Chapter 4 Vector Spaces

Section 4.1: Vector Spaces and Subspaces

Section 4.2: Null Spaces, Column Spaces,
Row Spaces, and Linear Transformation

Section 4.3: Linearly Independent Set; Bases

Vector Spaces

- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations,

  -- **addition**

  -- **and multiplication by scalars** (real numbers),
- The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

Vector Spaces

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ is in *V*.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There is a zero vector in *V* such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in *V*, there is a vector $-\mathbf{u}$ in *V* such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by *c*, $c\mathbf{u}$, is in *V*.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
9. $c(d\mathbf{u})=(cd)\mathbf{u}$
10. $1\mathbf{u}=\mathbf{u}$

Vector Spaces

- For each **u** in *V* and scalar *c*,

$$0\mathbf{u} = \mathbf{0}$$

$$c\mathbf{0} = \mathbf{0}$$

$$-\mathbf{u} = (-1)\mathbf{u}$$

Example: The set $\mathbb{R}^n$ is a vector space under the usual addition and scalar multiplication.

The Polynomials of Degree at Most n

Example: For $n \geq 0,$ the set $\mathbb{P}_n$ of polynomials of degree at most n, is a vector space.

- It consists of all polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

  where the coefficients $a_0, \ldots, a_n$ and the variable t are real numbers.

- The **degree** of $\mathbf{p}$ is the highest power of t whose coefficient is not zero. If all the coefficients are zero, $\mathbf{p}$ is called the **zero polynomial.**
- $\mathbb{P}_n$ is a vector space.

Subspaces

- **Definition:** A **subspace** of a vector space V is a subset H of V that has three properties:
  a. The **zero vector** of V is in H.
  b. H is closed under **vector addition**. That is, for each **u** and **v** in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
  c. H is closed under **multiplication by scalars**. That is, for each **u** in H and each scalar c, the vector $c\mathbf{u}$ is in H.

--Every subspace is a vector space.

--Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

Subspaces

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$.

- Recall that the term linear combination refers to any sum of scalar multiples of vectors, and

$$\text{Span}\ \{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$$

denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_p$.

A Subspaces Spanned by a Set

- **Example:** Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space *V*, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
  Show that *H* is a subspace of *V*.

- **Solution:**

a. The zero vector is in *H*, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$.

b. Take two arbitrary vectors in *H*, say

   $\mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ and $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$, then

   $\mathbf{u} + \mathbf{w} = (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2$ is in *H*.

c. If *c* is any scalar, $c\mathbf{u} = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$ is in *H*.

Thus *H* is a subspace of *V*.

A Subspaces Spanned by a Set

- **Theorem:** If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space *V*, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of *V*.

- We call Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

- Give any subspace *H* of *V*, a **spanning** (or **generating**) set for *H* is a set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in *H* such that

$$H = \text{Span}\,\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}.$$

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Null Space of a Matrix

- **Definition:** The null space of an $m \times n$ matrix *A*, Nul *A*, is the set of all solutions of the homogeneous equation $A\mathrm{x} = 0$.

$$\mathrm{Nul}\ A = \{\mathrm{x} : \mathrm{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathrm{x} = 0\}$$

- Nul *A* is a subset of $\mathbb{R}^n$ because *A* has *n* columns.

- **Theorem:** The null space of an $m \times n$ matrix *A* is a subspace of $\mathbb{R}^n$.

Null Space of a Matrix

- **Proof:** We need to show that Nul *A* satisfies the three properties of a subspace.

a. The vector **0** is in Nul *A*, since *A***0 = 0.**

b. Let u and v represent any two vectors in Nul A. Then $A\mathrm{v} = 0$ and $A\mathrm{u} = 0$, so $A(\mathrm{u} + \mathrm{v}) = A\mathrm{u} + A\mathrm{v} = 0$ showing $\mathrm{u} + \mathrm{v}$ is in Nul A.

c. If *c* is any scalar, then $A(c\mathrm{u}) = c(A\mathrm{u}) = c(0) = 0$ which shows that *c***u** is in Nul *A*.

Thus Nul *A* is a subspace of $\mathbb{R}^n$.

Null Space of a Matrix

- There is no obvious relation between vectors in Nul A and the entries in A.
- We say that Nul A is defined *implicitly*, because it is defined by a condition that must be checked.

Null Space of a Matrix

- **Example:** Find a spanning set for the null space of the matrix $A\mathrm{x} = 0$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Null Space of a Matrix

- **Solution:**

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$,
$x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free.

Null Space of a Matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\mathbf{u}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\mathbf{v}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\mathbf{w}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

Null Space of a Matrix

- Every linear combination of **u**, **v**, and **w** is an element of Nul *A*.
- Moreover {**u**, **v**, **w**} is a spanning set for Nul *A*.
- The spanning set produced by this method is automatically linearly independent because the free variables are the weights on the spanning vectors.
- When Nul *A* contains nonzero vectors, the number of vectors in the spanning set for Nul *A* equals the number of free variables in the equation $A\mathrm{x} = 0$.

Column Space of a Matrix

- **Definition:** The **column space** of an $m \times n$ matrix *A*, written as Col *A*, is the set of all linear combinations of the columns of *A*. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_\mathrm{n}]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ \ldots \ , \mathbf{a}_n\}$$

- A typical vector in Col *A* can be written as *A*$\mathbf{x}$ for some $\mathbf{x}$, because the *A*$\mathbf{x}$ is a linear combination of the columns of *A*. That is,

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Column Space of a Matrix

- **Theorem:** The column space of an $m \times n$ matrix *A* is a subspace of $\mathbb{R}^m$.
- The notation *A***x** for vectors in Col *A* also shows that Col *A* is the range of the linear transformation $\mathrm{x} \mapsto A\mathrm{x}$.
- The column space of an $m \times n$ matrix *A* is all of $\mathbb{R}^m$ if and only if the equation $A\mathrm{x} = \mathrm{b}$ has a solution for every **b** in $\mathbb{R}^m$

Column Space of a Matrix

- **Example:** Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$

  and $\mathrm{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

  a. Determine if **u** is in Nul *A*. Could **u** be in Col *A*?
  b. Determine if **v** is in Col *A*. Could **v** be in Nul *A*?

Column Space of a Matrix

a. An explicit description of Nul A is not needed here. Simply compute the product $A\mathbf{u}$.

$$A\mathrm{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector **u** is *not* a solution of $A\mathbf{x} = \mathbf{0}$, so **u** is not in Nul A.

Also, with four entries, **u** could not possibly be in Col A, since Col A is a subspace of $\mathbb{R}^3$.

Column Space of a Matrix

b. Reduce $[A \quad \mathbf{v}]$ to an echelon form.

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{v}$ is consistent, so **v** is in Col *A*. With only three entries, **v** could not possibly be in Nul *A*, since Nul *A* is a subspace of $\mathbb{R}^4$.

The Row Space

- If *A* is an $m \times n$ matrix, each row of *A* has *n* entries and thus can be identified with a vector in $\mathbb{R}^n$.
- The set of all linear combinations of the row vectors is called the **row space** of *A* and is denoted by Row *A*.
- Each row has *n* entries, so Row *A* is a subspace of $\mathbb{R}^n$.
- Since the rows of *A* are identified with the columns of A^T, we could also write Col A^T in place of Row *A*.

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Linearly Independent Set

- An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0} \quad (1)$$

  has **only** the trivial solution, $c_1 = 0, \ldots, c_p = 0$.
- The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights, $c_1, \ldots, c_p$, *not all zero*, such that (1) holds.
- In such a case, (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Linearly Independent Set

- **Theorem:** An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq 0$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

- **Definition:** Let *H* be a subspace of a vector space *V*. A set of vectors B in *V* is a basis for *H* if

  (i) B is a linearly independent set, and

  (ii) H = Span B

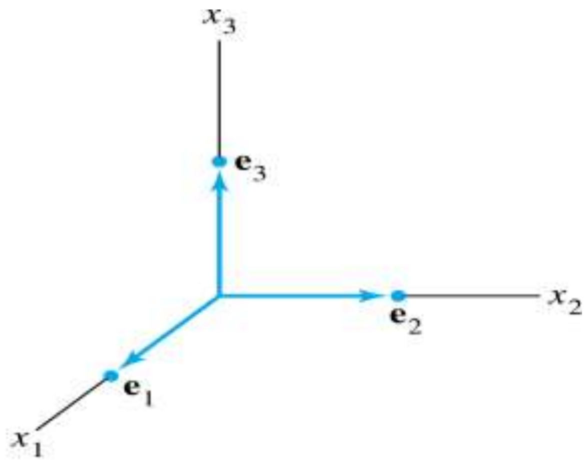
Linearly Independent Set; Bases

- A basis of V is a linearly independent set that spans V.
- The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself.
- When $H \neq V$, condition (ii) includes the requirement that each of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ must belong to H, because Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ contains $\mathbf{b}_1, \ldots, \mathbf{b}_p$.

Standard Basis

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$



The standard basis for $\mathbb{R}^3$.

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$.

The Spanning Set Theorem

- **Theorem:** Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_p\}$ be a set in *V*, and let

$$H = \mathrm{Span}\{\mathbf{v}_1, ..., \mathbf{v}_p\}$$

  a. If one of the vectors in *S*—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in *S*, then the set formed from *S* by removing $\mathbf{v}_k$ still spans *H*.

  b. If $H \neq \{\mathbf{0}\}$, some subset of *S* is a basis for *H*.

The Spanning Set Theorem

The set formed from S by removing $\mathbf{v}_k$ still spans H.

Proof:

a. By rearranging the list of vectors in S, if necessary, we may suppose that $\mathbf{v}_p$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}$

$$\mathbf{v}_p = a_1\mathbf{v}_1 + \ldots + a_{p-1}\mathbf{v}_{p-1}$$

- Given any $\mathbf{x}$ in H, write $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p$

for suitable scalars $c_1, \ldots, c_p$.

- Substituting the expression for $\mathbf{v}_p$ into the expression for $\mathbf{x}$, it is easy to see that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}$

- Thus $\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}\}$ spans H, because $\mathbf{x}$ was an arbitrary element of H.

The Spanning Set Theorem

If $H \neq \{\mathbf{0}\}$, some subset of *S* is a basis for *H*.

b. If the original spanning set *S* is linearly independent, then it is already a basis for *H*.

- Otherwise, one of the vectors in *S* depends on the others can be deleted.
- So long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and hence is a basis for *H*.
- If the spanning set is eventually reduced to one vector, that vector will be nonzero (and hence linearly independent) because

$$H \neq \{\mathbf{0}\}$$

The Spanning Set Theorem

- **Example:** Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

  and $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$.

- Show that $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, and then find a basis for the subspace *H*.

- **Solution:** Every vector in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to *H* because

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$$

The Spanning Set Theorem

- Now let **x** be any vector in *H*—say,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

- Since $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, we may substitute

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(5\mathbf{v}_1 + 3\mathbf{v}_2)$$

$$= (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2$$

- Thus **x** is in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, so every vector in *H* already belongs to Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- We conclude that *H* and Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ are actually the same set of vectors.
- It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of *H* since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Basis for Columns

- **Example:** Find a basis for Col *B*, where

$$B = \begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \cdots & \mathrm{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Solution:** Each nonpivot column of *B* is a linear combination of the pivot columns.
- In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.
- By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span Col *B*.

Basis for Columns

- Let

$$S = \{\mathrm{b}_1, \mathrm{b}_3, \mathrm{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Since $\mathbf{b}_1 \neq 0$ and no vector in *S* is a linear combination of the vectors that precede it, *S* is linearly independent.

- Thus *S* is a basis for Col *B*.

Basis for Columns

- **Theorem:** The pivot columns of a matrix A form a basis for Col A.
- **Proof:** Let B be the reduced echelon form of A.
- The set of pivot columns of B is linearly independent, for no vector in the set is a linear combination of the vectors that precede it.
- Since A is row equivalent to B, the pivot columns of A are linearly independent as well, because any linear dependence relation among the columns of A corresponds to a linear dependence relation among the columns of B.

Basis for Columns

- For this reason, every nonpivot column of A is a linear combination of the pivot columns of A.
- Thus the nonpivot columns of A may be discarded from the spanning set for Col A, by the Spanning Set Theorem.
- This leaves the pivot columns of A as a basis for Col A.

- **Warning:** The pivot columns of a matrix A are evident when A has been reduced only to echelon form. But, be careful to use the pivot columns of A itself for the basis of Col A.

The Row Space

- **Theorem:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

- **Proof:** If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A.

- It follows that any linear combination of the rows of B is automatically a linear combination of the rows of A.

The Row Space

- Thus the row space of B is contained in the row space of A.
- Since row operations are reversible, the same argument shows that the row space of A is a subset of the row space of B.
- So the two row spaces are the same.

The Row Space

- If B is in echelon form, its nonzero rows are linearly independent because no nonzero row is a linear combination of the nonzero rows below it.

- Thus the nonzero rows of B form a basis of the (common) row space of B and A.

Bases for Nul A, Col A, and Row A

- **Example:** Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

- **Solution:** To find bases for the row space and the column space, row reduce A to an echelon form:

Bases for Nul *A*, Col *A*, and Row *A*

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- By Theorem, the first three rows of *B* form a basis for the row space of *A* (as well as for the row space of *B*).

- Thus Basis for Row

$$A : \{(1,3,-5,1,5), (0,1,-2,2,-7), (0,0,0,-4,20)\}$$

Bases for Nul *A*, Col *A*, and Row *A*

- For the column space, observe from *B* that the pivots are in columns 1, 2, and 4.
- Hence columns 1, 2, and 4 of *A* (not *B*) form a basis for Col *A*:

$$\text{Basis for Col}\,A:\left\{\begin{bmatrix}-2\\1\\3\\1\end{bmatrix},\begin{bmatrix}-5\\3\\11\\7\end{bmatrix},\begin{bmatrix}0\\1\\7\\5\end{bmatrix}\right\}$$

- Notice that any echelon form of *A* provides (in its nonzero rows) a basis for Row *A* and also identifies the pivot columns of *A* for Col *A*.

Bases for Nul *A*, Col *A*, and Row *A*

- However, for Nul *A*, we need the *reduced echelon form*.

- Further row operations on *B* yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases for Nul A, Col A, and Row A

- The equation $A\mathbf{x} = \mathbf{0}$ is equivalent to $C\mathbf{x} = \mathbf{0}$, that is,

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 2x_3 + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

- So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables.

Bases for Nul A, Col A, and Row A

- The calculations show that

$$\text{Basis for Nul } A: \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

- Observe that, unlike the basis for Col A, the bases for Row A and Nul A have no simple connection with the entries in A itself.