Chapter 3 Determinants

Section 3.1: Introduction to Determinants

Section 3.2: Properties of Determinants

Section 3.3: Cramer’s rule, Volume

Introduction to Determinants

- **Definition**: Let A be a 2×2 matrix, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. The **determinant** of A is given by,

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

- **Definition**: The matrix A_{ij} is formed from the matrix A by removing the i-th row and j-th column of A.

Example If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$, and $A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$

Introduction to Determinants

- **Definition**: For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Introduction to Determinants

- **Example:** Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute

  det $A = a_{11}$det $A_{11} – a_{12}$det $A_{12} + a_{13}$det A_{13}:

$$= 1 \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$= 1(0 – 2) – 5(0 – 0) + 0(– 4 – 0) = – 2$

Introduction to Determinants

- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets.
- Thus the calculation in Example 1 can be written as

$$\det A = 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

- To state the next theorem, it is convenient to write the definition of det A in a slightly different form. Given
  $A = [a_{ij}]$, the **(*i*, *j*)-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Introduction to Determinants

- Then $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
- This formula is called a **cofactor expansion across the first row** of A.
- **Theorem**: The determinant of an $n \times n$ matrix A can be computed by a cofactor across any row or down any column.
- The expansion across the i th row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

- The cofactor expansion down the j th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Introduction to Determinants

- **Example:** Use a cofactor expansion across the third row to compute det *A*, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute det *A*

$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$= (-1)^{3\,+\,1}a_{31}\det A_{31} + (-1)^{3\,+\,2}a_{32}\det A_{32} + (-1)^{3\,+\,2}a_{33}\det A_{33}$

$= 0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$

$= 0 + 2(-1) + 0 = -2$

Introduction to Determinants

- **Theorem:** If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.
- **Example:** Use a cofactor down the first column to compute det A, where

$$A = \begin{bmatrix} 3 & 5 & 0 \\ 0 & -4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

- **Solution** Compute det A

$$= a_{11}C_{11} + 0C_{21} + 0C_{31}$$

$$= (-1)^{1+1}(3)\begin{vmatrix} -4 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= (3)(-4)(2) = -24$$

Section 3.1: Introduction to Determinants

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Properties of Determinants

- **Theorem**: Let A be a square matrix
  a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
  b) If two rows of A are interchanged to produce B, then
  $$\det B = -\det A.$$
  c) If one row of A is multiplied by k to produce B, then
  $$\det B = k \det A.$$

Properties of Determinants

- **Example：** Compute det *A*, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

- **Solution**

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

Properties of Determinants

- **Theorem**: A square matrix A is invertible if and only if $\det A \neq 0$.
- **Example** Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$
- **Solution**

$$\det A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 0 & 0 & 0 \\ -5 & -8 & 0 & 9 \end{vmatrix} = 0$$

Column Operations

- **Theorem:** If A is a $n \times n$ matrix, then det A^T = det A.
- **Proof**: The theorem is obvious for $n = 1$. Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$.

Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T, because the cofactors involve $k \times k$ determinants.

Hence the cofactor expansion of det A along the first row equals the cofactor expansion of det A^T down the first column. That is, A and A^T have equal determinants.

Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of n implies its truth for the next value of n. By the principle of induction, the theorem is true for all $n \geq 1$.

Determinants and Matrix Products

- **Theorem:** If A and B are $n \times n$ matrices, then det AB = (det A)(det B).
- **Example** Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

Solution

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

So det $AB = 25(13) - 20(14) = 325 - 280 = 45$

Since det A = 9 and det B = 5,

$$(\det A)(\det B) = 9(5) = 45 = \det AB$$

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Cramer's rule

- **Theorem**: Let A be an invertible $n \times n$ matrix. For any **b** in $\mathbb{R}^n$, the unique solution **x** of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

where $A_i(\mathbf{b})$ is formed by replacing column i of A by **b**.

- **Proof** Denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and the columns of the $n \times n$ identity matrix I by $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

Cramer's rule

- If $A\mathbf{x} = \mathbf{b}$, the definition of matrix multiplication shows that
$$A\, I_i(\mathbf{x}) = A[\mathbf{e}_1 \ \dots \ \mathbf{x} \ \dots \ \mathbf{e}_n] = [A\mathbf{e}_1 \ \dots \ A\mathbf{x} \ \dots \ A\mathbf{e}_n]$$
$$= [\mathbf{a}_1 \ \dots \ \mathbf{b} \ \dots \ \mathbf{a}_n] = A_i(\mathbf{b})$$
- By the multiplicative property of determinants,
$$(\det A)\left(\det I_i(\mathbf{x})\right) = \det Ai(b)$$
- The second determinant on the left is simply x_i. Hence $(\det A)x_i = \det A_i(\mathbf{b})$. This proves (1) because A is invertible and $\det A \neq 0$.

Cramer's rule

- **Example：** Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

- **Solution** View the system as $A\mathbf{x} = \mathbf{b}$. Using the notation introduced above, $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$,

$$A_1(\mathbf{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \qquad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

- Since det A = 2, the system has a unique solution.

Cramer’s rule

- By Cramer’s rule,

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{24+16}{2} = 20$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{24+30}{2} = 27$$

A Formula for A^{-1}

- The adjugate matrix is the *transpose* of the matrix of cofactors.

- **Theorem**: Let A be an invertible $n \times n$ matrix. Then
$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

A Formula for A^{-1}

- **Example:** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

- **Solution** The nine cofactors are

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

Thus

$$\text{adj } A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

A Formula for A^{-1}

- We could compute det A directly, but the following computation provides a check on the calculations above and produces det A:

$$(\text{adj } A)\, A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14\, I$$

A Formula for A^{-1}

- Since (adj A) A = 14 I, Theorem 8 shows that det A = 14 and

$$A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

Determinants as Area or Volume

- **Theorem**: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

- **Proof** The theorem is obviously true for any 2×2 diagonal matrix:

$$\left|\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right| = |ad| = \left\{ \begin{matrix} \textit{area of} \\ \textit{rectangle} \end{matrix} \right\}$$

Determinants as Area or Volume

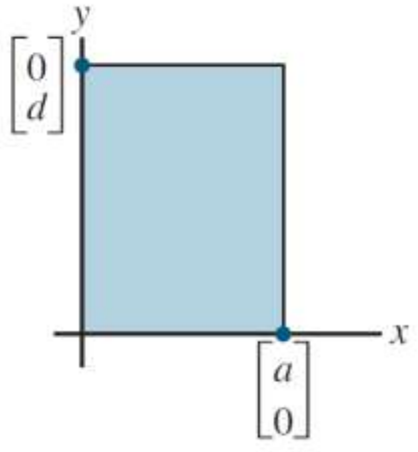

FIGURE 1
Area $= |ad|$.

- It will suffice to show that any 2×2 matrix $A = [\mathbf{a}_1 \ \ \mathbf{a}_2]$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$.

Determinants as Area or Volume

- It suffices to prove the following simple geometric observation that applies to vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$:
- Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2+c\mathbf{a}_1$.
- To prove this statement, we may assume that $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$, for otherwise the two parallelograms would be degenerate and have zero area.
- If L is the line through $\mathbf{0}$ and $\mathbf{a}_1$, then $\mathbf{a}_2 + L$ is the line through $\mathbf{a}_2$ parallel to L, and $\mathbf{a}_2 + c\mathbf{a}_1$ is on this line.

Determinants as Area or Volume

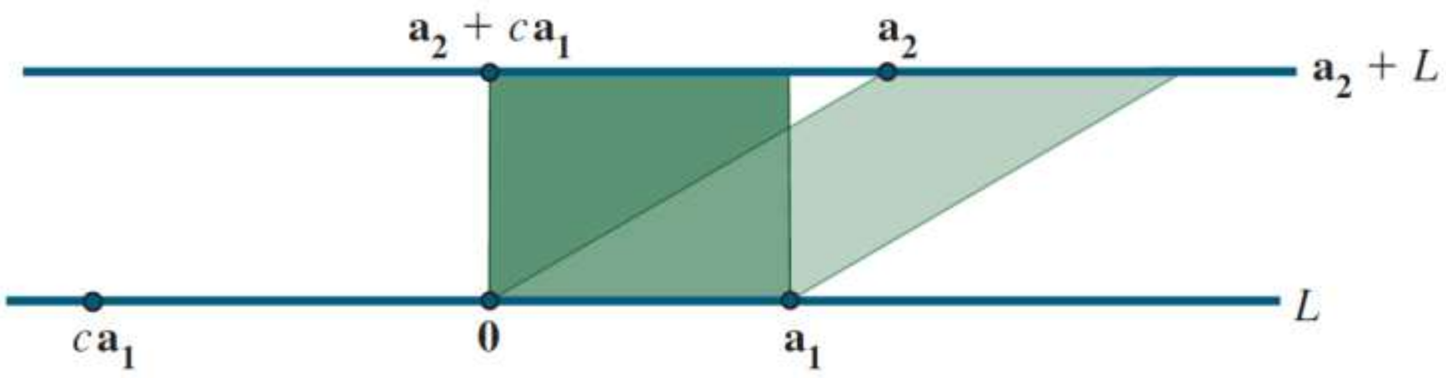

FIGURE 2 Two parallelograms of equal area.

- The points $\mathbf{a}_2$ and $\mathbf{a}_2 + c\mathbf{a}_1$ have the same perpendicular distance to L. Hence the two parallelograms have the same area, since they share the base from $\mathbf{0}$ to $\mathbf{a}_1$.

Determinants as Area or Volume

- The proof for $\mathbb{R}^3$ is similar. The theorem is obviously true for a 3×3 diagonal matrix.

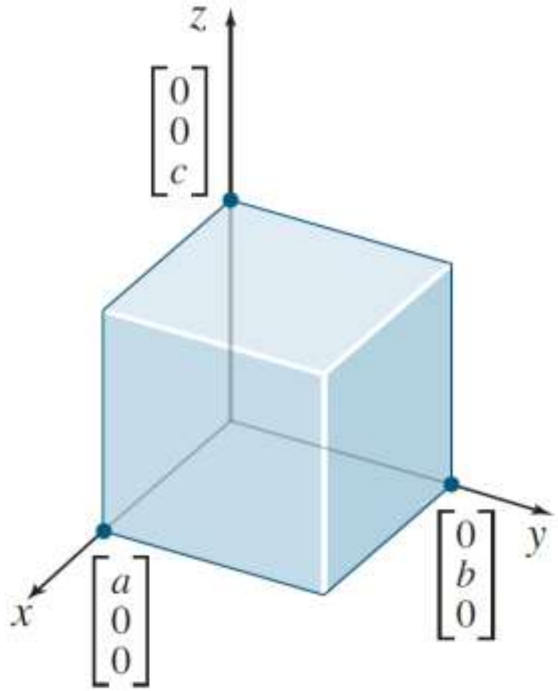

FIGURE 3

Volume $= |abc|$.

Determinants as Area or Volume

- And any 3×3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det A|$.
- A parallelepiped is shown below as a shaded box with two sloping sides.

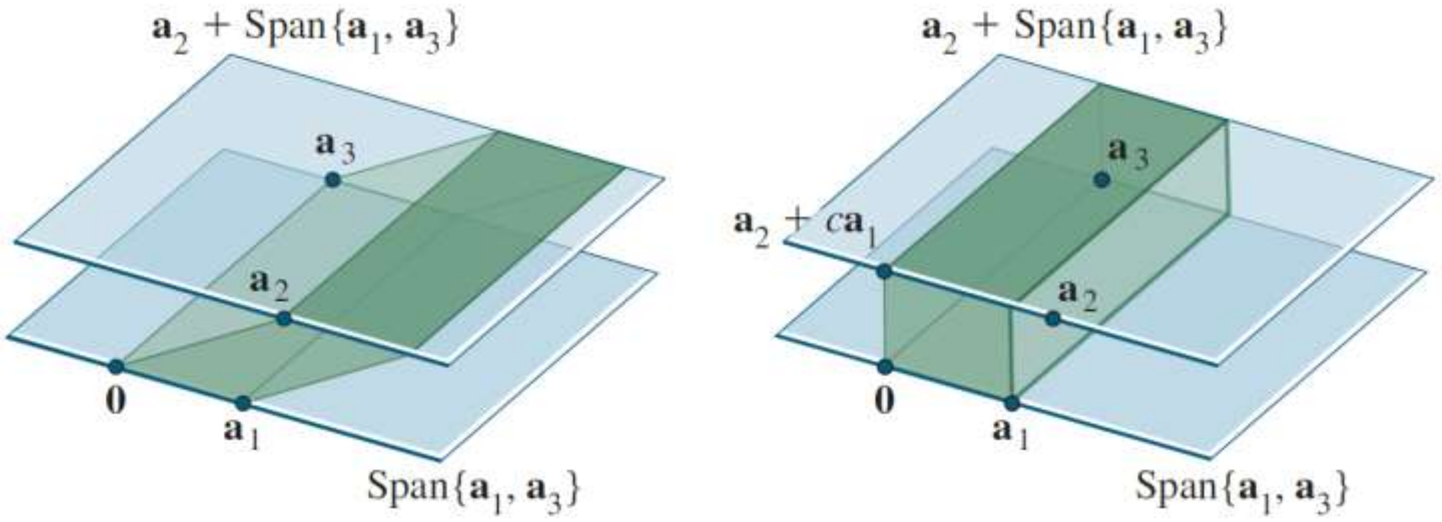

FIGURE 4 Two parallelepipeds of equal volume.

Determinants as Area or Volume

- Its volume is the area of the base in the plane Span{$\mathbf{a}_1$, $\mathbf{a}_3$} times the altitude of $\mathbf{a}_2$ above Span{$\mathbf{a}_1$, $\mathbf{a}_3$}. Any vector $\mathbf{a}_2 + c\mathbf{a}_1$ lies in the plane Span{$\mathbf{a}_1$, $\mathbf{a}_3$}, which is parallel to Span{$\mathbf{a}_1$, $\mathbf{a}_3$}.
- Hence the volume of the parallelepiped is unchanged when $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3]$ is changed to $[\mathbf{a}_{1,} \;\; \mathbf{a}_2 + c\mathbf{a}_{1,} \;\; \mathbf{a}_3]$.
- Thus a column replacement operation does not affect the volume of the parallelepiped. Since the column interchanges have no effect on the volume, the proof is complete.

Determinants as Area or Volume

- **Example 4** Calculate the area of the parallelogram determined by the points (-2, -2), (0, 3), (4, -1), and (6, 4).

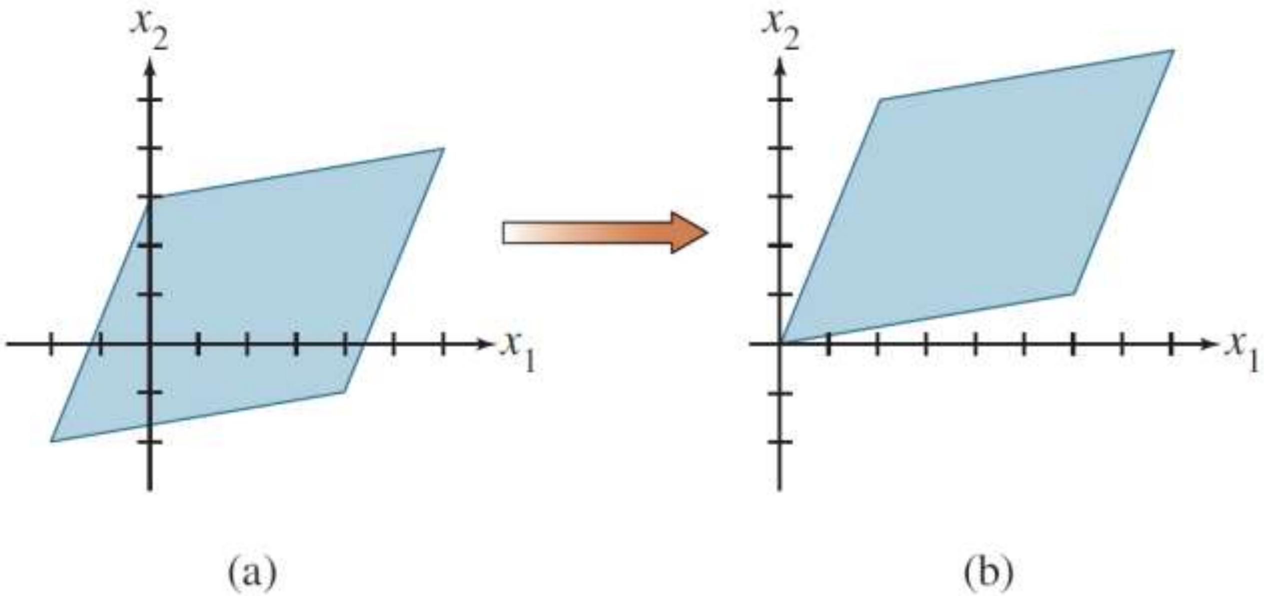

FIGURE 5 Translating a parallelogram does not change its area.

Determinants as Area or Volume

- **Solution** First translate the parallelogram to one having the origin as a vertex. For example, subtract the vertex (-2, -2) from each of the four vertices.
- The new parallelogram has the same area, and its vertices are (0, 0), (2, 5), (6, 1), and (8, 6).
- This parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

- Since $|\det A| = |-28|$, the area of the parallelogram is 28.