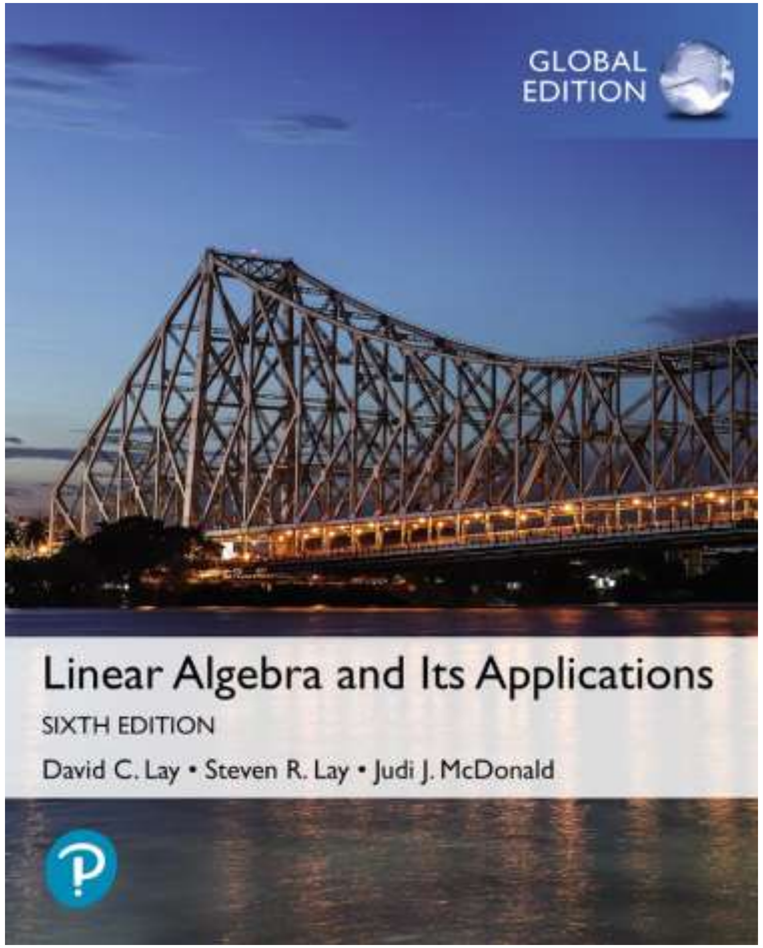
Linear Algebra



Sixth Edition, Global Edition

Chapter 1 Linear Equations in Linear Algebra

Section 1.1: Systems of Linear Equations

Section 1.2: Row Reduction and Echelon Forms

Section 1.3: Vector Equations

Linear Equation

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers that are usually known in advance.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

Linear Equation

- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

- The set of all possible solutions is called the **solution set** of the linear system.
- Two linear systems are called **equivalent** if they have the same solution set.

Linear Equation

- A system of linear equations has

  1. no solution, or
  2. exactly one solution, or
  3. infinitely many solutions.

- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system of linear equation is said to be **inconsistent** if it has no solution.

Matrix Notation

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.
- For the following system of equations,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned} \qquad \text{the matrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the **coefficient matrix** of the system.

Matrix Notation

- An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.
- For the given system of equations,

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the augmented matrix.

Matrix Size

- The size of a matrix tells how many rows and columns it has. If *m* and *n* are positive numbers, an $\mathrm{m} \times \mathrm{n}$ matrix is a rectangular array of numbers with *m* rows and *n* columns. (The number of rows always comes first.)

Solving System of Equations

- **Example:** Solve the given system of equations.

$$
\begin{array}{cr}
x_1 - 2x_2 + x_3 = 0 & ----(1) \\
2x_2 - 8x_3 = 8 & ----(2) \\
-4x_1 + 5x_2 + 9x_3 = -9 & ----(3)
\end{array}
$$

- **Solution:** The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison.

Solving System of Equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- Keep x_1 in the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3.
- The result of this calculation is written in place of the original third equation.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Solving System of Equations

- The result of this calculation is written in place of the original third equation.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Now, multiply equation $2 \text{ by } \frac{1}{2}$ in order to obtain 1 as the as the coefficient for x_2.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Solving System of Equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3.
- The new system has a triangular form.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solving System of Equations

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Elementary Row Operations

- Elementary row operations include the following:
  1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
  2. (Interchange) Interchange two rows.
  3. (Scaling) Multiply all entries in a row by a nonzero constant.

- Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

Elementary Row Operations

- Row operations are reversible.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Questions

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists, is it the only one; that is, is the solution unique?

Existence And Uniqueness of System of Equations

- **Example:** Determine if the following system is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned} \qquad ----(4)$$

- **Solution:** The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Existence And Uniqueness of System of Equations

- To obtain an x_1 in in the first equation, interchange rows 1 and 2.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

- To eliminate the $5x_1$ term in the third equation, add -5/2 times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix} \qquad ----(5)$$

Existence And Uniqueness of System of Equations

- Next, use the x_2 term in the second equation to eliminate the $-(1/2)x_2$ term from the third equation. Add $\frac{1}{2}$ times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \qquad ----(6)$$

- The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation.

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 5/2 \end{aligned} \qquad ----(7)$$

Existence And Uniqueness of System of Equations

- The equation $0 = \frac{5}{2}$ is a short form of $0x_1 + 0x_2 + 0x_3 = \frac{5}{2}$.
- There are no values of x_1, x_2, x_3 that satisfy (7) because the equation $0 = \frac{5}{2}$ is never true.
- Since (7) and (4) have the same solution set, the original system is inconsistent (i.e., has no solution).

Section 1.1: Systems of Linear Equations

Section 1.2: Row Reduction and Echelon Forms

Section 1.3: Vector Equations

Echelon Form

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Echelon Form

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

  4. The leading entry in each nonzero row is 1.
  5. Each leading 1 is the only nonzero entry in its column.

- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

Echelon Form

Theorem: Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

Pivot Position

- If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) **of A**; if U is in reduced echelon form, we call U **the reduced echelon form of A**.

- A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Pivot Position

- **Example**: Row reduce the matrix *A* below to echelon form, and locate the pivot columns of *A*.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Solution**: The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position.

Pivot Position

- Now, interchange rows 1 and 4.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

- Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.

Pivot Position

- Choose 2 in the second row as the next pivot.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Next pivot column} \end{array}$$

- Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

Pivot Position

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- There is no way a leading entry can be created in column 3. But, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot Position

$$\begin{array}{c}
\text{Pivot} \\
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\text{Pivot columns}
\end{array}$$

- The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of *A* are pivot columns.

Pivot Position

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot positions

Pivot columns

- The pivots in the example are 1, 2 and -5.

Row Reduction Algorithm

- **Example:** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Solution:**
- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Row Reduction Algorithm

$$\begin{array}{l}\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \\ \uparrow \text{Pivot column} \end{array}$$

- **STEP 2:** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Row Reduction Algorithm

- Interchange rows 1 and 3. (Rows 1 and 2 could have also been interchanged instead.)

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 3:** Use row replacement operations to create zeros in all positions below the pivot.

Row Reduction Algorithm

- We could have divided the top row by the pivot, 3, but with two 3s in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

- **STEP 4**: Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

Row Reduction Algorithm

- With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

- For step 3, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the “top” row to the row below.

Row Reduction Algorithm

- This produces the following matrix.

$$\begin{bmatrix} \boxed{3 & -9 & 12 & -9 & 6 & 15} \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row Reduction Algorithm

- Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. We perform one more step to obtain the reduced echelon form.
- **STEP 5**: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.
- The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \leftarrow \text{Row } 1 + (-6) \times \text{row } 3 \\ \leftarrow \text{Row } 2 + (-2) \times \text{row } 3 \\ \\ \end{matrix}$$

- The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{ Row scaled by } \frac{1}{2}$$

Row Reduction Algorithm

- Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row } 1 + (9) \times \text{row } 2$$

- Finally, scale row 1, dividing by the pivot, 3.

Row Reduction Algorithm

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{3}$$

- This is the reduced echelon form of the original matrix.
- The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

Solutions of Linear Systems

- Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solutions of Linear Systems

- There are 3 variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad ---(1)$$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable, x_3, is called a **free variable**.

Solutions of Linear Systems

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \qquad\qquad ---(2) \\ x_3 &\ \ \text{is free} \end{aligned}$$

- The statement "x_3 Is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (2) determine the values for x_1 and x_2. For instance, when $x_3 = 0,$ the solution is (1,4,0); when $x_3 = 1,$ the solution is (6,3,1).

- Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.

Parametric Descriptions of Solution Sets

- The description in (2) is a parametric description of solutions sets in which the free variables act as parameters.
- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

Parametric Descriptions of Solution Sets

- For instance, in system (1), add 5 times equation 2 to equation 1 and obtain the following equivalent system.

$$x_1 + 5x_2 = 21$$
$$x_2 + x_3 = 4$$

- We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set.
- When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

Existence and Uniqueness Theorem

Theorem: Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \ \ldots \ 0 \ b] \text{ with } b \text{ nonzero.}$$

Row Reduction to Solve A Linear System

- **Using Row Reduction to Solve a Linear System**

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

Row Reduction to Solve A Linear System

4. Write the system of equations corresponding to the matrix obtained in step 3.

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Section 1.1: Systems of Linear Equations

Section 1.2: Row Reduction and Echelon Forms

Section 1.3: Vector Equations

Vector Equations

Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.
- An example of a vector with two entries is

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where w_1 and w_2 are any real numbers.

- The set of all vectors with 2 entries is denoted by $\mathbb{R}^2$ (read “r-two”).

Vector Equations

- **Example**: Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$.

Solution: $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$, $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$ and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

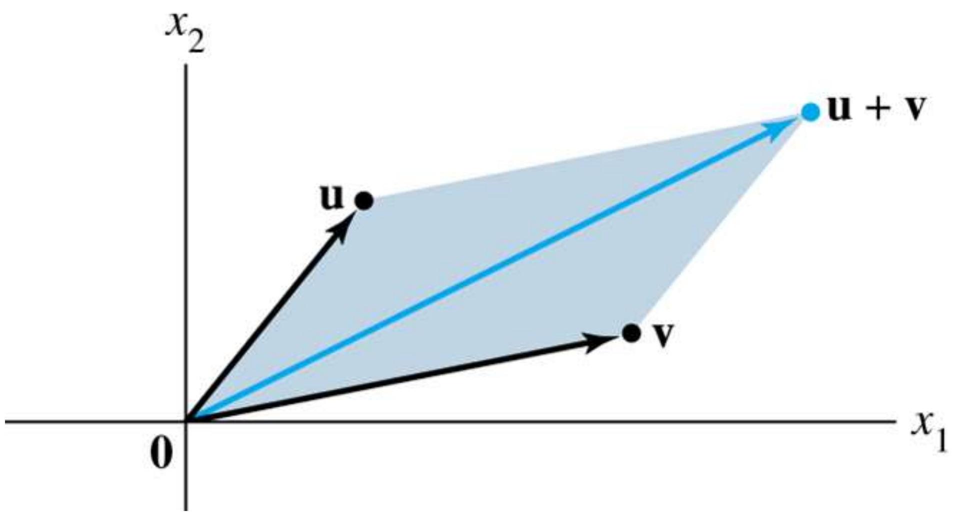
Geometric Descriptions of Double Lined R Squared

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point (a, b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

- So we may regard $\mathbb{R}^2$ as the set of all points in the plane.

Parallelogram Rule For Addition

- If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then **u** + **v** corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See the figure below.



Vectors In Double Lined R Cube and Double Lined R to the n Power

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If *n* is a positive integer, $\mathbb{R}^n$ (read “r-n”) denotes the collection of all lists (or **ordered *n*-tuples**) of *n* real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Algebraic Properties Of Double Lined R to the n Power

- The vector whose entries are all zero is called the **zero vector** and is denoted by **0**.
- For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars *c* and *d*:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ where $-\mathbf{u}$ denotes$(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=(cd)(\mathbf{u})$

(viii) $1\mathbf{u}=\mathbf{u}$

Linear Combinations

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p,$ the vector **y** defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

  is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.
- The weights in a linear combination can be any real numbers, including zero.

Linear Combinations

- Example: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

  Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathrm{b} \qquad --- (1)$$

  If vector equation (1) has a solution, find it.

Linear Combinations

Solution: Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$\underset{\mathbf{a}_1}{x_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}} + \underset{\mathbf{a}_2}{x_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}} = \underset{\mathbf{b}}{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}},$$

which is same as $$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

Linear Combinations

$$\text{and} \quad \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \qquad ---(2)$$

x_1 and x_2 make the vector equation (1) true if and only if satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \qquad ---(3)$$

Linear Combinations

- To solve this system, row reduce the augmented matrix of the system as follows.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3 \text{ and } x_2 = 2$. Hence **b** is a linear combination of $\mathbf{a}_1 \text{ and } \mathbf{a}_2,$ with weights $x_1 = 3 \text{ and } x_2 = 2.$ That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

Linear Combinations

- **A vector equation**

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

In Particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix.

Linear Combinations

- **Definition:** If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) by $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.