this figure and moving clockwise around the loop along with current  $i$ .

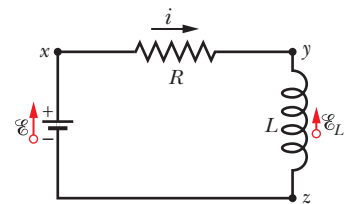
- 1. Resistor.** Because we move through the resistor in the direction of current  $i$ , the electric potential decreases by  $iR$ . Thus, as we move from point  $x$  to point  $y$ , we encounter a potential change of  $-iR$ .
- 2. Inductor.** Because current  $i$  is changing, there is a self-induced emf  $\mathcal{E}_L$  in the inductor. The magnitude of  $\mathcal{E}_L$  is given by Eq. 30-35 as  $L di/dt$ . The direction of  $\mathcal{E}_L$  is upward in Fig. 30-16 because current  $i$  is downward through the inductor and increasing. Thus, as we move from point  $y$  to point  $z$ , opposite the direction of  $\mathcal{E}_L$ , we encounter a potential change of  $-L di/dt$ .
- 3. Battery.** As we move from point  $z$  back to starting point  $x$ , we encounter a potential change of  $+\mathcal{E}$  due to the battery's emf.

Thus, the loop rule gives us

$$-iR - L \frac{di}{dt} + \mathcal{E} = 0$$



**Figure 30-15** An  $RL$  circuit. When switch  $S$  is closed on  $a$ , the current rises and approaches a limiting value  $\mathcal{E}/R$ .



**Figure 30-16** The circuit of Fig. 30-15 with the switch closed on  $a$ . We apply the loop rule for the circuit clockwise, starting at  $x$ .

$$\text{or} \quad L \frac{di}{dt} + Ri = \mathcal{E} \quad (RL \text{ circuit}). \quad (30-39)$$

Equation 30-39 is a differential equation involving the variable  $i$  and its first derivative  $di/dt$ . To solve it, we seek the function  $i(t)$  such that when  $i(t)$  and its first derivative are substituted in Eq. 30-39, the equation is satisfied and the initial condition  $i(0) = 0$  is satisfied.

Equation 30-39 and its initial condition are of exactly the form of Eq. 27-32 for an  $RC$  circuit, with  $i$  replacing  $q$ ,  $L$  replacing  $R$ , and  $R$  replacing  $1/C$ . The solution of Eq. 30-39 must then be of exactly the form of Eq. 27-33 with the same replacements. That solution is

$$i = \frac{\mathcal{E}}{R} (1 - e^{-Rt/L}), \quad (30-40)$$

which we can rewrite as

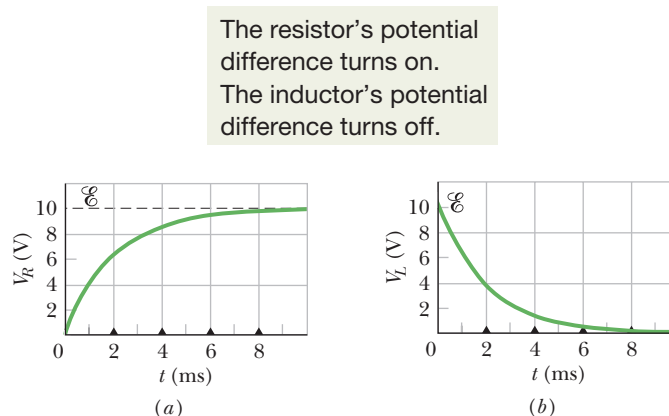
$$i = \frac{\mathcal{E}}{R} (1 - e^{-t/\tau_L}) \quad (\text{rise of current}). \quad (30-41)$$

Here  $\tau_L$ , the **inductive time constant**, is given by

$$\tau_L = \frac{L}{R} \quad (\text{time constant}). \quad (30-42)$$

Let's examine Eq. 30-41 for just after the switch is closed (at time  $t = 0$ ) and for a time long after the switch is closed ( $t \rightarrow \infty$ ). If we substitute  $t = 0$  into Eq. 30-41, the exponential becomes  $e^0 = 1$ . Thus, Eq. 30-41 tells us that the current is initially  $i = 0$ , as we expected. Next, if we let  $t$  go to  $\infty$ , then the exponential goes to  $e^{-\infty} = 0$ . Thus, Eq. 30-41 tells us that the current goes to its equilibrium value of  $\mathcal{E}/R$ .

We can also examine the potential differences in the circuit. For example, Fig. 30-17 shows how the potential differences  $V_R (= iR)$  across the resistor and  $V_L (= L di/dt)$  across the inductor vary with time for particular values of  $\mathcal{E}$ ,  $L$ , and  $R$ . Compare this figure carefully with the corresponding figure for an  $RC$  circuit (Fig. 27-16).



**Figure 30-17** The variation with time of (a)  $V_R$ , the potential difference across the resistor in the circuit of Fig. 30-16, and (b)  $V_L$ , the potential difference across the inductor in that circuit. The small triangles represent successive intervals of one inductive time constant  $\tau_L = L/R$ . The figure is plotted for  $R = 2000 \, \Omega$ ,  $L = 4.0 \, \text{H}$ , and  $\mathcal{E} = 10 \, \text{V}$ .

To show that the quantity  $\tau_L (= L/R)$  has the dimension of time (as it must, because the argument of the exponential function in Eq. 30-41 must be dimensionless), we convert from henries per ohm as follows:

$$1 \frac{\text{H}}{\Omega} = 1 \frac{\text{H}}{\Omega} \left( \frac{1 \text{ V} \cdot \text{s}}{1 \text{ H} \cdot \text{A}} \right) \left( \frac{1 \Omega \cdot \text{A}}{1 \text{ V}} \right) = 1 \text{ s}.$$

The first quantity in parentheses is a conversion factor based on Eq. 30-35, and the second one is a conversion factor based on the relation  $V = iR$ .

**Time Constant.** The physical significance of the time constant follows from Eq. 30-41. If we put  $t = \tau_L = L/R$  in this equation, it reduces to

$$i = \frac{\mathcal{E}}{R} (1 - e^{-1}) = 0.63 \frac{\mathcal{E}}{R}. \quad (30-43)$$

Thus, the time constant  $\tau_L$  is the time it takes the current in the circuit to reach about 63% of its final equilibrium value  $\mathcal{E}/R$ . Since the potential difference  $V_R$  across the resistor is proportional to the current  $i$ , a graph of the increasing current versus time has the same shape as that of  $V_R$  in Fig. 30-17*a*.

**Current Decay.** If the switch  $S$  in Fig. 30-15 is closed on  $a$  long enough for the equilibrium current  $\mathcal{E}/R$  to be established and then is thrown to  $b$ , the effect will be to remove the battery from the circuit. (The connection to  $b$  must actually be made an instant before the connection to  $a$  is broken. A switch that does this is called a *make-before-break* switch.) With the battery gone, the current through the resistor will decrease. However, it cannot drop immediately to zero but must decay to zero over time. The differential equation that governs the decay can be found by putting  $\mathcal{E} = 0$  in Eq. 30-39:

$$L \frac{di}{dt} + iR = 0. \quad (30-44)$$

By analogy with Eqs. 27-38 and 27-39, the solution of this differential equation that satisfies the initial condition  $i(0) = i_0 = \mathcal{E}/R$  is

$$i = \frac{\mathcal{E}}{R} e^{-t/\tau_L} = i_0 e^{-t/\tau_L} \quad (\text{decay of current}). \quad (30-45)$$

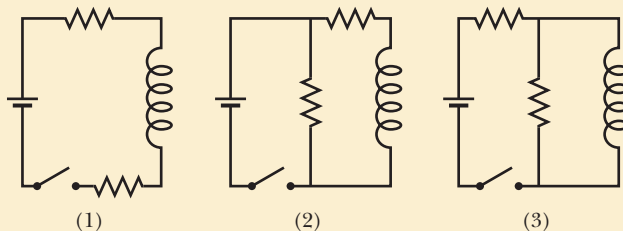
We see that both current rise (Eq. 30-41) and current decay (Eq. 30-45) in an *RL* circuit are governed by the same inductive time constant,  $\tau_L$ .

We have used  $i_0$  in Eq. 30-45 to represent the current at time  $t = 0$ . In our case that happened to be  $\mathcal{E}/R$ , but it could be any other initial value.



### Checkpoint 6

The figure shows three circuits with identical batteries, inductors, and resistors. Rank the circuits according to the current through the battery (a) just after the switch is closed and (b) a long time later, greatest first. (If you have trouble here, work through the next sample problem and then try again.)



**Sample Problem 30.05** *RL* circuit, immediately after switching and after a long time

Figure 30-18*a* shows a circuit that contains three identical resistors with resistance  $R = 9.0\ \Omega$ , two identical inductors with inductance  $L = 2.0\ \text{mH}$ , and an ideal battery with emf  $\mathcal{E} = 18\ \text{V}$ .

(a) What is the current  $i$  through the battery just after the switch is closed?

**KEY IDEA**

Just after the switch is closed, the inductor acts to oppose a change in the current through it.

**Calculations:** Because the current through each inductor is zero before the switch is closed, it will also be zero just afterward. Thus, immediately after the switch is closed, the inductors act as broken wires, as indicated in Fig. 30-18*b*. We then have a single-loop circuit for which the loop rule gives us

$$\mathcal{E} - iR = 0.$$

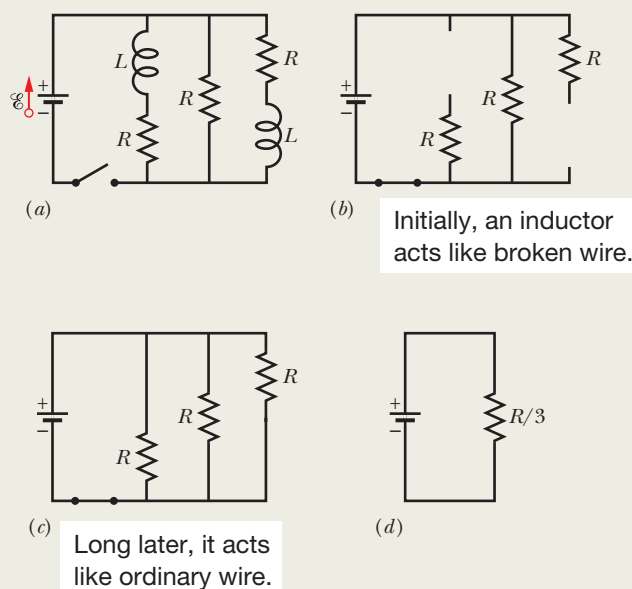
Substituting given data, we find that

$$i = \frac{\mathcal{E}}{R} = \frac{18\ \text{V}}{9.0\ \Omega} = 2.0\ \text{A}. \quad (\text{Answer})$$

(b) What is the current  $i$  through the battery long after the switch has been closed?

**KEY IDEA**

Long after the switch has been closed, the currents in the circuit have reached their equilibrium values, and the inductors act as simple connecting wires, as indicated in Fig. 30-18*c*.



**Figure 30-18** (a) A multiloop *RL* circuit with an open switch. (b) The equivalent circuit just after the switch has been closed. (c) The equivalent circuit a long time later. (d) The single-loop circuit that is equivalent to circuit (c).

**Calculations:** We now have a circuit with three identical resistors in parallel; from Eq. 27-23, their equivalent resistance is  $R_{\text{eq}} = R/3 = (9.0\ \Omega)/3 = 3.0\ \Omega$ . The equivalent circuit shown in Fig. 30-18*d* then yields the loop equation  $\mathcal{E} - iR_{\text{eq}} = 0$ , or

$$i = \frac{\mathcal{E}}{R_{\text{eq}}} = \frac{18\ \text{V}}{3.0\ \Omega} = 6.0\ \text{A}. \quad (\text{Answer})$$

**Sample Problem 30.06** *RL* circuit, current during the transition

A solenoid has an inductance of  $53\ \text{mH}$  and a resistance of  $0.37\ \Omega$ . If the solenoid is connected to a battery, how long will the current take to reach half its final equilibrium value? (This is a *real solenoid* because we are considering its small, but nonzero, internal resistance.)

**KEY IDEA**

We can mentally separate the solenoid into a resistance and an inductance that are wired in series with a battery, as in Fig. 30-16. Then application of the loop rule leads to Eq. 30-39, which has the solution of Eq. 30-41 for the current  $i$  in the circuit.

**Calculations:** According to that solution, current  $i$  increases exponentially from zero to its final equilibrium value of  $\mathcal{E}/R$ . Let  $t_0$  be the time that current  $i$  takes to reach half its equilibrium value. Then Eq. 30-41 gives us

$$\frac{1}{2} \frac{\mathcal{E}}{R} = \frac{\mathcal{E}}{R} (1 - e^{-t_0/\tau_L}).$$

We solve for  $t_0$  by canceling  $\mathcal{E}/R$ , isolating the exponential, and taking the natural logarithm of each side. We find

$$\begin{aligned} t_0 &= \tau_L \ln 2 = \frac{L}{R} \ln 2 = \frac{53 \times 10^{-3}\ \text{H}}{0.37\ \Omega} \ln 2 \\ &= 0.10\ \text{s}. \end{aligned} \quad (\text{Answer})$$



## 30-7 ENERGY STORED IN A MAGNETIC FIELD

### Learning Objectives

After reading this module, you should be able to . . .

**30.35** Describe the derivation of the equation for the magnetic field energy of an inductor in an  $RL$  circuit with a constant emf source.

**30.36** For an inductor in an  $RL$  circuit, apply the relationship between the magnetic field energy  $U$ , the inductance  $L$ , and the current  $i$ .

### Key Idea

- If an inductor  $L$  carries a current  $i$ , the inductor's magnetic field stores an energy given by

$$U_B = \frac{1}{2} Li^2 \quad (\text{magnetic energy}).$$

### Energy Stored in a Magnetic Field

When we pull two charged particles of opposite signs away from each other, we say that the resulting electric potential energy is stored in the electric field of the particles. We get it back from the field by letting the particles move closer together again. In the same way we say energy is stored in a magnetic field, but now we deal with current instead of electric charges.

To derive a quantitative expression for that stored energy, consider again Fig. 30-16, which shows a source of emf  $\mathcal{E}$  connected to a resistor  $R$  and an inductor  $L$ . Equation 30-39, restated here for convenience,

$$\mathcal{E} = L \frac{di}{dt} + iR, \quad (30-46)$$

is the differential equation that describes the growth of current in this circuit. Recall that this equation follows immediately from the loop rule and that the loop rule in turn is an expression of the principle of conservation of energy for single-loop circuits. If we multiply each side of Eq. 30-46 by  $i$ , we obtain

$$\mathcal{E}i = Li \frac{di}{dt} + i^2 R, \quad (30-47)$$

which has the following physical interpretation in terms of the work done by the battery and the resulting energy transfers:

1. If a differential amount of charge  $dq$  passes through the battery of emf  $\mathcal{E}$  in Fig. 30-16 in time  $dt$ , the battery does work on it in the amount  $\mathcal{E} dq$ . The rate at which the battery does work is  $(\mathcal{E} dq)/dt$ , or  $\mathcal{E}i$ . Thus, the left side of Eq. 30-47 represents the rate at which the emf device delivers energy to the rest of the circuit.
2. The rightmost term in Eq. 30-47 represents the rate at which energy appears as thermal energy in the resistor.
3. Energy that is delivered to the circuit but does not appear as thermal energy must, by the conservation-of-energy hypothesis, be stored in the magnetic field of the inductor. Because Eq. 30-47 represents the principle of conservation of energy for  $RL$  circuits, the middle term must represent the rate  $dU_B/dt$  at which magnetic potential energy  $U_B$  is stored in the magnetic field.

Thus

$$\frac{dU_B}{dt} = Li \frac{di}{dt}. \quad (30-48)$$

We can write this as

$$dU_B = Li \, di.$$

Integrating yields

$$\int_0^{U_B} dU_B = \int_0^i Li \, di$$

$$\text{or} \quad U_B = \frac{1}{2}Li^2 \quad (\text{magnetic energy}), \quad (30-49)$$

which represents the total energy stored by an inductor  $L$  carrying a current  $i$ . Note the similarity in form between this expression for the energy stored in a magnetic field and the expression for the energy stored in an electric field by a capacitor with capacitance  $C$  and charge  $q$ ; namely,

$$U_E = \frac{q^2}{2C}. \quad (30-50)$$

(The variable  $i^2$  corresponds to  $q^2$ , and the constant  $L$  corresponds to  $1/C$ .)



### Sample Problem 30.07 Energy stored in a magnetic field

A coil has an inductance of 53 mH and a resistance of 0.35  $\Omega$ .

(a) If a 12 V emf is applied across the coil, how much energy is stored in the magnetic field after the current has built up to its equilibrium value?

#### KEY IDEA

The energy stored in the magnetic field of a coil at any time depends on the current through the coil at that time, according to Eq. 30-49 ( $U_B = \frac{1}{2}Li^2$ ).

**Calculations:** Thus, to find the energy  $U_{B\infty}$  stored at equilibrium, we must first find the equilibrium current. From Eq. 30-41, the equilibrium current is

$$i_{\infty} = \frac{\mathcal{E}}{R} = \frac{12 \text{ V}}{0.35 \, \Omega} = 34.3 \text{ A}. \quad (30-51)$$

Then substitution yields

$$\begin{aligned} U_{B\infty} &= \frac{1}{2}Li_{\infty}^2 = \left(\frac{1}{2}\right)(53 \times 10^{-3} \text{ H})(34.3 \text{ A})^2 \\ &= 31 \text{ J}. \end{aligned} \quad (\text{Answer})$$

(b) After how many time constants will half this equilibrium energy be stored in the magnetic field?

**Calculations:** Now we are being asked: At what time  $t$  will the relation

$$U_B = \frac{1}{2}U_{B\infty}$$

be satisfied? Using Eq. 30-49 twice allows us to rewrite this energy condition as

$$\frac{1}{2}Li^2 = \left(\frac{1}{2}\right)\frac{1}{2}Li_{\infty}^2$$

$$\text{or} \quad i = \left(\frac{1}{\sqrt{2}}\right)i_{\infty}. \quad (30-52)$$

This equation tells us that, as the current increases from its initial value of 0 to its final value of  $i_{\infty}$ , the magnetic field will have half its final stored energy when the current has increased to this value. In general, we know that  $i$  is given by Eq. 30-41, and here  $i_{\infty}$  (see Eq. 30-51) is  $\mathcal{E}/R$ ; so Eq. 30-52 becomes

$$\frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}) = \frac{\mathcal{E}}{\sqrt{2}R}.$$

By canceling  $\mathcal{E}/R$  and rearranging, we can write this as

$$e^{-t/\tau_L} = 1 - \frac{1}{\sqrt{2}} = 0.293,$$

which yields

$$\frac{t}{\tau_L} = -\ln 0.293 = 1.23$$

$$\text{or} \quad t \approx 1.2\tau_L. \quad (\text{Answer})$$

Thus, the energy stored in the magnetic field of the coil by the current will reach half its equilibrium value 1.2 time constants after the emf is applied.



## 30-8 ENERGY DENSITY OF A MAGNETIC FIELD

### Learning Objectives

After reading this module, you should be able to . . .

**30.37** Identify that energy is associated with any magnetic field.

**30.38** Apply the relationship between energy density  $u_B$  of a magnetic field and the magnetic field magnitude  $B$ .

### Key Idea

● If  $B$  is the magnitude of a magnetic field at any point (in an inductor or anywhere else), the density of stored magnetic energy at that point is

$$u_B = \frac{B^2}{2\mu_0} \quad (\text{magnetic energy density}).$$

### Energy Density of a Magnetic Field

Consider a length  $l$  near the middle of a long solenoid of cross-sectional area  $A$  carrying current  $i$ ; the volume associated with this length is  $Al$ . The energy  $U_B$  stored by the length  $l$  of the solenoid must lie entirely within this volume because the magnetic field outside such a solenoid is approximately zero. Moreover, the stored energy must be uniformly distributed within the solenoid because the magnetic field is (approximately) uniform everywhere inside.

Thus, the energy stored per unit volume of the field is

$$u_B = \frac{U_B}{Al}$$

or, since

$$U_B = \frac{1}{2} Li^2,$$

we have

$$u_B = \frac{Li^2}{2Al} = \frac{L}{l} \frac{i^2}{2A}. \quad (30-53)$$

Here  $L$  is the inductance of length  $l$  of the solenoid.

Substituting for  $L/l$  from Eq. 30-31, we find

$$u_B = \frac{1}{2} \mu_0 n^2 i^2, \quad (30-54)$$

where  $n$  is the number of turns per unit length. From Eq. 29-23 ( $B = \mu_0 n i$ ) we can write this *energy density* as

$$u_B = \frac{B^2}{2\mu_0} \quad (\text{magnetic energy density}). \quad (30-55)$$

This equation gives the density of stored energy at any point where the magnitude of the magnetic field is  $B$ . Even though we derived it by considering the special case of a solenoid, Eq. 30-55 holds for all magnetic fields, no matter how they are generated. The equation is comparable to Eq. 25-25,

$$u_E = \frac{1}{2} \epsilon_0 E^2, \quad (30-56)$$

which gives the energy density (in a vacuum) at any point in an electric field. Note that both  $u_B$  and  $u_E$  are proportional to the square of the appropriate field magnitude,  $B$  or  $E$ .

**Checkpoint 7**

The table lists the number of turns per unit length, current, and cross-sectional area for three solenoids. Rank the solenoids according to the magnetic energy density within them, greatest first.

Solenoid	Turns per Unit Length	Current	Area
<i>a</i>	$2n_1$	$i_1$	$2A_1$
<i>b</i>	$n_1$	$2i_1$	$A_1$
<i>c</i>	$n_1$	$i_1$	$6A_1$

## 30-9 MUTUAL INDUCTION

### Learning Objectives

After reading this module, you should be able to . . .

**30.39** Describe the mutual induction of two coils and sketch the arrangement.

**30.40** Calculate the mutual inductance of one coil with respect to a second coil (or some second current that is changing).

**30.41** Calculate the emf induced in one coil by a second coil in terms of the mutual inductance and the rate of change of the current in the second coil.

### Key Idea

● If coils 1 and 2 are near each other, a changing current in either coil can induce an emf in the other. This mutual induction is described by

$$\mathcal{E}_2 = -M \frac{di_1}{dt}$$

and

$$\mathcal{E}_1 = -M \frac{di_2}{dt},$$

where  $M$  (measured in henries) is the mutual inductance.

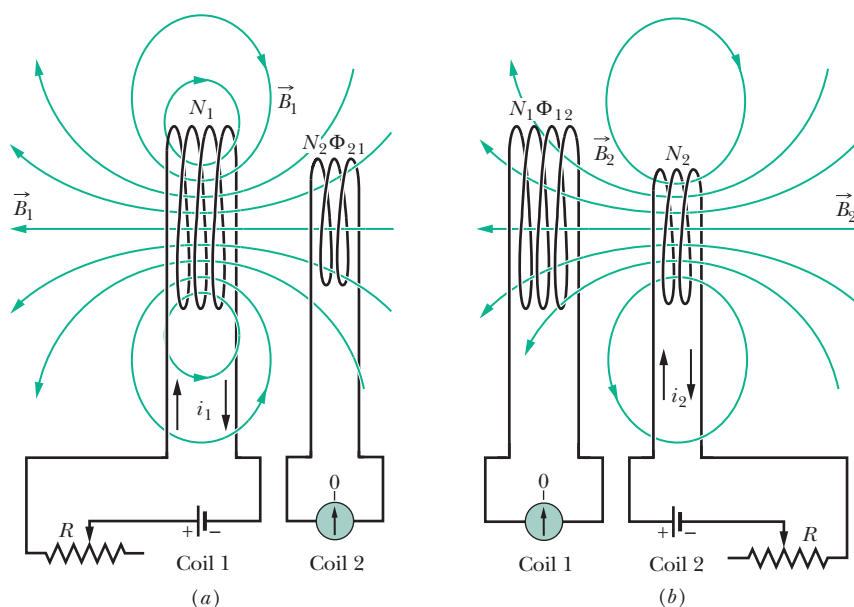
## Mutual Induction

In this section we return to the case of two interacting coils, which we first discussed in Module 30-1, and we treat it in a somewhat more formal manner. We saw earlier that if two coils are close together as in Fig. 30-2, a steady current  $i$  in one coil will set up a magnetic flux  $\Phi$  through the other coil (*linking* the other coil). If we change  $i$  with time, an emf  $\mathcal{E}$  given by Faraday's law appears in the second coil; we called this process *induction*. We could better have called it **mutual induction**, to suggest the mutual interaction of the two coils and to distinguish it from *self-induction*, in which only one coil is involved.

Let us look a little more quantitatively at mutual induction. Figure 30-19a shows two circular close-packed coils near each other and sharing a common central axis. With the variable resistor set at a particular resistance  $R$ , the battery produces a steady current  $i_1$  in coil 1. This current creates a magnetic field represented by the lines of  $\vec{B}_1$  in the figure. Coil 2 is connected to a sensitive meter but contains no battery; a magnetic flux  $\Phi_{21}$  (the flux through coil 2 associated with the current in coil 1) links the  $N_2$  turns of coil 2.

We define the mutual inductance  $M_{21}$  of coil 2 with respect to coil 1 as

$$M_{21} = \frac{N_2 \Phi_{21}}{i_1}, \quad (30-57)$$



**Figure 30-19** Mutual induction. (a) The magnetic field  $\vec{B}_1$  produced by current  $i_1$  in coil 1 extends through coil 2. If  $i_1$  is varied (by varying resistance  $R$ ), an emf is induced in coil 2 and current registers on the meter connected to coil 2. (b) The roles of the coils interchanged.

which has the same form as Eq. 30-28,

$$L = N\Phi/i, \quad (30-58)$$

the definition of inductance. We can recast Eq. 30-57 as

$$M_{21}i_1 = N_2\Phi_{21}. \quad (30-59)$$

If we cause  $i_1$  to vary with time by varying  $R$ , we have

$$M_{21} \frac{di_1}{dt} = N_2 \frac{d\Phi_{21}}{dt}. \quad (30-60)$$

The right side of this equation is, according to Faraday's law, just the magnitude of the emf  $\mathcal{E}_2$  appearing in coil 2 due to the changing current in coil 1. Thus, with a minus sign to indicate direction,

$$\mathcal{E}_2 = -M_{21} \frac{di_1}{dt}, \quad (30-61)$$

which you should compare with Eq. 30-35 for self-induction ( $\mathcal{E} = -L di/dt$ ).

**Interchange.** Let us now interchange the roles of coils 1 and 2, as in Fig. 30-19b; that is, we set up a current  $i_2$  in coil 2 by means of a battery, and this produces a magnetic flux  $\Phi_{12}$  that links coil 1. If we change  $i_2$  with time by varying  $R$ , we then have, by the argument given above,

$$\mathcal{E}_1 = -M_{12} \frac{di_2}{dt}. \quad (30-62)$$

Thus, we see that the emf induced in either coil is proportional to the rate of change of current in the other coil. The proportionality constants  $M_{21}$  and  $M_{12}$  seem to be different. However, they turn out to be the same, although we cannot prove that fact here. Thus, we have

$$M_{21} = M_{12} = M, \quad (30-63)$$

and we can rewrite Eqs. 30-61 and 30-62 as

$$\mathcal{E}_2 = -M \frac{di_1}{dt} \quad (30-64)$$

and

$$\mathcal{E}_1 = -M \frac{di_2}{dt}. \quad (30-65)$$



### Sample Problem 30.08 Mutual inductance of two parallel coils

Figure 30-20 shows two circular close-packed coils, the smaller (radius  $R_2$ , with  $N_2$  turns) being coaxial with the larger (radius  $R_1$ , with  $N_1$  turns) and in the same plane.

(a) Derive an expression for the mutual inductance  $M$  for this arrangement of these two coils, assuming that  $R_1 \gg R_2$ .

#### KEY IDEA

The mutual inductance  $M$  for these coils is the ratio of the flux linkage ( $N\Phi$ ) through one coil to the current  $i$  in the other coil, which produces that flux linkage. Thus, we need to assume that currents exist in the coils; then we need to calculate the flux linkage in one of the coils.

**Calculations:** The magnetic field through the larger coil due to the smaller coil is nonuniform in both magnitude and direction; so the flux through the larger coil due to the smaller coil is nonuniform and difficult to calculate. However, the smaller coil is small enough for us to assume that the magnetic field through it due to the larger coil is approximately uniform. Thus, the flux through it due to the larger coil is also approximately uniform. Hence, to find  $M$  we shall assume a current  $i_1$  in the larger coil and calculate the flux linkage  $N_2\Phi_{21}$  in the smaller coil:

$$M = \frac{N_2\Phi_{21}}{i_1}. \quad (30-66)$$

The flux  $\Phi_{21}$  through each turn of the smaller coil is, from Eq. 30-2,

$$\Phi_{21} = B_1 A_2,$$

where  $B_1$  is the magnitude of the magnetic field at points within the small coil due to the larger coil and  $A_2 (= \pi R_2^2)$  is the area enclosed by the turn. Thus, the flux linkage in the smaller coil (with its  $N_2$  turns) is

$$N_2\Phi_{21} = N_2 B_1 A_2. \quad (30-67)$$

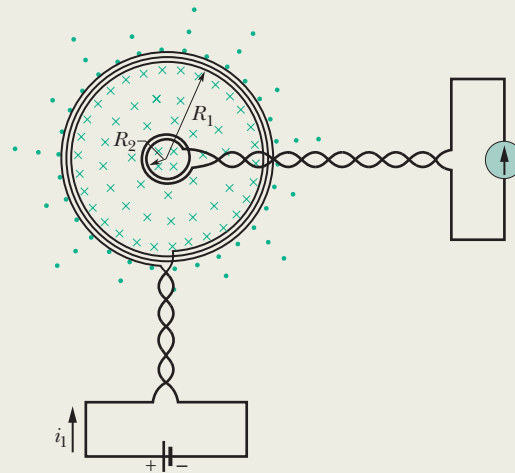
To find  $B_1$  at points within the smaller coil, we can use Eq. 29-26,

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}},$$

with  $z$  set to 0 because the smaller coil is in the plane of the larger coil. That equation tells us that each turn of the larger coil produces a magnetic field of magnitude  $\mu_0 i_1 / 2R_1$  at points within the smaller coil. Thus, the larger coil (with its  $N_1$  turns) produces a total magnetic field of magnitude

$$B_1 = N_1 \frac{\mu_0 i_1}{2R_1} \quad (30-68)$$

at points within the smaller coil.



**Figure 30-20** A small coil is located at the center of a large coil. The mutual inductance of the coils can be determined by sending current  $i_1$  through the large coil.

Substituting Eq. 30-68 for  $B_1$  and  $\pi R_2^2$  for  $A_2$  in Eq. 30-67 yields

$$N_2\Phi_{21} = \frac{\pi\mu_0 N_1 N_2 R_2^2 i_1}{2R_1}.$$

Substituting this result into Eq. 30-66, we find

$$M = \frac{N_2\Phi_{21}}{i_1} = \frac{\pi\mu_0 N_1 N_2 R_2^2}{2R_1}. \quad (\text{Answer}) \quad (30-69)$$

(b) What is the value of  $M$  for  $N_1 = N_2 = 1200$  turns,  $R_2 = 1.1$  cm, and  $R_1 = 15$  cm?

**Calculations:** Equation 30-69 yields

$$\begin{aligned} M &= \frac{(\pi)(4\pi \times 10^{-7} \text{ H/m})(1200)(1200)(0.011 \text{ m})^2}{(2)(0.15 \text{ m})} \\ &= 2.29 \times 10^{-3} \text{ H} \approx 2.3 \text{ mH}. \end{aligned} \quad (\text{Answer})$$

Consider the situation if we reverse the roles of the two coils — that is, if we produce a current  $i_2$  in the smaller coil and try to calculate  $M$  from Eq. 30-57 in the form

$$M = \frac{N_1\Phi_{12}}{i_2}.$$

The calculation of  $\Phi_{12}$  (the nonuniform flux of the smaller coil's magnetic field encompassed by the larger coil) is not simple. If we were to do the calculation numerically using a computer, we would find  $M$  to be 2.3 mH, as above! This emphasizes that Eq. 30-63 ( $M_{21} = M_{12} = M$ ) is not obvious.

## Review & Summary

**Magnetic Flux** The *magnetic flux*  $\Phi_B$  through an area  $A$  in a magnetic field  $\vec{B}$  is defined as

$$\Phi_B = \int \vec{B} \cdot d\vec{A}, \quad (30-1)$$

where the integral is taken over the area. The SI unit of magnetic flux is the weber, where  $1 \text{ Wb} = 1 \text{ T} \cdot \text{m}^2$ . If  $\vec{B}$  is perpendicular to the area and uniform over it, Eq. 30-1 becomes

$$\Phi_B = BA \quad (\vec{B} \perp A, \vec{B} \text{ uniform}). \quad (30-2)$$

**Faraday's Law of Induction** If the magnetic flux  $\Phi_B$  through an area bounded by a closed conducting loop changes with time, a current and an emf are produced in the loop; this process is called *induction*. The induced emf is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad (\text{Faraday's law}). \quad (30-4)$$

If the loop is replaced by a closely packed coil of  $N$  turns, the induced emf is

$$\mathcal{E} = -N \frac{d\Phi_B}{dt}. \quad (30-5)$$

**Lenz's Law** An induced current has a direction such that the magnetic field *due to the current* opposes the change in the magnetic flux that induces the current. The induced emf has the same direction as the induced current.

**Emf and the Induced Electric Field** An emf is induced by a changing magnetic flux even if the loop through which the flux is changing is not a physical conductor but an imaginary line. The changing magnetic field induces an electric field  $\vec{E}$  at every point of such a loop; the induced emf is related to  $\vec{E}$  by

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s}, \quad (30-19)$$

where the integration is taken around the loop. From Eq. 30-19 we can write Faraday's law in its most general form,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad (\text{Faraday's law}). \quad (30-20)$$

A changing magnetic field induces an electric field  $\vec{E}$ .

**Inductors** An **inductor** is a device that can be used to produce a known magnetic field in a specified region. If a current  $i$  is established through each of the  $N$  windings of an inductor, a magnetic flux  $\Phi_B$  links those windings. The **inductance**  $L$  of the inductor is

$$L = \frac{N\Phi_B}{i} \quad (\text{inductance defined}). \quad (30-28)$$

The SI unit of inductance is the **henry** (H), where  $1 \text{ henry} = 1 \text{ H} = 1 \text{ T} \cdot \text{m}^2/\text{A}$ . The inductance per unit length near the middle of a long solenoid of cross-sectional area  $A$  and  $n$  turns per unit length is

$$\frac{L}{l} = \mu_0 n^2 A \quad (\text{solenoid}). \quad (30-31)$$

**Self-Induction** If a current  $i$  in a coil changes with time, an emf is induced in the coil. This self-induced emf is

$$\mathcal{E}_L = -L \frac{di}{dt}. \quad (30-35)$$

The direction of  $\mathcal{E}_L$  is found from Lenz's law: The self-induced emf acts to oppose the change that produces it.

**Series RL Circuits** If a constant emf  $\mathcal{E}$  is introduced into a single-loop circuit containing a resistance  $R$  and an inductance  $L$ , the current rises to an equilibrium value of  $\mathcal{E}/R$ :

$$i = \frac{\mathcal{E}}{R} (1 - e^{-t/\tau_L}) \quad (\text{rise of current}). \quad (30-41)$$

Here  $\tau_L (= L/R)$  is the **inductive time constant**. When the source of constant emf is removed, the current decays from a value  $i_0$  according to

$$i = i_0 e^{-t/\tau_L} \quad (\text{decay of current}). \quad (30-45)$$

**Magnetic Energy** If an inductor  $L$  carries a current  $i$ , the inductor's magnetic field stores an energy given by

$$U_B = \frac{1}{2} Li^2 \quad (\text{magnetic energy}). \quad (30-49)$$

If  $B$  is the magnitude of a magnetic field at any point (in an inductor or anywhere else), the density of stored magnetic energy at that point is

$$u_B = \frac{B^2}{2\mu_0} \quad (\text{magnetic energy density}). \quad (30-55)$$

**Mutual Induction** If coils 1 and 2 are near each other, a changing current in either coil can induce an emf in the other. This mutual induction is described by

$$\mathcal{E}_2 = -M \frac{di_1}{dt} \quad (30-64)$$

$$\text{and} \quad \mathcal{E}_1 = -M \frac{di_2}{dt}, \quad (30-65)$$

where  $M$  (measured in henries) is the mutual inductance.

## Questions

**1** If the circular conductor in Fig. 30-21 undergoes thermal expansion while it is in a uniform magnetic field, a current is induced clockwise around it. Is the magnetic field directed into or out of the page?

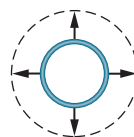


Figure 30-21 Question 1.

**2** The wire loop in Fig. 30-22a is subjected, in turn, to six uniform magnetic fields, each directed parallel to the  $z$  axis, which is directed out of the plane of the figure. Figure 30-22b

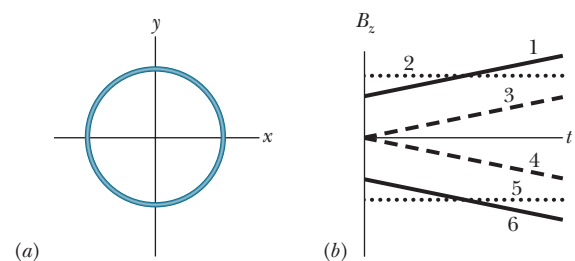


Figure 30-22 Question 2.

gives the  $z$  components  $B_z$  of the fields versus time  $t$ . (Plots 1 and 3 are parallel; so are plots 4 and 6. Plots 2 and 5 are parallel to the time axis.) Rank the six plots according to the emf induced in the loop, greatest clockwise emf first, greatest counterclockwise emf last.

**3** In Fig. 30-23, a long straight wire with current  $i$  passes (without touching) three rectangular wire loops with edge lengths  $L$ ,  $1.5L$ , and  $2L$ . The loops are widely spaced (so as not to affect one another). Loops 1 and 3 are symmetric about the long wire. Rank the loops according to the size of the current induced in them if current  $i$  is (a) constant and (b) increasing, greatest first.

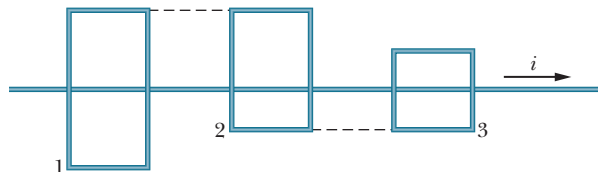


Figure 30-23 Question 3.

**4** Figure 30-24 shows two circuits in which a conducting bar is slid at the same speed  $v$  through the same uniform magnetic field and along a U-shaped wire. The parallel lengths of the wire are separated by  $2L$  in circuit 1 and by  $L$  in circuit 2. The current induced in circuit 1 is counterclockwise. (a) Is the magnetic field into or out of the page? (b) Is the current induced in circuit 2 clockwise or counterclockwise? (c) Is the emf induced in circuit 1 larger than, smaller than, or the same as that in circuit 2?

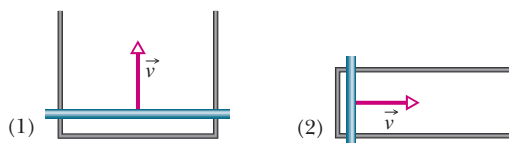


Figure 30-24 Question 4.

**5** Figure 30-25 shows a circular region in which a decreasing uniform magnetic field is directed out of the page, as well as four concentric circular paths. Rank the paths according to the magnitude of  $\oint \vec{E} \cdot d\vec{s}$  evaluated along them, greatest first.

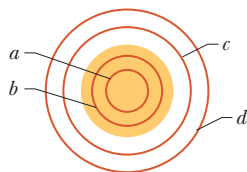


Figure 30-25 Question 5.

**6** In Fig. 30-26, a wire loop has been bent so that it has three segments: segment  $bc$  (a quarter-circle),  $ac$  (a square corner), and  $ab$  (straight). Here are three choices for a magnetic field through the loop:

- (1)  $\vec{B}_1 = 3\hat{i} + 7\hat{j} - 5\hat{k}$ ,
- (2)  $\vec{B}_2 = 5\hat{i} - 4\hat{j} - 15\hat{k}$ ,
- (3)  $\vec{B}_3 = 2\hat{i} - 5\hat{j} - 12\hat{k}$ ,

where  $\vec{B}$  is in milliteslas and  $t$  is in seconds. Without written calculation, rank the choices according to (a) the work done per unit

charge in setting up the induced current and (b) that induced current, greatest first. (c) For each choice, what is the direction of the induced current in the figure?

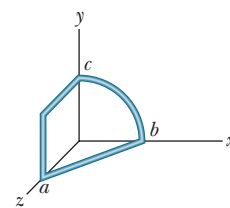


Figure 30-26 Question 6.

**7** Figure 30-27 shows a circuit with two identical resistors and an ideal inductor. Is the current through the central resistor more than, less than, or the same as that through the other resistor (a) just after the closing of switch  $S$ , (b) a long time after that, (c) just after  $S$  is reopened a long time later, and (d) a long time after that?

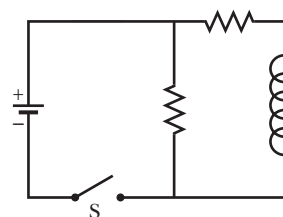


Figure 30-27 Question 7.

**8** The switch in the circuit of Fig. 30-15 has been closed on  $a$  for a very long time when it is then thrown to  $b$ . The resulting current through the inductor is indicated in Fig. 30-28 for four sets of values for the resistance  $R$  and inductance  $L$ : (1)  $R_0$  and  $L_0$ , (2)  $2R_0$  and  $L_0$ , (3)  $R_0$  and  $2L_0$ , (4)  $2R_0$  and  $2L_0$ . Which set goes with which curve?

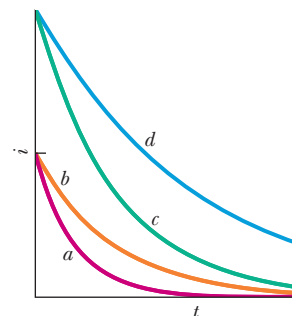


Figure 30-28 Question 8.

**9** Figure 30-29 shows three circuits with identical batteries, inductors, and resistors. Rank the circuits, greatest first, according to the current through the resistor labeled  $R$  (a) long after the switch is closed, (b) just after the switch is reopened a long time later, and (c) long after it is reopened.

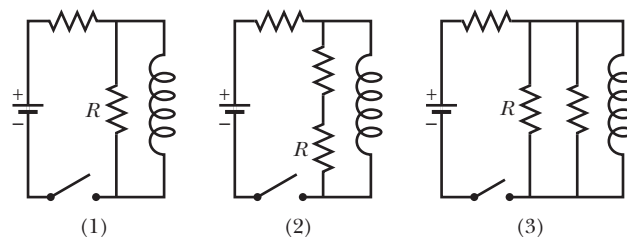


Figure 30-29 Question 9.

**10** Figure 30-30 gives the variation with time of the potential difference  $V_R$  across a resistor in three circuits wired as shown in Fig. 30-16. The circuits contain the same resistance  $R$  and emf  $\mathcal{E}$  but differ in the inductance  $L$ . Rank the circuits according to the value of  $L$ , greatest first.

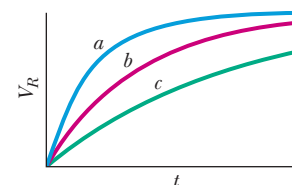


Figure 30-30 Question 10.

**11** Figure 30-31 shows three situations in which a wire loop lies partially in a magnetic field. The magnitude of the field is either increasing or decreasing, as indicated. In each situation, a battery is part of the loop. In which situations are the induced emf and the battery emf in the same direction along the loop?

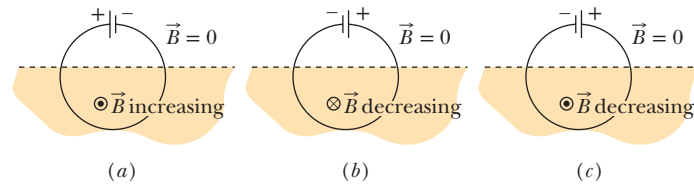


Figure 30-31 Question 11.

**12** Figure 30-32 gives four situations in which we pull rectangular wire loops out of identical magnetic fields (directed into the page).

page) at the same constant speed. The loops have edge lengths of either  $L$  or  $2L$ , as drawn. Rank the situations according to (a) the magnitude of the force required of us and (b) the rate at which energy is transferred from us to thermal energy of the loop, greatest first.

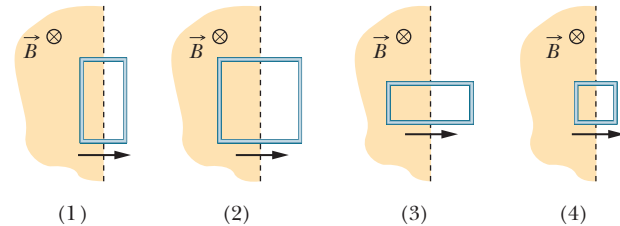


Figure 30-32 Question 12.

## Problems

**GO** Tutoring problem available (at instructor's discretion) in WileyPLUS and WebAssign

**SSM** Worked-out solution available in Student Solutions Manual

**•••** Number of dots indicates level of problem difficulty



Additional information available in *The Flying Circus of Physics* and at [flyingcircusofphysics.com](http://flyingcircusofphysics.com)

**WWW** Worked-out solution is at

**ILW** Interactive solution is at

<http://www.wiley.com/college/halliday>

### Module 30-1 Faraday's Law and Lenz's Law

**•1** In Fig. 30-33, a circular loop of wire 10 cm in diameter (seen edge-on) is placed with its normal  $\vec{N}$  at an angle  $\theta = 30^\circ$  with the direction of a uniform magnetic field  $\vec{B}$  of magnitude 0.50 T. The loop is then rotated such that  $\vec{N}$  rotates in a cone about the field direction at the rate 100 rev/min; angle  $\theta$  remains unchanged during the process. What is the emf induced in the loop?

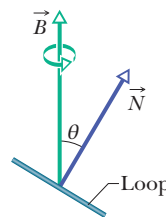


Figure 30-33 Problem 1.

**•2** A certain elastic conducting material is stretched into a circular loop of 12.0 cm radius. It is placed with its plane perpendicular to a uniform 0.800 T magnetic field. When released, the radius of the loop starts to shrink at an instantaneous rate of 75.0 cm/s. What emf is induced in the loop at that instant?

**•3 SSM WWW** In Fig. 30-34, a 120-turn coil of radius 1.8 cm and resistance  $5.3 \Omega$  is coaxial with a solenoid of 220 turns/cm and diameter 3.2 cm. The solenoid current drops from 1.5 A to zero in time interval  $\Delta t = 25$  ms. What current is induced in the coil during  $\Delta t$ ?

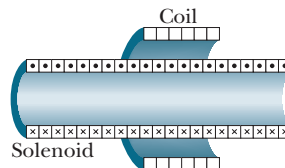


Figure 30-34 Problem 3.

**•4** A wire loop of radius 12 cm and resistance  $8.5 \Omega$  is located in a uniform magnetic field  $\vec{B}$  that changes in magnitude as given in Fig. 30-35. The vertical axis scale is set by  $B_s = 0.50$  T, and the horizontal axis scale is set by  $t_s = 6.00$  s. The loop's plane is perpendicular to  $\vec{B}$ . What emf is induced in the loop during time intervals (a) 0 to 2.0 s, (b) 2.0 s to 4.0 s, and (c) 4.0 s to 6.0 s?

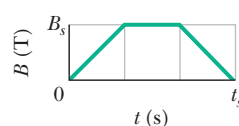


Figure 30-35 Problem 4.

**•5** In Fig. 30-36, a wire forms a closed circular loop, of radius  $R = 2.0$  m and resistance  $4.0 \Omega$ . The circle is centered on a long straight wire; at time  $t = 0$ , the current in the long straight wire is 5.0 A rightward. Thereafter, the current changes according to  $i = 5.0 \text{ A} - (2.0 \text{ A/s}^2)t^2$ . (The straight wire is insulated; so there is no electrical contact between it and the wire of the loop.) What is the magnitude of the current induced in the loop at times  $t > 0$ ?

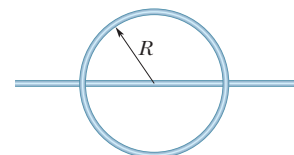


Figure 30-36 Problem 5.

**•6** Figure 30-37a shows a circuit consisting of an ideal battery with emf  $\mathcal{E} = 6.00 \mu\text{V}$ , a resistance  $R$ , and a small wire loop of area  $5.0 \text{ cm}^2$ . For the time interval  $t = 10$  s to  $t = 20$  s, an external magnetic field is set up throughout the loop. The field is uniform, its direction is into the page in Fig. 30-37a, and the field magnitude is given by  $B = at$ , where  $B$  is in teslas,  $a$  is a constant, and  $t$  is in seconds. Figure 30-37b gives the current  $i$  in the circuit before, during, and after the external field is set up. The vertical axis scale is set by  $i_s = 2.0$  mA. Find the constant  $a$  in the equation for the field magnitude.

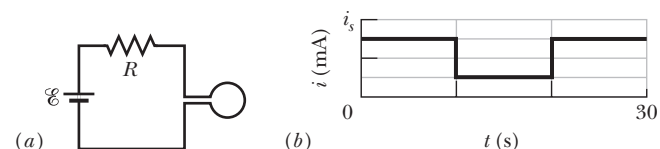


Figure 30-37 Problem 6.

•7 In Fig. 30-38, the magnetic flux through the loop increases according to the relation  $\Phi_B = 6.0t^2 + 7.0t$ , where  $\Phi_B$  is in milliwebers and  $t$  is in seconds. (a) What is the magnitude of the emf induced in the loop when  $t = 2.0$  s? (b) Is the direction of the current through  $R$  to the right or left?

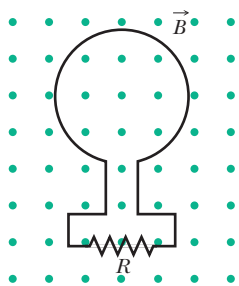


Figure 30-38 Problem 7.

•8 A uniform magnetic field  $\vec{B}$  is perpendicular to the plane of a circular loop of diameter 10 cm formed from wire of diameter 2.5 mm and resistivity  $1.69 \times 10^{-8} \Omega \cdot \text{m}$ . At what rate must the magnitude of  $\vec{B}$  change to induce a 10 A current in the loop?

•9 A small loop of area  $6.8 \text{ mm}^2$  is placed inside a long solenoid that has 854 turns/cm and carries a sinusoidally varying current  $i$  of amplitude 1.28 A and angular frequency 212 rad/s. The central axes of the loop and solenoid coincide. What is the amplitude of the emf induced in the loop?

••10 Figure 30-39 shows a closed loop of wire that consists of a pair of equal semicircles, of radius 3.7 cm, lying in mutually perpendicular planes. The loop was formed by folding a flat circular loop along a diameter until the two halves became perpendicular to each other. A uniform magnetic field  $\vec{B}$  of magnitude 76 mT is directed perpendicular to the fold diameter and makes equal angles (of  $45^\circ$ ) with the planes of the semicircles. The magnetic field is reduced to zero at a uniform rate during a time interval of 4.5 ms. During this interval, what are the (a) magnitude and (b) direction (clockwise or counterclockwise when viewed along the direction of  $\vec{B}$ ) of the emf induced in the loop?

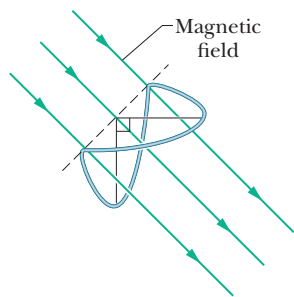


Figure 30-39 Problem 10.

••11 A rectangular coil of  $N$  turns and of length  $a$  and width  $b$  is rotated at frequency  $f$  in a uniform magnetic field  $\vec{B}$ , as indicated in Fig. 30-40. The coil is connected to co-rotating cylinders, against which metal brushes slide to make contact. (a) Show that the emf induced in the coil is given (as a function of time  $t$ ) by

$$\mathcal{E} = 2\pi f NabB \sin(2\pi ft) = \mathcal{E}_0 \sin(2\pi ft).$$

This is the principle of the commercial alternating-current generator. (b) What value of  $Nab$  gives an emf with  $\mathcal{E}_0 = 150$  V when the loop is rotated at 60.0 rev/s in a uniform magnetic field of 0.500 T?

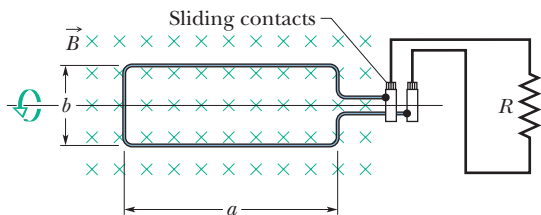


Figure 30-40 Problem 11.

••12 In Fig. 30-41, a wire loop of lengths  $L = 40.0$  cm and  $W = 25.0$  cm lies in a magnetic field  $\vec{B}$ . What are the (a) magnitude  $\mathcal{E}$  and (b) direction (clockwise or counterclockwise – or “none” if  $\mathcal{E} = 0$ ) of

the emf induced in the loop if  $\vec{B} = (4.00 \times 10^{-2} \text{ T/m})y\hat{k}$ ? What are (c)  $\mathcal{E}$  and (d) the direction if  $\vec{B} = (6.00 \times 10^{-2} \text{ T/s})t\hat{k}$ ? What are (e)  $\mathcal{E}$  and (f) the direction if  $\vec{B} = (8.00 \times 10^{-2} \text{ T/m} \cdot \text{s})y\hat{k}$ ? What are (g)  $\mathcal{E}$  and (h) the direction if  $\vec{B} = (3.00 \times 10^{-2} \text{ T/m} \cdot \text{s})x\hat{j}$ ? What are (i)  $\mathcal{E}$  and (j) the direction if  $\vec{B} = (5.00 \times 10^{-2} \text{ T/m} \cdot \text{s})y\hat{i}$ ?

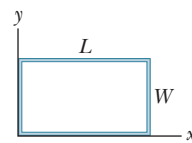


Figure 30-41 Problem 12.

••13 ILW One hundred turns of (insulated) copper wire are wrapped around a wooden cylindrical core of cross-sectional area  $1.20 \times 10^{-3} \text{ m}^2$ . The two ends of the wire are connected to a resistor. The total resistance in the circuit is  $13.0 \Omega$ . If an externally applied uniform longitudinal magnetic field in the core changes from 1.60 T in one direction to 1.60 T in the opposite direction, how much charge flows through a point in the circuit during the change?

••14 GO In Fig. 30-42a, a uniform magnetic field  $\vec{B}$  increases in magnitude with time  $t$  as given by Fig. 30-42b, where the vertical axis scale is set by  $B_s = 9.0$  mT and the horizontal scale is set by  $t_s = 3.0$  s. A circular conducting loop of area  $8.0 \times 10^{-4} \text{ m}^2$  lies in the field, in the plane of the page. The amount of charge  $q$  passing point A on the loop is given in Fig. 30-42c as a function of  $t$ , with the vertical axis scale set by  $q_s = 6.0$  mC and the horizontal axis scale again set by  $t_s = 3.0$  s. What is the loop's resistance?

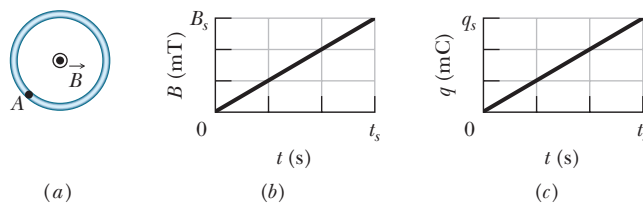


Figure 30-42 Problem 14.

••15 GO A square wire loop with 2.00 m sides is perpendicular to a uniform magnetic field, with half the area of the loop in the field as shown in Fig. 30-43. The loop contains an ideal battery with emf  $\mathcal{E} = 20.0$  V. If the magnitude of the field varies with time according to  $B = 0.0420 - 0.870t$ , with  $B$  in teslas and  $t$  in seconds, what are (a) the net emf in the circuit and (b) the direction of the (net) current around the loop?

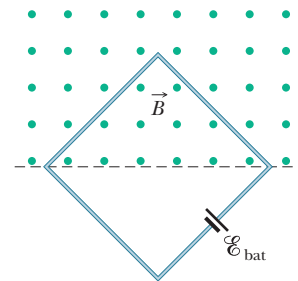


Figure 30-43 Problem 15.

••16 GO Figure 30-44a shows a wire that forms a rectangle ( $W = 20$  cm,  $H = 30$  cm) and has a resistance of  $5.0 \text{ m}\Omega$ . Its interior

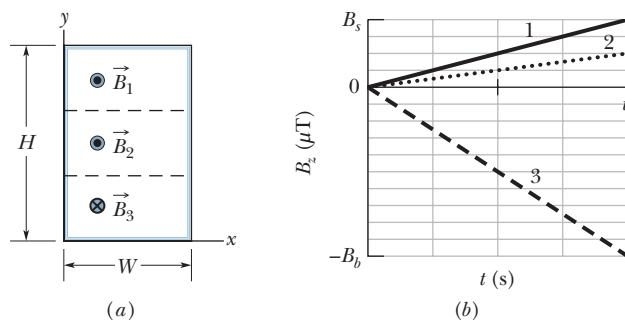


Figure 30-44 Problem 16.



is split into three equal areas, with magnetic fields  $\vec{B}_1$ ,  $\vec{B}_2$ , and  $\vec{B}_3$ . The fields are uniform within each region and directly out of or into the page as indicated. Figure 30-44b gives the change in the  $z$  components  $B_z$  of the three fields with time  $t$ ; the vertical axis scale is set by  $B_s = 4.0 \mu\text{T}$  and  $B_b = -2.5B_s$ , and the horizontal axis scale is set by  $t_s = 2.0 \text{ s}$ . What are the (a) magnitude and (b) direction of the current induced in the wire?

**••17** A small circular loop of area  $2.00 \text{ cm}^2$  is placed in the plane of, and concentric with, a large circular loop of radius  $1.00 \text{ m}$ . The current in the large loop is changed at a constant rate from  $200 \text{ A}$  to  $-200 \text{ A}$  (a change in direction) in a time of  $1.00 \text{ s}$ , starting at  $t = 0$ . What is the magnitude of the magnetic field  $\vec{B}$  at the center of the small loop due to the current in the large loop at (a)  $t = 0$ , (b)  $t = 0.500 \text{ s}$ , and (c)  $t = 1.00 \text{ s}$ ? (d) From  $t = 0$  to  $t = 1.00 \text{ s}$ , is  $\vec{B}$  reversed? Because the inner loop is small, assume  $\vec{B}$  is uniform over its area. (e) What emf is induced in the small loop at  $t = 0.500 \text{ s}$ ?

**••18** In Fig. 30-45, two straight conducting rails form a right angle. A conducting bar in contact with the rails starts at the vertex at time  $t = 0$  and moves with a constant velocity of  $5.20 \text{ m/s}$  along them. A magnetic field with  $B = 0.350 \text{ T}$  is directed out of the page. Calculate (a) the flux through the triangle formed by the rails and bar at  $t = 3.00 \text{ s}$  and (b) the emf around the triangle at that time. (c) If the emf is  $\mathcal{E} = at^n$ , where  $a$  and  $n$  are constants, what is the value of  $n$ ?

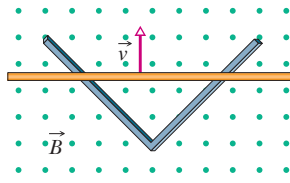


Figure 30-45 Problem 18.

**••19 ILW** An electric generator contains a coil of 100 turns of wire, each forming a rectangular loop  $50.0 \text{ cm}$  by  $30.0 \text{ cm}$ . The coil is placed entirely in a uniform magnetic field with magnitude  $B = 3.50 \text{ T}$  and with  $\vec{B}$  initially perpendicular to the coil's plane. What is the maximum value of the emf produced when the coil is spun at  $1000 \text{ rev/min}$  about an axis perpendicular to  $\vec{B}$ ?

**••20** At a certain place, Earth's magnetic field has magnitude  $B = 0.590 \text{ gauss}$  and is inclined downward at an angle of  $70.0^\circ$  to the horizontal. A flat horizontal circular coil of wire with a radius of  $10.0 \text{ cm}$  has 1000 turns and a total resistance of  $85.0 \Omega$ . It is connected in series to a meter with  $140 \Omega$  resistance. The coil is flipped through a half-revolution about a diameter, so that it is again horizontal. How much charge flows through the meter during the flip?

**••21** In Fig. 30-46, a stiff wire bent into a semicircle of radius  $a = 2.0 \text{ cm}$  is rotated at constant angular speed  $40 \text{ rev/s}$  in a uniform  $20 \text{ mT}$  magnetic field. What are the (a) frequency and (b) amplitude of the emf induced in the loop?

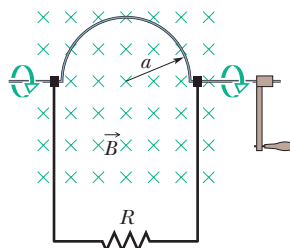


Figure 30-46 Problem 21.

**••22** A rectangular loop (area =  $0.15 \text{ m}^2$ ) turns in a uniform magnetic field,  $B = 0.20 \text{ T}$ . When the angle between the field and the normal to the plane of the loop is  $\pi/2 \text{ rad}$  and increasing at  $0.60 \text{ rad/s}$ , what emf is induced in the loop?

**••23 SSM** Figure 30-47 shows two parallel loops of wire having a common axis. The smaller loop (radius  $r$ ) is above the larger loop (radius  $R$ )

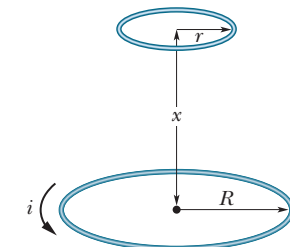


Figure 30-47 Problem 23.

by a distance  $x \gg R$ . Consequently, the magnetic field due to the counterclockwise current  $i$  in the larger loop is nearly uniform throughout the smaller loop. Suppose that  $x$  is increasing at the constant rate  $dx/dt = v$ . (a) Find an expression for the magnetic flux through the area of the smaller loop as a function of  $x$ . (Hint: See Eq. 29-27.) In the smaller loop, find (b) an expression for the induced emf and (c) the direction of the induced current.

**••24** A wire is bent into three circular segments, each of radius  $r = 10 \text{ cm}$ , as shown in Fig. 30-48. Each segment is a quadrant of a circle,  $ab$  lying in the  $xy$  plane,  $bc$  lying in the  $yz$  plane, and  $ca$  lying in the  $xz$  plane. (a) If a uniform magnetic field  $\vec{B}$  points in the positive  $x$  direction, what is the magnitude of the emf developed in the wire when  $B$  increases at the rate of  $3.0 \text{ mT/s}$ ? (b) What is the direction of the current in segment  $bc$ ?

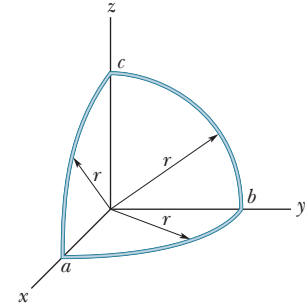


Figure 30-48 Problem 24.

**••25 GO** Two long, parallel copper wires of diameter  $2.5 \text{ mm}$  carry currents of  $10 \text{ A}$  in opposite directions. (a) Assuming that their central axes are  $20 \text{ mm}$  apart, calculate the magnetic flux per meter of wire that exists in the space between those axes. (b) What percentage of this flux lies inside the wires? (c) Repeat part (a) for parallel currents.

**••26 GO** For the wire arrangement in Fig. 30-49,  $a = 12.0 \text{ cm}$  and  $b = 16.0 \text{ cm}$ . The current in the long straight wire is  $i = 4.50t^2 - 10.0t$ , where  $i$  is in amperes and  $t$  is in seconds. (a) Find the emf in the square loop at  $t = 3.00 \text{ s}$ . (b) What is the direction of the induced current in the loop?

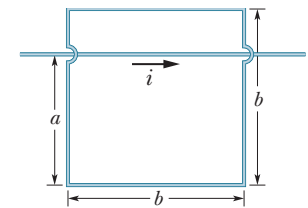


Figure 30-49 Problem 26.

**••27 ILW** As seen in Fig. 30-50, a square loop of wire has sides of length  $2.0 \text{ cm}$ . A magnetic field is directed out of the page; its magnitude is given by  $B = 4.0t^2y$ , where  $B$  is in teslas,  $t$  is in seconds, and  $y$  is in meters. At  $t = 2.5 \text{ s}$ , what are the (a) magnitude and (b) direction of the emf induced in the loop?

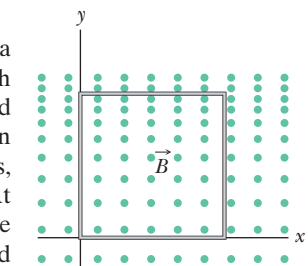


Figure 30-50 Problem 27.

**••28 GO** In Fig. 30-51, a rectangular loop of wire with length  $a = 2.2 \text{ cm}$ , width  $b = 0.80 \text{ cm}$ , and resistance  $R = 0.40 \text{ m}\Omega$  is placed near an infinitely long wire carrying current  $i = 4.7 \text{ A}$ . The loop is then moved away from the wire at constant speed  $v = 3.2 \text{ mm/s}$ . When the center of the loop is at distance  $r = 1.5b$ , what are (a) the magnitude of the magnetic flux through the loop and (b) the current induced in the loop?

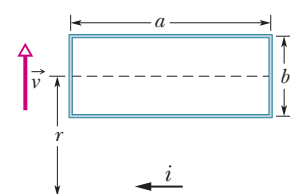


Figure 30-51 Problem 28.



**Module 30-2 Induction and Energy Transfers**

•29 In Fig. 30-52, a metal rod is forced to move with constant velocity  $\vec{v}$  along two parallel metal rails, connected with a strip of metal at one end. A magnetic field of magnitude  $B = 0.350$  T points out of the page. (a) If the rails are separated by  $L = 25.0$  cm and the speed of the rod is  $55.0$  cm/s, what emf is generated? (b) If the rod has a resistance of  $18.0\ \Omega$  and the rails and connector have negligible resistance, what is the current in the rod? (c) At what rate is energy being transferred to thermal energy?

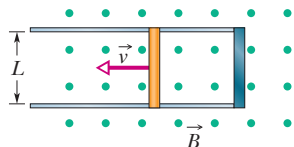


Figure 30-52  
Problems 29 and 35.

•30 In Fig. 30-53a, a circular loop of wire is concentric with a solenoid and lies in a plane perpendicular to the solenoid's central axis. The loop has radius  $6.00$  cm. The solenoid has radius  $2.00$  cm, consists of  $8000$  turns/m, and has a current  $i_{\text{sol}}$  varying with time  $t$  as given in Fig. 30-53b, where the vertical axis scale is set by  $i_s = 1.00$  A and the horizontal axis scale is set by  $t_s = 2.0$  s. Figure 30-53c shows, as a function of time, the energy  $E_{\text{th}}$  that is transferred to thermal energy of the loop; the vertical axis scale is set by  $E_s = 100.0$  nJ. What is the loop's resistance?

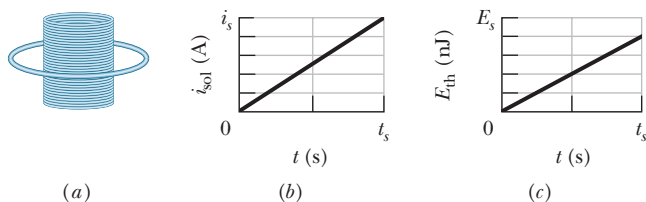


Figure 30-53 Problem 30.

•31 **SSM ILW** If  $50.0$  cm of copper wire (diameter =  $1.00$  mm) is formed into a circular loop and placed perpendicular to a uniform magnetic field that is increasing at the constant rate of  $10.0$  mT/s, at what rate is thermal energy generated in the loop?

•32 A loop antenna of area  $2.00$  cm<sup>2</sup> and resistance  $5.21\ \mu\Omega$  is perpendicular to a uniform magnetic field of magnitude  $17.0\ \mu\text{T}$ . The field magnitude drops to zero in  $2.96$  ms. How much thermal energy is produced in the loop by the change in field?

•33 **GO** Figure 30-54 shows a rod of length  $L = 10.0$  cm that is forced to move at constant speed  $v = 5.00$  m/s along horizontal rails. The rod, rails, and connecting strip at the right form a conducting loop. The rod has resistance  $0.400\ \Omega$ ; the rest of the loop has negligible resistance. A current  $i = 100$  A through the long straight wire at distance  $a = 10.0$  mm from the loop sets up a (nonuniform) magnetic field through the loop. Find the (a) emf and (b) current induced in the loop. (c) At what rate is thermal energy generated in the rod? (d) What is the magnitude of the force that must be applied to the rod to make it move at constant speed? (e) At what rate does this force do work on the rod?

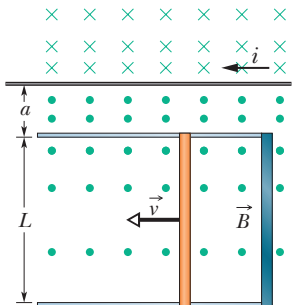


Figure 30-54 Problem 33.

•34 In Fig. 30-55, a long rectangular conducting loop, of width  $L$ , resistance  $R$ , and mass  $m$ , is hung in a horizontal, uniform magnetic

field  $\vec{B}$  that is directed into the page and that exists only above line  $aa$ . The loop is then dropped; during its fall, it accelerates until it reaches a certain terminal speed  $v_t$ . Ignoring air drag, find an expression for  $v_t$ .

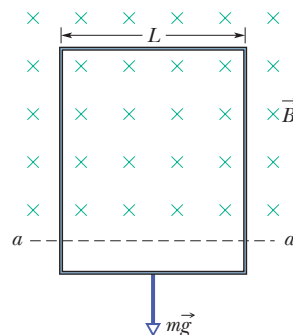


Figure 30-55 Problem 34.

•35 The conducting rod shown in Fig. 30-52 has length  $L$  and is being pulled along horizontal, frictionless conducting rails at a constant velocity  $\vec{v}$ . The rails are connected at one end with a metal strip. A uniform magnetic field  $\vec{B}$ , directed out of the page, fills the region in which the rod moves. Assume that  $L = 10$  cm,  $v = 5.0$  m/s, and  $B = 1.2$  T. What are the (a) magnitude and (b) direction (up or down the page) of the emf induced in the rod? What are the (c) size and (d) direction of the current in the conducting loop? Assume that the resistance of the rod is  $0.40\ \Omega$  and that the resistance of the rails and metal strip is negligibly small. (e) At what rate is thermal energy being generated in the rod? (f) What external force on the rod is needed to maintain  $\vec{v}$ ? (g) At what rate does this force do work on the rod?

**Module 30-3 Induced Electric Fields**

•36 Figure 30-56 shows two circular regions  $R_1$  and  $R_2$  with radii  $r_1 = 20.0$  cm and  $r_2 = 30.0$  cm. In  $R_1$  there is a uniform magnetic field of magnitude  $B_1 = 50.0$  mT directed into the page, and in  $R_2$  there is a uniform magnetic field of magnitude  $B_2 = 75.0$  mT directed out of the page (ignore fringing). Both fields are decreasing at the rate of  $8.50$  mT/s. Calculate  $\oint \vec{E} \cdot d\vec{s}$  for (a) path 1, (b) path 2, and (c) path 3.

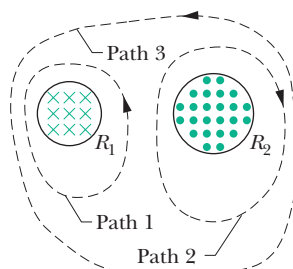


Figure 30-56 Problem 36.

•37 **SSM ILW** A long solenoid has a diameter of  $12.0$  cm. When a current  $i$  exists in its windings, a uniform magnetic field of magnitude  $B = 30.0$  mT is produced in its interior. By decreasing  $i$ , the field is caused to decrease at the rate of  $6.50$  mT/s. Calculate the magnitude of the induced electric field (a)  $2.20$  cm and (b)  $8.20$  cm from the axis of the solenoid.

•38 **GO** A circular region in an  $xy$  plane is penetrated by a uniform magnetic field in the positive direction of the  $z$  axis. The field's magnitude  $B$  (in teslas) increases with time  $t$  (in seconds) according to  $B = at$ , where  $a$  is a constant. The magnitude  $E$  of the electric field set up by that increase in the magnetic field is given by Fig. 30-57 versus radial distance  $r$ ; the vertical axis scale is set by  $E_s = 300\ \mu\text{N/C}$ , and the horizontal axis scale is set by  $r_s = 4.00$  cm. Find  $a$ .

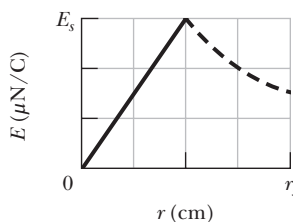


Figure 30-57 Problem 38.

•39 The magnetic field of a cylindrical magnet that has a pole-face diameter of  $3.3$  cm can be varied sinusoidally between  $29.6$  T and  $30.0$  T at a frequency of  $15$  Hz. (The current in a wire wrapped around a permanent magnet is varied to give this variation in the net field.) At a radial distance of  $1.6$  cm, what is the amplitude of the electric field induced by the variation?

**Module 30-4 Inductors and Inductance**

•40 The inductance of a closely packed coil of 400 turns is 8.0 mH. Calculate the magnetic flux through the coil when the current is 5.0 mA.

•41 A circular coil has a 10.0 cm radius and consists of 30.0 closely wound turns of wire. An externally produced magnetic field of magnitude 2.60 mT is perpendicular to the coil. (a) If no current is in the coil, what magnetic flux links its turns? (b) When the current in the coil is 3.80 A in a certain direction, the net flux through the coil is found to vanish. What is the inductance of the coil?

•42 Figure 30-58 shows a copper strip of width  $W = 16.0$  cm that has been bent to form a shape that consists of a tube of radius  $R = 1.8$  cm plus two parallel flat extensions. Current  $i = 35$  mA is distributed uniformly across the width so that the tube is effectively a one-turn solenoid. Assume that the magnetic field outside the tube is negligible and the field inside the tube is uniform. What are (a) the magnetic field magnitude inside the tube and (b) the inductance of the tube (excluding the flat extensions)?

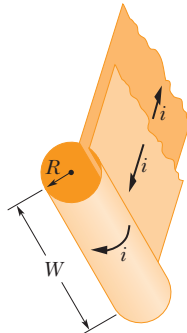


Figure 30-58  
Problem 42.

•43 GO Two identical long wires of radius  $a = 1.53$  mm are parallel and carry identical currents in opposite directions. Their center-to-center separation is  $d = 14.2$  cm. Neglect the flux within the wires but consider the flux in the region between the wires. What is the inductance per unit length of the wires?

**Module 30-5 Self-Induction**

•44 A 12 H inductor carries a current of 2.0 A. At what rate must the current be changed to produce a 60 V emf in the inductor?

•45 At a given instant the current and self-induced emf in an inductor are directed as indicated in Fig. 30-59. (a) Is the current increasing or decreasing? (b) The induced emf is 17 V, and the rate of change of the current is 25 kA/s; find the inductance.

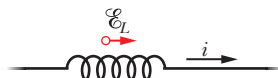


Figure 30-59 Problem 45.

•46 The current  $i$  through a 4.6 H inductor varies with time  $t$  as shown by the graph of Fig. 30-60, where the vertical axis scale is set by  $i_s = 8.0$  A and the horizontal axis scale is set by  $t_s = 6.0$  ms. The inductor has a resistance of 12  $\Omega$ . Find the magnitude of the induced emf  $\mathcal{E}$  during time intervals (a) 0 to 2 ms, (b) 2 ms to 5 ms, and (c) 5 ms to 6 ms. (Ignore the behavior at the ends of the intervals.)

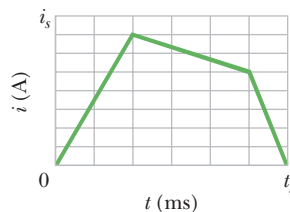


Figure 30-60 Problem 46.

•47 Inductors in series. Two inductors  $L_1$  and  $L_2$  are connected in series and are separated by a large distance so that the magnetic field of one cannot affect the other. (a) Show that the equivalent inductance is given by

$$L_{\text{eq}} = L_1 + L_2.$$

(Hint: Review the derivations for resistors in series and capacitors in series. Which is similar here?) (b) What is the generalization of (a) for  $N$  inductors in series?

•48 Inductors in parallel. Two inductors  $L_1$  and  $L_2$  are connected in parallel and separated by a large distance so that the magnetic field of one cannot affect the other. (a) Show that the equivalent inductance is given by

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2}.$$

(Hint: Review the derivations for resistors in parallel and capacitors in parallel. Which is similar here?) (b) What is the generalization of (a) for  $N$  inductors in parallel?

•49 The inductor arrangement of Fig. 30-61, with  $L_1 = 30.0$  mH,  $L_2 = 50.0$  mH,  $L_3 = 20.0$  mH, and  $L_4 = 15.0$  mH, is to be connected to a varying current source. What is the equivalent inductance of the arrangement? (First see Problems 47 and 48.)

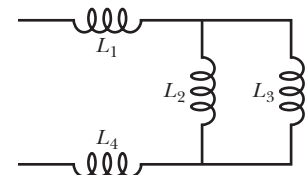


Figure 30-61 Problem 49.

**Module 30-6 RL Circuits**

•50 The current in an  $RL$  circuit builds up to one-third of its steady-state value in 5.00 s. Find the inductive time constant.

•51 ILW The current in an  $RL$  circuit drops from 1.0 A to 10 mA in the first second following removal of the battery from the circuit. If  $L$  is 10 H, find the resistance  $R$  in the circuit.

•52 The switch in Fig. 30-15 is closed on  $a$  at time  $t = 0$ . What is the ratio  $\mathcal{E}_L/\mathcal{E}$  of the inductor's self-induced emf to the battery's emf (a) just after  $t = 0$  and (b) at  $t = 2.00\tau_L$ ? (c) At what multiple of  $\tau_L$  will  $\mathcal{E}_L/\mathcal{E} = 0.500$ ?

•53 SSM A solenoid having an inductance of  $6.30 \mu\text{H}$  is connected in series with a 1.20 k $\Omega$  resistor. (a) If a 14.0 V battery is connected across the pair, how long will it take for the current through the resistor to reach 80.0% of its final value? (b) What is the current through the resistor at time  $t = 1.0\tau_L$ ?

•54 In Fig. 30-62,  $\mathcal{E} = 100$  V,  $R_1 = 10.0 \Omega$ ,  $R_2 = 20.0 \Omega$ ,  $R_3 = 30.0 \Omega$ , and  $L = 2.00$  H. Immediately after switch  $S$  is closed, what are (a)  $i_1$  and (b)  $i_2$ ? (Let currents in the indicated directions have positive values and currents in the opposite directions have negative values.) A long time later, what are (c)  $i_1$  and (d)  $i_2$ ? The switch is then reopened. Just then, what are (e)  $i_1$  and (f)  $i_2$ ? A long time later, what are (g)  $i_1$  and (h)  $i_2$ ?

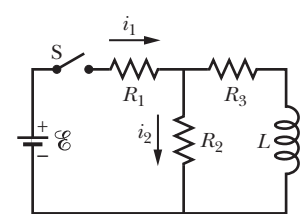


Figure 30-62 Problem 54.

•55 SSM A battery is connected to a series  $RL$  circuit at time  $t = 0$ . At what multiple of  $\tau_L$  will the current be 0.100% less than its equilibrium value?

•56 In Fig. 30-63, the inductor has 25 turns and the ideal battery has an emf of 16 V. Figure 30-64 gives the magnetic flux  $\Phi$  through each turn versus the current  $i$  through the inductor. The vertical

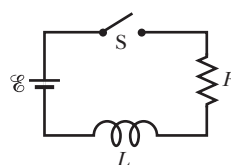


Figure 30-63 Problems 56, 80, 83, and 93.

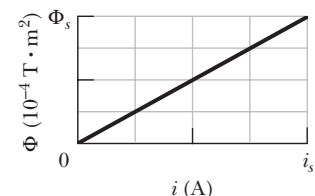


Figure 30-64 Problem 56.

axis scale is set by  $\Phi_s = 4.0 \times 10^{-4} \text{ T} \cdot \text{m}^2$ , and the horizontal axis scale is set by  $i_s = 2.00 \text{ A}$ . If switch  $S$  is closed at time  $t = 0$ , at what rate  $di/dt$  will the current be changing at  $t = 1.5\tau_L$ ?

••57 **GO** In Fig. 30-65,  $R = 15 \, \Omega$ ,  $L = 5.0 \text{ H}$ , the ideal battery has  $\mathcal{E} = 10 \text{ V}$ , and the fuse in the upper branch is an ideal  $3.0 \text{ A}$  fuse. It has zero resistance as long as the current through it remains less than  $3.0 \text{ A}$ . If the current reaches  $3.0 \text{ A}$ , the fuse “blows” and thereafter has infinite resistance. Switch  $S$  is closed at time  $t = 0$ . (a) When does the fuse blow? (*Hint:* Equation 30-41 does not apply. Rethink Eq. 30-39.) (b) Sketch a graph of the current  $i$  through the inductor as a function of time. Mark the time at which the fuse blows.

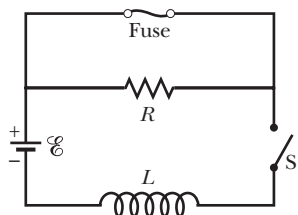


Figure 30-65 Problem 57.

••58 **GO** Suppose the emf of the battery in the circuit shown in Fig. 30-16 varies with time  $t$  so that the current is given by  $i(t) = 3.0 + 5.0t$ , where  $i$  is in amperes and  $t$  is in seconds. Take  $R = 4.0 \, \Omega$  and  $L = 6.0 \text{ H}$ , and find an expression for the battery emf as a function of  $t$ . (*Hint:* Apply the loop rule.)

•••59 **SSM WWW** In Fig. 30-66, after switch  $S$  is closed at time  $t = 0$ , the emf of the source is automatically adjusted to maintain a constant current  $i$  through  $S$ . (a) Find the current through the inductor as a function of time. (b) At what time is the current through the resistor equal to the current through the inductor?

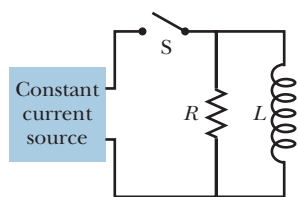


Figure 30-66 Problem 59.

•••60 A wooden toroidal core with a square cross section has an inner radius of  $10 \text{ cm}$  and an outer radius of  $12 \text{ cm}$ . It is wound with one layer of wire (of diameter  $1.0 \text{ mm}$  and resistance per meter  $0.020 \, \Omega/\text{m}$ ). What are (a) the inductance and (b) the inductive time constant of the resulting toroid? Ignore the thickness of the insulation on the wire.

### Module 30-7 Energy Stored in a Magnetic Field

•61 **SSM** A coil is connected in series with a  $10.0 \text{ k}\Omega$  resistor. An ideal  $50.0 \text{ V}$  battery is applied across the two devices, and the current reaches a value of  $2.00 \text{ mA}$  after  $5.00 \text{ ms}$ . (a) Find the inductance of the coil. (b) How much energy is stored in the coil at this same moment?

•62 A coil with an inductance of  $2.0 \text{ H}$  and a resistance of  $10 \, \Omega$  is suddenly connected to an ideal battery with  $\mathcal{E} = 100 \text{ V}$ . At  $0.10 \text{ s}$  after the connection is made, what is the rate at which (a) energy is being stored in the magnetic field, (b) thermal energy is appearing in the resistance, and (c) energy is being delivered by the battery?

•63 **ILW** At  $t = 0$ , a battery is connected to a series arrangement of a resistor and an inductor. If the inductive time constant is  $37.0 \text{ ms}$ , at what time is the rate at which energy is dissipated in the resistor equal to the rate at which energy is stored in the inductor's magnetic field?

•64 At  $t = 0$ , a battery is connected to a series arrangement of a resistor and an inductor. At what multiple of the inductive time constant will the energy stored in the inductor's magnetic field be  $0.500$  its steady-state value?

••65 **GO** For the circuit of Fig. 30-16, assume that  $\mathcal{E} = 10.0 \text{ V}$ ,  $R = 6.70 \, \Omega$ , and  $L = 5.50 \text{ H}$ . The ideal battery is connected at time

$t = 0$ . (a) How much energy is delivered by the battery during the first  $2.00 \text{ s}$ ? (b) How much of this energy is stored in the magnetic field of the inductor? (c) How much of this energy is dissipated in the resistor?

### Module 30-8 Energy Density of a Magnetic Field

•66 A circular loop of wire  $50 \text{ mm}$  in radius carries a current of  $100 \text{ A}$ . Find the (a) magnetic field strength and (b) energy density at the center of the loop.

•67 **SSM** A solenoid that is  $85.0 \text{ cm}$  long has a cross-sectional area of  $17.0 \text{ cm}^2$ . There are  $950$  turns of wire carrying a current of  $6.60 \text{ A}$ . (a) Calculate the energy density of the magnetic field inside the solenoid. (b) Find the total energy stored in the magnetic field there (neglect end effects).

•68 A toroidal inductor with an inductance of  $90.0 \text{ mH}$  encloses a volume of  $0.0200 \text{ m}^3$ . If the average energy density in the toroid is  $70.0 \text{ J/m}^3$ , what is the current through the inductor?

•69 **ILW** What must be the magnitude of a uniform electric field if it is to have the same energy density as that possessed by a  $0.50 \text{ T}$  magnetic field?

••70 **GO** Figure 30-67a shows, in cross section, two wires that are straight, parallel, and very long. The ratio  $i_1/i_2$  of the current carried by wire 1 to that carried by wire 2 is  $1/3$ . Wire 1 is fixed in place. Wire 2 can be moved along the positive side of the  $x$  axis so as to change the magnetic energy density  $u_B$  set up by the two currents at the origin. Figure 30-67b gives  $u_B$  as a function of the position  $x$  of wire 2. The curve has an asymptote of  $u_B = 1.96 \text{ nJ/m}^3$  as  $x \rightarrow \infty$ , and the horizontal axis scale is set by  $x_s = 60.0 \text{ cm}$ . What is the value of (a)  $i_1$  and (b)  $i_2$ ?

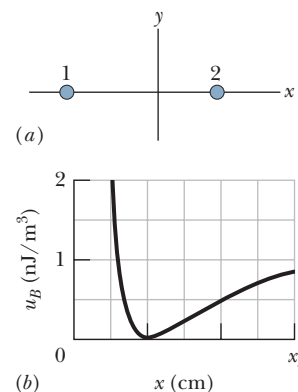


Figure 30-67 Problem 70.

••71 A length of copper wire carries a current of  $10 \text{ A}$  uniformly distributed through its cross section. Calculate the energy density of (a) the magnetic field and (b) the electric field at the surface of the wire. The wire diameter is  $2.5 \text{ mm}$ , and its resistance per unit length is  $3.3 \, \Omega/\text{km}$ .

### Module 30-9 Mutual Induction

•72 Coil 1 has  $L_1 = 25 \text{ mH}$  and  $N_1 = 100$  turns. Coil 2 has  $L_2 = 40 \text{ mH}$  and  $N_2 = 200$  turns. The coils are fixed in place; their mutual inductance  $M$  is  $3.0 \text{ mH}$ . A  $6.0 \text{ mA}$  current in coil 1 is changing at the rate of  $4.0 \text{ A/s}$ . (a) What magnetic flux  $\Phi_{12}$  links coil 1, and (b) what self-induced emf appears in that coil? (c) What magnetic flux  $\Phi_{21}$  links coil 2, and (d) what mutually induced emf appears in that coil?

•73 **SSM** Two coils are at fixed locations. When coil 1 has no current and the current in coil 2 increases at the rate  $15.0 \text{ A/s}$ , the emf in coil 1 is  $25.0 \text{ mV}$ . (a) What is their mutual inductance? (b) When coil 2 has no current and coil 1 has a current of  $3.60 \text{ A}$ , what is the flux linkage in coil 2?

•74 Two solenoids are part of the spark coil of an automobile. When the current in one solenoid falls from  $6.0 \text{ A}$  to zero in  $2.5 \text{ ms}$ , an emf of  $30 \text{ kV}$  is induced in the other solenoid. What is the mutual inductance  $M$  of the solenoids?



••75 **ILW** A rectangular loop of  $N$  closely packed turns is positioned near a long straight wire as shown in Fig. 30-68. What is the mutual inductance  $M$  for the loop–wire combination if  $N = 100$ ,  $a = 1.0$  cm,  $b = 8.0$  cm, and  $l = 30$  cm?

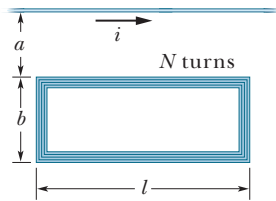


Figure 30-68 Problem 75.

••76 A coil  $C$  of  $N$  turns is placed around a long solenoid  $S$  of radius  $R$  and  $n$  turns per unit length, as in Fig. 30-69. (a) Show that the mutual inductance for the coil–solenoid combination is given by  $M = \mu_0 \pi R^2 n N$ . (b) Explain why  $M$  does not depend on the shape, size, or possible lack of close packing of the coil.

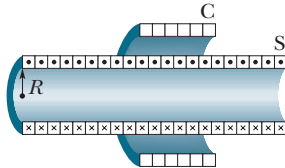


Figure 30-69 Problem 76.

••77 **SSM** Two coils connected as shown in Fig. 30-70 separately have inductances  $L_1$  and  $L_2$ . Their mutual inductance is  $M$ . (a) Show that this combination can be replaced by a single coil of equivalent inductance given by

$$L_{\text{eq}} = L_1 + L_2 + 2M.$$

(b) How could the coils in Fig. 30-70 be reconnected to yield an equivalent inductance of

$$L_{\text{eq}} = L_1 + L_2 - 2M?$$

(This problem is an extension of Problem 47, but the requirement that the coils be far apart has been removed.)

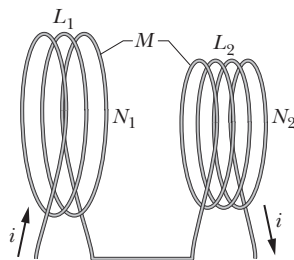


Figure 30-70 Problem 77.

### Additional Problems

78 At time  $t = 0$ , a 12.0 V potential difference is suddenly applied to the leads of a coil of inductance 23.0 mH and a certain resistance  $R$ . At time  $t = 0.150$  ms, the current through the inductor is changing at the rate of 280 A/s. Evaluate  $R$ .

79 **SSM** In Fig. 30-71, the battery is ideal and  $\mathcal{E} = 10$  V,  $R_1 = 5.0$   $\Omega$ ,  $R_2 = 10$   $\Omega$ , and  $L = 5.0$  H. Switch  $S$  is closed at time  $t = 0$ . Just afterwards, what are (a)  $i_1$ , (b)  $i_2$ , (c) the current  $i_s$  through the switch, (d) the potential difference  $V_2$  across resistor 2, (e) the potential difference  $V_L$  across the inductor, and (f) the rate of change  $di_2/dt$ ? A long time later, what are (g)  $i_1$ , (h)  $i_2$ , (i)  $i_s$ , (j)  $V_2$ , (k)  $V_L$ , and (l)  $di_2/dt$ ?

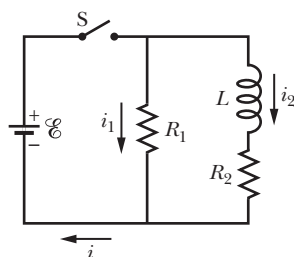


Figure 30-71 Problem 79.

80 In Fig. 30-63,  $R = 4.0$  k $\Omega$ ,  $L = 8.0$   $\mu$ H, and the ideal battery has  $\mathcal{E} = 20$  V. How long after switch  $S$  is closed is the current 2.0 mA?

81 **SSM** Figure 30-72a shows a rectangular conducting loop of resistance  $R = 0.020$   $\Omega$ , height  $H = 1.5$  cm, and length  $D = 2.5$  cm being pulled at constant speed  $v = 40$  cm/s through two regions of uniform magnetic field. Figure 30-72b gives the current  $i$  induced in the loop as a function of the position  $x$  of the right side of the loop. The vertical axis scale is set by  $i_s = 3.0$   $\mu$ A. For example, a current equal to  $i_s$  is induced clockwise as the loop enters region 1. What are the (a) magnitude and (b) direction (into or out of the page) of the magnetic field in region 1? What are the (c) magnitude and (d) direction of the magnetic field in region 2?

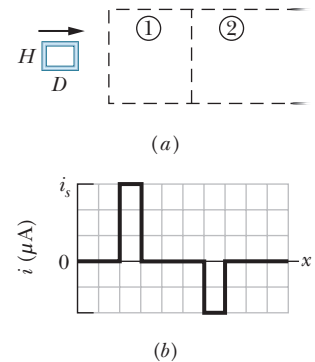


Figure 30-72 Problem 81.

82 A uniform magnetic field  $\vec{B}$  is perpendicular to the plane of a circular wire loop of radius  $r$ . The magnitude of the field varies with time according to  $B = B_0 e^{-t/\tau}$ , where  $B_0$  and  $\tau$  are constants. Find an expression for the emf in the loop as a function of time.

83 Switch  $S$  in Fig. 30-63 is closed at time  $t = 0$ , initiating the buildup of current in the 15.0 mH inductor and the 20.0  $\Omega$  resistor. At what time is the emf across the inductor equal to the potential difference across the resistor?

84 **GO** Figure 30-73a shows two concentric circular regions in which uniform magnetic fields can change. Region 1, with radius  $r_1 = 1.0$  cm, has an outward magnetic field  $\vec{B}_1$  that is increasing in magnitude. Region 2, with radius  $r_2 = 2.0$  cm, has an outward magnetic field  $\vec{B}_2$  that may also be changing. Imagine that a conducting ring of radius  $R$  is centered on the two regions and then the emf  $\mathcal{E}$  around the ring is determined. Figure 30-73b gives emf  $\mathcal{E}$  as a function of the square  $R^2$  of the ring's radius, to the outer edge of region 2. The vertical axis scale is set by  $\mathcal{E}_s = 20.0$  nV. What are the rates (a)  $dB_1/dt$  and (b)  $dB_2/dt$ ? (c) Is the magnitude of  $\vec{B}_2$  increasing, decreasing, or remaining constant?

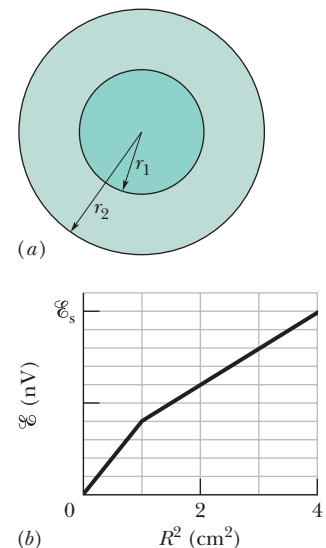


Figure 30-73 Problem 84.

85 **SSM** Figure 30-74 shows a uniform magnetic field  $\vec{B}$  confined to a cylindrical volume of radius  $R$ . The magnitude of  $\vec{B}$  is decreasing at a constant rate of 10 mT/s. In unit-vector notation, what is the initial acceleration of an electron released at (a) point  $a$  (radial distance  $r = 5.0$  cm), (b) point  $b$  ( $r = 0$ ), and (c) point  $c$  ( $r = 5.0$  cm)?

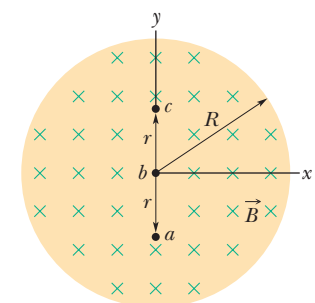


Figure 30-74 Problem 85.

86 **GO** In Fig. 30-75a, switch  $S$  has been closed on  $A$  long enough to establish a steady current in the inductor of inductance

$L_1 = 5.00$  mH and the resistor of resistance  $R_1 = 25.0\ \Omega$ . Similarly, in Fig. 30-75b, switch S has been closed on A long enough to establish a steady current in the inductor of inductance  $L_2 = 3.00$  mH and the resistor of resistance  $R_2 = 30.0\ \Omega$ . The ratio  $\Phi_{02}/\Phi_{01}$  of the magnetic flux through a turn in inductor 2 to that in inductor 1 is 1.50. At time  $t = 0$ , the two switches are closed on B. At what time  $t$  is the flux through a turn in the two inductors equal?

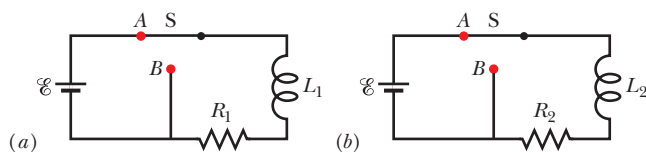


Figure 30-75 Problem 86.

**87 SSM** A square wire loop 20 cm on a side, with resistance  $20\ \text{m}\Omega$ , has its plane normal to a uniform magnetic field of magnitude  $B = 2.0$  T. If you pull two opposite sides of the loop away from each other, the other two sides automatically draw toward each other, reducing the area enclosed by the loop. If the area is reduced to zero in time  $\Delta t = 0.20$  s, what are (a) the average emf and (b) the average current induced in the loop during  $\Delta t$ ?

**88** A coil with 150 turns has a magnetic flux of  $50.0\ \text{nT}\cdot\text{m}^2$  through each turn when the current is  $2.00$  mA. (a) What is the inductance of the coil? What are the (b) inductance and (c) flux through each turn when the current is increased to  $4.00$  mA? (d) What is the maximum emf  $\mathcal{E}$  across the coil when the current through it is given by  $i = (3.00\ \text{mA}) \cos(377t)$ , with  $t$  in seconds?

**89** A coil with an inductance of  $2.0$  H and a resistance of  $10\ \Omega$  is suddenly connected to an ideal battery with  $\mathcal{E} = 100$  V. (a) What is the equilibrium current? (b) How much energy is stored in the magnetic field when this current exists in the coil?

**90** How long would it take, following the removal of the battery, for the potential difference across the resistor in an  $RL$  circuit (with  $L = 2.00$  H,  $R = 3.00\ \Omega$ ) to decay to 10.0% of its initial value?

**91 SSM** In the circuit of Fig. 30-76,  $R_1 = 20\ \text{k}\Omega$ ,  $R_2 = 20\ \Omega$ ,  $L = 50$  mH, and the ideal battery has  $\mathcal{E} = 40$  V. Switch S has been open for a long time when it is closed at time  $t = 0$ . Just after the switch is closed, what are (a) the current  $i_{\text{bat}}$  through the battery and (b) the rate  $di_{\text{bat}}/dt$ ? At  $t = 3.0\ \mu\text{s}$ , what are (c)  $i_{\text{bat}}$  and (d)  $di_{\text{bat}}/dt$ ? A long time later, what are (e)  $i_{\text{bat}}$  and (f)  $di_{\text{bat}}/dt$ ?

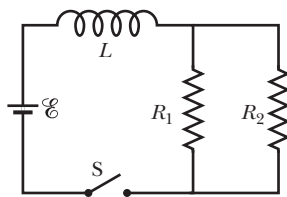


Figure 30-76 Problem 91.

**92** The flux linkage through a certain coil of  $0.75\ \Omega$  resistance would be  $26\ \text{mWb}$  if there were a current of  $5.5$  A in it. (a) Calculate the inductance of the coil. (b) If a  $6.0$  V ideal battery were suddenly connected across the coil, how long would it take for the current to rise from 0 to  $2.5$  A?

**93** In Fig. 30-63, a  $12.0$  V ideal battery, a  $20.0\ \Omega$  resistor, and an inductor are connected by a switch at time  $t = 0$ . At what rate is the battery transferring energy to the inductor's field at  $t = 1.61\tau_L$ ?

**94** A long cylindrical solenoid with 100 turns/cm has a radius of  $1.6$  cm. Assume that the magnetic field it produces is parallel to its axis and is uniform in its interior. (a) What is its inductance per

meter of length? (b) If the current changes at the rate of  $13$  A/s, what emf is induced per meter?

**95** In Fig. 30-77,  $R_1 = 8.0\ \Omega$ ,  $R_2 = 10\ \Omega$ ,  $L_1 = 0.30$  H,  $L_2 = 0.20$  H, and the ideal battery has  $\mathcal{E} = 6.0$  V. (a) Just after switch S is closed, at what rate is the current in inductor 1 changing? (b) When the circuit is in the steady state, what is the current in inductor 1?

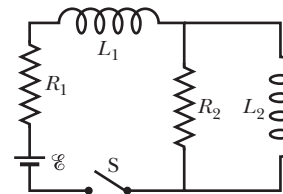


Figure 30-77 Problem 95.

**96** A square loop of wire is held in a uniform  $0.24$  T magnetic field directed perpendicular to the plane of the loop. The length of each side of the square is decreasing at a constant rate of  $5.0$  cm/s. What emf is induced in the loop when the length is  $12$  cm?

**97** At time  $t = 0$ , a  $45$  V potential difference is suddenly applied to the leads of a coil with inductance  $L = 50$  mH and resistance  $R = 180\ \Omega$ . At what rate is the current through the coil increasing at  $t = 1.2$  ms?

**98** The inductance of a closely wound coil is such that an emf of  $3.00$  mV is induced when the current changes at the rate of  $5.00$  A/s. A steady current of  $8.00$  A produces a magnetic flux of  $40.0\ \mu\text{Wb}$  through each turn. (a) Calculate the inductance of the coil. (b) How many turns does the coil have?

**99** The magnetic field in the interstellar space of our galaxy has a magnitude of about  $10^{-10}$  T. How much energy is stored in this field in a cube 10 light-years on edge? (For scale, note that the nearest star is 4.3 light-years distant and the radius of the galaxy is about  $8 \times 10^4$  light-years.)

**100** Figure 30-78 shows a wire that has been bent into a circular arc of radius  $r = 24.0$  cm, centered at  $O$ . A straight wire  $OP$  can be rotated about  $O$  and makes sliding contact with the arc at  $P$ . Another straight wire  $OQ$  completes the conducting loop. The three wires have cross-sectional area  $1.20\ \text{mm}^2$  and resistivity  $1.70 \times 10^{-8}\ \Omega\cdot\text{m}$ , and the apparatus lies in a uniform magnetic field of magnitude  $B = 0.150$  T directed out of the figure. Wire  $OP$  begins from rest at angle  $\theta = 0$  and has constant angular acceleration of  $12\ \text{rad/s}^2$ . As functions of  $\theta$  (in rad), find (a) the loop's resistance and (b) the magnetic flux through the loop. (c) For what  $\theta$  is the induced current maximum and (d) what is that maximum?

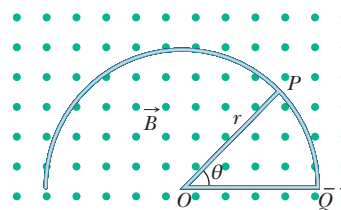


Figure 30-78 Problem 100.

**101** A toroid has a  $5.00$  cm square cross section, an inside radius of  $15.0$  cm, 500 turns of wire, and a current of  $0.800$  A. What is the magnetic flux through the cross section?

# Electromagnetic Oscillations and Alternating Current

## 31-1 LC OSCILLATIONS

### Learning Objectives

After reading this module, you should be able to . . .

- 31.01** Sketch an  $LC$  oscillator and explain which quantities oscillate and what constitutes one period of the oscillation.
- 31.02** For an  $LC$  oscillator, sketch graphs of the potential difference across the capacitor and the current through the inductor as functions of time, and indicate the period  $T$  on each graph.
- 31.03** Explain the analogy between a block–spring oscillator and an  $LC$  oscillator.
- 31.04** For an  $LC$  oscillator, apply the relationships between the angular frequency  $\omega$  (and the related frequency  $f$  and period  $T$ ) and the values of the inductance and capacitance.
- 31.05** Starting with the energy of a block–spring system, explain the derivation of the differential equation for charge  $q$  in an  $LC$  oscillator and then identify the solution for  $q(t)$ .
- 31.06** For an  $LC$  oscillator, calculate the charge  $q$  on the capacitor for any given time and identify the amplitude  $Q$  of the charge oscillations.
- 31.07** Starting from the equation giving the charge  $q(t)$  on the capacitor in an  $LC$  oscillator, find the current  $i(t)$  in the inductor as a function of time.
- 31.08** For an  $LC$  oscillator, calculate the current  $i$  in the inductor for any given time and identify the amplitude  $I$  of the current oscillations.
- 31.09** For an  $LC$  oscillator, apply the relationship between the charge amplitude  $Q$ , the current amplitude  $I$ , and the angular frequency  $\omega$ .
- 31.10** From the expressions for the charge  $q$  and the current  $i$  in an  $LC$  oscillator, find the magnetic field energy  $U_B(t)$  and the electric field energy  $U_E(t)$  and the total energy.
- 31.11** For an  $LC$  oscillator, sketch graphs of the magnetic field energy  $U_B(t)$ , the electric field energy  $U_E(t)$ , and the total energy, all as functions of time.
- 31.12** Calculate the maximum values of the magnetic field energy  $U_B$  and the electric field energy  $U_E$  and also calculate the total energy.

### Key Ideas

- In an oscillating  $LC$  circuit, energy is shuttled periodically between the electric field of the capacitor and the magnetic field of the inductor; instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2},$$

where  $q$  is the instantaneous charge on the capacitor and  $i$  is the instantaneous current through the inductor.

- The total energy  $U (= U_E + U_B)$  remains constant.
- The principle of conservation of energy leads to

$$L \frac{d^2q}{dt^2} + \frac{1}{C} q = 0 \quad (LC \text{ oscillations})$$

as the differential equation of  $LC$  oscillations (with no resistance).

- The solution of this differential equation is

$$q = Q \cos(\omega t + \phi) \quad (\text{charge}),$$

in which  $Q$  is the charge amplitude (maximum charge on the capacitor) and the angular frequency  $\omega$  of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}.$$

- The phase constant  $\phi$  is determined by the initial conditions (at  $t = 0$ ) of the system.
- The current  $i$  in the system at any time  $t$  is

$$i = -\omega Q \sin(\omega t + \phi) \quad (\text{current}),$$

in which  $\omega Q$  is the current amplitude  $I$ .



## What Is Physics?

We have explored the basic physics of electric and magnetic fields and how energy can be stored in capacitors and inductors. We next turn to the associated applied physics, in which the energy stored in one location can be transferred to another location so that it can be put to use. For example, energy produced at a power plant can show up at your home to run a computer. The total worth of this applied physics is now so high that its estimation is almost impossible. Indeed, modern civilization would be impossible without this applied physics.

In most parts of the world, electrical energy is transferred not as a direct current but as a sinusoidally oscillating current (alternating current, or ac). The challenge to both physicists and engineers is to design ac systems that transfer energy efficiently and to build appliances that make use of that energy. Our first step here is to study the oscillations in a circuit with inductance  $L$  and capacitance  $C$ .

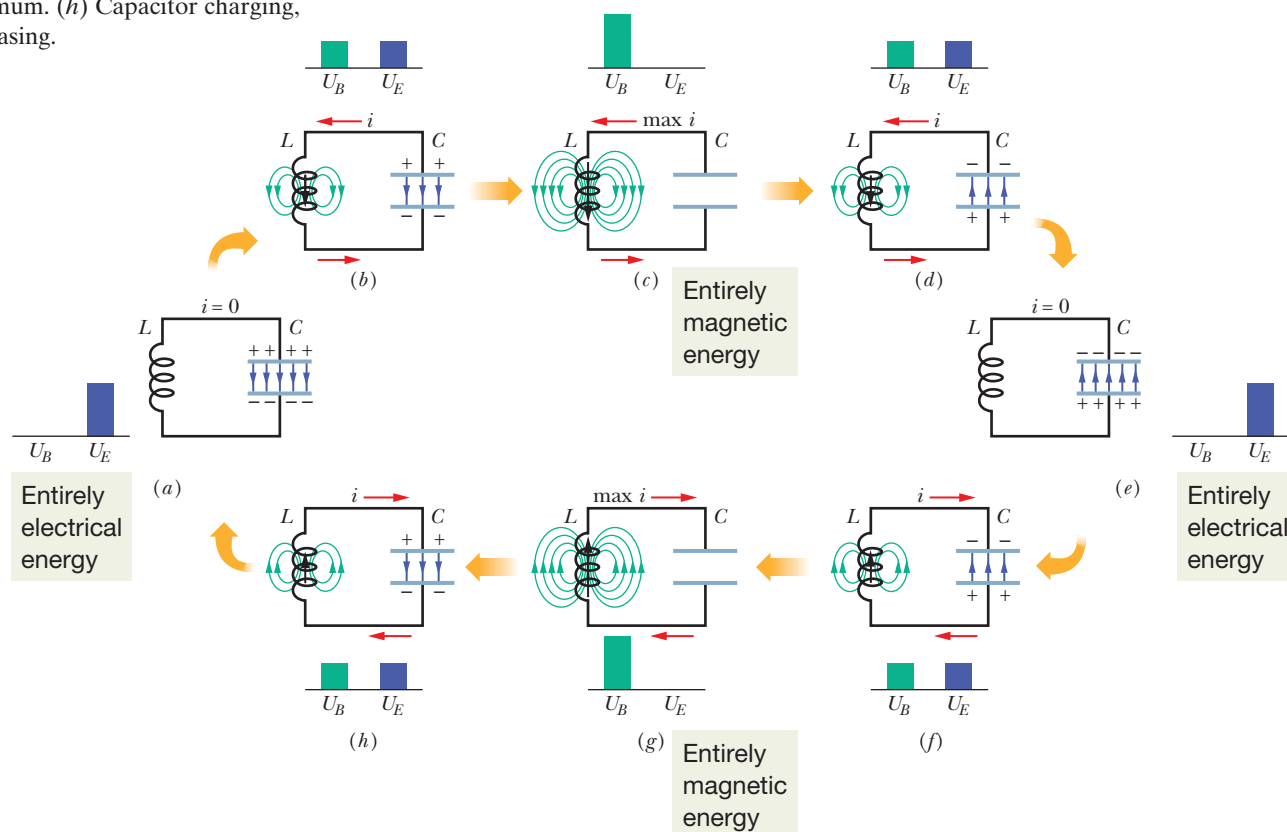
**Figure 31-1** Eight stages in a single cycle of oscillation of a resistanceless  $LC$  circuit. The bar graphs by each figure show the stored magnetic and electrical energies. The magnetic field lines of the inductor and the electric field lines of the capacitor are shown. (a) Capacitor with maximum charge, no current. (b) Capacitor discharging, current increasing. (c) Capacitor fully discharged, current maximum. (d) Capacitor charging but with polarity opposite that in (a), current decreasing. (e) Capacitor with maximum charge having polarity opposite that in (a), no current. (f) Capacitor discharging, current increasing with direction opposite that in (b). (g) Capacitor fully discharged, current maximum. (h) Capacitor charging, current decreasing.

## $LC$ Oscillations, Qualitatively

Of the three circuit elements, resistance  $R$ , capacitance  $C$ , and inductance  $L$ , we have so far discussed the series combinations  $RC$  (in Module 27-4) and  $RL$  (in Module 30-6). In these two kinds of circuit we found that the charge, current, and potential difference grow and decay exponentially. The time scale of the growth or decay is given by a *time constant*  $\tau$ , which is either capacitive or inductive.

We now examine the remaining two-element circuit combination  $LC$ . You will see that in this case the charge, current, and potential difference do not decay exponentially with time but vary sinusoidally (with period  $T$  and angular frequency  $\omega$ ). The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations**. Such a circuit is said to oscillate.

Parts *a* through *h* of Fig. 31-1 show succeeding stages of the oscillations in a simple  $LC$  circuit. From Eq. 25-21, the energy stored in the electric field of the



capacitor at any time is

$$U_E = \frac{q^2}{2C}, \quad (31-1)$$

where  $q$  is the charge on the capacitor at that time. From Eq. 30-49, the energy stored in the magnetic field of the inductor at any time is

$$U_B = \frac{Li^2}{2}, \quad (31-2)$$

where  $i$  is the current through the inductor at that time.

We now adopt the convention of representing *instantaneous values* of the electrical quantities of a sinusoidally oscillating circuit with small letters, such as  $q$ , and the *amplitudes* of those quantities with capital letters, such as  $Q$ . With this convention in mind, let us assume that initially the charge  $q$  on the capacitor in Fig. 31-1 is at its maximum value  $Q$  and that the current  $i$  through the inductor is zero. This initial state of the circuit is shown in Fig. 31-1a. The bar graphs for energy included there indicate that at this instant, with zero current through the inductor and maximum charge on the capacitor, the energy  $U_B$  of the magnetic field is zero and the energy  $U_E$  of the electric field is a maximum. As the circuit oscillates, energy shifts back and forth from one type of stored energy to the other, but the total amount is conserved.

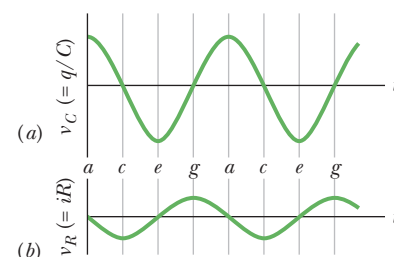
The capacitor now starts to discharge through the inductor, positive charge carriers moving counterclockwise, as shown in Fig. 31-1b. This means that a current  $i$ , given by  $dq/dt$  and pointing down in the inductor, is established. As the capacitor's charge decreases, the energy stored in the electric field within the capacitor also decreases. This energy is transferred to the magnetic field that appears around the inductor because of the current  $i$  that is building up there. Thus, the electric field decreases and the magnetic field builds up as energy is transferred from the electric field to the magnetic field.

The capacitor eventually loses all its charge (Fig. 31-1c) and thus also loses its electric field and the energy stored in that field. The energy has then been fully transferred to the magnetic field of the inductor. The magnetic field is then at its maximum magnitude, and the current through the inductor is then at its maximum value  $I$ .

Although the charge on the capacitor is now zero, the counterclockwise current must continue because the inductor does not allow it to change suddenly to zero. The current continues to transfer positive charge from the top plate to the bottom plate through the circuit (Fig. 31-1d). Energy now flows from the inductor back to the capacitor as the electric field within the capacitor builds up again. The current gradually decreases during this energy transfer. When, eventually, the energy has been transferred completely back to the capacitor (Fig. 31-1e), the current has decreased to zero (momentarily). The situation of Fig. 31-1e is like the initial situation, except that the capacitor is now charged oppositely.

The capacitor then starts to discharge again but now with a clockwise current (Fig. 31-1f). Reasoning as before, we see that the clockwise current builds to a maximum (Fig. 31-1g) and then decreases (Fig. 31-1h), until the circuit eventually returns to its initial situation (Fig. 31-1a). The process then repeats at some frequency  $f$  and thus at an angular frequency  $\omega = 2\pi f$ . In the ideal  $LC$  circuit with no resistance, there are no energy transfers other than that between the electric field of the capacitor and the magnetic field of the inductor. Because of the conservation of energy, the oscillations continue indefinitely. The oscillations need not begin with the energy all in the electric field; the initial situation could be any other stage of the oscillation.

**Figure 31-2** (a) The potential difference across the capacitor in the circuit of Fig. 31-1 as a function of time. This quantity is proportional to the charge on the capacitor. (b) A potential proportional to the current in the circuit of Fig. 31-1. The letters refer to the correspondingly labeled oscillation stages in Fig. 31-1.



To determine the charge  $q$  on the capacitor as a function of time, we can put in a voltmeter to measure the time-varying potential difference (or *voltage*)  $v_C$  that exists across the capacitor  $C$ . From Eq. 25-1 we can write

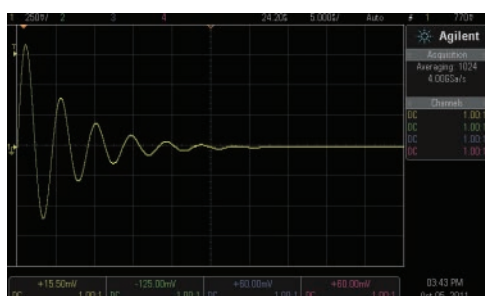
$$v_C = \left( \frac{1}{C} \right) q,$$

which allows us to find  $q$ . To measure the current, we can connect a small resistance  $R$  in series with the capacitor and inductor and measure the time-varying potential difference  $v_R$  across it;  $v_R$  is proportional to  $i$  through the relation

$$v_R = iR.$$

We assume here that  $R$  is so small that its effect on the behavior of the circuit is negligible. The variations in time of  $v_C$  and  $v_R$ , and thus of  $q$  and  $i$ , are shown in Fig. 31-2. All four quantities vary sinusoidally.

In an actual  $LC$  circuit, the oscillations will not continue indefinitely because there is always some resistance present that will drain energy from the electric and magnetic fields and dissipate it as thermal energy (the circuit may become warmer). The oscillations, once started, will die away as Fig. 31-3 suggests. Compare this figure with Fig. 15-17, which shows the decay of mechanical oscillations caused by frictional damping in a block–spring system.



Courtesy Agilent Technologies

**Figure 31-3** An oscilloscope trace showing how the oscillations in an  $RLC$  circuit actually die away because energy is dissipated in the resistor as thermal energy.



### Checkpoint 1

A charged capacitor and an inductor are connected in series at time  $t = 0$ . In terms of the period  $T$  of the resulting oscillations, determine how much later the following reach their maximum value: (a) the charge on the capacitor; (b) the voltage across the capacitor, with its original polarity; (c) the energy stored in the electric field; and (d) the current.

## The Electrical–Mechanical Analogy

Let us look a little closer at the analogy between the oscillating  $LC$  system of Fig. 31-1 and an oscillating block–spring system. Two kinds of energy are involved in the block–spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block. These two energies are given by the formulas in the first energy column in Table 31-1.

**Table 31-1** Comparison of the Energy in Two Oscillating Systems

Block–Spring System		$LC$ Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
$v = dx/dt$		$i = dq/dt$	

The table also shows, in the second energy column, the two kinds of energy involved in  $LC$  oscillations. By looking across the table, we can see an analogy between the forms of the two pairs of energies — the mechanical energies of the block–spring system and the electromagnetic energies of the  $LC$  oscillator. The equations for  $v$  and  $i$  at the bottom of the table help us see the details of the analogy. They tell us that  $q$  corresponds to  $x$  and  $i$  corresponds to  $v$  (in both equations, the former is differentiated to obtain the latter). These correspondences then suggest that, in the energy expressions,  $1/C$  corresponds to  $k$  and  $L$  corresponds to  $m$ . Thus,

$$\begin{aligned} q &\text{ corresponds to } x, & 1/C &\text{ corresponds to } k, \\ i &\text{ corresponds to } v, & L &\text{ corresponds to } m. \end{aligned}$$

These correspondences suggest that in an  $LC$  oscillator, the capacitor is mathematically like the spring in a block–spring system and the inductor is like the block.

In Module 15-1 we saw that the angular frequency of oscillation of a (frictionless) block–spring system is

$$\omega = \sqrt{\frac{k}{m}} \quad (\text{block–spring system}). \quad (31-3)$$

The correspondences listed above suggest that to find the angular frequency of oscillation for an ideal (resistanceless)  $LC$  circuit,  $k$  should be replaced by  $1/C$  and  $m$  by  $L$ , yielding

$$\omega = \frac{1}{\sqrt{LC}} \quad (LC \text{ circuit}). \quad (31-4)$$

## LC Oscillations, Quantitatively

Here we want to show explicitly that Eq. 31-4 for the angular frequency of  $LC$  oscillations is correct. At the same time, we want to examine even more closely the analogy between  $LC$  oscillations and block–spring oscillations. We start by extending somewhat our earlier treatment of the mechanical block–spring oscillator.

### The Block–Spring Oscillator

We analyzed block–spring oscillations in Chapter 15 in terms of energy transfers and did not — at that early stage — derive the fundamental differential equation that governs those oscillations. We do so now.

We can write, for the total energy  $U$  of a block–spring oscillator at any instant,

$$U = U_b + U_s = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \quad (31-5)$$

where  $U_b$  and  $U_s$  are, respectively, the kinetic energy of the moving block and the potential energy of the stretched or compressed spring. If there is no friction — which we assume — the total energy  $U$  remains constant with time, even though  $v$  and  $x$  vary. In more formal language,  $dU/dt = 0$ . This leads to

$$\frac{dU}{dt} = \frac{d}{dt} \left( \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \right) = mv \frac{dv}{dt} + kx \frac{dx}{dt} = 0. \quad (31-6)$$

Substituting  $v = dx/dt$  and  $dv/dt = d^2x/dt^2$ , we find

$$m \frac{d^2x}{dt^2} + kx = 0 \quad (\text{block–spring oscillations}). \quad (31-7)$$

Equation 31-7 is the fundamental *differential equation* that governs the frictionless block–spring oscillations.

The general solution to Eq. 31-7 is (as we saw in Eq. 15-3)

$$x = X \cos(\omega t + \phi) \quad (\text{displacement}), \quad (31-8)$$

in which  $X$  is the amplitude of the mechanical oscillations ( $x_m$  in Chapter 15),  $\omega$  is the angular frequency of the oscillations, and  $\phi$  is a phase constant.

### The LC Oscillator

Now let us analyze the oscillations of a resistanceless  $LC$  circuit, proceeding exactly as we just did for the block–spring oscillator. The total energy  $U$  present at any instant in an oscillating  $LC$  circuit is given by

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}, \quad (31-9)$$

in which  $U_B$  is the energy stored in the magnetic field of the inductor and  $U_E$  is the energy stored in the electric field of the capacitor. Since we have assumed the circuit resistance to be zero, no energy is transferred to thermal energy and  $U$  remains constant with time. In more formal language,  $dU/dt$  must be zero. This leads to

$$\frac{dU}{dt} = \frac{d}{dt} \left( \frac{Li^2}{2} + \frac{q^2}{2C} \right) = Li \frac{di}{dt} + \frac{q}{C} \frac{dq}{dt} = 0. \quad (31-10)$$

However,  $i = dq/dt$  and  $di/dt = d^2q/dt^2$ . With these substitutions, Eq. 31-10 becomes

$$L \frac{d^2q}{dt^2} + \frac{1}{C} q = 0 \quad (LC \text{ oscillations}). \quad (31-11)$$

This is the *differential equation* that describes the oscillations of a resistanceless  $LC$  circuit. Equations 31-11 and 31-7 are exactly of the same mathematical form.

### Charge and Current Oscillations

Since the differential equations are mathematically identical, their solutions must also be mathematically identical. Because  $q$  corresponds to  $x$ , we can write the general solution of Eq. 31-11, by analogy to Eq. 31-8, as

$$q = Q \cos(\omega t + \phi) \quad (\text{charge}), \quad (31-12)$$

where  $Q$  is the amplitude of the charge variations,  $\omega$  is the angular frequency of the electromagnetic oscillations, and  $\phi$  is the phase constant. Taking the first derivative of Eq. 31-12 with respect to time gives us the current:

$$i = \frac{dq}{dt} = -\omega Q \sin(\omega t + \phi) \quad (\text{current}). \quad (31-13)$$

The amplitude  $I$  of this sinusoidally varying current is

$$I = \omega Q, \quad (31-14)$$

and so we can rewrite Eq. 31-13 as

$$i = -I \sin(\omega t + \phi). \quad (31-15)$$

### Angular Frequencies

We can test whether Eq. 31-12 is a solution of Eq. 31-11 by substituting Eq. 31-12 and its second derivative with respect to time into Eq. 31-11. The first derivative of Eq. 31-12 is Eq. 31-13. The second derivative is then

$$\frac{d^2q}{dt^2} = -\omega^2 Q \cos(\omega t + \phi).$$

Substituting for  $q$  and  $d^2q/dt^2$  in Eq. 31-11, we obtain

$$-L\omega^2 Q \cos(\omega t + \phi) + \frac{1}{C} Q \cos(\omega t + \phi) = 0.$$

Canceling  $Q \cos(\omega t + \phi)$  and rearranging lead to

$$\omega = \frac{1}{\sqrt{LC}}.$$

Thus, Eq. 31-12 is indeed a solution of Eq. 31-11 if  $\omega$  has the constant value  $1/\sqrt{LC}$ . Note that this expression for  $\omega$  is exactly that given by Eq. 31-4.

The phase constant  $\phi$  in Eq. 31-12 is determined by the conditions that exist at any certain time — say,  $t = 0$ . If the conditions yield  $\phi = 0$  at  $t = 0$ , Eq. 31-12 requires that  $q = Q$  and Eq. 31-13 requires that  $i = 0$ ; these are the initial conditions represented by Fig. 31-1a.

### Electrical and Magnetic Energy Oscillations

The electrical energy stored in the  $LC$  circuit at time  $t$  is, from Eqs. 31-1 and 31-12,

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C} \cos^2(\omega t + \phi). \quad (31-16)$$

The magnetic energy is, from Eqs. 31-2 and 31-13,

$$U_B = \frac{1}{2} Li^2 = \frac{1}{2} L \omega^2 Q^2 \sin^2(\omega t + \phi).$$

Substituting for  $\omega$  from Eq. 31-4 then gives us

$$U_B = \frac{Q^2}{2C} \sin^2(\omega t + \phi). \quad (31-17)$$

Figure 31-4 shows plots of  $U_E(t)$  and  $U_B(t)$  for the case of  $\phi = 0$ . Note that

1. The maximum values of  $U_E$  and  $U_B$  are both  $Q^2/2C$ .
2. At any instant the sum of  $U_E$  and  $U_B$  is equal to  $Q^2/2C$ , a constant.
3. When  $U_E$  is maximum,  $U_B$  is zero, and conversely.



#### Checkpoint 2

A capacitor in an  $LC$  oscillator has a maximum potential difference of 17 V and a maximum energy of  $160 \mu\text{J}$ . When the capacitor has a potential difference of 5 V and an energy of  $10 \mu\text{J}$ , what are (a) the emf across the inductor and (b) the energy stored in the magnetic field?

### Sample Problem 31.01 LC oscillator: potential change, rate of current change

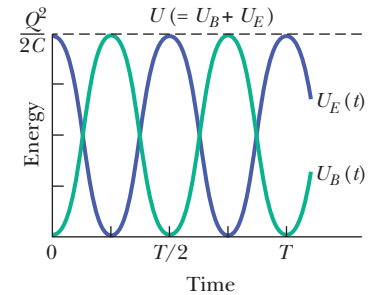
A  $1.5 \mu\text{F}$  capacitor is charged to 57 V by a battery, which is then removed. At time  $t = 0$ , a 12 mH coil is connected in series with the capacitor to form an  $LC$  oscillator (Fig. 31-1).

(a) What is the potential difference  $v_L(t)$  across the inductor as a function of time?

#### KEY IDEAS

- (1) The current and potential differences of the circuit (both the potential difference of the capacitor and the potential difference of the coil) undergo sinusoidal oscillations.
- (2) We can still apply the loop rule to these oscillating potential differences, just as we did for the nonoscillating circuits of Chapter 27.

The electrical and magnetic energies vary but the total is constant.



**Figure 31-4** The stored magnetic energy and electrical energy in the circuit of Fig. 31-1 as a function of time. Note that their sum remains constant.  $T$  is the period of oscillation.

**Calculations:** At any time  $t$  during the oscillations, the loop rule and Fig. 31-1 give us

$$v_L(t) = v_C(t); \quad (31-18)$$

that is, the potential difference  $v_L$  across the inductor must always be equal to the potential difference  $v_C$  across the capacitor, so that the net potential difference around the circuit is zero. Thus, we will find  $v_L(t)$  if we can find  $v_C(t)$ , and we can find  $v_C(t)$  from  $q(t)$  with Eq. 25-1 ( $q = CV$ ).

Because the potential difference  $v_C(t)$  is maximum when the oscillations begin at time  $t = 0$ , the charge  $q$  on the capacitor must also be maximum then. Thus, phase constant  $\phi$  must be zero; so Eq. 31-12 gives us

$$q = Q \cos \omega t. \quad (31-19)$$



(Note that this cosine function does indeed yield maximum  $q (= Q)$  when  $t = 0$ .) To get the potential difference  $v_C(t)$ , we divide both sides of Eq. 31-19 by  $C$  to write

$$\frac{q}{C} = \frac{Q}{C} \cos \omega t,$$

and then use Eq. 25-1 to write

$$v_C = V_C \cos \omega t. \quad (31-20)$$

Here,  $V_C$  is the amplitude of the oscillations in the potential difference  $v_C$  across the capacitor.

Next, substituting  $v_C = v_L$  from Eq. 31-18, we find

$$v_L = V_C \cos \omega t. \quad (31-21)$$

We can evaluate the right side of this equation by first noting that the amplitude  $V_C$  is equal to the initial (maximum) potential difference of 57 V across the capacitor. Then we find  $\omega$  with Eq. 31-4:

$$\begin{aligned} \omega &= \frac{1}{\sqrt{LC}} = \frac{1}{[(0.012 \text{ H})(1.5 \times 10^{-6} \text{ F})]^{0.5}} \\ &= 7454 \text{ rad/s} \approx 7500 \text{ rad/s}. \end{aligned}$$

Thus, Eq. 31-21 becomes

$$v_L = (57 \text{ V}) \cos(7500 \text{ rad/s})t. \quad (\text{Answer})$$

(b) What is the maximum rate  $(di/dt)_{\max}$  at which the current  $i$  changes in the circuit?

### KEY IDEA

With the charge on the capacitor oscillating as in Eq. 31-12, the current is in the form of Eq. 31-13. Because  $\phi = 0$ , that equation gives us

$$i = -\omega Q \sin \omega t.$$

**Calculations:** Taking the derivative, we have

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q \sin \omega t) = -\omega^2 Q \cos \omega t.$$

We can simplify this equation by substituting  $CV_C$  for  $Q$  (because we know  $C$  and  $V_C$  but not  $Q$ ) and  $1/\sqrt{LC}$  for  $\omega$  according to Eq. 31-4. We get

$$\frac{di}{dt} = -\frac{1}{LC} CV_C \cos \omega t = -\frac{V_C}{L} \cos \omega t.$$

This tells us that the current changes at a varying (sinusoidal) rate, with its maximum rate of change being

$$\frac{V_C}{L} = \frac{57 \text{ V}}{0.012 \text{ H}} = 4750 \text{ A/s} \approx 4800 \text{ A/s}. \quad (\text{Answer})$$



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## 31-2 DAMPED OSCILLATIONS IN AN *RLC* CIRCUIT

### Learning Objectives

After reading this module, you should be able to . . .

**31.13** Draw the schematic of a damped *RLC* circuit and explain why the oscillations are damped.

**31.14** Starting with the expressions for the field energies and the rate of energy loss in a damped *RLC* circuit, write the differential equation for the charge  $q$  on the capacitor.

**31.15** For a damped *RLC* circuit, apply the expression for charge  $q(t)$ .

**31.16** Identify that in a damped *RLC* circuit, the charge amplitude and the amplitude of the electric field energy decrease exponentially with time.

**31.17** Apply the relationship between the angular frequency  $\omega'$  of a given damped *RLC* oscillator and the angular frequency  $\omega$  of the circuit if  $R$  is removed.

**31.18** For a damped *RLC* circuit, apply the expression for the electric field energy  $U_E$  as a function of time.

### Key Ideas

● Oscillations in an *LC* circuit are damped when a dissipative element  $R$  is also present in the circuit. Then

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0 \quad (RLC \text{ circuit}).$$

● The solution of this differential equation is

$$q = Q e^{-Rt/2L} \cos(\omega' t + \phi),$$

where  $\omega' = \sqrt{\omega^2 - (R/2L)^2}$ .

We consider only situations with small  $R$  and thus small damping; then  $\omega' \approx \omega$ .

## Damped Oscillations in an *RLC* Circuit

A circuit containing resistance, inductance, and capacitance is called an *RLC circuit*. We shall here discuss only *series RLC circuits* like that shown in Fig. 31-5. With a resistance  $R$  present, the total *electromagnetic energy*  $U$  of the circuit (the sum of the electrical energy and magnetic energy) is no longer constant; instead, it decreases with time as energy is transferred to thermal energy in the resistance. Because of this loss of energy, the oscillations of charge, current, and potential difference continuously decrease in amplitude, and the oscillations are said to be *damped*, just as with the damped block–spring oscillator of Module 15-5.

To analyze the oscillations of this circuit, we write an equation for the total electromagnetic energy  $U$  in the circuit at any instant. Because the resistance does not store electromagnetic energy, we can use Eq. 31-9:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}. \quad (31-22)$$

Now, however, this total energy decreases as energy is transferred to thermal energy. The rate of that transfer is, from Eq. 26-27,

$$\frac{dU}{dt} = -i^2 R, \quad (31-23)$$

where the minus sign indicates that  $U$  decreases. By differentiating Eq. 31-22 with respect to time and then substituting the result in Eq. 31-23, we obtain

$$\frac{dU}{dt} = Li \frac{di}{dt} + \frac{q}{C} \frac{dq}{dt} = -i^2 R.$$

Substituting  $dq/dt$  for  $i$  and  $d^2q/dt^2$  for  $di/dt$ , we obtain

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0 \quad (RLC \text{ circuit}), \quad (31-24)$$

which is the differential equation for damped oscillations in an *RLC* circuit.

**Charge Decay.** The solution to Eq. 31-24 is

$$q = Qe^{-Rt/2L} \cos(\omega' t + \phi), \quad (31-25)$$

in which

$$\omega' = \sqrt{\omega^2 - (R/2L)^2}, \quad (31-26)$$

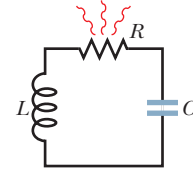
where  $\omega = 1/\sqrt{LC}$ , as with an undamped oscillator. Equation 31-25 tells us how the charge on the capacitor oscillates in a damped *RLC* circuit; that equation is the electromagnetic counterpart of Eq. 15-42, which gives the displacement of a damped block–spring oscillator.

Equation 31-25 describes a sinusoidal oscillation (the cosine function) with an *exponentially decaying amplitude*  $Qe^{-Rt/2L}$  (the factor that multiplies the cosine). The angular frequency  $\omega'$  of the damped oscillations is always less than the angular frequency  $\omega$  of the undamped oscillations; however, we shall here consider only situations in which  $R$  is small enough for us to replace  $\omega'$  with  $\omega$ .

**Energy Decay.** Let us next find an expression for the total electromagnetic energy  $U$  of the circuit as a function of time. One way to do so is to monitor the energy of the electric field in the capacitor, which is given by Eq. 31-1 ( $U_E = q^2/2C$ ). By substituting Eq. 31-25 into Eq. 31-1, we obtain

$$U_E = \frac{q^2}{2C} = \frac{[Qe^{-Rt/2L} \cos(\omega' t + \phi)]^2}{2C} = \frac{Q^2}{2C} e^{-Rt/L} \cos^2(\omega' t + \phi). \quad (31-27)$$

Thus, the energy of the electric field oscillates according to a cosine-squared term, and the amplitude of that oscillation decreases exponentially with time.



**Figure 31-5** A series *RLC* circuit. As the charge contained in the circuit oscillates back and forth through the resistance, electromagnetic energy is dissipated as thermal energy, damping (decreasing the amplitude of) the oscillations.



### Sample Problem 31.02 Damped $RLC$ circuit: charge amplitude

A series  $RLC$  circuit has inductance  $L = 12$  mH, capacitance  $C = 1.6$   $\mu$ F, and resistance  $R = 1.5$   $\Omega$  and begins to oscillate at time  $t = 0$ .

(a) At what time  $t$  will the amplitude of the charge oscillations in the circuit be 50% of its initial value? (Note that we do not know that initial value.)

#### KEY IDEA

The amplitude of the charge oscillations decreases exponentially with time  $t$ : According to Eq. 31-25, the charge amplitude at any time  $t$  is  $Qe^{-Rt/2L}$ , in which  $Q$  is the amplitude at time  $t = 0$ .

**Calculations:** We want the time when the charge amplitude has decreased to  $0.50Q$ —that is, when

$$Qe^{-Rt/2L} = 0.50Q.$$

We can now cancel  $Q$  (which also means that we can answer the question without knowing the initial charge). Taking the natural logarithms of both sides (to eliminate the exponential function), we have

$$-\frac{Rt}{2L} = \ln 0.50.$$

Solving for  $t$  and then substituting given data yield

$$t = -\frac{2L}{R} \ln 0.50 = -\frac{(2)(12 \times 10^{-3} \text{ H})(\ln 0.50)}{1.5 \Omega} = 0.0111 \text{ s} \approx 11 \text{ ms.} \quad (\text{Answer})$$

(b) How many oscillations are completed within this time?

#### KEY IDEA

The time for one complete oscillation is the period  $T = 2\pi/\omega$ , where the angular frequency for  $LC$  oscillations is given by Eq. 31-4 ( $\omega = 1/\sqrt{LC}$ ).

**Calculation:** In the time interval  $\Delta t = 0.0111$  s, the number of complete oscillations is

$$\begin{aligned} \frac{\Delta t}{T} &= \frac{\Delta t}{2\pi\sqrt{LC}} \\ &= \frac{0.0111 \text{ s}}{2\pi[(12 \times 10^{-3} \text{ H})(1.6 \times 10^{-6} \text{ F})]^{1/2}} \approx 13. \end{aligned} \quad (\text{Answer})$$

Thus, the amplitude decays by 50% in about 13 complete oscillations. This damping is less severe than that shown in Fig. 31-3, where the amplitude decays by a little more than 50% in one oscillation.



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## 31-3 FORCED OSCILLATIONS OF THREE SIMPLE CIRCUITS

### Learning Objectives

After reading this module, you should be able to . . .

- 31.19** Distinguish alternating current from direct current.
- 31.20** For an ac generator, write the emf as a function of time, identifying the emf amplitude and driving angular frequency.
- 31.21** For an ac generator, write the current as a function of time, identifying its amplitude and its phase constant with respect to the emf.
- 31.22** Draw a schematic diagram of a (series)  $RLC$  circuit that is driven by a generator.
- 31.23** Distinguish driving angular frequency  $\omega_d$  from natural angular frequency  $\omega$ .
- 31.24** In a driven (series)  $RLC$  circuit, identify the conditions for resonance and the effect of resonance on the current amplitude.
- 31.25** For each of the three basic circuits (purely resistive load, purely capacitive load, and purely inductive load),

draw the circuit and sketch graphs and phasor diagrams for voltage  $v(t)$  and current  $i(t)$ .

- 31.26** For the three basic circuits, apply equations for voltage  $v(t)$  and current  $i(t)$ .
- 31.27** On a phasor diagram for each of the basic circuits, identify angular speed, amplitude, projection on the vertical axis, and rotation angle.
- 31.28** For each basic circuit, identify the phase constant, and interpret it in terms of the relative orientations of the current phasor and voltage phasor and also in terms of leading and lagging.
- 31.29** Apply the mnemonic “*ELI* positively is the *ICE* man.”
- 31.30** For each basic circuit, apply the relationships between the voltage amplitude  $V$  and the current amplitude  $I$ .
- 31.31** Calculate capacitive reactance  $X_C$  and inductive reactance  $X_L$ .

## Key Ideas

- A series  $RLC$  circuit may be set into forced oscillation at a driving angular frequency  $\omega_d$  by an external alternating emf

$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t.$$

- The current driven in the circuit is

$$i = I \sin(\omega_d t - \phi),$$

where  $\phi$  is the phase constant of the current.

- The alternating potential difference across a resistor

has amplitude  $V_R = IR$ ; the current is in phase with the potential difference.

- For a capacitor,  $V_C = IX_C$ , in which  $X_C = 1/\omega_d C$  is the capacitive reactance; the current here leads the potential difference by  $90^\circ$  ( $\phi = -90^\circ = -\pi/2$  rad).

- For an inductor,  $V_L = IX_L$ , in which  $X_L = \omega_d L$  is the inductive reactance; the current here lags the potential difference by  $90^\circ$  ( $\phi = +90^\circ = +\pi/2$  rad).

## Alternating Current

The oscillations in an  $RLC$  circuit will not damp out if an external emf device supplies enough energy to make up for the energy dissipated as thermal energy in the resistance  $R$ . Circuits in homes, offices, and factories, including countless  $RLC$  circuits, receive such energy from local power companies. In most countries the energy is supplied via oscillating emfs and currents — the current is said to be an **alternating current**, or **ac** for short. (The nonoscillating current from a battery is said to be a **direct current**, or **dc**.) These oscillating emfs and currents vary sinusoidally with time, reversing direction (in North America) 120 times per second and thus having frequency  $f = 60$  Hz.

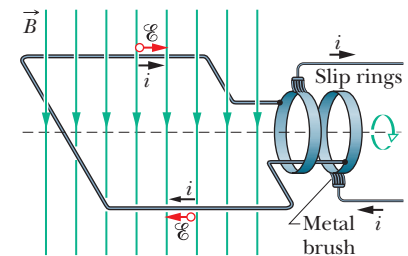
**Electron Oscillations.** At first sight this may seem to be a strange arrangement. We have seen that the drift speed of the conduction electrons in household wiring may typically be  $4 \times 10^{-5}$  m/s. If we now reverse their direction every  $\frac{1}{120}$  s, such electrons can move only about  $3 \times 10^{-7}$  m in a half-cycle. At this rate, a typical electron can drift past no more than about 10 atoms in the wiring before it is required to reverse its direction. How, you may wonder, can the electron ever get anywhere?

Although this question may be worrisome, it is a needless concern. The conduction electrons do not have to “get anywhere.” When we say that the current in a wire is one ampere, we mean that charge passes through any plane cutting across that wire at the rate of one coulomb per second. The speed at which the charge carriers cross that plane does not matter directly; one ampere may correspond to many charge carriers moving very slowly or to a few moving very rapidly. Furthermore, the signal to the electrons to reverse directions — which originates in the alternating emf provided by the power company’s generator — is propagated along the conductor at a speed close to that of light. All electrons, no matter where they are located, get their reversal instructions at about the same instant. Finally, we note that for many devices, such as lightbulbs and toasters, the direction of motion is unimportant as long as the electrons do move so as to transfer energy to the device via collisions with atoms in the device.

**Why ac?** The basic advantage of alternating current is this: *As the current alternates, so does the magnetic field that surrounds the conductor.* This makes possible the use of Faraday’s law of induction, which, among other things, means that we can step up (increase) or step down (decrease) the magnitude of an alternating potential difference at will, using a device called a transformer, as we shall discuss later. Moreover, alternating current is more readily adaptable to rotating machinery such as generators and motors than is (nonalternating) direct current.

**Emf and Current.** Figure 31-6 shows a simple model of an ac generator. As the conducting loop is forced to rotate through the external magnetic field  $\vec{B}$ , a sinusoidally oscillating emf  $\mathcal{E}$  is induced in the loop:

$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t. \quad (31-28)$$



**Figure 31-6** The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field. In practice, the alternating emf induced in a coil of many turns of wire is made accessible by means of slip rings attached to the rotating loop. Each ring is connected to one end of the loop wire and is electrically connected to the rest of the generator circuit by a conducting brush against which the ring slips as the loop (and it) rotates.

The *angular frequency*  $\omega_d$  of the emf is equal to the angular speed with which the loop rotates in the magnetic field, the *phase* of the emf is  $\omega_d t$ , and the *amplitude* of the emf is  $\mathcal{E}_m$  (where the subscript stands for maximum). When the rotating loop is part of a closed conducting path, this emf produces (*drives*) a sinusoidal (alternating) current along the path with the same angular frequency  $\omega_d$ , which then is called the **driving angular frequency**. We can write the current as

$$i = I \sin(\omega_d t - \phi), \quad (31-29)$$

in which  $I$  is the amplitude of the driven current. (The phase  $\omega_d t - \phi$  of the current is traditionally written with a minus sign instead of as  $\omega_d t + \phi$ .) We include a phase constant  $\phi$  in Eq. 31-29 because the current  $i$  may not be in phase with the emf  $\mathcal{E}$ . (As you will see, the phase constant depends on the circuit to which the generator is connected.) We can also write the current  $i$  in terms of the **driving frequency**  $f_d$  of the emf, by substituting  $2\pi f_d$  for  $\omega_d$  in Eq. 31-29.

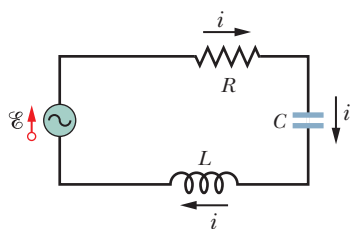
## Forced Oscillations

We have seen that once started, the charge, potential difference, and current in both undamped  $LC$  circuits and damped  $RLC$  circuits (with small enough  $R$ ) oscillate at angular frequency  $\omega = 1/\sqrt{LC}$ . Such oscillations are said to be *free oscillations* (free of any external emf), and the angular frequency  $\omega$  is said to be the circuit's **natural angular frequency**.

When the external alternating emf of Eq. 31-28 is connected to an  $RLC$  circuit, the oscillations of charge, potential difference, and current are said to be *driven oscillations* or *forced oscillations*. These oscillations always occur at the driving angular frequency  $\omega_d$ :



Whatever the natural angular frequency  $\omega$  of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency  $\omega_d$ .



**Figure 31-7** A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.

However, as you will see in Module 31-4, the amplitudes of the oscillations very much depend on how close  $\omega_d$  is to  $\omega$ . When the two angular frequencies match — a condition known as **resonance** — the amplitude  $I$  of the current in the circuit is maximum.

## Three Simple Circuits

Later in this chapter, we shall connect an external alternating emf device to a series  $RLC$  circuit as in Fig. 31-7. We shall then find expressions for the amplitude  $I$  and phase constant  $\phi$  of the sinusoidally oscillating current in terms of the amplitude  $\mathcal{E}_m$  and angular frequency  $\omega_d$  of the external emf. First, let's consider three simpler circuits, each having an external emf and only one other circuit element:  $R$ ,  $C$ , or  $L$ . We start with a resistive element (a purely *resistive load*).

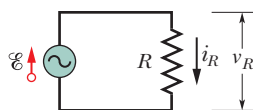
### A Resistive Load

Figure 31-8 shows a circuit containing a resistance element of value  $R$  and an ac generator with the alternating emf of Eq. 31-28. By the loop rule, we have

$$\mathcal{E} - v_R = 0.$$

With Eq. 31-28, this gives us

$$v_R = \mathcal{E}_m \sin \omega_d t.$$



**Figure 31-8** A resistor is connected across an alternating-current generator.

Because the amplitude  $V_R$  of the alternating potential difference (or voltage) across the resistance is equal to the amplitude  $\mathcal{E}_m$  of the alternating emf, we can

write this as

$$v_R = V_R \sin \omega_d t. \quad (31-30)$$

From the definition of resistance ( $R = V/i$ ), we can now write the current  $i_R$  in the resistance as

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t. \quad (31-31)$$

From Eq. 31-29, we can also write this current as

$$i_R = I_R \sin(\omega_d t - \phi), \quad (31-32)$$

where  $I_R$  is the amplitude of the current  $i_R$  in the resistance. Comparing Eqs. 31-31 and 31-32, we see that for a purely resistive load the phase constant  $\phi = 0^\circ$ . We also see that the voltage amplitude and current amplitude are related by

$$V_R = I_R R \quad (\text{resistor}). \quad (31-33)$$

Although we found this relation for the circuit of Fig. 31-8, it applies to any resistance in any ac circuit.

By comparing Eqs. 31-30 and 31-31, we see that the time-varying quantities  $v_R$  and  $i_R$  are both functions of  $\sin \omega_d t$  with  $\phi = 0^\circ$ . Thus, these two quantities are *in phase*, which means that their corresponding maxima (and minima) occur at the same times. Figure 31-9a, which is a plot of  $v_R(t)$  and  $i_R(t)$ , illustrates this fact. Note that  $v_R$  and  $i_R$  do not decay here because the generator supplies energy to the circuit to make up for the energy dissipated in  $R$ .

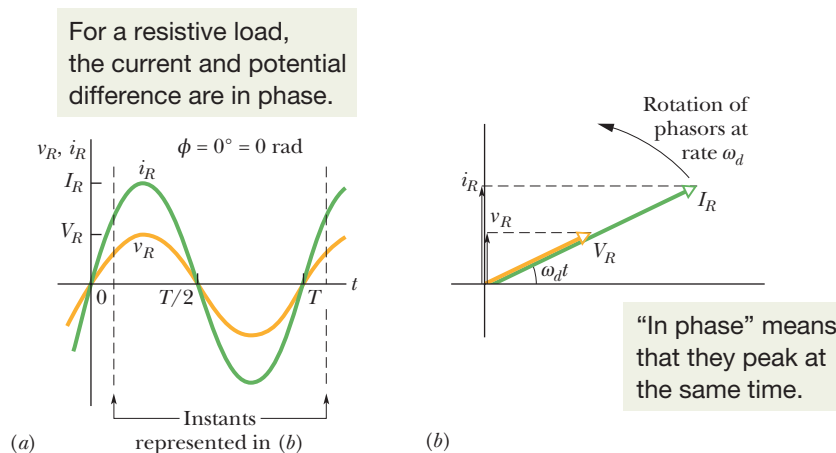
The time-varying quantities  $v_R$  and  $i_R$  can also be represented geometrically by *phasors*. Recall from Module 16-6 that phasors are vectors that rotate around an origin. Those that represent the voltage across and current in the resistor of Fig. 31-8 are shown in Fig. 31-9b at an arbitrary time  $t$ . Such phasors have the following properties:

**Angular speed:** Both phasors rotate counterclockwise about the origin with an angular speed equal to the angular frequency  $\omega_d$  of  $v_R$  and  $i_R$ .

**Length:** The length of each phasor represents the amplitude of the alternating quantity:  $V_R$  for the voltage and  $I_R$  for the current.

**Projection:** The projection of each phasor on the *vertical* axis represents the value of the alternating quantity at time  $t$ :  $v_R$  for the voltage and  $i_R$  for the current.

**Rotation angle:** The rotation angle of each phasor is equal to the phase of the



**Figure 31-9** (a) The current  $i_R$  and the potential difference  $v_R$  across the resistor are plotted on the same graph, both versus time  $t$ . They are in phase and complete one cycle in one period  $T$ . (b) A phasor diagram shows the same thing as (a).



alternating quantity at time  $t$ . In Fig. 31-9b, the voltage and current are in phase; so their phasors always have the same phase  $\omega_d t$  and the same rotation angle, and thus they rotate together.

Mentally follow the rotation. Can you see that when the phasors have rotated so that  $\omega_d t = 90^\circ$  (they point vertically upward), they indicate that just then  $v_R = V_R$  and  $i_R = I_R$ ? Equations 31-30 and 31-32 give the same results.



### Checkpoint 3

If we increase the driving frequency in a circuit with a purely resistive load, do (a) amplitude  $V_R$  and (b) amplitude  $I_R$  increase, decrease, or remain the same?



### Sample Problem 31.03 Purely resistive load: potential difference and current

In Fig. 31-8, resistance  $R$  is  $200\ \Omega$  and the sinusoidal alternating emf device operates at amplitude  $\mathcal{E}_m = 36.0\ \text{V}$  and frequency  $f_d = 60.0\ \text{Hz}$ .

(a) What is the potential difference  $v_R(t)$  across the resistance as a function of time  $t$ , and what is the amplitude  $V_R$  of  $v_R(t)$ ?

#### KEY IDEA

In a circuit with a purely resistive load, the potential difference  $v_R(t)$  across the resistance is always equal to the potential difference  $\mathcal{E}(t)$  across the emf device.

**Calculations:** For our situation,  $v_R(t) = \mathcal{E}(t)$  and  $V_R = \mathcal{E}_m$ . Since  $\mathcal{E}_m$  is given, we can write

$$V_R = \mathcal{E}_m = 36.0\ \text{V}. \quad (\text{Answer})$$

To find  $v_R(t)$ , we use Eq. 31-28 to write

$$v_R(t) = \mathcal{E}(t) = \mathcal{E}_m \sin \omega_d t \quad (31-34)$$

and then substitute  $\mathcal{E}_m = 36.0\ \text{V}$  and

$$\omega_d = 2\pi f_d = 2\pi(60\ \text{Hz}) = 120\pi$$

to obtain

$$v_R = (36.0\ \text{V}) \sin(120\pi t). \quad (\text{Answer})$$

We can leave the argument of the sine in this form for convenience, or we can write it as  $(377\ \text{rad/s})t$  or as  $(377\ \text{s}^{-1})t$ .

(b) What are the current  $i_R(t)$  in the resistance and the amplitude  $I_R$  of  $i_R(t)$ ?

#### KEY IDEA

In an ac circuit with a purely resistive load, the alternating current  $i_R(t)$  in the resistance is *in phase* with the alternating potential difference  $v_R(t)$  across the resistance; that is, the phase constant  $\phi$  for the current is zero.

**Calculations:** Here we can write Eq. 31-29 as

$$i_R = I_R \sin(\omega_d t - \phi) = I_R \sin \omega_d t. \quad (31-35)$$

From Eq. 31-33, the amplitude  $I_R$  is

$$I_R = \frac{V_R}{R} = \frac{36.0\ \text{V}}{200\ \Omega} = 0.180\ \text{A}. \quad (\text{Answer})$$

Substituting this and  $\omega_d = 2\pi f_d = 120\pi$  into Eq. 31-35, we have

$$i_R = (0.180\ \text{A}) \sin(120\pi t). \quad (\text{Answer})$$



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### A Capacitive Load

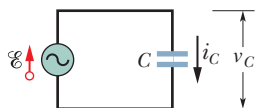
Figure 31-10 shows a circuit containing a capacitance and a generator with the alternating emf of Eq. 31-28. Using the loop rule and proceeding as we did when we obtained Eq. 31-30, we find that the potential difference across the capacitor is

$$v_C = V_C \sin \omega_d t, \quad (31-36)$$

where  $V_C$  is the amplitude of the alternating voltage across the capacitor. From the definition of capacitance we can also write

$$q_C = C v_C = C V_C \sin \omega_d t. \quad (31-37)$$

Our concern, however, is with the current rather than the charge. Thus, we



**Figure 31-10** A capacitor is connected across an alternating-current generator.

differentiate Eq. 31-37 to find

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t. \quad (31-38)$$

We now modify Eq. 31-38 in two ways. First, for reasons of symmetry of notation, we introduce the quantity  $X_C$ , called the **capacitive reactance** of a capacitor, defined as

$$X_C = \frac{1}{\omega_d C} \quad (\text{capacitive reactance}). \quad (31-39)$$

Its value depends not only on the capacitance but also on the driving angular frequency  $\omega_d$ . We know from the definition of the capacitive time constant ( $\tau = RC$ ) that the SI unit for  $C$  can be expressed as seconds per ohm. Applying this to Eq. 31-39 shows that the SI unit of  $X_C$  is the *ohm*, just as for resistance  $R$ .

Second, we replace  $\cos \omega_d t$  in Eq. 31-38 with a phase-shifted sine:

$$\cos \omega_d t = \sin(\omega_d t + 90^\circ).$$

You can verify this identity by shifting a sine curve  $90^\circ$  in the negative direction.

With these two modifications, Eq. 31-38 becomes

$$i_C = \left( \frac{V_C}{X_C} \right) \sin(\omega_d t + 90^\circ). \quad (31-40)$$

From Eq. 31-29, we can also write the current  $i_C$  in the capacitor of Fig. 31-10 as

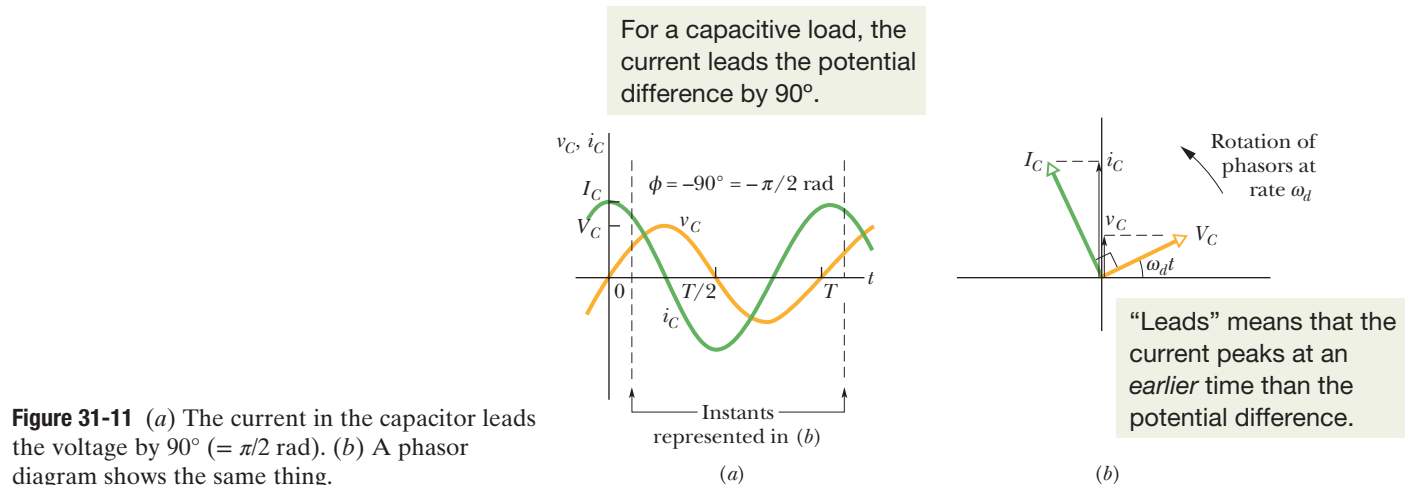
$$i_C = I_C \sin(\omega_d t - \phi), \quad (31-41)$$

where  $I_C$  is the amplitude of  $i_C$ . Comparing Eqs. 31-40 and 31-41, we see that for a purely capacitive load the phase constant  $\phi$  for the current is  $-90^\circ$ . We also see that the voltage amplitude and current amplitude are related by

$$V_C = I_C X_C \quad (\text{capacitor}). \quad (31-42)$$

Although we found this relation for the circuit of Fig. 31-10, it applies to any capacitance in any ac circuit.

Comparison of Eqs. 31-36 and 31-40, or inspection of Fig. 31-11a, shows that the quantities  $v_C$  and  $i_C$  are  $90^\circ$ ,  $\pi/2$  rad, or one-quarter cycle, out of phase. Furthermore, we see that  $i_C$  *leads*  $v_C$ , which means that, if you monitored the current  $i_C$  and the potential difference  $v_C$  in the circuit of Fig. 31-10, you would find that  $i_C$  reaches its maximum *before*  $v_C$  does, by one-quarter cycle.

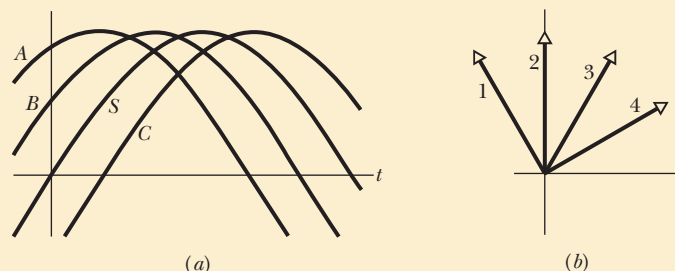


This relation between  $i_C$  and  $v_C$  is illustrated by the phasor diagram of Fig. 31-11b. As the phasors representing these two quantities rotate counterclockwise together, the phasor labeled  $I_C$  does indeed lead that labeled  $V_C$ , and by an angle of  $90^\circ$ ; that is, the phasor  $I_C$  coincides with the vertical axis one-quarter cycle before the phasor  $V_C$  does. Be sure to convince yourself that the phasor diagram of Fig. 31-11b is consistent with Eqs. 31-36 and 31-40.



### Checkpoint 4

The figure shows, in (a), a sine curve  $S(t) = \sin(\omega_d t)$  and three other sinusoidal curves  $A(t)$ ,  $B(t)$ , and  $C(t)$ , each of the form  $\sin(\omega_d t - \phi)$ . (a) Rank the three other curves according to the value of  $\phi$ , most positive first and most negative last. (b) Which curve corresponds to which phasor in (b) of the figure? (c) Which curve leads the others?



### Sample Problem 31.04 Purely capacitive load: potential difference and current

In Fig. 31-10, capacitance  $C$  is  $15.0 \mu\text{F}$  and the sinusoidal alternating emf device operates at amplitude  $\mathcal{E}_m = 36.0 \text{ V}$  and frequency  $f_d = 60.0 \text{ Hz}$ .

(a) What are the potential difference  $v_C(t)$  across the capacitance and the amplitude  $V_C$  of  $v_C(t)$ ?

#### KEY IDEA

In a circuit with a purely capacitive load, the potential difference  $v_C(t)$  across the capacitance is always equal to the potential difference  $\mathcal{E}(t)$  across the emf device.

**Calculations:** Here we have  $v_C(t) = \mathcal{E}(t)$  and  $V_C = \mathcal{E}_m$ . Since  $\mathcal{E}_m$  is given, we have

$$V_C = \mathcal{E}_m = 36.0 \text{ V.} \quad (\text{Answer})$$

To find  $v_C(t)$ , we use Eq. 31-28 to write

$$v_C(t) = \mathcal{E}(t) = \mathcal{E}_m \sin \omega_d t. \quad (31-43)$$

Then, substituting  $\mathcal{E}_m = 36.0 \text{ V}$  and  $\omega_d = 2\pi f_d = 120\pi$  into Eq. 31-43, we have

$$v_C = (36.0 \text{ V}) \sin(120\pi t). \quad (\text{Answer})$$

(b) What are the current  $i_C(t)$  in the circuit as a function of time and the amplitude  $I_C$  of  $i_C(t)$ ?

#### KEY IDEA

In an ac circuit with a purely capacitive load, the alternating current  $i_C(t)$  in the capacitance leads the alternating potential difference  $v_C(t)$  by  $90^\circ$ ; that is, the phase constant  $\phi$  for the current is  $-90^\circ$ , or  $-\pi/2$  rad.

**Calculations:** Thus, we can write Eq. 31-29 as

$$i_C = I_C \sin(\omega_d t - \phi) = I_C \sin(\omega_d t + \pi/2). \quad (31-44)$$

We can find the amplitude  $I_C$  from Eq. 31-42 ( $V_C = I_C X_C$ ) if we first find the capacitive reactance  $X_C$ . From Eq. 31-39 ( $X_C = 1/\omega_d C$ ), with  $\omega_d = 2\pi f_d$ , we can write

$$X_C = \frac{1}{2\pi f_d C} = \frac{1}{(2\pi)(60.0 \text{ Hz})(15.0 \times 10^{-6} \text{ F})} = 177 \Omega.$$

Then Eq. 31-42 tells us that the current amplitude is

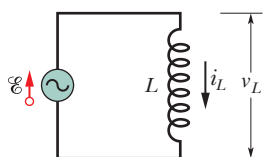
$$I_C = \frac{V_C}{X_C} = \frac{36.0 \text{ V}}{177 \Omega} = 0.203 \text{ A.} \quad (\text{Answer})$$

Substituting this and  $\omega_d = 2\pi f_d = 120\pi$  into Eq. 31-44, we have

$$i_C = (0.203 \text{ A}) \sin(120\pi t + \pi/2). \quad (\text{Answer})$$



Additional examples, video, and practice available at WileyPLUS



**Figure 31-12** An inductor is connected across an alternating-current generator.

### An Inductive Load

Figure 31-12 shows a circuit containing an inductance and a generator with the alternating emf of Eq. 31-28. Using the loop rule and proceeding as we did to obtain Eq. 31-30, we find that the potential difference across the inductance is

$$v_L = V_L \sin \omega_d t, \quad (31-45)$$

where  $V_L$  is the amplitude of  $v_L$ . From Eq. 30-35 ( $\mathcal{E}_L = -L di/dt$ ), we can write the potential difference across an inductance  $L$  in which the current is changing at the rate  $di_L/dt$  as

$$v_L = L \frac{di_L}{dt}. \quad (31-46)$$

If we combine Eqs. 31-45 and 31-46, we have

$$\frac{di_L}{dt} = \frac{V_L}{L} \sin \omega_d t. \quad (31-47)$$

Our concern, however, is with the current, so we integrate:

$$i_L = \int di_L = \frac{V_L}{L} \int \sin \omega_d t \, dt = -\left(\frac{V_L}{\omega_d L}\right) \cos \omega_d t. \quad (31-48)$$

We now modify this equation in two ways. First, for reasons of symmetry of notation, we introduce the quantity  $X_L$ , called the **inductive reactance** of an inductor, which is defined as

$$X_L = \omega_d L \quad (\text{inductive reactance}). \quad (31-49)$$

The value of  $X_L$  depends on the driving angular frequency  $\omega_d$ . The unit of the inductive time constant  $\tau_L$  indicates that the SI unit of  $X_L$  is the *ohm*, just as it is for  $X_C$  and for  $R$ .

Second, we replace  $-\cos \omega_d t$  in Eq. 31-48 with a phase-shifted sine:

$$-\cos \omega_d t = \sin(\omega_d t - 90^\circ).$$

You can verify this identity by shifting a sine curve  $90^\circ$  in the positive direction.

With these two changes, Eq. 31-48 becomes

$$i_L = \left(\frac{V_L}{X_L}\right) \sin(\omega_d t - 90^\circ). \quad (31-50)$$

From Eq. 31-29, we can also write this current in the inductance as

$$i_L = I_L \sin(\omega_d t - \phi), \quad (31-51)$$

where  $I_L$  is the amplitude of the current  $i_L$ . Comparing Eqs. 31-50 and 31-51, we see that for a purely inductive load the phase constant  $\phi$  for the current is  $+90^\circ$ . We also see that the voltage amplitude and current amplitude are related by

$$V_L = I_L X_L \quad (\text{inductor}). \quad (31-52)$$

Although we found this relation for the circuit of Fig. 31-12, it applies to any inductance in any ac circuit.

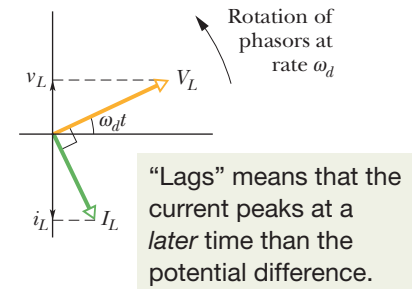
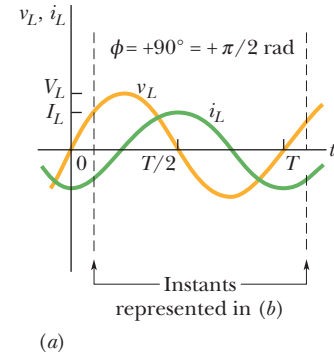
Comparison of Eqs. 31-45 and 31-50, or inspection of Fig. 31-13a, shows that the quantities  $i_L$  and  $v_L$  are  $90^\circ$  out of phase. In this case, however,  $i_L$  *lags*  $v_L$ ; that is, monitoring the current  $i_L$  and the potential difference  $v_L$  in the circuit of Fig. 31-12 shows that  $i_L$  reaches its maximum value *after*  $v_L$  does, by one-quarter cycle. The phasor diagram of Fig. 31-13b also contains this information. As the phasors rotate counterclockwise in the figure, the phasor labeled  $I_L$  does indeed lag that labeled  $V_L$ , and by an angle of  $90^\circ$ . Be sure to convince yourself that Fig. 31-13b represents Eqs. 31-45 and 31-50.



### Checkpoint 5

If we increase the driving frequency in a circuit with a purely capacitive load, do (a) amplitude  $V_C$  and (b) amplitude  $I_C$  increase, decrease, or remain the same? If, instead, the circuit has a purely inductive load, do (c) amplitude  $V_L$  and (d) amplitude  $I_L$  increase, decrease, or remain the same?

For an inductive load, the current lags the potential difference by  $90^\circ$ .



**Figure 31-13** (a) The current in the inductor lags the voltage by  $90^\circ$  ( $= \pi/2$  rad). (b) A phasor diagram shows the same thing.



## Problem-Solving Tactics

**Leading and Lagging in AC Circuits:** Table 31-2 summarizes the relations between the current  $i$  and the voltage  $v$  for each of the three kinds of circuit elements we have considered. When an applied alternating voltage produces an alternating current in these elements, the current is always in phase with the voltage across a resistor, always leads the voltage across a capacitor, and always lags the voltage across an inductor.

Many students remember these results with the mnemonic “*ELI* the *ICE* man.” *ELI* contains the letter  $L$

(for inductor), and in it the letter  $I$  (for current) comes *after* the letter  $E$  (for emf or voltage). Thus, for an inductor, the current *lags* (comes after) the voltage. Similarly, *ICE* (which contains a  $C$  for capacitor) means that the current *leads* (comes before) the voltage. You might also use the modified mnemonic “*ELI* positively is the *ICE* man” to remember that the phase constant  $\phi$  is positive for an inductor.

If you have difficulty in remembering whether  $X_C$  is equal to  $\omega_d C$  (wrong) or  $1/\omega_d C$  (right), try remembering that  $C$  is in the “cellar” — that is, in the denominator.

**Table 31-2** Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) $\phi$	Amplitude Relation
Resistor	$R$	$R$	In phase with $v_R$	$0^\circ (= 0 \text{ rad})$	$V_R = I_R R$
Capacitor	$C$	$X_C = 1/\omega_d C$	Leads $v_C$ by $90^\circ (= \pi/2 \text{ rad})$	$-90^\circ (= -\pi/2 \text{ rad})$	$V_C = I_C X_C$
Inductor	$L$	$X_L = \omega_d L$	Lags $v_L$ by $90^\circ (= \pi/2 \text{ rad})$	$+90^\circ (= +\pi/2 \text{ rad})$	$V_L = I_L X_L$

## Sample Problem 31.05 Purely inductive load: potential difference and current

In Fig. 31-12, inductance  $L$  is 230 mH and the sinusoidal alternating emf device operates at amplitude  $\mathcal{E}_m = 36.0 \text{ V}$  and frequency  $f_d = 60.0 \text{ Hz}$ .

(a) What are the potential difference  $v_L(t)$  across the inductance and the amplitude  $V_L$  of  $v_L(t)$ ?

### KEY IDEA

In a circuit with a purely inductive load, the potential difference  $v_L(t)$  across the inductance is always equal to the potential difference  $\mathcal{E}(t)$  across the emf device.

**Calculations:** Here we have  $v_L(t) = \mathcal{E}(t)$  and  $V_L = \mathcal{E}_m$ . Since  $\mathcal{E}_m$  is given, we know that

$$V_L = \mathcal{E}_m = 36.0 \text{ V.} \quad (\text{Answer})$$

To find  $v_L(t)$ , we use Eq. 31-28 to write

$$v_L(t) = \mathcal{E}(t) = \mathcal{E}_m \sin \omega_d t. \quad (31-53)$$

Then, substituting  $\mathcal{E}_m = 36.0 \text{ V}$  and  $\omega_d = 2\pi f_d = 120\pi$  into Eq. 31-53, we have

$$v_L = (36.0 \text{ V}) \sin(120\pi t). \quad (\text{Answer})$$

(b) What are the current  $i_L(t)$  in the circuit as a function of time and the amplitude  $I_L$  of  $i_L(t)$ ?

### KEY IDEA

In an ac circuit with a purely inductive load, the alternating current  $i_L(t)$  in the inductance lags the alternating potential difference  $v_L(t)$  by  $90^\circ$ . (In the mnemonic of the problem-solving tactic, this circuit is “positively an *ELI* circuit,” which tells us that the emf  $E$  leads the current  $I$  and that  $\phi$  is *positive*.)

**Calculations:** Because the phase constant  $\phi$  for the current is  $+90^\circ$ , or  $+\pi/2 \text{ rad}$ , we can write Eq. 31-29 as

$$i_L = I_L \sin(\omega_d t - \phi) = I_L \sin(\omega_d t - \pi/2). \quad (31-54)$$

We can find the amplitude  $I_L$  from Eq. 31-52 ( $V_L = I_L X_L$ ) if we first find the inductive reactance  $X_L$ . From Eq. 31-49 ( $X_L = \omega_d L$ ), with  $\omega_d = 2\pi f_d$ , we can write

$$\begin{aligned} X_L &= 2\pi f_d L = (2\pi)(60.0 \text{ Hz})(230 \times 10^{-3} \text{ H}) \\ &= 86.7 \Omega. \end{aligned}$$

Then Eq. 31-52 tells us that the current amplitude is

$$I_L = \frac{V_L}{X_L} = \frac{36.0 \text{ V}}{86.7 \Omega} = 0.415 \text{ A.} \quad (\text{Answer})$$

Substituting this and  $\omega_d = 2\pi f_d = 120\pi$  into Eq. 31-54, we have

$$i_L = (0.415 \text{ A}) \sin(120\pi t - \pi/2). \quad (\text{Answer})$$





## 31-4 THE SERIES *RLC* CIRCUIT

### Learning Objectives

After reading this module, you should be able to . . .

**31.32** Draw the schematic diagram of a series *RLC* circuit.

**31.33** Identify the conditions for a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit.

**31.34** For a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit, sketch graphs for voltage  $v(t)$  and current  $i(t)$  and sketch phasor diagrams, indicating leading, lagging, or resonance.

**31.35** Calculate impedance  $Z$ .

**31.36** Apply the relationship between current amplitude  $I$ , impedance  $Z$ , and emf amplitude  $\mathcal{E}_m$ .

**31.37** Apply the relationships between phase constant  $\phi$  and voltage amplitudes  $V_L$  and  $V_C$ , and also between

phase constant  $\phi$ , resistance  $R$ , and reactances  $X_L$  and  $X_C$ .

**31.38** Identify the values of the phase constant  $\phi$  corresponding to a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit.

**31.39** For resonance, apply the relationship between the driving angular frequency  $\omega_d$ , the natural angular frequency  $\omega$ , the inductance  $L$ , and the capacitance  $C$ .

**31.40** Sketch a graph of current amplitude versus the ratio  $\omega_d/\omega$ , identifying the portions corresponding to a mainly inductive circuit, a mainly capacitive circuit, and a resonant circuit and indicating what happens to the curve for an increase in the resistance.

### Key Ideas

- For a series *RLC* circuit with an external emf given by

$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t,$$

and current given by

$$i = I \sin(\omega_d t - \phi),$$

the current amplitude is given by

$$\begin{aligned} I &= \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} \\ &= \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad (\text{current amplitude}). \end{aligned}$$

- The phase constant is given by

$$\tan \phi = \frac{X_L - X_C}{R} \quad (\text{phase constant}).$$

- The impedance  $Z$  of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad (\text{impedance}).$$

- We relate the current amplitude and the impedance with

$$I = \mathcal{E}_m / Z.$$

- The current amplitude  $I$  is maximum ( $I = \mathcal{E}_m / R$ ) when the driving angular frequency  $\omega_d$  equals the natural angular frequency  $\omega$  of the circuit, a condition known as resonance. Then  $X_C = X_L$ ,  $\phi = 0$ , and the current is in phase with the emf.

## The Series *RLC* Circuit

We are now ready to apply the alternating emf of Eq. 31-28,

$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t \quad (\text{applied emf}), \quad (31-55)$$

to the full *RLC* circuit of Fig. 31-7. Because  $R$ ,  $L$ , and  $C$  are in series, the same current

$$i = I \sin(\omega_d t - \phi) \quad (31-56)$$

is driven in all three of them. We wish to find the current amplitude  $I$  and the phase constant  $\phi$  and to investigate how these quantities depend on the driving angular frequency  $\omega_d$ . The solution is simplified by the use of phasor diagrams as introduced for the three basic circuits of Module 31-3: capacitive load, inductive load, and resistive load. In particular we shall make use of how the voltage phasor is related to the current phasor for each of those basic circuits. We shall find that series *RLC* circuits can be separated into three types: mainly capacitive circuits, mainly inductive circuits, and circuits that are in resonance.

### The Current Amplitude

We start with Fig. 31-14*a*, which shows the phasor representing the current of Eq. 31-56 at an arbitrary time  $t$ . The length of the phasor is the current amplitude  $I$ , the projection of the phasor on the vertical axis is the current  $i$  at time  $t$ , and the angle of rotation of the phasor is the phase  $\omega_d t - \phi$  of the current at time  $t$ .

Figure 31-14*b* shows the phasors representing the voltages across  $R$ ,  $L$ , and  $C$  at the same time  $t$ . Each phasor is oriented relative to the angle of rotation of current phasor  $I$  in Fig. 31-14*a*, based on the information in Table 31-2:

**Resistor:** Here current and voltage are in phase; so the angle of rotation of voltage phasor  $V_R$  is the same as that of phasor  $I$ .

**Capacitor:** Here current leads voltage by  $90^\circ$ ; so the angle of rotation of voltage phasor  $V_C$  is  $90^\circ$  less than that of phasor  $I$ .

**Inductor:** Here current lags voltage by  $90^\circ$ ; so the angle of rotation of voltage phasor  $V_L$  is  $90^\circ$  greater than that of phasor  $I$ .

Figure 31-14*b* also shows the instantaneous voltages  $v_R$ ,  $v_C$ , and  $v_L$  across  $R$ ,  $C$ , and  $L$  at time  $t$ ; those voltages are the projections of the corresponding phasors on the vertical axis of the figure.

Figure 31-14*c* shows the phasor representing the applied emf of Eq. 31-55. The length of the phasor is the emf amplitude  $\mathcal{E}_m$ , the projection of the phasor on the vertical axis is the emf  $\mathcal{E}$  at time  $t$ , and the angle of rotation of the phasor is the phase  $\omega_d t$  of the emf at time  $t$ .

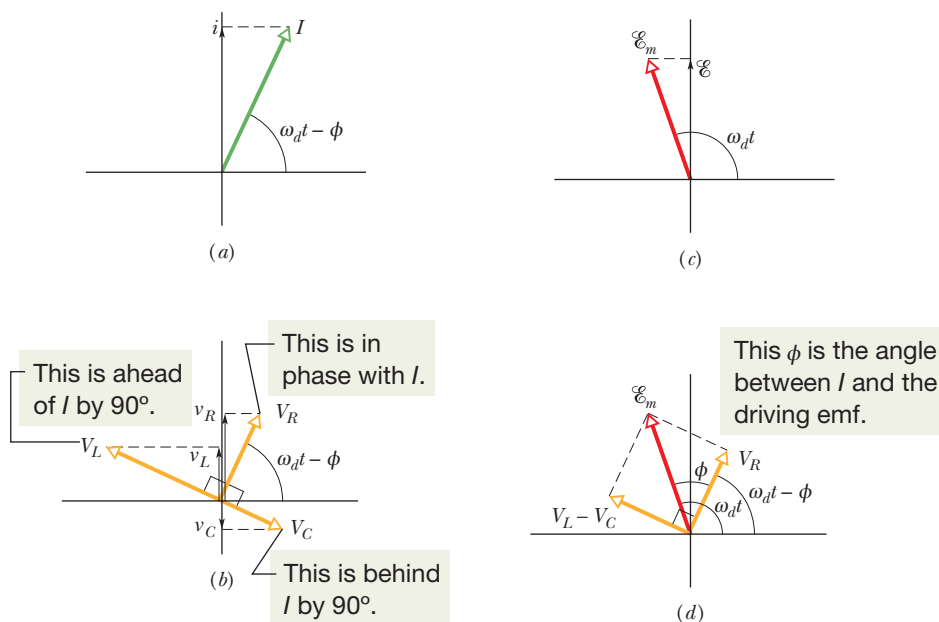
From the loop rule we know that at any instant the sum of the voltages  $v_R$ ,  $v_C$ , and  $v_L$  is equal to the applied emf  $\mathcal{E}$ :

$$\mathcal{E} = v_R + v_C + v_L. \quad (31-57)$$

Thus, at time  $t$  the projection  $\mathcal{E}$  in Fig. 31-14*c* is equal to the algebraic sum of the projections  $v_R$ ,  $v_C$ , and  $v_L$  in Fig. 31-14*b*. In fact, as the phasors rotate together, this equality always holds. This means that phasor  $\mathcal{E}_m$  in Fig. 31-14*c* must be equal to the vector sum of the three voltage phasors  $V_R$ ,  $V_C$ , and  $V_L$  in Fig. 31-14*b*.

That requirement is indicated in Fig. 31-14*d*, where phasor  $\mathcal{E}_m$  is drawn as the sum of phasors  $V_R$ ,  $V_L$ , and  $V_C$ . Because phasors  $V_L$  and  $V_C$  have opposite directions in the figure, we simplify the vector sum by first combining  $V_L$  and  $V_C$  to form the single phasor  $V_L - V_C$ . Then we combine that single phasor with  $V_R$  to find the net phasor. Again, the net phasor must coincide with phasor  $\mathcal{E}_m$ , as shown.

**Figure 31-14** (a) A phasor representing the alternating current in the driven  $RLC$  circuit of Fig. 31-7 at time  $t$ . The amplitude  $I$ , the instantaneous value  $i$ , and the phase  $(\omega_d t - \phi)$  are shown. (b) Phasors representing the voltages across the inductor, resistor, and capacitor, oriented with respect to the current phasor in (a). (c) A phasor representing the alternating emf that drives the current of (a). (d) The emf phasor is equal to the vector sum of the three voltage phasors of (b). Here, voltage phasors  $V_L$  and  $V_C$  have been added vectorially to yield their net phasor  $(V_L - V_C)$ .



Both triangles in Fig. 31-14*d* are right triangles. Applying the Pythagorean theorem to either one yields

$$\mathcal{E}_m^2 = V_R^2 + (V_L - V_C)^2. \quad (31-58)$$

From the voltage amplitude information displayed in the rightmost column of Table 31-2, we can rewrite this as

$$\mathcal{E}_m^2 = (IR)^2 + (IX_L - IX_C)^2, \quad (31-59)$$

and then rearrange it to the form

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}. \quad (31-60)$$

The denominator in Eq. 31-60 is called the **impedance**  $Z$  of the circuit for the driving angular frequency  $\omega_d$ :

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad (\text{impedance defined}). \quad (31-61)$$

We can then write Eq. 31-60 as

$$I = \frac{\mathcal{E}_m}{Z}. \quad (31-62)$$

If we substitute for  $X_C$  and  $X_L$  from Eqs. 31-39 and 31-49, we can write Eq. 31-60 more explicitly as

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad (\text{current amplitude}). \quad (31-63)$$

We have now accomplished half our goal: We have obtained an expression for the current amplitude  $I$  in terms of the sinusoidal driving emf and the circuit elements in a series *RLC* circuit.

The value of  $I$  depends on the difference between  $\omega_d L$  and  $1/\omega_d C$  in Eq. 31-63 or, equivalently, the difference between  $X_L$  and  $X_C$  in Eq. 31-60. In either equation, it does not matter which of the two quantities is greater because the difference is always squared.

The current that we have been describing in this module is the *steady-state current* that occurs after the alternating emf has been applied for some time. When the emf is first applied to a circuit, a brief *transient current* occurs. Its duration (before settling down into the steady-state current) is determined by the time constants  $\tau_L = L/R$  and  $\tau_C = RC$  as the inductive and capacitive elements “turn on.” This transient current can, for example, destroy a motor on start-up if it is not properly taken into account in the motor’s circuit design.

### The Phase Constant

From the right-hand phasor triangle in Fig. 31-14*d* and from Table 31-2 we can write

$$\tan \phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR}, \quad (31-64)$$

which gives us

$$\tan \phi = \frac{X_L - X_C}{R} \quad (\text{phase constant}). \quad (31-65)$$

This is the other half of our goal: an equation for the phase constant  $\phi$  in the sinusoidally driven series *RLC* circuit of Fig. 31-7. In essence, it gives us three

different results for the phase constant, depending on the relative values of the reactances  $X_L$  and  $X_C$ :

$X_L > X_C$ : The circuit is said to be *more inductive than capacitive*. Equation 31-65 tells us that  $\phi$  is positive for such a circuit, which means that phasor  $I$  rotates behind phasor  $\mathcal{E}_m$  (Fig. 31-15a). A plot of  $\mathcal{E}$  and  $i$  versus time is like that in Fig. 31-15b. (Figures 31-14c and d were drawn assuming  $X_L > X_C$ .)

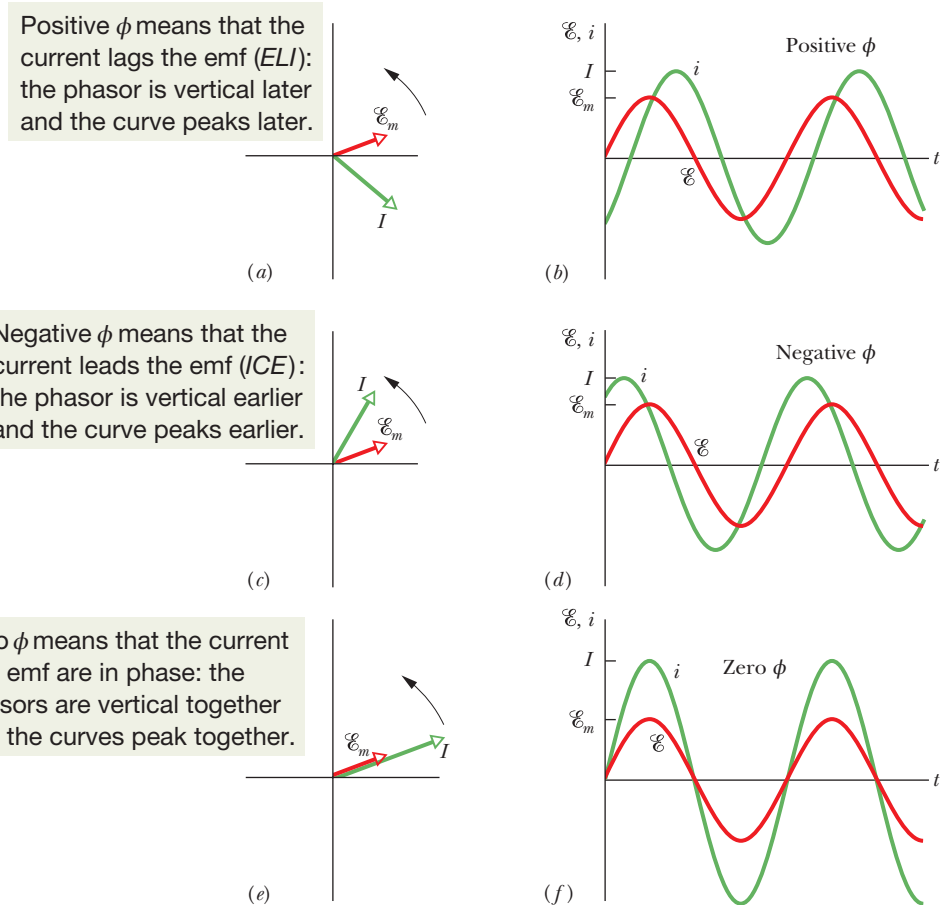
$X_C > X_L$ : The circuit is said to be *more capacitive than inductive*. Equation 31-65 tells us that  $\phi$  is negative for such a circuit, which means that phasor  $I$  rotates ahead of phasor  $\mathcal{E}_m$  (Fig. 31-15c). A plot of  $\mathcal{E}$  and  $i$  versus time is like that in Fig. 31-15d.

$X_C = X_L$ : The circuit is said to be in *resonance*, a state that is discussed next. Equation 31-65 tells us that  $\phi = 0^\circ$  for such a circuit, which means that phasors  $\mathcal{E}_m$  and  $I$  rotate together (Fig. 31-15e). A plot of  $\mathcal{E}$  and  $i$  versus time is like that in Fig. 31-15f.

As illustration, let us reconsider two extreme circuits: In the *purely inductive circuit* of Fig. 31-12, where  $X_L$  is nonzero and  $X_C = R = 0$ , Eq. 31-65 tells us that the circuit's phase constant is  $\phi = +90^\circ$  (the greatest value of  $\phi$ ), consistent with Fig. 31-13b. In the *purely capacitive circuit* of Fig. 31-10, where  $X_C$  is nonzero and  $X_L = R = 0$ , Eq. 31-65 tells us that the circuit's phase constant is  $\phi = -90^\circ$  (the least value of  $\phi$ ), consistent with Fig. 31-11b.

### Resonance

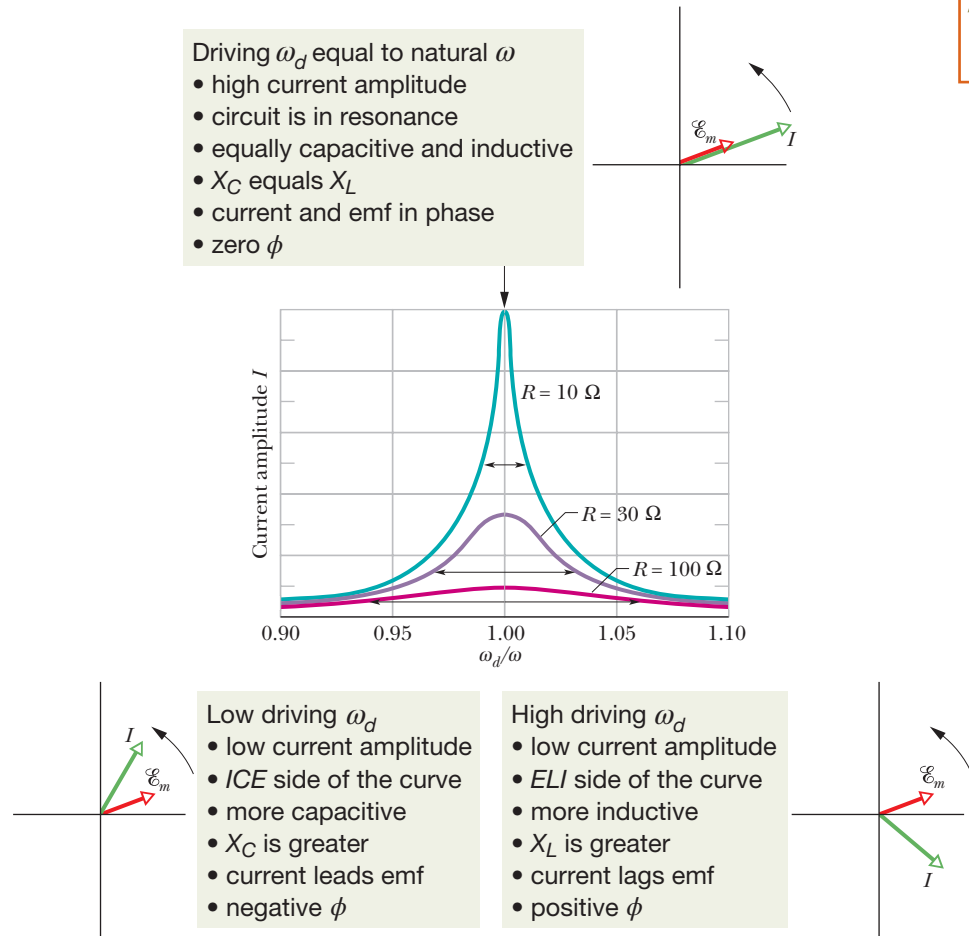
Equation 31-63 gives the current amplitude  $I$  in an  $RLC$  circuit as a function of the driving angular frequency  $\omega_d$  of the external alternating emf. For a given resistance  $R$ , that amplitude is a maximum when the quantity  $\omega_d L - 1/\omega_d C$  in the



**Figure 31-15** Phasor diagrams and graphs of the alternating emf  $\mathcal{E}$  and current  $i$  for the driven  $RLC$  circuit of Fig. 31-7. In the phasor diagram of (a) and the graph of (b), the current  $i$  lags the driving emf  $\mathcal{E}$  and its phase constant  $\phi$  is positive. In (c) and (d), the current  $i$  leads the driving emf  $\mathcal{E}$  and its phase constant  $\phi$  is negative. In (e) and (f), the current  $i$  is in phase with the driving emf  $\mathcal{E}$  and its phase constant  $\phi$  is zero.



**Figure 31-16** Resonance curves for the driven  $RLC$  circuit of Fig. 31-7 with  $L = 100 \mu\text{H}$ ,  $C = 100 \text{ pF}$ , and three values of  $R$ . The current amplitude  $I$  of the alternating current depends on how close the driving angular frequency  $\omega_d$  is to the natural angular frequency  $\omega$ . The horizontal arrow on each curve measures the curve's *half-width*, which is the width at the half-maximum level and is a measure of the sharpness of the resonance. To the left of  $\omega_d/\omega = 1.00$ , the circuit is mainly capacitive, with  $X_C > X_L$ ; to the right, it is mainly inductive, with  $X_L > X_C$ .



denominator is zero — that is, when

$$\omega_d L = \frac{1}{\omega_d C}$$

or

$$\omega_d = \frac{1}{\sqrt{LC}} \quad (\text{maximum } I). \quad (31-66)$$

Because the natural angular frequency  $\omega$  of the  $RLC$  circuit is also equal to  $1/\sqrt{LC}$ , the maximum value of  $I$  occurs when the driving angular frequency matches the natural angular frequency — that is, at resonance. Thus, in an  $RLC$  circuit, resonance and maximum current amplitude  $I$  occur when

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \quad (\text{resonance}). \quad (31-67)$$

**Resonance Curves.** Figure 31-16 shows three *resonance curves* for sinusoidally driven oscillations in three series  $RLC$  circuits differing only in  $R$ . Each curve peaks at its maximum current amplitude  $I$  when the ratio  $\omega_d/\omega$  is 1.00, but the maximum value of  $I$  decreases with increasing  $R$ . (The maximum  $I$  is always  $\mathcal{E}_m/R$ ; to see why, combine Eqs. 31-61 and 31-62.) In addition, the curves increase in width (measured in Fig. 31-16 at half the maximum value of  $I$ ) with increasing  $R$ .



To make physical sense of Fig. 31-16, consider how the reactances  $X_L$  and  $X_C$  change as we increase the driving angular frequency  $\omega_d$ , starting with a value much less than the natural frequency  $\omega$ . For small  $\omega_d$ , reactance  $X_L (= \omega_d L)$  is small and reactance  $X_C (= 1/\omega_d C)$  is large. Thus, the circuit is mainly capacitive and the impedance is dominated by the large  $X_C$ , which keeps the current low.

As we increase  $\omega_d$ , reactance  $X_C$  remains dominant but decreases while reactance  $X_L$  increases. The decrease in  $X_C$  decreases the impedance, allowing the current to increase, as we see on the left side of any resonance curve in Fig. 31-16. When the increasing  $X_L$  and the decreasing  $X_C$  reach equal values, the current is greatest and the circuit is in resonance, with  $\omega_d = \omega$ .

As we continue to increase  $\omega_d$ , the increasing reactance  $X_L$  becomes progressively more dominant over the decreasing reactance  $X_C$ . The impedance increases because of  $X_L$  and the current decreases, as on the right side of any resonance curve in Fig. 31-16. In summary, then: The low-angular-frequency side of a resonance curve is dominated by the capacitor's reactance, the high-angular-frequency side is dominated by the inductor's reactance, and resonance occurs in the middle.



### Checkpoint 6

Here are the capacitive reactance and inductive reactance, respectively, for three sinusoidally driven series  $RLC$  circuits: (1)  $50\ \Omega$ ,  $100\ \Omega$ ; (2)  $100\ \Omega$ ,  $50\ \Omega$ ; (3)  $50\ \Omega$ ,  $50\ \Omega$ .  
 (a) For each, does the current lead or lag the applied emf, or are the two in phase?  
 (b) Which circuit is in resonance?



### Sample Problem 31.06 Current amplitude, impedance, and phase constant

In Fig. 31-7, let  $R = 200\ \Omega$ ,  $C = 15.0\ \mu\text{F}$ ,  $L = 230\ \text{mH}$ ,  $f_d = 60.0\ \text{Hz}$ , and  $\mathcal{E}_m = 36.0\ \text{V}$ . (These parameters are those used in the earlier sample problems.)

(a) What is the current amplitude  $I$ ?

#### KEY IDEA

The current amplitude  $I$  depends on the amplitude  $\mathcal{E}_m$  of the driving emf and on the impedance  $Z$  of the circuit, according to Eq. 31-62 ( $I = \mathcal{E}_m/Z$ ).

**Calculations:** So, we need to find  $Z$ , which depends on resistance  $R$ , capacitive reactance  $X_C$ , and inductive reactance  $X_L$ . The circuit's resistance is the given resistance  $R$ . Its capacitive reactance is due to the given capacitance and, from an earlier sample problem,  $X_C = 177\ \Omega$ . Its inductive reactance is due to the given inductance and, from another sample problem,  $X_L = 86.7\ \Omega$ . Thus, the circuit's impedance is

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(200\ \Omega)^2 + (86.7\ \Omega - 177\ \Omega)^2} \\ &= 219\ \Omega. \end{aligned}$$

We then find

$$I = \frac{\mathcal{E}_m}{Z} = \frac{36.0\ \text{V}}{219\ \Omega} = 0.164\ \text{A}. \quad (\text{Answer})$$

(b) What is the phase constant  $\phi$  of the current in the circuit relative to the driving emf?

#### KEY IDEA

The phase constant depends on the inductive reactance, the capacitive reactance, and the resistance of the circuit, according to Eq. 31-65.

**Calculation:** Solving Eq. 31-65 for  $\phi$  leads to

$$\begin{aligned} \phi &= \tan^{-1} \frac{X_L - X_C}{R} = \tan^{-1} \frac{86.7\ \Omega - 177\ \Omega}{200\ \Omega} \\ &= -24.3^\circ = -0.424\ \text{rad}. \quad (\text{Answer}) \end{aligned}$$

The negative phase constant is consistent with the fact that the load is mainly capacitive; that is,  $X_C > X_L$ . In the common mnemonic for driven series  $RLC$  circuits, this circuit is an *ICE* circuit — the current *leads* the driving emf.



## 31-5 POWER IN ALTERNATING-CURRENT CIRCUITS

### Learning Objectives

After reading this module, you should be able to . . .

- 31.41** For the current, voltage, and emf in an ac circuit, apply the relationship between the rms values and the amplitudes.
- 31.42** For an alternating emf connected across a capacitor, an inductor, or a resistor, sketch graphs of the sinusoidal variation of the current and voltage and indicate the peak and rms values.
- 31.43** Apply the relationship between average power  $P_{\text{avg}}$ , rms current  $I_{\text{rms}}$ , and resistance  $R$ .
- 31.44** In a driven  $RLC$  circuit, calculate the power of each element.
- 31.45** For a driven  $RLC$  circuit in steady state, explain what happens to (a) the value of the average stored energy with time and (b) the energy that the generator puts into the circuit.
- 31.46** Apply the relationship between the power factor  $\cos \phi$ , the resistance  $R$ , and the impedance  $Z$ .
- 31.47** Apply the relationship between the average power  $P_{\text{avg}}$ , the rms emf  $\mathcal{E}_{\text{rms}}$ , the rms current  $I_{\text{rms}}$ , and the power factor  $\cos \phi$ .
- 31.48** Identify what power factor is required in order to maximize the rate at which energy is supplied to a resistive load.

### Key Ideas

● In a series  $RLC$  circuit, the average power  $P_{\text{avg}}$  of the generator is equal to the production rate of thermal energy in the resistor:

$$P_{\text{avg}} = I_{\text{rms}}^2 R = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos \phi.$$

● The abbreviation rms stands for root-mean-square; the rms quantities are related to the maximum quantities by  $I_{\text{rms}} = I/\sqrt{2}$ ,  $V_{\text{rms}} = V/\sqrt{2}$ , and  $\mathcal{E}_{\text{rms}} = \mathcal{E}_m/\sqrt{2}$ . The term  $\cos \phi$  is called the power factor of the circuit.

## Power in Alternating-Current Circuits

In the  $RLC$  circuit of Fig. 31-7, the source of energy is the alternating-current generator. Some of the energy that it provides is stored in the electric field in the capacitor, some is stored in the magnetic field in the inductor, and some is dissipated as thermal energy in the resistor. In steady-state operation, the average stored energy remains constant. The net transfer of energy is thus from the generator to the resistor, where energy is dissipated.

The instantaneous rate at which energy is dissipated in the resistor can be written, with the help of Eqs. 26-27 and 31-29, as

$$P = i^2 R = [I \sin(\omega_d t - \phi)]^2 R = I^2 R \sin^2(\omega_d t - \phi). \quad (31-68)$$

The *average* rate at which energy is dissipated in the resistor, however, is the average of Eq. 31-68 over time. Over one complete cycle, the average value of  $\sin \theta$ , where  $\theta$  is any variable, is zero (Fig. 31-17a) but the average value of  $\sin^2 \theta$  is  $\frac{1}{2}$  (Fig. 31-17b). (Note in Fig. 31-17b how the shaded areas under the curve but above the horizontal line marked  $+\frac{1}{2}$  exactly fill in the unshaded spaces below that line.) Thus, we can write, from Eq. 31-68,

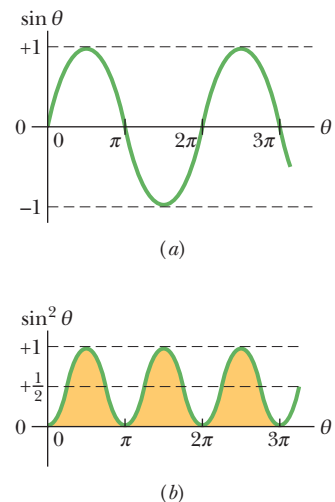
$$P_{\text{avg}} = \frac{I^2 R}{2} = \left( \frac{I}{\sqrt{2}} \right)^2 R. \quad (31-69)$$

The quantity  $I/\sqrt{2}$  is called the **root-mean-square**, or **rms**, value of the current  $i$ :

$$I_{\text{rms}} = \frac{I}{\sqrt{2}} \quad (\text{rms current}). \quad (31-70)$$

We can now rewrite Eq. 31-69 as

$$P_{\text{avg}} = I_{\text{rms}}^2 R \quad (\text{average power}). \quad (31-71)$$



**Figure 31-17** (a) A plot of  $\sin \theta$  versus  $\theta$ . The average value over one cycle is zero. (b) A plot of  $\sin^2 \theta$  versus  $\theta$ . The average value over one cycle is  $\frac{1}{2}$ .

Equation 31-71 has the same mathematical form as Eq. 26-27 ( $P = i^2 R$ ); the message here is that if we switch to the rms current, we can compute the average rate of energy dissipation for alternating-current circuits just as for direct-current circuits.

We can also define rms values of voltages and emfs for alternating-current circuits:

$$V_{\text{rms}} = \frac{V}{\sqrt{2}} \quad \text{and} \quad \mathcal{E}_{\text{rms}} = \frac{\mathcal{E}_m}{\sqrt{2}} \quad (\text{rms voltage; rms emf}). \quad (31-72)$$

Alternating-current instruments, such as ammeters and voltmeters, are usually calibrated to read  $I_{\text{rms}}$ ,  $V_{\text{rms}}$ , and  $\mathcal{E}_{\text{rms}}$ . Thus, if you plug an alternating-current voltmeter into a household electrical outlet and it reads 120 V, that is an rms voltage. The *maximum* value of the potential difference at the outlet is  $\sqrt{2} \times (120 \text{ V})$ , or 170 V. Generally scientists and engineers report rms values instead of maximum values.

Because the proportionality factor  $1/\sqrt{2}$  in Eqs. 31-70 and 31-72 is the same for all three variables, we can write Eqs. 31-62 and 31-60 as

$$I_{\text{rms}} = \frac{\mathcal{E}_{\text{rms}}}{Z} = \frac{\mathcal{E}_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}}, \quad (31-73)$$

and, indeed, this is the form that we almost always use.

We can use the relationship  $I_{\text{rms}} = \mathcal{E}_{\text{rms}}/Z$  to recast Eq. 31-71 in a useful equivalent way. We write

$$P_{\text{avg}} = \frac{\mathcal{E}_{\text{rms}}}{Z} I_{\text{rms}} R = \mathcal{E}_{\text{rms}} I_{\text{rms}} \frac{R}{Z}. \quad (31-74)$$

From Fig. 31-14d, Table 31-2, and Eq. 31-62, however, we see that  $R/Z$  is just the cosine of the phase constant  $\phi$ :

$$\cos \phi = \frac{V_R}{\mathcal{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}. \quad (31-75)$$

Equation 31-74 then becomes

$$P_{\text{avg}} = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos \phi \quad (\text{average power}), \quad (31-76)$$

in which the term  $\cos \phi$  is called the **power factor**. Because  $\cos \phi = \cos(-\phi)$ , Eq. 31-76 is independent of the sign of the phase constant  $\phi$ .

To maximize the rate at which energy is supplied to a resistive load in an *RLC* circuit, we should keep the power factor  $\cos \phi$  as close to unity as possible. This is equivalent to keeping the phase constant  $\phi$  in Eq. 31-29 as close to zero as possible. If, for example, the circuit is highly inductive, it can be made less so by putting more capacitance in the circuit, connected in series. (Recall that putting an additional capacitance into a series of capacitances decreases the equivalent capacitance  $C_{\text{eq}}$  of the series.) Thus, the resulting decrease in  $C_{\text{eq}}$  in the circuit reduces the phase constant and increases the power factor in Eq. 31-76. Power companies place series-connected capacitors throughout their transmission systems to get these results.



### Checkpoint 7

- (a) If the current in a sinusoidally driven series *RLC* circuit leads the emf, would we increase or decrease the capacitance to increase the rate at which energy is supplied to the resistance? (b) Would this change bring the resonant angular frequency of the circuit closer to the angular frequency of the emf or put it farther away?

**Sample Problem 31.07** Driven *RLC* circuit: power factor and average power

A series *RLC* circuit, driven with  $\mathcal{E}_{\text{rms}} = 120 \text{ V}$  at frequency  $f_d = 60.0 \text{ Hz}$ , contains a resistance  $R = 200 \Omega$ , an inductance with inductive reactance  $X_L = 80.0 \Omega$ , and a capacitance with capacitive reactance  $X_C = 150 \Omega$ .

(a) What are the power factor  $\cos \phi$  and phase constant  $\phi$  of the circuit?

**KEY IDEA**

The power factor  $\cos \phi$  can be found from the resistance  $R$  and impedance  $Z$  via Eq. 31-75 ( $\cos \phi = R/Z$ ).

**Calculations:** To calculate  $Z$ , we use Eq. 31-61:

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(200 \Omega)^2 + (80.0 \Omega - 150 \Omega)^2} = 211.90 \Omega. \end{aligned}$$

Equation 31-75 then gives us

$$\cos \phi = \frac{R}{Z} = \frac{200 \Omega}{211.90 \Omega} = 0.9438 \approx 0.944. \quad (\text{Answer})$$

Taking the inverse cosine then yields

$$\phi = \cos^{-1} 0.944 = \pm 19.3^\circ.$$

The inverse cosine on a calculator gives only the positive answer here, but both  $+19.3^\circ$  and  $-19.3^\circ$  have a cosine of 0.944. To determine which sign is correct, we must consider whether the current leads or lags the driving emf. Because  $X_C > X_L$ , this circuit is mainly capacitive, with the current leading the emf. Thus,  $\phi$  must be negative:

$$\phi = -19.3^\circ. \quad (\text{Answer})$$

We could, instead, have found  $\phi$  with Eq. 31-65. A calculator would then have given us the answer with the minus sign.

(b) What is the average rate  $P_{\text{avg}}$  at which energy is dissipated in the resistance?

**KEY IDEAS**

There are two ways and two ideas to use: (1) Because the circuit is assumed to be in steady-state operation, the rate at which energy is dissipated in the resistance is equal to the rate at which energy is supplied to the circuit, as given by Eq. 31-76 ( $P_{\text{avg}} = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos \phi$ ). (2) The rate at which energy is dissipated in a resistance  $R$  depends on the square of the rms current  $I_{\text{rms}}$  through it, according to Eq. 31-71 ( $P_{\text{avg}} = I_{\text{rms}}^2 R$ ).

**First way:** We are given the rms driving emf  $\mathcal{E}_{\text{rms}}$  and we already know  $\cos \phi$  from part (a). The rms current  $I_{\text{rms}}$  is

determined by the rms value of the driving emf and the circuit's impedance  $Z$  (which we know), according to Eq. 31-73:

$$I_{\text{rms}} = \frac{\mathcal{E}_{\text{rms}}}{Z}.$$

Substituting this into Eq. 31-76 then leads to

$$\begin{aligned} P_{\text{avg}} &= \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos \phi = \frac{\mathcal{E}_{\text{rms}}^2}{Z} \cos \phi \\ &= \frac{(120 \text{ V})^2}{211.90 \Omega} (0.9438) = 64.1 \text{ W}. \quad (\text{Answer}) \end{aligned}$$

**Second way:** Instead, we can write

$$\begin{aligned} P_{\text{avg}} &= I_{\text{rms}}^2 R = \frac{\mathcal{E}_{\text{rms}}^2 Z^2}{R} \\ &= \frac{(120 \text{ V})^2}{(211.90 \Omega)^2} (200 \Omega) = 64.1 \text{ W}. \quad (\text{Answer}) \end{aligned}$$

(c) What new capacitance  $C_{\text{new}}$  is needed to maximize  $P_{\text{avg}}$  if the other parameters of the circuit are not changed?

**KEY IDEAS**

(1) The average rate  $P_{\text{avg}}$  at which energy is supplied and dissipated is maximized if the circuit is brought into resonance with the driving emf. (2) Resonance occurs when  $X_C = X_L$ .

**Calculations:** From the given data, we have  $X_C > X_L$ . Thus, we must decrease  $X_C$  to reach resonance. From Eq. 31-39 ( $X_C = 1/\omega_d C$ ), we see that this means we must increase  $C$  to the new value  $C_{\text{new}}$ .

Using Eq. 31-39, we can write the resonance condition  $X_C = X_L$  as

$$\frac{1}{\omega_d C_{\text{new}}} = X_L.$$

Substituting  $2\pi f_d$  for  $\omega_d$  (because we are given  $f_d$  and not  $\omega_d$ ) and then solving for  $C_{\text{new}}$ , we find

$$\begin{aligned} C_{\text{new}} &= \frac{1}{2\pi f_d X_L} = \frac{1}{(2\pi)(60 \text{ Hz})(80.0 \Omega)} \\ &= 3.32 \times 10^{-5} \text{ F} = 33.2 \mu\text{F}. \quad (\text{Answer}) \end{aligned}$$

Following the procedure of part (b), you can show that with  $C_{\text{new}}$ , the average power of energy dissipation  $P_{\text{avg}}$  would then be at its maximum value of

$$P_{\text{avg, max}} = 72.0 \text{ W}.$$



## 31-6 TRANSFORMERS

### Learning Objectives

After reading this module, you should be able to . . .

- 31.49** For power transmission lines, identify why the transmission should be at low current and high voltage.
- 31.50** Identify the role of transformers at the two ends of a transmission line.
- 31.51** Calculate the energy dissipation in a transmission line.
- 31.52** Identify a transformer's primary and secondary.
- 31.53** Apply the relationship between the voltage and number of turns on the two sides of a transformer.
- 31.54** Distinguish between a step-down transformer and a step-up transformer.

- 31.55** Apply the relationship between the current and number of turns on the two sides of a transformer.
- 31.56** Apply the relationship between the power into and out of an ideal transformer.
- 31.57** Identify the equivalent resistance as seen from the primary side of a transformer.
- 31.58** Apply the relationship between the equivalent resistance and the actual resistance.
- 31.59** Explain the role of a transformer in impedance matching.

### Key Ideas

- A transformer (assumed to be ideal) is an iron core on which are wound a primary coil of  $N_p$  turns and a secondary coil of  $N_s$  turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_s = V_p \frac{N_s}{N_p} \quad (\text{transformation of voltage}).$$

- The currents through the coils are related by

$$I_s = I_p \frac{N_p}{N_s} \quad (\text{transformation of currents}).$$

- The equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{\text{eq}} = \left( \frac{N_p}{N_s} \right)^2 R,$$

where  $R$  is the resistive load in the secondary circuit. The ratio  $N_p/N_s$  is called the transformer's turns ratio.

## Transformers

### Energy Transmission Requirements

When an ac circuit has only a resistive load, the power factor in Eq. 31-76 is  $\cos 0^\circ = 1$  and the applied rms emf  $\mathcal{E}_{\text{rms}}$  is equal to the rms voltage  $V_{\text{rms}}$  across the load. Thus, with an rms current  $I_{\text{rms}}$  in the load, energy is supplied and dissipated at the average rate of

$$P_{\text{avg}} = \mathcal{E}I = IV. \quad (31-77)$$

(In Eq. 31-77 and the rest of this module, we follow conventional practice and drop the subscripts identifying rms quantities. Engineers and scientists assume that all time-varying currents and voltages are reported as rms values; that is what the meters read.) Equation 31-77 tells us that, to satisfy a given power requirement, we have a range of choices for  $I$  and  $V$ , provided only that the product  $IV$  is as required.

In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory). Nobody wants an electric toaster to operate at, say, 10 kV. However, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize  $I^2R$  losses (often called *ohmic losses*) in the transmission line.

As an example, consider the 735 kV line used to transmit electrical energy from the La Grande 2 hydroelectric plant in Quebec to Montreal, 1000 km away. Suppose that the current is 500 A and the power factor is close to unity. Then from Eq. 31-77, energy is supplied at the average rate

$$P_{\text{avg}} = \mathcal{E}I = (7.35 \times 10^5 \text{ V})(500 \text{ A}) = 368 \text{ MW}.$$



The resistance of the transmission line is about  $0.220\ \Omega/\text{km}$ ; thus, there is a total resistance of about  $220\ \Omega$  for the 1000 km stretch. Energy is dissipated due to that resistance at a rate of about

$$P_{\text{avg}} = I^2 R = (500\ \text{A})^2 (220\ \Omega) = 55.0\ \text{MW},$$

which is nearly 15% of the supply rate.

Imagine what would happen if we doubled the current and halved the voltage. Energy would be supplied by the plant at the same average rate of 368 MW as previously, but now energy would be dissipated at the rate of about

$$P_{\text{avg}} = I^2 R = (1000\ \text{A})^2 (220\ \Omega) = 220\ \text{MW},$$

which is *almost 60% of the supply rate*. Hence the general energy transmission rule: Transmit at the highest possible voltage and the lowest possible current.

### The Ideal Transformer

The transmission rule leads to a fundamental mismatch between the requirement for efficient high-voltage transmission and the need for safe low-voltage generation and consumption. We need a device with which we can raise (for transmission) and lower (for use) the ac voltage in a circuit, keeping the product current  $\times$  voltage essentially constant. The **transformer** is such a device. It has no moving parts, operates by Faraday's law of induction, and has no simple direct-current counterpart.

The *ideal transformer* in Fig. 31-18 consists of two coils, with different numbers of turns, wound around an iron core. (The coils are insulated from the core.) In use, the primary winding, of  $N_p$  turns, is connected to an alternating-current generator whose emf  $\mathcal{E}$  at any time  $t$  is given by

$$\mathcal{E} = \mathcal{E}_m \sin \omega t. \quad (31-78)$$

The secondary winding, of  $N_s$  turns, is connected to load resistance  $R$ , but its circuit is an open circuit as long as switch  $S$  is open (which we assume for the present). Thus, there can be no current through the secondary coil. We assume further for this ideal transformer that the resistances of the primary and secondary windings are negligible. Well-designed, high-capacity transformers can have energy losses as low as 1%; so our assumptions are reasonable.

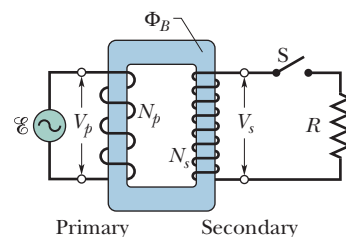
For the assumed conditions, the primary winding (or *primary*) is a pure inductance and the primary circuit is like that in Fig. 31-12. Thus, the (very small) primary current, also called the *magnetizing current*  $I_{\text{mag}}$ , lags the primary voltage  $V_p$  by  $90^\circ$ ; the primary's power factor ( $= \cos \phi$  in Eq. 31-76) is zero; so no power is delivered from the generator to the transformer.

However, the small sinusoidally changing primary current  $I_{\text{mag}}$  produces a sinusoidally changing magnetic flux  $\Phi_B$  in the iron core. The core acts to strengthen the flux and to bring it through the secondary winding (or *secondary*). Because  $\Phi_B$  varies, it induces an emf  $\mathcal{E}_{\text{turn}}$  ( $= d\Phi_B/dt$ ) in each turn of the secondary. In fact, this emf per turn  $\mathcal{E}_{\text{turn}}$  is the same in the primary and the secondary. Across the primary, the voltage  $V_p$  is the product of  $\mathcal{E}_{\text{turn}}$  and the number of turns  $N_p$ ; that is,  $V_p = \mathcal{E}_{\text{turn}} N_p$ . Similarly, across the secondary the voltage is  $V_s = \mathcal{E}_{\text{turn}} N_s$ . Thus, we can write

$$\mathcal{E}_{\text{turn}} = \frac{V_p}{N_p} = \frac{V_s}{N_s},$$

$$\text{or} \quad V_s = V_p \frac{N_s}{N_p} \quad (\text{transformation of voltage}). \quad (31-79)$$

If  $N_s > N_p$ , the device is a *step-up transformer* because it steps the primary's voltage  $V_p$  up to a higher voltage  $V_s$ . Similarly, if  $N_s < N_p$ , it is a *step-down transformer*.



**Figure 31-18** An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load  $R$  when switch  $S$  is closed.

With switch S open, no energy is transferred from the generator to the rest of the circuit, but when we close S to connect the secondary to the resistive load  $R$ , energy *is* transferred. (In general, the load would also contain inductive and capacitive elements, but here we consider just resistance  $R$ .) Here is the process:

1. An alternating current  $I_s$  appears in the secondary circuit, with corresponding energy dissipation rate  $I_s^2 R (= V_s^2/R)$  in the resistive load.
2. This current produces its own alternating magnetic flux in the iron core, and this flux induces an opposing emf in the primary windings.
3. The voltage  $V_p$  of the primary, however, cannot change in response to this opposing emf because it must always be equal to the emf  $\mathcal{E}$  that is provided by the generator; closing switch S cannot change this fact.
4. To maintain  $V_p$ , the generator now produces (in addition to  $I_{\text{mag}}$ ) an alternating current  $I_p$  in the primary circuit; the magnitude and phase constant of  $I_p$  are just those required for the emf induced by  $I_p$  in the primary to exactly cancel the emf induced there by  $I_s$ . Because the phase constant of  $I_p$  is not  $90^\circ$  like that of  $I_{\text{mag}}$ , this current  $I_p$  can transfer energy to the primary.

**Energy Transfers.** We want to relate  $I_s$  to  $I_p$ . However, rather than analyze the foregoing complex process in detail, let us just apply the principle of conservation of energy. The rate at which the generator transfers energy to the primary is equal to  $I_p V_p$ . The rate at which the primary then transfers energy to the secondary (via the alternating magnetic field linking the two coils) is  $I_s V_s$ . Because we assume that no energy is lost along the way, conservation of energy requires that

$$I_p V_p = I_s V_s.$$

Substituting for  $V_s$  from Eq. 31-79, we find that

$$I_s = I_p \frac{N_p}{N_s} \quad (\text{transformation of currents}). \quad (31-80)$$

This equation tells us that the current  $I_s$  in the secondary can differ from the current  $I_p$  in the primary, depending on the *turns ratio*  $N_p/N_s$ .

Current  $I_p$  appears in the primary circuit because of the resistive load  $R$  in the secondary circuit. To find  $I_p$ , we substitute  $I_s = V_s/R$  into Eq. 31-80 and then we substitute for  $V_s$  from Eq. 31-79. We find

$$I_p = \frac{1}{R} \left( \frac{N_s}{N_p} \right)^2 V_p. \quad (31-81)$$

This equation has the form  $I_p = V_p/R_{\text{eq}}$ , where equivalent resistance  $R_{\text{eq}}$  is

$$R_{\text{eq}} = \left( \frac{N_p}{N_s} \right)^2 R. \quad (31-82)$$

This  $R_{\text{eq}}$  is the value of the load resistance as “seen” by the generator; the generator produces the current  $I_p$  and voltage  $V_p$  as if the generator were connected to a resistance  $R_{\text{eq}}$ .

### Impedance Matching

Equation 31-82 suggests still another function for the transformer. For maximum transfer of energy from an emf device to a resistive load, the resistance of the emf device must equal the resistance of the load. The same relation holds for ac circuits except that the *impedance* (rather than just the resistance) of the generator must equal that of the load. Often this condition is not met. For example, in a music-playing system, the amplifier has high impedance and the speaker set has low impedance. We can match the impedances of the two devices by coupling them through a transformer that has a suitable turns ratio  $N_p/N_s$ .



### Checkpoint 8

An alternating-current emf device in a certain circuit has a smaller resistance than that of the resistive load in the circuit; to increase the transfer of energy from the device to the load, a transformer will be connected between the two. (a) Should  $N_s$  be greater than or less than  $N_p$ ? (b) Will that make it a step-up or step-down transformer?

### Sample Problem 31.08 Transformer: turns ratio, average power, rms currents

A transformer on a utility pole operates at  $V_p = 8.5$  kV on the primary side and supplies electrical energy to a number of nearby houses at  $V_s = 120$  V, both quantities being rms values. Assume an ideal step-down transformer, a purely resistive load, and a power factor of unity.

(a) What is the turns ratio  $N_p/N_s$  of the transformer?

#### KEY IDEA

The turns ratio  $N_p/N_s$  is related to the (given) rms primary and secondary voltages via Eq. 31-79 ( $V_s = V_p N_s/N_p$ ).

**Calculation:** We can write Eq. 31-79 as

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}. \quad (31-83)$$

(Note that the right side of this equation is the *inverse* of the turns ratio.) Inverting both sides of Eq. 31-83 gives us

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5 \times 10^3 \text{ V}}{120 \text{ V}} = 70.83 \approx 71. \quad (\text{Answer})$$

(b) The average rate of energy consumption (or dissipation) in the houses served by the transformer is 78 kW. What are the rms currents in the primary and secondary of the transformer?

#### KEY IDEA

For a purely resistive load, the power factor  $\cos \phi$  is unity; thus, the average rate at which energy is supplied and dissipated is given by Eq. 31-77 ( $P_{\text{avg}} = \mathcal{E}I = IV$ ).

**Calculations:** In the primary circuit, with  $V_p = 8.5$  kV,

Eq. 31-77 yields

$$I_p = \frac{P_{\text{avg}}}{V_p} = \frac{78 \times 10^3 \text{ W}}{8.5 \times 10^3 \text{ V}} = 9.176 \text{ A} \approx 9.2 \text{ A}. \quad (\text{Answer})$$

Similarly, in the secondary circuit,

$$I_s = \frac{P_{\text{avg}}}{V_s} = \frac{78 \times 10^3 \text{ W}}{120 \text{ V}} = 650 \text{ A}. \quad (\text{Answer})$$

You can check that  $I_s = I_p(N_p/N_s)$  as required by Eq. 31-80.

(c) What is the resistive load  $R_s$  in the secondary circuit? What is the corresponding resistive load  $R_p$  in the primary circuit?

**One way:** We can use  $V = IR$  to relate the resistive load to the rms voltage and current. For the secondary circuit, we find

$$R_s = \frac{V_s}{I_s} = \frac{120 \text{ V}}{650 \text{ A}} = 0.1846 \Omega \approx 0.18 \Omega. \quad (\text{Answer})$$

Similarly, for the primary circuit we find

$$R_p = \frac{V_p}{I_p} = \frac{8.5 \times 10^3 \text{ V}}{9.176 \text{ A}} = 926 \Omega \approx 930 \Omega. \quad (\text{Answer})$$

**Second way:** We use the fact that  $R_p$  equals the equivalent resistive load “seen” from the primary side of the transformer, which is a resistance modified by the turns ratio and given by Eq. 31-82 ( $R_{\text{eq}} = (N_p/N_s)^2 R$ ). If we substitute  $R_p$  for  $R_{\text{eq}}$  and  $R_s$  for  $R$ , that equation yields

$$\begin{aligned} R_p &= \left( \frac{N_p}{N_s} \right)^2 R_s = (70.83)^2 (0.1846 \Omega) \\ &= 926 \Omega \approx 930 \Omega. \end{aligned} \quad (\text{Answer})$$



Additional examples, video, and practice available at WileyPLUS



## Review & Summary

**LC Energy Transfers** In an oscillating  $LC$  circuit, energy is shuttled periodically between the electric field of the capacitor and the magnetic field of the inductor; instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2}, \quad (31-1, 31-2)$$

where  $q$  is the instantaneous charge on the capacitor and  $i$  is the

instantaneous current through the inductor. The total energy  $U (= U_E + U_B)$  remains constant.

**LC Charge and Current Oscillations** The principle of conservation of energy leads to

$$L \frac{d^2 q}{dt^2} + \frac{1}{C} q = 0 \quad (LC \text{ oscillations}) \quad (31-11)$$

as the differential equation of  $LC$  oscillations (with no resistance). The solution of Eq. 31-11 is

$$q = Q \cos(\omega t + \phi) \quad (\text{charge}), \quad (31-12)$$

in which  $Q$  is the *charge amplitude* (maximum charge on the capacitor) and the angular frequency  $\omega$  of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}. \quad (31-4)$$

The phase constant  $\phi$  in Eq. 31-12 is determined by the initial conditions (at  $t = 0$ ) of the system.

The current  $i$  in the system at any time  $t$  is

$$i = -\omega Q \sin(\omega t + \phi) \quad (\text{current}), \quad (31-13)$$

in which  $\omega Q$  is the *current amplitude*  $I$ .

**Damped Oscillations** Oscillations in an  $LC$  circuit are damped when a dissipative element  $R$  is also present in the circuit. Then

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0 \quad (RLC \text{ circuit}). \quad (31-24)$$

The solution of this differential equation is

$$q = Q e^{-Rt/2L} \cos(\omega' t + \phi), \quad (31-25)$$

$$\text{where} \quad \omega' = \sqrt{\omega^2 - (R/2L)^2}. \quad (31-26)$$

We consider only situations with small  $R$  and thus small damping; then  $\omega' \approx \omega$ .

**Alternating Currents; Forced Oscillations** A series  $RLC$  circuit may be set into *forced oscillation* at a *driving angular frequency*  $\omega_d$  by an external alternating emf

$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t. \quad (31-28)$$

The current driven in the circuit is

$$i = I \sin(\omega_d t - \phi), \quad (31-29)$$

where  $\phi$  is the phase constant of the current.

**Resonance** The current amplitude  $I$  in a series  $RLC$  circuit driven by a sinusoidal external emf is a maximum ( $I = \mathcal{E}_m/R$ ) when the driving angular frequency  $\omega_d$  equals the natural angular frequency  $\omega$  of the circuit (that is, at *resonance*). Then  $X_C = X_L$ ,  $\phi = 0$ , and the current is in phase with the emf.

**Single Circuit Elements** The alternating potential difference across a resistor has amplitude  $V_R = IR$ ; the current is in phase with the potential difference.

For a *capacitor*,  $V_C = IX_C$ , in which  $X_C = 1/\omega_d C$  is the **capacitive reactance**; the current here leads the potential difference by  $90^\circ$  ( $\phi = -90^\circ = -\pi/2$  rad).

For an *inductor*,  $V_L = IX_L$ , in which  $X_L = \omega_d L$  is the **inductive reactance**; the current here lags the potential difference by  $90^\circ$  ( $\phi = +90^\circ = +\pi/2$  rad).

**Series RLC Circuits** For a series  $RLC$  circuit with an alternating external emf given by Eq. 31-28 and a resulting alternating current given by Eq. 31-29,

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} \\ = \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad (\text{current amplitude}) \quad (31-60, 31-63)$$

$$\text{and} \quad \tan \phi = \frac{X_L - X_C}{R} \quad (\text{phase constant}). \quad (31-65)$$

Defining the impedance  $Z$  of the circuit as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad (\text{impedance}) \quad (31-61)$$

allows us to write Eq. 31-60 as  $I = \mathcal{E}_m/Z$ .

**Power** In a series  $RLC$  circuit, the **average power**  $P_{\text{avg}}$  of the generator is equal to the production rate of thermal energy in the resistor:

$$P_{\text{avg}} = I_{\text{rms}}^2 R = \mathcal{E}_{\text{rms}} I_{\text{rms}} \cos \phi. \quad (31-71, 31-76)$$

Here rms stands for **root-mean-square**; the rms quantities are related to the maximum quantities by  $I_{\text{rms}} = I/\sqrt{2}$ ,  $V_{\text{rms}} = V/\sqrt{2}$ , and  $\mathcal{E}_{\text{rms}} = \mathcal{E}_m/\sqrt{2}$ . The term  $\cos \phi$  is called the **power factor** of the circuit.

**Transformers** A *transformer* (assumed to be ideal) is an iron core on which are wound a primary coil of  $N_p$  turns and a secondary coil of  $N_s$  turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_s = V_p \frac{N_s}{N_p} \quad (\text{transformation of voltage}). \quad (31-79)$$

The currents through the coils are related by

$$I_s = I_p \frac{N_p}{N_s} \quad (\text{transformation of currents}), \quad (31-80)$$

and the equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{\text{eq}} = \left( \frac{N_p}{N_s} \right)^2 R, \quad (31-82)$$

where  $R$  is the resistive load in the secondary circuit. The ratio  $N_p/N_s$  is called the transformer's *turns ratio*.

## Questions

**1** Figure 31-19 shows three oscillating  $LC$  circuits with identical inductors and capacitors. At a particular time, the charges on the capacitor plates (and thus the electric fields between the plates) are all at their maximum values. Rank the circuits according to the time taken to fully discharge the capacitors during the oscillations, greatest first.

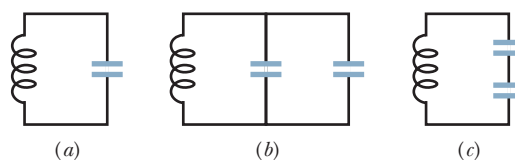


Figure 31-19 Question 1.

**2** Figure 31-20 shows graphs of capacitor voltage  $v_C$  for  $LC$  circuits 1 and 2, which contain identical capacitances and have the same maximum charge  $Q$ . Are (a) the inductance  $L$  and (b) the maximum current  $I$  in circuit 1 greater than, less than, or the same as those in circuit 2?

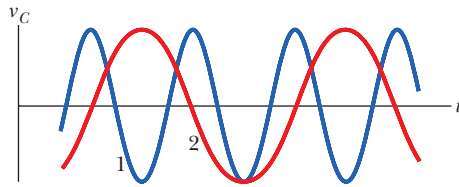


Figure 31-20 Question 2.

**3** A charged capacitor and an inductor are connected at time  $t = 0$ . In terms of the period  $T$  of the resulting oscillations, what is the first later time at which the following reach a maximum: (a)  $U_B$ , (b) the magnetic flux through the inductor, (c)  $di/dt$ , and (d) the emf of the inductor?

**4** What values of phase constant  $\phi$  in Eq. 31-12 allow situations (a), (c), (e), and (g) of Fig. 31-1 to occur at  $t = 0$ ?

**5** Curve  $a$  in Fig. 31-21 gives the impedance  $Z$  of a driven  $RC$  circuit versus the driving angular frequency  $\omega_d$ . The other two curves are similar but for different values of resistance  $R$  and capacitance  $C$ . Rank the three curves according to the corresponding value of  $R$ , greatest first.

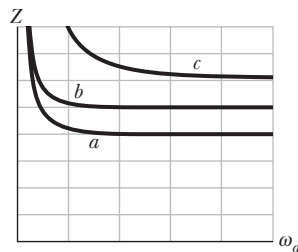


Figure 31-21 Question 5.

**6** Charges on the capacitors in three oscillating  $LC$  circuits vary as: (1)  $q = 2 \cos 4t$ , (2)  $q = 4 \cos t$ , (3)  $q = 3 \cos 4t$  (with  $q$  in coulombs and  $t$  in seconds). Rank the circuits according to (a) the current amplitude and (b) the period, greatest first.

**7** An alternating emf source with a certain emf amplitude is connected, in turn, to a resistor, a capacitor, and then an inductor. Once connected to one of the devices, the driving frequency  $f_d$  is varied and the amplitude  $I$  of the resulting current through the device is measured and plotted. Which of the three plots in Fig. 31-22 corresponds to which of the three devices?

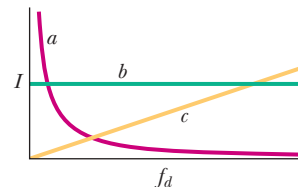


Figure 31-22 Question 7.

**8** The values of the phase constant  $\phi$  for four sinusoidally driven series  $RLC$  circuits are (1)  $-15^\circ$ , (2)  $+35^\circ$ , (3)  $\pi/3$  rad, and (4)  $-\pi/6$  rad. (a) In which is the load primarily capacitive? (b) In which does the current lag the alternating emf?

**9** Figure 31-23 shows the current  $i$  and driving emf  $\mathcal{E}$  for a series  $RLC$  circuit. (a) Is the phase constant positive or negative? (b) To increase the rate at which energy is transferred to the resistive load, should  $L$  be increased or decreased? (c) Should, instead,  $C$  be increased or decreased?

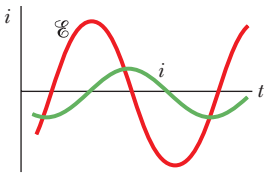


Figure 31-23 Question 9.

**10** Figure 31-24 shows three situations like those of Fig. 31-15. Is the driving angular frequency greater than, less than, or equal to the resonant angular frequency of the circuit in (a) situation 1, (b) situation 2, and (c) situation 3?

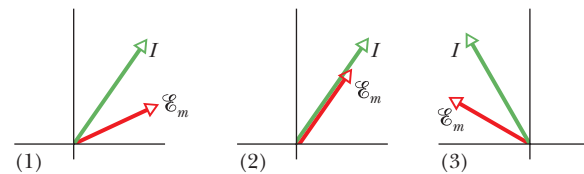


Figure 31-24 Question 10.

**11** Figure 31-25 shows the current  $i$  and driving emf  $\mathcal{E}$  for a series  $RLC$  circuit. Relative to the emf curve, does the current curve shift leftward or rightward and does the amplitude of that curve increase or decrease if we slightly increase (a)  $L$ , (b)  $C$ , and (c)  $\omega_d$ ?

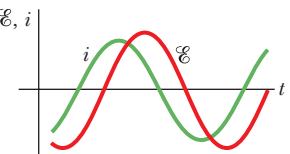


Figure 31-25 Questions 11 and 12.

**12** Figure 31-25 shows the current  $i$  and driving emf  $\mathcal{E}$  for a series  $RLC$  circuit. (a) Does the current lead or lag the emf? (b) Is the circuit's load mainly capacitive or mainly inductive? (c) Is the angular frequency  $\omega_d$  of the emf greater than or less than the natural angular frequency  $\omega$ ?

**13** (a) Does the phasor diagram of Fig. 31-26 correspond to an alternating emf source connected to a resistor, a capacitor, or an inductor? (b) If the angular speed of the phasors is increased, does the current phasor length increase or decrease when the scale of the diagram is maintained?

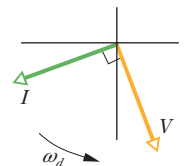


Figure 31-26 Question 13.

## Problems



Tutoring problem available (at instructor's discretion) in WileyPLUS and WebAssign



Worked-out solution available in Student Solutions Manual



Worked-out solution is at



Number of dots indicates level of problem difficulty



Interactive solution is at



Additional information available in The Flying Circus of Physics and at flyingcircusofphysics.com

<http://www.wiley.com/college/halliday>

### Module 31-1 LC Oscillations

**•1** An oscillating  $LC$  circuit consists of a  $75.0$  mH inductor and a  $3.60$   $\mu\text{F}$  capacitor. If the maximum charge on the capacitor is  $2.90$   $\mu\text{C}$ , what are (a) the total energy in the circuit and (b) the maximum current?

**•2** The frequency of oscillation of a certain  $LC$  circuit is  $200$  kHz. At time  $t = 0$ , plate  $A$  of the capacitor has maximum positive charge. At what earliest time  $t > 0$  will (a) plate  $A$  again have maximum positive charge, (b) the other plate of the capacitor have maximum positive charge, and (c) the inductor have maximum magnetic field?



•3 In a certain oscillating  $LC$  circuit, the total energy is converted from electrical energy in the capacitor to magnetic energy in the inductor in  $1.50\ \mu\text{s}$ . What are (a) the period of oscillation and (b) the frequency of oscillation? (c) How long after the magnetic energy is a maximum will it be a maximum again?

•4 What is the capacitance of an oscillating  $LC$  circuit if the maximum charge on the capacitor is  $1.60\ \mu\text{C}$  and the total energy is  $140\ \mu\text{J}$ ?

•5 In an oscillating  $LC$  circuit,  $L = 1.10\ \text{mH}$  and  $C = 4.00\ \mu\text{F}$ . The maximum charge on the capacitor is  $3.00\ \mu\text{C}$ . Find the maximum current.

•6 A  $0.50\ \text{kg}$  body oscillates in SHM on a spring that, when extended  $2.0\ \text{mm}$  from its equilibrium position, has an  $8.0\ \text{N}$  restoring force. What are (a) the angular frequency of oscillation, (b) the period of oscillation, and (c) the capacitance of an  $LC$  circuit with the same period if  $L$  is  $5.0\ \text{H}$ ?

•7 **SSM** The energy in an oscillating  $LC$  circuit containing a  $1.25\ \text{H}$  inductor is  $5.70\ \mu\text{J}$ . The maximum charge on the capacitor is  $175\ \mu\text{C}$ . For a mechanical system with the same period, find the (a) mass, (b) spring constant, (c) maximum displacement, and (d) maximum speed.

•8 A single loop consists of inductors ( $L_1, L_2, \dots$ ), capacitors ( $C_1, C_2, \dots$ ), and resistors ( $R_1, R_2, \dots$ ) connected in series as shown, for example, in Fig. 31-27a. Show that regardless of the sequence of these circuit elements in the loop, the behavior of this circuit is identical to that of the simple  $LC$  circuit shown in Fig. 31-27b. (Hint: Consider the loop rule and see Problem 47 in Chapter 30.)

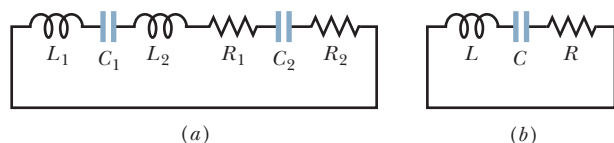


Figure 31-27 Problem 8.

•9 **ILW** In an oscillating  $LC$  circuit with  $L = 50\ \text{mH}$  and  $C = 4.0\ \mu\text{F}$ , the current is initially a maximum. How long will it take before the capacitor is fully charged for the first time?

•10  $LC$  oscillators have been used in circuits connected to loudspeakers to create some of the sounds of electronic music. What inductance must be used with a  $6.7\ \mu\text{F}$  capacitor to produce a frequency of  $10\ \text{kHz}$ , which is near the middle of the audible range of frequencies?

••11 **SSM WWW** A variable capacitor with a range from  $10$  to  $365\ \text{pF}$  is used with a coil to form a variable-frequency  $LC$  circuit to tune the input to a radio. (a) What is the ratio of maximum frequency to minimum frequency that can be obtained with such a capacitor? If this circuit is to obtain frequencies from  $0.54\ \text{MHz}$  to  $1.60\ \text{MHz}$ , the ratio computed in (a) is too large. By adding a capacitor in parallel to the variable capacitor, this range can be adjusted. To obtain the desired frequency range, (b) what capacitance should be added and (c) what inductance should the coil have?

••12 In an oscillating  $LC$  circuit, when  $75.0\%$  of the total energy is stored in the inductor's magnetic field, (a) what multiple of the maximum charge is on the capacitor and (b) what multiple of the maximum current is in the inductor?

••13 In an oscillating  $LC$  circuit,  $L = 3.00\ \text{mH}$  and  $C = 2.70\ \mu\text{F}$ . At  $t = 0$  the charge on the capacitor is zero and the current is  $2.00\ \text{A}$ . (a) What is the maximum charge that will appear on the capacitor? (b) At what earliest time  $t > 0$  is the rate at which energy is stored in the capacitor greatest, and (c) what is that greatest rate?

••14 To construct an oscillating  $LC$  system, you can choose from a  $10\ \text{mH}$  inductor, a  $5.0\ \mu\text{F}$  capacitor, and a  $2.0\ \mu\text{F}$  capacitor. What are the (a) smallest, (b) second smallest, (c) second largest, and (d) largest oscillation frequency that can be set up by these elements in various combinations?

••15 **ILW** An oscillating  $LC$  circuit consisting of a  $1.0\ \text{nF}$  capacitor and a  $3.0\ \text{mH}$  coil has a maximum voltage of  $3.0\ \text{V}$ . What are (a) the maximum charge on the capacitor, (b) the maximum current through the circuit, and (c) the maximum energy stored in the magnetic field of the coil?

••16 An inductor is connected across a capacitor whose capacitance can be varied by turning a knob. We wish to make the frequency of oscillation of this  $LC$  circuit vary linearly with the angle of rotation of the knob, going from  $2 \times 10^5$  to  $4 \times 10^5\ \text{Hz}$  as the knob turns through  $180^\circ$ . If  $L = 1.0\ \text{mH}$ , plot the required capacitance  $C$  as a function of the angle of rotation of the knob.

••17 **GO ILW** In Fig. 31-28,  $R = 14.0\ \Omega$ ,  $C = 6.20\ \mu\text{F}$ , and  $L = 54.0\ \text{mH}$ , and the ideal battery has emf  $\mathcal{E} = 34.0\ \text{V}$ . The switch is kept at  $a$  for a long time and then thrown to position  $b$ . What are the (a) frequency and (b) current amplitude of the resulting oscillations?

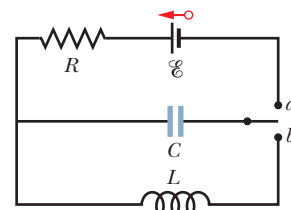


Figure 31-28 Problem 17.

••18 An oscillating  $LC$  circuit has a current amplitude of  $7.50\ \text{mA}$ , a potential amplitude of  $250\ \text{mV}$ , and a capacitance of  $220\ \text{nF}$ . What are (a) the period of oscillation, (b) the maximum energy stored in the capacitor, (c) the maximum energy stored in the inductor, (d) the maximum rate at which the current changes, and (e) the maximum rate at which the inductor gains energy?

••19 Using the loop rule, derive the differential equation for an  $LC$  circuit (Eq. 31-11).

••20 **GO** In an oscillating  $LC$  circuit in which  $C = 4.00\ \mu\text{F}$ , the maximum potential difference across the capacitor during the oscillations is  $1.50\ \text{V}$  and the maximum current through the inductor is  $50.0\ \text{mA}$ . What are (a) the inductance  $L$  and (b) the frequency of the oscillations? (c) How much time is required for the charge on the capacitor to rise from zero to its maximum value?

••21 **ILW** In an oscillating  $LC$  circuit with  $C = 64.0\ \mu\text{F}$ , the current is given by  $i = (1.60) \sin(2500t + 0.680)$ , where  $t$  is in seconds,  $i$  in amperes, and the phase constant in radians. (a) How soon after  $t = 0$  will the current reach its maximum value? What are (b) the inductance  $L$  and (c) the total energy?

••22 A series circuit containing inductance  $L_1$  and capacitance  $C_1$  oscillates at angular frequency  $\omega$ . A second series circuit, containing inductance  $L_2$  and capacitance  $C_2$ , oscillates at the same angular frequency. In terms of  $\omega$ , what is the angular frequency of oscillation of a series circuit containing all four of these elements? Neglect resistance. (Hint: Use the formulas for equivalent capacitance and equivalent inductance; see Module 25-3 and Problem 47 in Chapter 30.)

••23 **GO** In an oscillating  $LC$  circuit,  $L = 25.0$  mH and  $C = 7.80$   $\mu$ F. At time  $t = 0$  the current is 9.20 mA, the charge on the capacitor is 3.80  $\mu$ C, and the capacitor is charging. What are (a) the total energy in the circuit, (b) the maximum charge on the capacitor, and (c) the maximum current? (d) If the charge on the capacitor is given by  $q = Q \cos(\omega t + \phi)$ , what is the phase angle  $\phi$ ? (e) Suppose the data are the same, except that the capacitor is discharging at  $t = 0$ . What then is  $\phi$ ?

### Module 31-2 Damped Oscillations in an $RLC$ Circuit

••24 **GO** A single-loop circuit consists of a 7.20  $\Omega$  resistor, a 12.0 H inductor, and a 3.20  $\mu$ F capacitor. Initially the capacitor has a charge of 6.20  $\mu$ C and the current is zero. Calculate the charge on the capacitor  $N$  complete cycles later for (a)  $N = 5$ , (b)  $N = 10$ , and (c)  $N = 100$ .

••25 **ILW** What resistance  $R$  should be connected in series with an inductance  $L = 220$  mH and capacitance  $C = 12.0$   $\mu$ F for the maximum charge on the capacitor to decay to 99.0% of its initial value in 50.0 cycles? (Assume  $\omega' \approx \omega$ .)

••26 **GO** In an oscillating series  $RLC$  circuit, find the time required for the maximum energy present in the capacitor during an oscillation to fall to half its initial value. Assume  $q = Q$  at  $t = 0$ .

•••27 **SSM** In an oscillating series  $RLC$  circuit, show that  $\Delta U/U$ , the fraction of the energy lost per cycle of oscillation, is given to a close approximation by  $2\pi R/\omega L$ . The quantity  $\omega L/R$  is often called the  $Q$  of the circuit (for *quality*). A high- $Q$  circuit has low resistance and a low fractional energy loss ( $= 2\pi/Q$ ) per cycle.

### Module 31-3 Forced Oscillations of Three Simple Circuits

•28 A 1.50  $\mu$ F capacitor is connected as in Fig. 31-10 to an ac generator with  $\mathcal{E}_m = 30.0$  V. What is the amplitude of the resulting alternating current if the frequency of the emf is (a) 1.00 kHz and (b) 8.00 kHz?

•29 **ILW** A 50.0 mH inductor is connected as in Fig. 31-12 to an ac generator with  $\mathcal{E}_m = 30.0$  V. What is the amplitude of the resulting alternating current if the frequency of the emf is (a) 1.00 kHz and (b) 8.00 kHz?

•30 A 50.0  $\Omega$  resistor is connected as in Fig. 31-8 to an ac generator with  $\mathcal{E}_m = 30.0$  V. What is the amplitude of the resulting alternating current if the frequency of the emf is (a) 1.00 kHz and (b) 8.00 kHz?

•31 (a) At what frequency would a 6.0 mH inductor and a 10  $\mu$ F capacitor have the same reactance? (b) What would the reactance be? (c) Show that this frequency would be the natural frequency of an oscillating circuit with the same  $L$  and  $C$ .

••32 **GO** An ac generator has emf  $\mathcal{E} = \mathcal{E}_m \sin \omega_d t$ , with  $\mathcal{E}_m = 25.0$  V and  $\omega_d = 377$  rad/s. It is connected to a 12.7 H inductor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is  $-12.5$  V and increasing in magnitude, what is the current?

••33 **SSM** An ac generator has emf  $\mathcal{E} = \mathcal{E}_m \sin(\omega_d t - \pi/4)$ , where  $\mathcal{E}_m = 30.0$  V and  $\omega_d = 350$  rad/s. The current produced in a connected circuit is  $i(t) = I \sin(\omega_d t - 3\pi/4)$ , where  $I = 620$  mA. At what time after  $t = 0$  does (a) the generator emf first reach a maximum and (b) the current first reach a maximum? (c) The circuit contains a single element other than the generator. Is it a capacitor, an inductor, or a resistor? Justify your answer. (d) What is

the value of the capacitance, inductance, or resistance, as the case may be?

••34 **GO** An ac generator with emf  $\mathcal{E} = \mathcal{E}_m \sin \omega_d t$ , where  $\mathcal{E}_m = 25.0$  V and  $\omega_d = 377$  rad/s, is connected to a 4.15  $\mu$ F capacitor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is  $-12.5$  V and increasing in magnitude, what is the current?

### Module 31-4 The Series $RLC$ Circuit

•35 **ILW** A coil of inductance 88 mH and unknown resistance and a 0.94  $\mu$ F capacitor are connected in series with an alternating emf of frequency 930 Hz. If the phase constant between the applied voltage and the current is  $75^\circ$ , what is the resistance of the coil?

•36 An alternating source with a variable frequency, a capacitor with capacitance  $C$ , and a resistor with resistance  $R$  are connected in series. Figure 31-29 gives the impedance  $Z$  of the circuit versus the driving angular frequency  $\omega_d$ ; the curve reaches an asymptote of 500  $\Omega$ , and the horizontal scale is set by  $\omega_{ds} = 300$  rad/s. The figure also gives the reactance  $X_C$  for the capacitor versus  $\omega_d$ . What are (a)  $R$  and (b)  $C$ ?

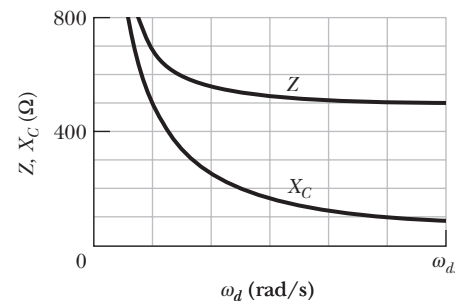


Figure 31-29 Problem 36.

•37 An electric motor has an effective resistance of 32.0  $\Omega$  and an inductive reactance of 45.0  $\Omega$  when working under load. The voltage amplitude across the alternating source is 420 V. Calculate the current amplitude.

•38 The current amplitude  $I$  versus driving angular frequency  $\omega_d$  for a driven  $RLC$  circuit is given in Fig. 31-30, where the vertical axis scale is set by  $I_s = 4.00$  A. The inductance is 200  $\mu$ H, and the emf amplitude is 8.0 V. What are (a)  $C$  and (b)  $R$ ?

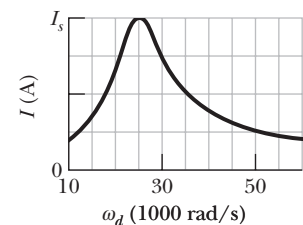


Figure 31-30 Problem 38.

•39 Remove the inductor from the circuit in Fig. 31-7 and set  $R = 200$   $\Omega$ ,  $C = 15.0$   $\mu$ F,  $f_d = 60.0$  Hz, and  $\mathcal{E}_m = 36.0$  V. What are (a)  $Z$ , (b)  $\phi$ , and (c)  $I$ ? (d) Draw a phasor diagram.

•40 An alternating source drives a series  $RLC$  circuit with an emf amplitude of 6.00 V, at a phase angle of  $+30.0^\circ$ . When the potential difference across the capacitor reaches its maximum positive value of +5.00 V, what is the potential difference across the inductor (sign included)?

•41 **SSM** In Fig. 31-7, set  $R = 200$   $\Omega$ ,  $C = 70.0$   $\mu$ F,  $L = 230$  mH,  $f_d = 60.0$  Hz, and  $\mathcal{E}_m = 36.0$  V. What are (a)  $Z$ , (b)  $\phi$ , and (c)  $I$ ? (d) Draw a phasor diagram.

•42 An alternating source with a variable frequency, an inductor

with inductance  $L$ , and a resistor with resistance  $R$  are connected in series. Figure 31-31 gives the impedance  $Z$  of the circuit versus the driving angular frequency  $\omega_d$ , with the horizontal axis scale set by  $\omega_{ds} = 1600 \text{ rad/s}$ . The figure also gives the reactance  $X_L$  for the inductor versus  $\omega_d$ . What are (a)  $R$  and (b)  $L$ ?

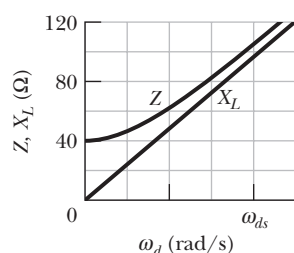


Figure 31-31 Problem 42.

••43 Remove the capacitor from the circuit in Fig. 31-7 and set  $R = 200 \, \Omega$ ,  $L = 230 \text{ mH}$ ,  $f_d = 60.0 \text{ Hz}$ , and  $\mathcal{E}_m = 36.0 \text{ V}$ . What are (a)  $Z$ , (b)  $\phi$ , and (c)  $I$ ? (d) Draw a phasor diagram.

••44 GO An ac generator with emf amplitude  $\mathcal{E}_m = 220 \text{ V}$  and operating at frequency  $400 \text{ Hz}$  causes oscillations in a series  $RLC$  circuit having  $R = 220 \, \Omega$ ,  $L = 150 \text{ mH}$ , and  $C = 24.0 \, \mu\text{F}$ . Find (a) the capacitive reactance  $X_C$ , (b) the impedance  $Z$ , and (c) the current amplitude  $I$ . A second capacitor of the same capacitance is then connected in series with the other components. Determine whether the values of (d)  $X_C$ , (e)  $Z$ , and (f)  $I$  increase, decrease, or remain the same.

••45 GO ILW (a) In an  $RLC$  circuit, can the amplitude of the voltage across an inductor be greater than the amplitude of the generator emf? (b) Consider an  $RLC$  circuit with emf amplitude  $\mathcal{E}_m = 10 \text{ V}$ , resistance  $R = 10 \, \Omega$ , inductance  $L = 1.0 \text{ H}$ , and capacitance  $C = 1.0 \, \mu\text{F}$ . Find the amplitude of the voltage across the inductor at resonance.

••46 GO An alternating emf source with a variable frequency  $f_d$  is connected in series with a  $50.0 \, \Omega$  resistor and a  $20.0 \, \mu\text{F}$  capacitor. The emf amplitude is  $12.0 \text{ V}$ . (a) Draw a phasor diagram for phasor  $V_R$  (the potential across the resistor) and phasor  $V_C$  (the potential across the capacitor). (b) At what driving frequency  $f_d$  do the two phasors have the same length? At that driving frequency, what are (c) the phase angle in degrees, (d) the angular speed at which the phasors rotate, and (e) the current amplitude?

••47 SSM WWW An  $RLC$  circuit such as that of Fig. 31-7 has  $R = 5.00 \, \Omega$ ,  $C = 20.0 \, \mu\text{F}$ ,  $L = 1.00 \text{ H}$ , and  $\mathcal{E}_m = 30.0 \text{ V}$ . (a) At what angular frequency  $\omega_d$  will the current amplitude have its maximum value, as in the resonance curves of Fig. 31-16? (b) What is this maximum value? At what (c) lower angular frequency  $\omega_{d1}$  and (d) higher angular frequency  $\omega_{d2}$  will the current amplitude be half this maximum value? (e) For the resonance curve for this circuit, what is the fractional half-width  $(\omega_{d1} - \omega_{d2})/\omega$ ?

••48 GO Figure 31-32 shows a driven  $RLC$  circuit that contains two identical capacitors and two switches. The emf amplitude is set at  $12.0 \text{ V}$ , and the driving frequency is set at  $60.0 \text{ Hz}$ . With both switches open, the current leads the emf by  $30.9^\circ$ . With switch  $S_1$  closed and switch  $S_2$  still open, the emf leads the current by  $15.0^\circ$ . With both switches closed, the current amplitude is  $447 \text{ mA}$ . What are (a)  $R$ , (b)  $C$ , and (c)  $L$ ?

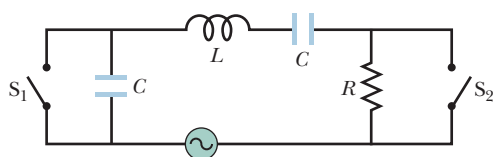


Figure 31-32 Problem 48.

••49 GO In Fig. 31-33, a generator with an adjustable frequency of oscillation is connected to resistance  $R = 100 \, \Omega$ , inductances  $L_1 = 1.70 \text{ mH}$  and  $L_2 = 2.30 \text{ mH}$ , and capacitances  $C_1 = 4.00 \, \mu\text{F}$ ,  $C_2 = 2.50 \, \mu\text{F}$ , and  $C_3 = 3.50 \, \mu\text{F}$ . (a) What is the resonant frequency of the circuit? (Hint: See Problem 47 in Chapter 30.) What happens to the resonant frequency if (b)  $R$  is increased, (c)  $L_1$  is increased, and (d)  $C_3$  is removed from the circuit?

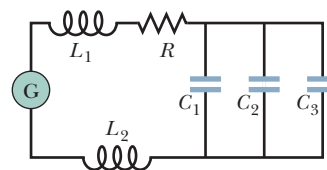


Figure 31-33 Problem 49.

••50 An alternating emf source with a variable frequency  $f_d$  is connected in series with an  $80.0 \, \Omega$  resistor and a  $40.0 \text{ mH}$  inductor. The emf amplitude is  $6.00 \text{ V}$ . (a) Draw a phasor diagram for phasor  $V_R$  (the potential across the resistor) and phasor  $V_L$  (the potential across the inductor). (b) At what driving frequency  $f_d$  do the two phasors have the same length? At that driving frequency, what are (c) the phase angle in degrees, (d) the angular speed at which the phasors rotate, and (e) the current amplitude?

••51 SSM The fractional half-width  $\Delta\omega_d$  of a resonance curve, such as the ones in Fig. 31-16, is the width of the curve at half the maximum value of  $I$ . Show that  $\Delta\omega_d/\omega = R(3C/L)^{1/2}$ , where  $\omega$  is the angular frequency at resonance. Note that the ratio  $\Delta\omega_d/\omega$  increases with  $R$ , as Fig. 31-16 shows.

### Module 31-5 Power in Alternating-Current Circuits

•52 An ac voltmeter with large impedance is connected in turn across the inductor, the capacitor, and the resistor in a series circuit having an alternating emf of  $100 \text{ V (rms)}$ ; the meter gives the same reading in volts in each case. What is this reading?

•53 SSM An air conditioner connected to a  $120 \text{ V rms}$  ac line is equivalent to a  $12.0 \, \Omega$  resistance and a  $1.30 \, \Omega$  inductive reactance in series. Calculate (a) the impedance of the air conditioner and (b) the average rate at which energy is supplied to the appliance.

•54 What is the maximum value of an ac voltage whose rms value is  $100 \text{ V}$ ?

•55 What direct current will produce the same amount of thermal energy, in a particular resistor, as an alternating current that has a maximum value of  $2.60 \text{ A}$ ?

••56 A typical light dimmer used to dim the stage lights in a theater consists of a variable inductor  $L$  (whose inductance is adjustable between zero and  $L_{\text{max}}$ ) connected in series with a lightbulb B, as shown in Fig. 31-34. The electrical supply is  $120 \text{ V (rms)}$  at  $60.0 \text{ Hz}$ ; the lightbulb is rated at  $120 \text{ V}$ ,  $1000 \text{ W}$ . (a) What  $L_{\text{max}}$  is required if the rate of energy dissipation in the lightbulb is to be varied by a factor of 5 from its upper limit of  $1000 \text{ W}$ ? Assume that the resistance of the lightbulb is independent of its temperature. (b) Could one use a variable resistor (adjustable between zero and  $R_{\text{max}}$ ) instead of an inductor? (c) If so, what  $R_{\text{max}}$  is required? (d) Why isn't this done?

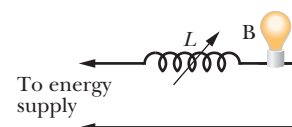


Figure 31-34 Problem 56.

••57 In an  $RLC$  circuit such as that of Fig. 31-7 assume that  $R = 5.00 \, \Omega$ ,  $L = 60.0 \text{ mH}$ ,  $f_d = 60.0 \text{ Hz}$ , and  $\mathcal{E}_m = 30.0 \text{ V}$ . For what values of the capacitance would the average rate at which energy is dissipated in the resistance be (a) a maximum and (b) a minimum? What are (c) the maximum dissipation rate and the corresponding



(d) phase angle and (e) power factor? What are (f) the minimum dissipation rate and the corresponding (g) phase angle and (h) power factor?

••58 For Fig. 31-35, show that the average rate at which energy is dissipated in resistance  $R$  is a maximum when  $R$  is equal to the internal resistance  $r$  of the ac generator. (In the text discussion we tacitly assumed that  $r = 0$ .)

••59 GO In Fig. 31-7,  $R = 15.0\ \Omega$ ,  $C = 4.70\ \mu\text{F}$ , and  $L = 25.0\ \text{mH}$ . The generator provides an emf with rms voltage  $75.0\ \text{V}$  and frequency  $550\ \text{Hz}$ . (a) What is the rms current? What is the rms voltage across (b)  $R$ , (c)  $C$ , (d)  $L$ , (e)  $C$  and  $L$  together, and (f)  $R$ ,  $C$ , and  $L$  together? At what average rate is energy dissipated by (g)  $R$ , (h)  $C$ , and (i)  $L$ ?

••60 GO In a series oscillating  $RLC$  circuit,  $R = 16.0\ \Omega$ ,  $C = 31.2\ \mu\text{F}$ ,  $L = 9.20\ \text{mH}$ , and  $\mathcal{E}_m = \mathcal{E}_m \sin \omega_d t$  with  $\mathcal{E}_m = 45.0\ \text{V}$  and  $\omega_d = 3000\ \text{rad/s}$ . For time  $t = 0.442\ \text{ms}$  find (a) the rate  $P_g$  at which energy is being supplied by the generator, (b) the rate  $P_C$  at which the energy in the capacitor is changing, (c) the rate  $P_L$  at which the energy in the inductor is changing, and (d) the rate  $P_R$  at which energy is being dissipated in the resistor. (e) Is the sum of  $P_C$ ,  $P_L$ , and  $P_R$  greater than, less than, or equal to  $P_g$ ?

••61 SSM WWW Figure 31-36 shows an ac generator connected to a “black box” through a pair of terminals. The box contains an  $RLC$  circuit, possibly even a multiloop circuit, whose elements and connections we do not know. Measurements outside the box reveal that

$$\mathcal{E}(t) = (75.0\ \text{V}) \sin \omega_d t$$

and

$$i(t) = (1.20\ \text{A}) \sin(\omega_d t + 42.0^\circ).$$

(a) What is the power factor? (b) Does the current lead or lag the emf? (c) Is the circuit in the box largely inductive or largely capacitive? (d) Is the circuit in the box in resonance? (e) Must there be a capacitor in the box? (f) An inductor? (g) A resistor? (h) At what average rate is energy delivered to the box by the generator? (i) Why don't you need to know  $\omega_d$  to answer all these questions?

### Module 31-6 Transformers

•62 A generator supplies  $100\ \text{V}$  to a transformer's primary coil, which has 50 turns. If the secondary coil has 500 turns, what is the secondary voltage?

•63 SSM ILW A transformer has 500 primary turns and 10 secondary turns. (a) If  $V_p$  is  $120\ \text{V}$  (rms), what is  $V_s$  with an open circuit? If the secondary now has a resistive load of  $15\ \Omega$ , what is the current in the (b) primary and (c) secondary?

•64 Figure 31-37 shows an “autotransformer.” It consists of a single coil (with an iron core). Three taps  $T_i$  are provided. Between taps  $T_1$  and  $T_2$  there are 200 turns, and between taps  $T_2$  and  $T_3$  there are 800 turns. Any two taps can be chosen as the

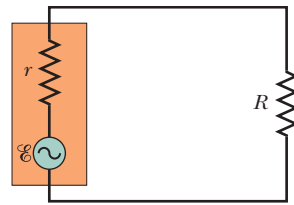


Figure 31-35 Problems 58 and 66.

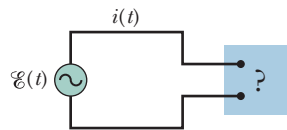


Figure 31-36 Problem 61.

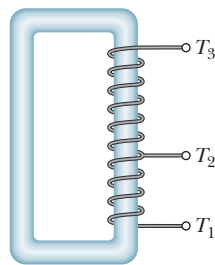


Figure 31-37 Problem 64.

primary terminals, and any two taps can be chosen as the secondary terminals. For choices producing a step-up transformer, what are the (a) smallest, (b) second smallest, and (c) largest values of the ratio  $V_s/V_p$ ? For a step-down transformer, what are the (d) smallest, (e) second smallest, and (f) largest values of  $V_s/V_p$ ?

••65 An ac generator provides emf to a resistive load in a remote factory over a two-cable transmission line. At the factory a step-down transformer reduces the voltage from its (rms) transmission value  $V_t$  to a much lower value that is safe and convenient for use in the factory. The transmission line resistance is  $0.30\ \Omega/\text{cable}$ , and the power of the generator is  $250\ \text{kW}$ . If  $V_t = 80\ \text{kV}$ , what are (a) the voltage decrease  $\Delta V$  along the transmission line and (b) the rate  $P_d$  at which energy is dissipated in the line as thermal energy? If  $V_t = 8.0\ \text{kV}$ , what are (c)  $\Delta V$  and (d)  $P_d$ ? If  $V_t = 0.80\ \text{kV}$ , what are (e)  $\Delta V$  and (f)  $P_d$ ?

### Additional Problems

66 In Fig. 31-35, let the rectangular box on the left represent the (high-impedance) output of an audio amplifier, with  $r = 1000\ \Omega$ . Let  $R = 10\ \Omega$  represent the (low-impedance) coil of a loudspeaker. For maximum transfer of energy to the load  $R$  we must have  $R = r$ , and that is not true in this case. However, a transformer can be used to “transform” resistances, making them behave electrically as if they were larger or smaller than they actually are. (a) Sketch the primary and secondary coils of a transformer that can be introduced between the amplifier and the speaker in Fig. 31-35 to match the impedances. (b) What must be the turns ratio?

67 GO An ac generator produces emf  $\mathcal{E} = \mathcal{E}_m \sin(\omega_d t - \pi/4)$ , where  $\mathcal{E}_m = 30.0\ \text{V}$  and  $\omega_d = 350\ \text{rad/s}$ . The current in the circuit attached to the generator is  $i(t) = I \sin(\omega_d t + \pi/4)$ , where  $I = 620\ \text{mA}$ . (a) At what time after  $t = 0$  does the generator emf first reach a maximum? (b) At what time after  $t = 0$  does the current first reach a maximum? (c) The circuit contains a single element other than the generator. Is it a capacitor, an inductor, or a resistor? Justify your answer. (d) What is the value of the capacitance, inductance, or resistance, as the case may be?

68 A series  $RLC$  circuit is driven by a generator at a frequency of  $2000\ \text{Hz}$  and an emf amplitude of  $170\ \text{V}$ . The inductance is  $60.0\ \text{mH}$ , the capacitance is  $0.400\ \mu\text{F}$ , and the resistance is  $200\ \Omega$ . (a) What is the phase constant in radians? (b) What is the current amplitude?

69 A generator of frequency  $3000\ \text{Hz}$  drives a series  $RLC$  circuit with an emf amplitude of  $120\ \text{V}$ . The resistance is  $40.0\ \Omega$ , the capacitance is  $1.60\ \mu\text{F}$ , and the inductance is  $850\ \mu\text{H}$ . What are (a) the phase constant in radians and (b) the current amplitude? (c) Is the circuit capacitive, inductive, or in resonance?

70 A  $45.0\ \text{mH}$  inductor has a reactance of  $1.30\ \text{k}\Omega$ . (a) What is its operating frequency? (b) What is the capacitance of a capacitor with the same reactance at that frequency? If the frequency is doubled, what is the new reactance of (c) the inductor and (d) the capacitor?

71 An  $RLC$  circuit is driven by a generator with an emf amplitude of  $80.0\ \text{V}$  and a current amplitude of  $1.25\ \text{A}$ . The current leads the emf by  $0.650\ \text{rad}$ . What are the (a) impedance and (b) resistance of the circuit? (c) Is the circuit inductive, capacitive, or in resonance?

72 A series  $RLC$  circuit is driven in such a way that the maximum voltage across the inductor is  $1.50$  times the maximum voltage across the capacitor and  $2.00$  times the maximum voltage across the resistor. (a) What is  $\phi$  for the circuit? (b) Is the circuit inductive, capacitive, or in resonance? The resistance is  $49.9\ \Omega$ .

and the current amplitude is 200 mA. (c) What is the amplitude of the driving emf?

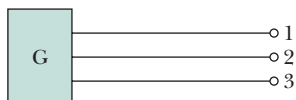
**73** A capacitor of capacitance  $158\ \mu\text{F}$  and an inductor form an  $LC$  circuit that oscillates at  $8.15\ \text{kHz}$ , with a current amplitude of  $4.21\ \text{mA}$ . What are (a) the inductance, (b) the total energy in the circuit, and (c) the maximum charge on the capacitor?

**74** An oscillating  $LC$  circuit has an inductance of  $3.00\ \text{mH}$  and a capacitance of  $10.0\ \mu\text{F}$ . Calculate the (a) angular frequency and (b) period of the oscillation. (c) At time  $t = 0$ , the capacitor is charged to  $200\ \mu\text{C}$  and the current is zero. Roughly sketch the charge on the capacitor as a function of time.

**75** For a certain driven series  $RLC$  circuit, the maximum generator emf is  $125\ \text{V}$  and the maximum current is  $3.20\ \text{A}$ . If the current leads the generator emf by  $0.982\ \text{rad}$ , what are the (a) impedance and (b) resistance of the circuit? (c) Is the circuit predominantly capacitive or inductive?

**76** A  $1.50\ \mu\text{F}$  capacitor has a capacitive reactance of  $12.0\ \Omega$ . (a) What must be its operating frequency? (b) What will be the capacitive reactance if the frequency is doubled?

**77 SSM** In Fig. 31-38, a three-phase generator  $G$  produces electrical power that is transmitted by means of three wires. The electric potentials (each relative to a common reference level) are  $V_1 = A \sin \omega_d t$  for wire 1,  $V_2 = A \sin(\omega_d t - 120^\circ)$  for wire 2, and  $V_3 = A \sin(\omega_d t - 240^\circ)$  for wire 3. Some types of industrial equipment (for example, motors) have three terminals and are designed to be connected directly to these three wires. To use a more conventional two-terminal device (for example, a lightbulb), one connects it to any two of the three wires. Show that the potential difference between *any two* of the wires (a) oscillates sinusoidally with angular frequency  $\omega_d$  and (b) has an amplitude of  $A\sqrt{3}$ .



Three-wire transmission line  
**Figure 31-38** Problem 77.

**78** An electric motor connected to a  $120\ \text{V}$ ,  $60.0\ \text{Hz}$  ac outlet does mechanical work at the rate of  $0.100\ \text{hp}$  ( $1\ \text{hp} = 746\ \text{W}$ ). (a) If the motor draws an rms current of  $0.650\ \text{A}$ , what is its effective resistance, relative to power transfer? (b) Is this the same as the resistance of the motor's coils, as measured with an ohmmeter with the motor disconnected from the outlet?

**79 SSM** (a) In an oscillating  $LC$  circuit, in terms of the maximum charge  $Q$  on the capacitor, what is the charge there when the energy in the electric field is  $50.0\%$  of that in the magnetic field? (b) What fraction of a period must elapse following the time the capacitor is fully charged for this condition to occur?

**80** A series  $RLC$  circuit is driven by an alternating source at a frequency of  $400\ \text{Hz}$  and an emf amplitude of  $90.0\ \text{V}$ . The resistance is  $20.0\ \Omega$ , the capacitance is  $12.1\ \mu\text{F}$ , and the inductance is  $24.2\ \text{mH}$ . What is the rms potential difference across (a) the resistor, (b) the capacitor, and (c) the inductor? (d) What is the average rate at which energy is dissipated?

**81 SSM** In a certain series  $RLC$  circuit being driven at a frequency of  $60.0\ \text{Hz}$ , the maximum voltage across the inductor is  $2.00$  times the maximum voltage across the resistor and  $2.00$  times the maximum voltage across the capacitor. (a) By what angle does the current lag the generator emf? (b) If the maximum generator emf is  $30.0\ \text{V}$ , what should be the resistance of the circuit to obtain a maximum current of  $300\ \text{mA}$ ?

**82** A  $1.50\ \text{mH}$  inductor in an oscillating  $LC$  circuit stores a maximum energy of  $10.0\ \mu\text{J}$ . What is the maximum current?

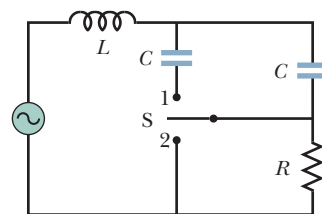
**83** A generator with an adjustable frequency of oscillation is wired in series to an inductor of  $L = 2.50\ \text{mH}$  and a capacitor of  $C = 3.00\ \mu\text{F}$ . At what frequency does the generator produce the largest possible current amplitude in the circuit?

**84** A series  $RLC$  circuit has a resonant frequency of  $6.00\ \text{kHz}$ . When it is driven at  $8.00\ \text{kHz}$ , it has an impedance of  $1.00\ \text{k}\Omega$  and a phase constant of  $45^\circ$ . What are (a)  $R$ , (b)  $L$ , and (c)  $C$  for this circuit?

**85 SSM** An  $LC$  circuit oscillates at a frequency of  $10.4\ \text{kHz}$ . (a) If the capacitance is  $340\ \mu\text{F}$ , what is the inductance? (b) If the maximum current is  $7.20\ \text{mA}$ , what is the total energy in the circuit? (c) What is the maximum charge on the capacitor?

**86** When under load and operating at an rms voltage of  $220\ \text{V}$ , a certain electric motor draws an rms current of  $3.00\ \text{A}$ . It has a resistance of  $24.0\ \Omega$  and no capacitive reactance. What is its inductive reactance?

**87** The ac generator in Fig. 31-39 supplies  $120\ \text{V}$  at  $60.0\ \text{Hz}$ . With the switch open as in the diagram, the current leads the generator emf by  $20.0^\circ$ . With the switch in position 1, the current lags the generator emf by  $10.0^\circ$ . When the switch is in position 2, the current amplitude is  $2.00\ \text{A}$ . What are (a)  $R$ , (b)  $L$ , and (c)  $C$ ?



**Figure 31-39** Problem 87.

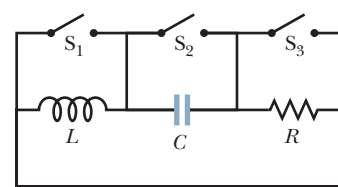
**88** In an oscillating  $LC$  circuit,  $L = 8.00\ \text{mH}$  and  $C = 1.40\ \mu\text{F}$ . At time  $t = 0$ , the current is maximum at  $12.0\ \text{mA}$ . (a) What is the maximum charge on the capacitor during the oscillations? (b) At what earliest time  $t > 0$  is the rate of change of energy in the capacitor maximum? (c) What is that maximum rate of change?

**89 SSM** For a sinusoidally driven series  $RLC$  circuit, show that over one complete cycle with period  $T$  (a) the energy stored in the capacitor does not change; (b) the energy stored in the inductor does not change; (c) the driving emf device supplies energy  $(\frac{1}{2}T)\mathcal{E}_m I \cos \phi$ ; and (d) the resistor dissipates energy  $(\frac{1}{2}T)RI^2$ . (e) Show that the quantities found in (c) and (d) are equal.

**90** What capacitance would you connect across a  $1.30\ \text{mH}$  inductor to make the resulting oscillator resonate at  $3.50\ \text{kHz}$ ?

**91** A series circuit with resistor-inductor-capacitor combination  $R_1, L_1, C_1$  has the same resonant frequency as a second circuit with a different combination  $R_2, L_2, C_2$ . You now connect the two combinations in series. Show that this new circuit has the same resonant frequency as the separate circuits.

**92** Consider the circuit shown in Fig. 31-40. With switch  $S_1$  closed and the other two switches open, the circuit has a time constant  $\tau_C$ . With switch  $S_2$  closed and the other two switches open, the circuit has a time constant  $\tau_L$ . With switch  $S_3$  closed and the other two switches open, the circuit oscillates with a period  $T$ . Show that  $T = 2\pi\sqrt{\tau_C\tau_L}$ .



**Figure 31-40** Problem 92.

**93** When the generator emf in Sample Problem 31.07 is a maximum, what is the voltage across (a) the generator, (b) the resistance, (c) the capacitance, and (d) the inductance? (e) By summing these with appropriate signs, verify that the loop rule is satisfied.