香港科技大学(广州)
THE HONG KONG
UNIVERSITY OF SCIENCE AND
TECHNOLOGY (GUANGZHOU)

Design and Analysis of Algorithms

Jing Tang | DSAA 2043 Fall 2024

Greedy Algorithms

Today

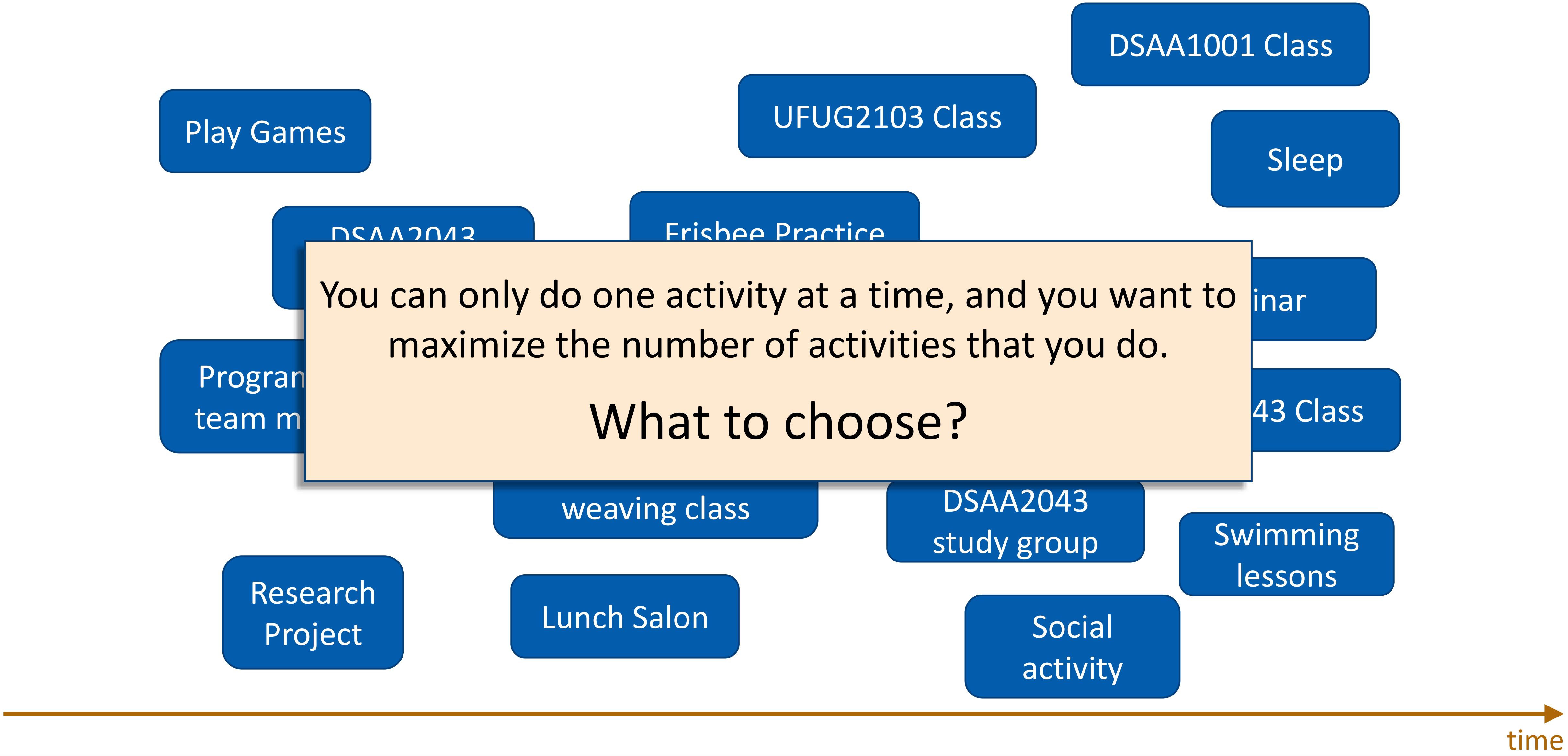
One example of a greedy algorithm that **does not work**:

- Knapsack

Three examples of greedy algorithms that **do work**:

- Activity Selection
- Job Scheduling
- Minimum Spanning Tree

Example where greedy works

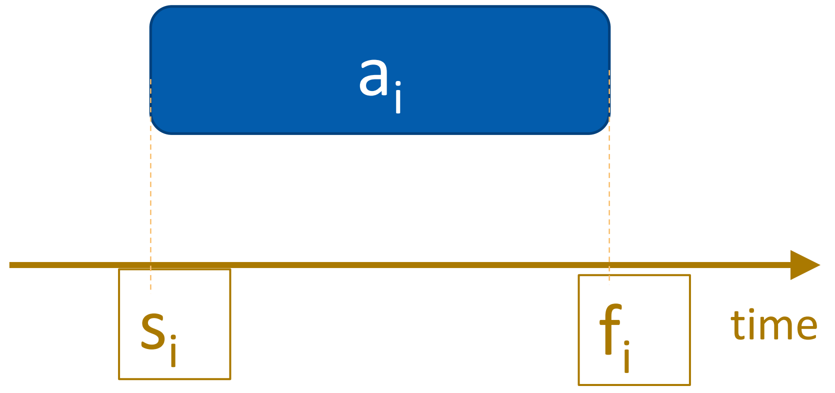


You can only do one activity at a time, and you want to maximize the number of activities that you do.

What to choose?

Activity selection

- Input:
  - Activities $a_1, a_2, \ldots, a_n$
  - Start times $s_1, s_2, \ldots, s_n$
  - Finish times $f_1, f_2, \ldots, f_n$

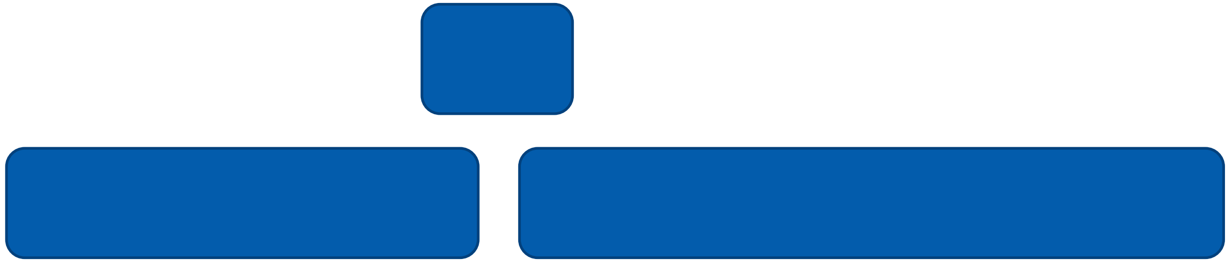


- Output:
  - A way to maximize the number of activities you can do today.

In what order should you greedily add activities?

In what order?

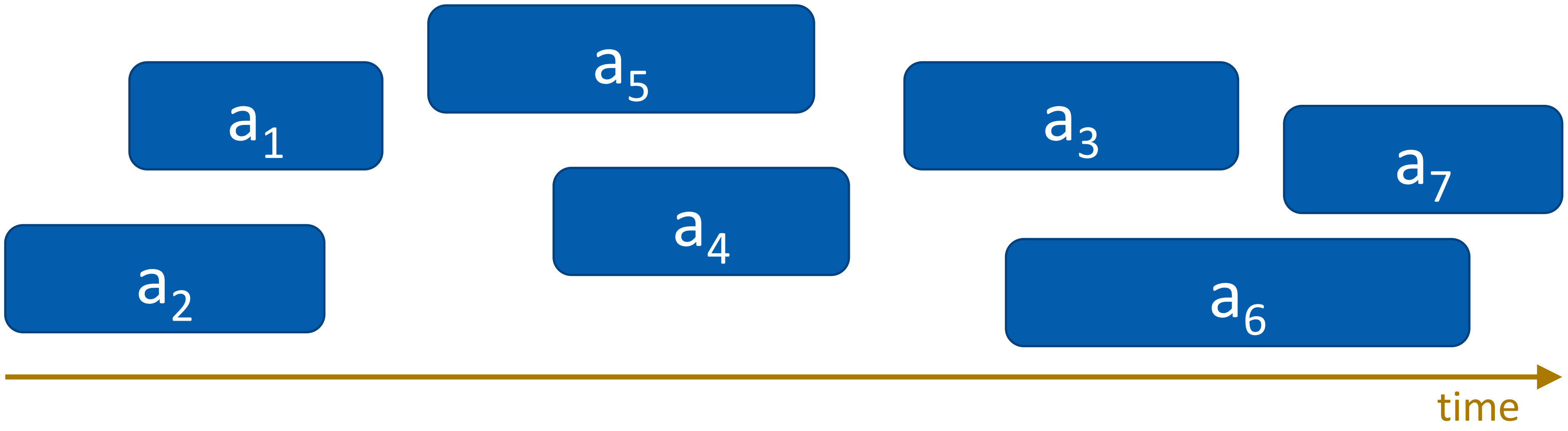
- Shortest job first?



- Earliest start time?

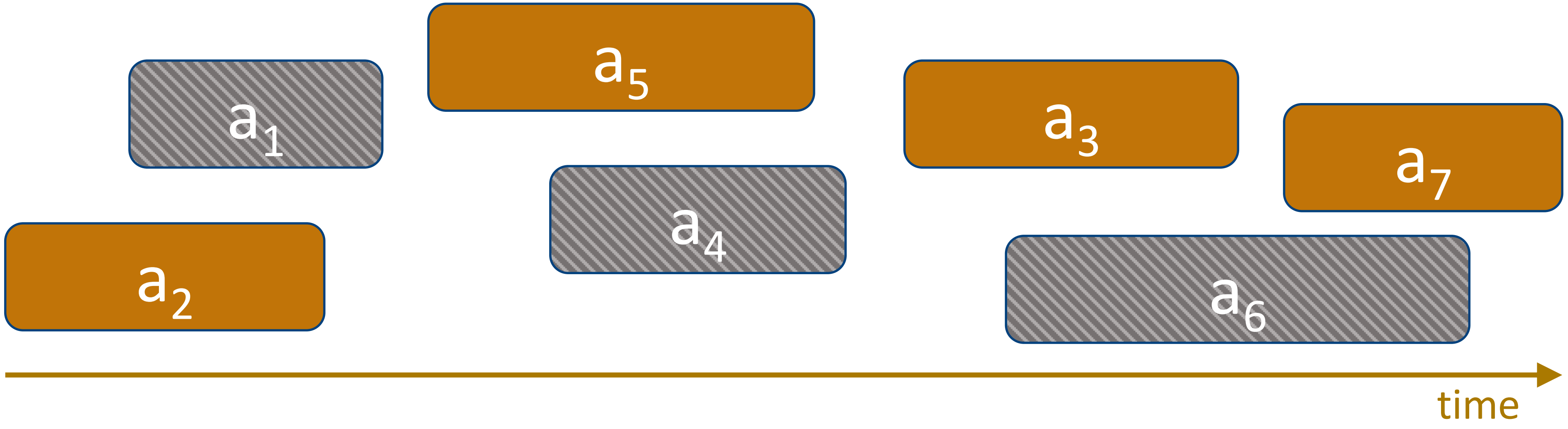
- Earliest finish time?

Greedy Algorithm



- Pick activity you can add with the smallest finish time.
- Repeat.

Greedy Algorithm



- Pick activity you can add with the smallest finish time.
- Repeat.

Efficiency

- Running time:
  - O(n) if the activities are already sorted by finish time.
  - Otherwise, O(n log(n)) if you have to sort them first.

Back to Activity Selection

Why does it work?

- **We never rule out an optimal solution**
- At the end of the algorithm, we've got some solution.
- So it must be optimal.

A Common Strategy

A common strategy for proving the correctness of greedy algorithms:

- Make a **series of choices**.
- Show that, at each step, our choice **won't rule out an optimal solution** at the end of the day.
- After we've made all our choices, we haven't ruled out an optimal solution, **so we must have found one.**

A Common Strategy

- Inductive Hypothesis:
  - After greedy choice t, you haven't ruled out success.
- Base case:
  - Success is possible before you make any choices.
- Inductive step:
  - If you haven't ruled out success after choice t, then you won't rule out success after choice t+1.
- Conclusion:
  - If you reach the end of the algorithm and haven't ruled out success then you must have succeeded.

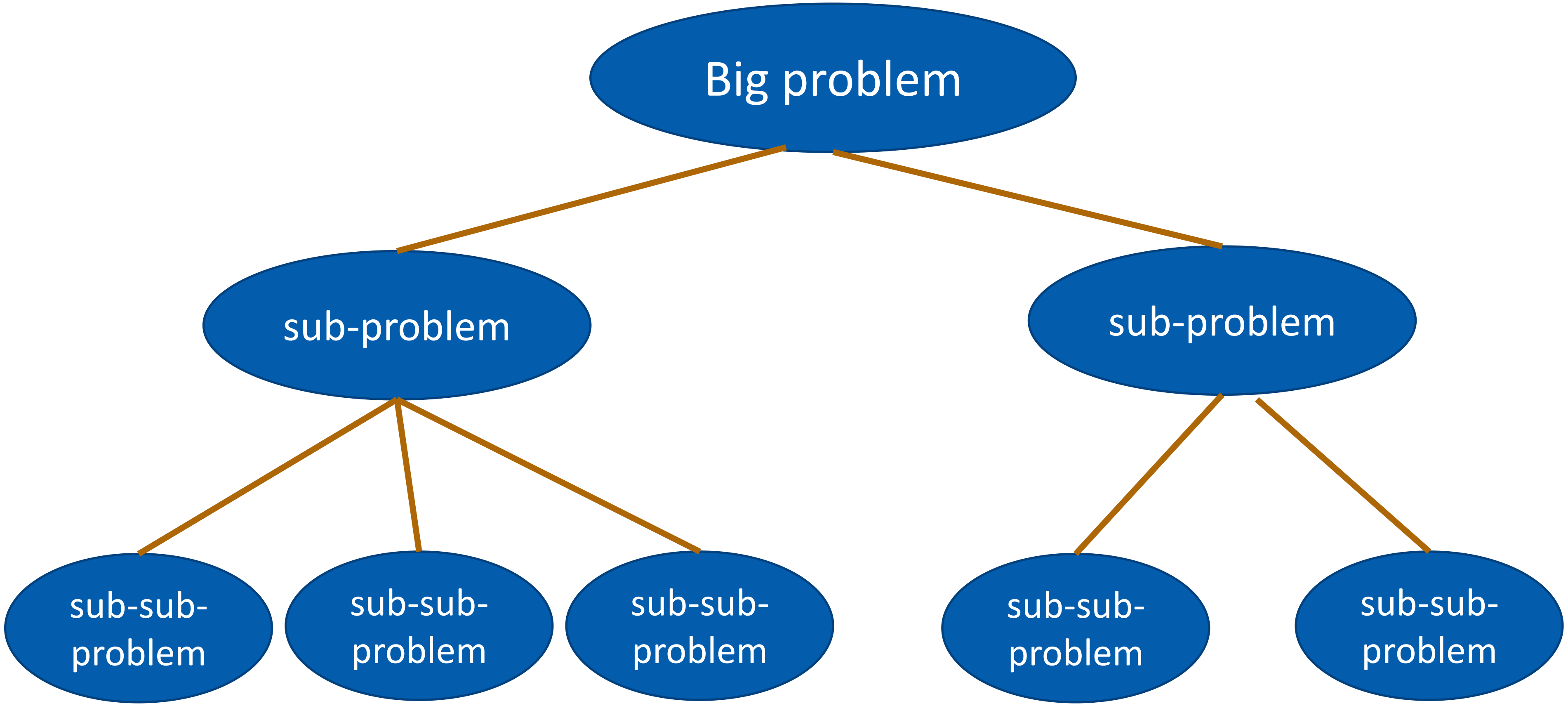
A Common Strategy

A common strategy for showing we don’t rule out the optimal solution:

- Suppose that you’re on track to make an optimal solution T*.
  - E.g., after you’ve picked activity i, you’re still on track.
- Suppose that T* *disagrees* with your next greedy choice.
  - E.g., it *doesn’t* involve activity k.
- Manipulate T* in order to make a solution T that’s not worse but that *agrees* with your greedy choice.
  - E.g., swap whatever activity T* did pick next with activity k.

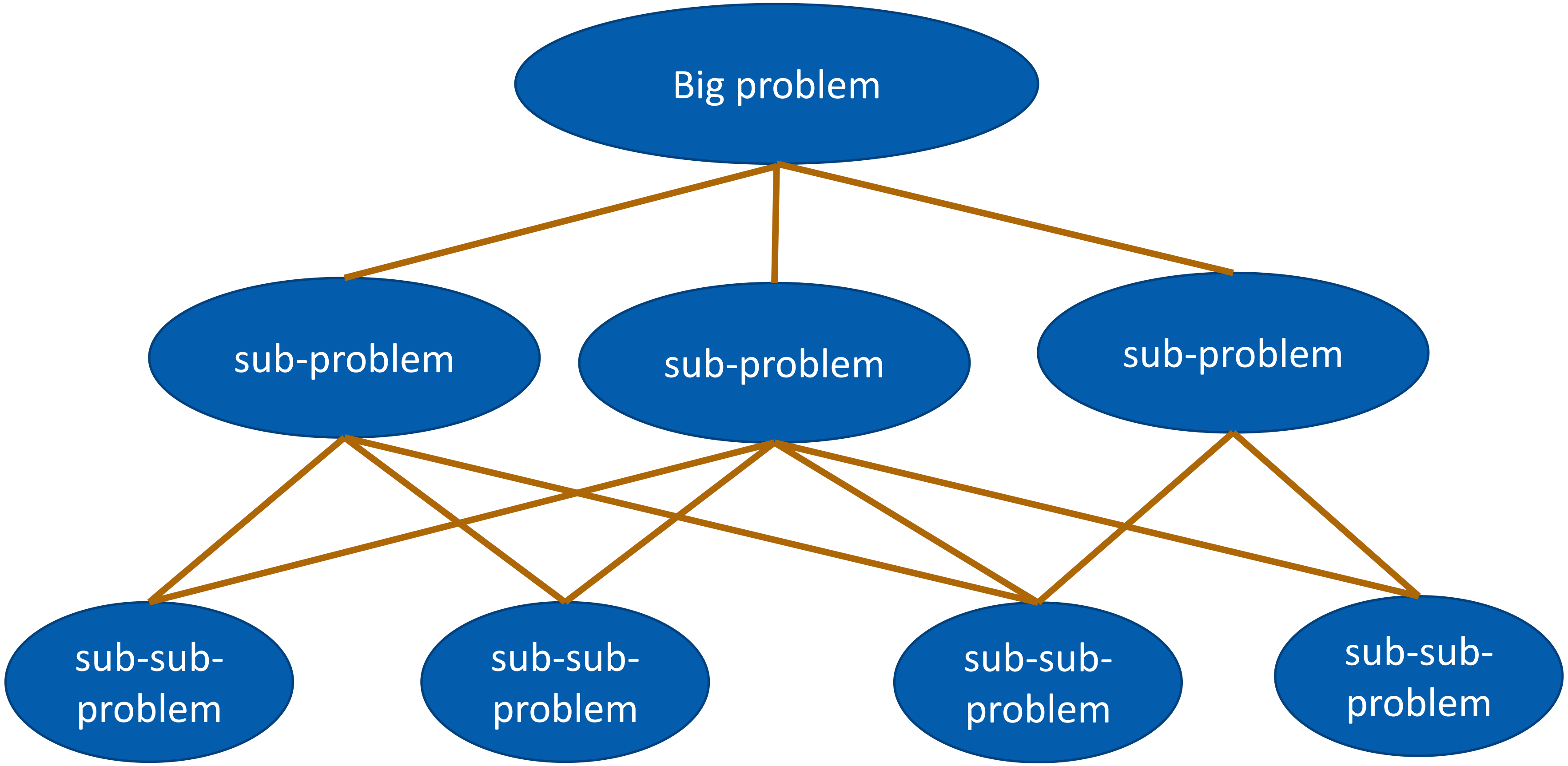
Sub-problem graph view

- Divide-and-conquer:



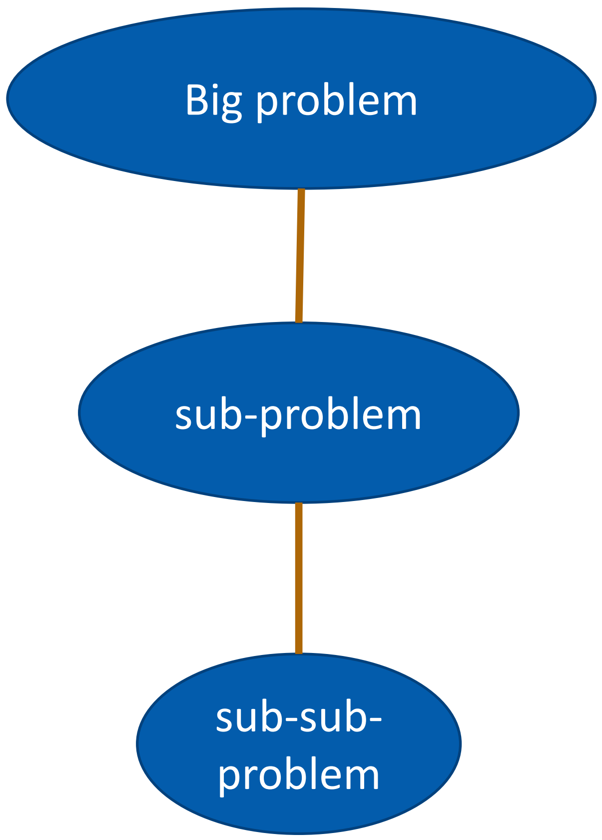
Sub-problem graph view

- Dynamic Programming:



Sub-problem graph view

- Greedy algorithms:



- Not only is there **optimal sub-structure:**
  - optimal solutions to a problem are made up from optimal solutions of sub-problems

- but each problem **depends on only one sub-problem**.

Another Example: Scheduling

- DSAA2043 HW
- Personal hygiene
- Math HW
- Administrative stuff for student club
- Econ HW
- Do laundry
- Sports
- Practice musical instrument
- Read lecture notes
- Have a social life
- Sleep

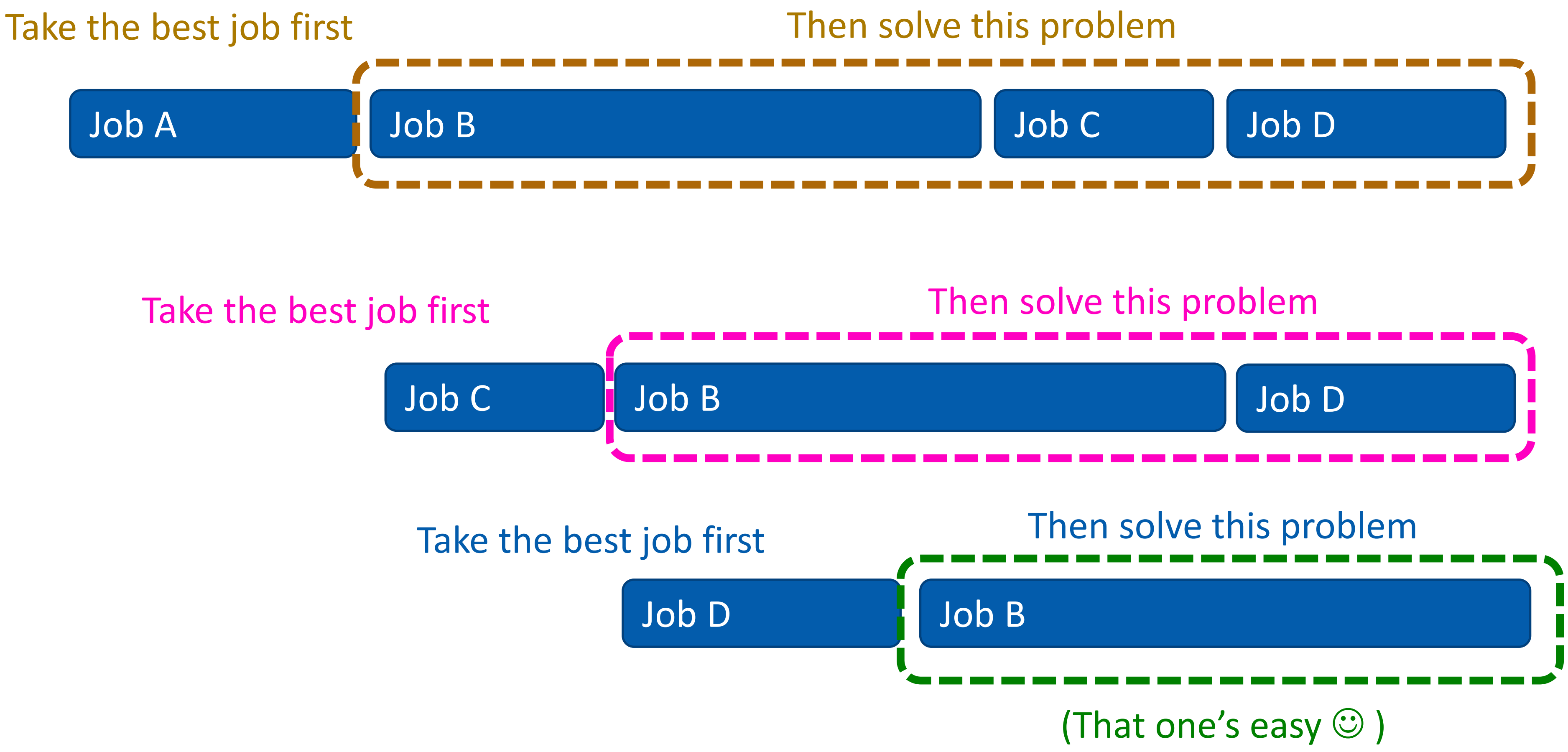
Scheduling

- n tasks
- Task i takes t_i hours
- For every hour that passes until task i is done, pay c_i

- DSAA2043 HW, then Sleep: costs $\mathbf{10 \cdot 2 + (10 + 8) \cdot 3 = 74}$ **units**
- Sleep, then DSAA2043 HW: costs $\mathbf{8 \cdot 3 + (10 + 8) \cdot 2 = 60}$ **units**

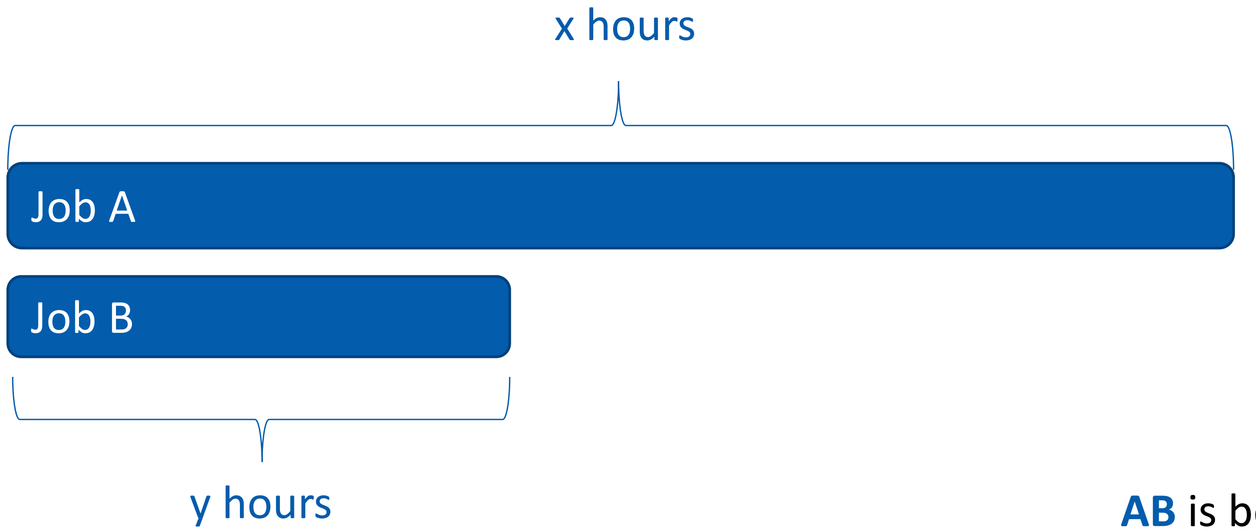
Scheduling

- Seems amenable to a greedy algorithm:



What does "best" mean?

- Of these two jobs, which should we do first?



Cost: z units per hour until it's done.

Cost: w units per hour until it's done.

- Cost(**A then B**) = x · z + (x + y) · w
- Cost(**B then A**) = y · w + (x + y) · z

AB is better than **BA** when:

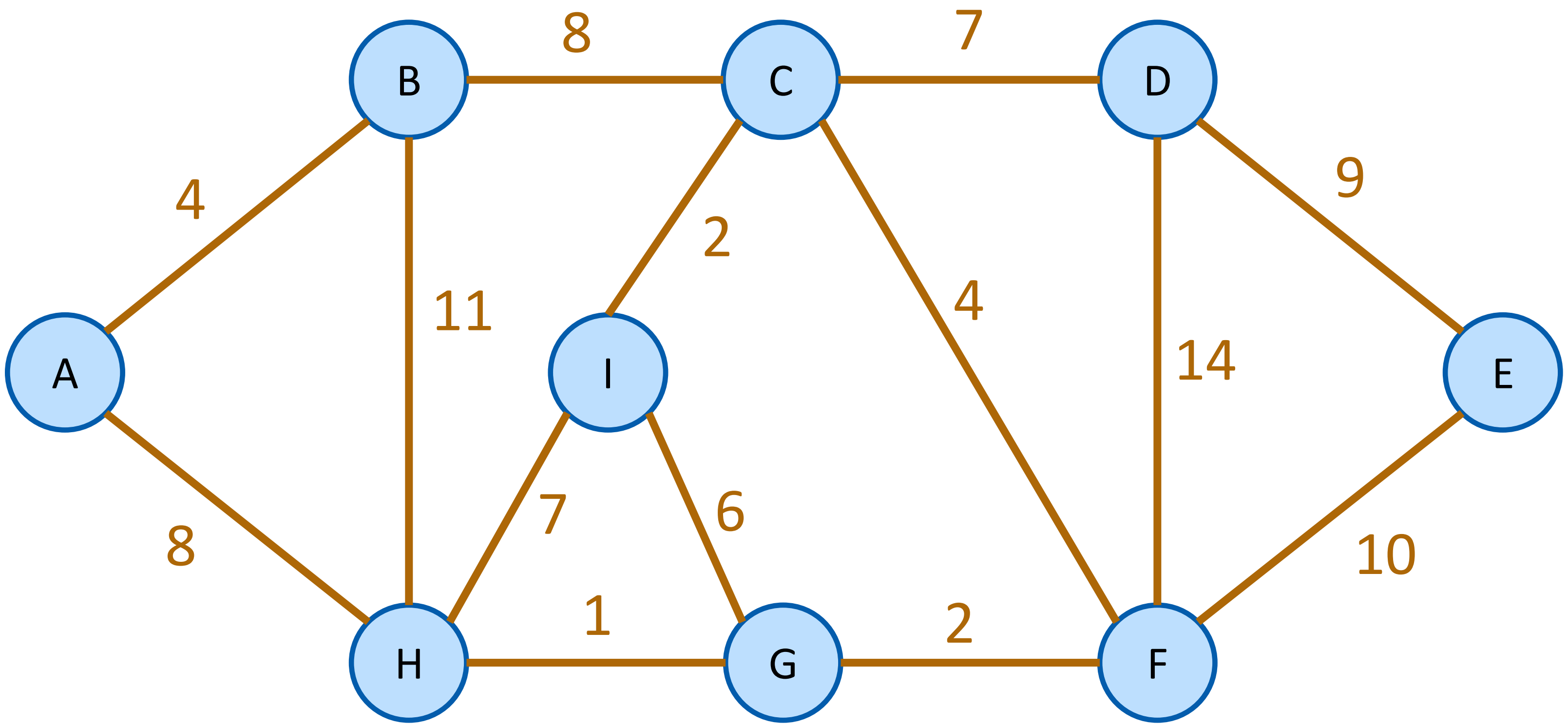
$$xz + (x+y)w \leq yw + (x+y)z$$
$$xz + xw + yw \leq yw + xz + yz$$
$$wx \leq yz$$
$$\frac{w}{y} \leq \frac{z}{x}$$

- Choose the job with the biggest $\dfrac{\text{cost of delay}}{\text{time it takes}}$ ratio.

Minimum Spanning Trees

Minimum Spanning Trees

- Say we have an undirected weighted graph



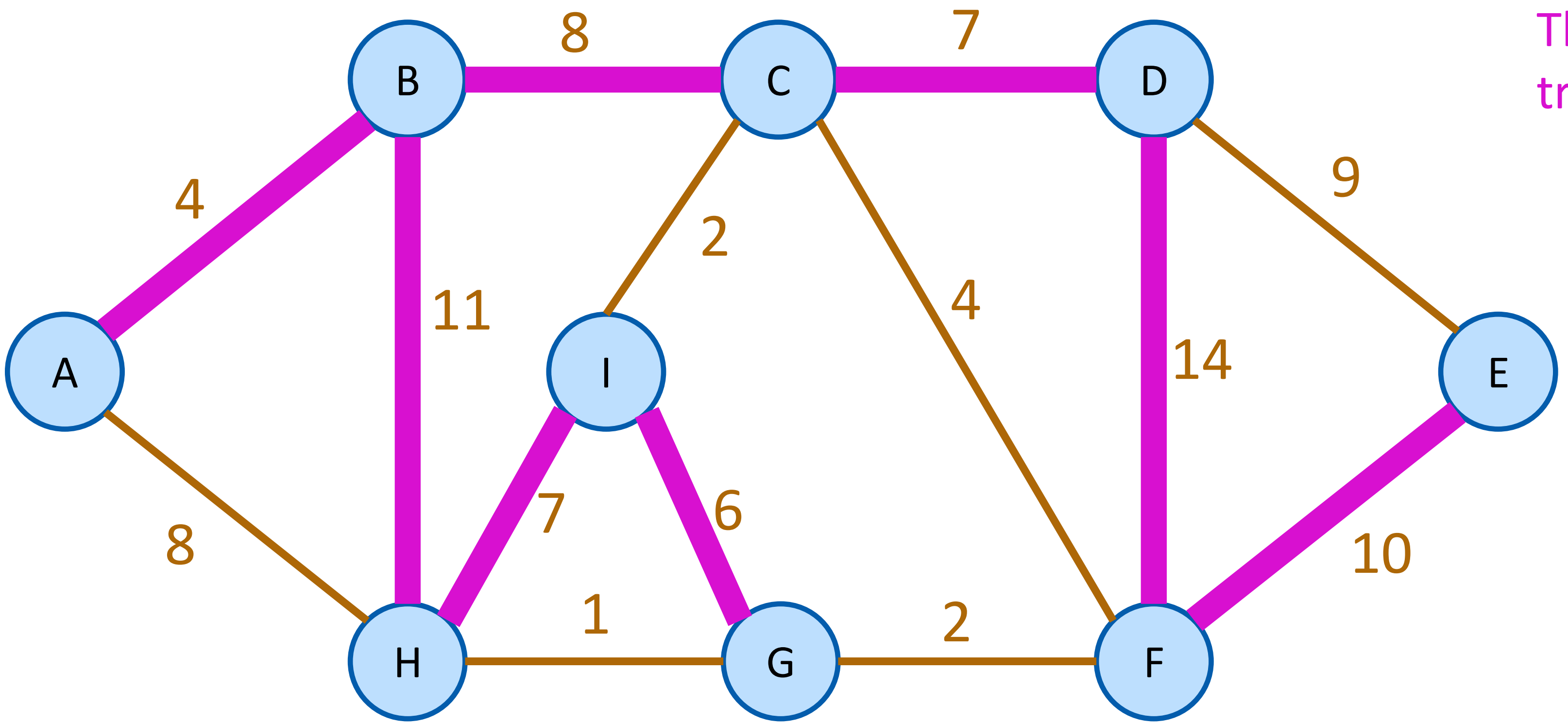
A **spanning tree** is a **tree** that connects all of the vertices.

A **tree** is a connected graph with no cycles!

Minimum Spanning Trees

- Say we have an undirected weighted graph

The **cost** of a spanning tree is the sum of the weights on the edges.

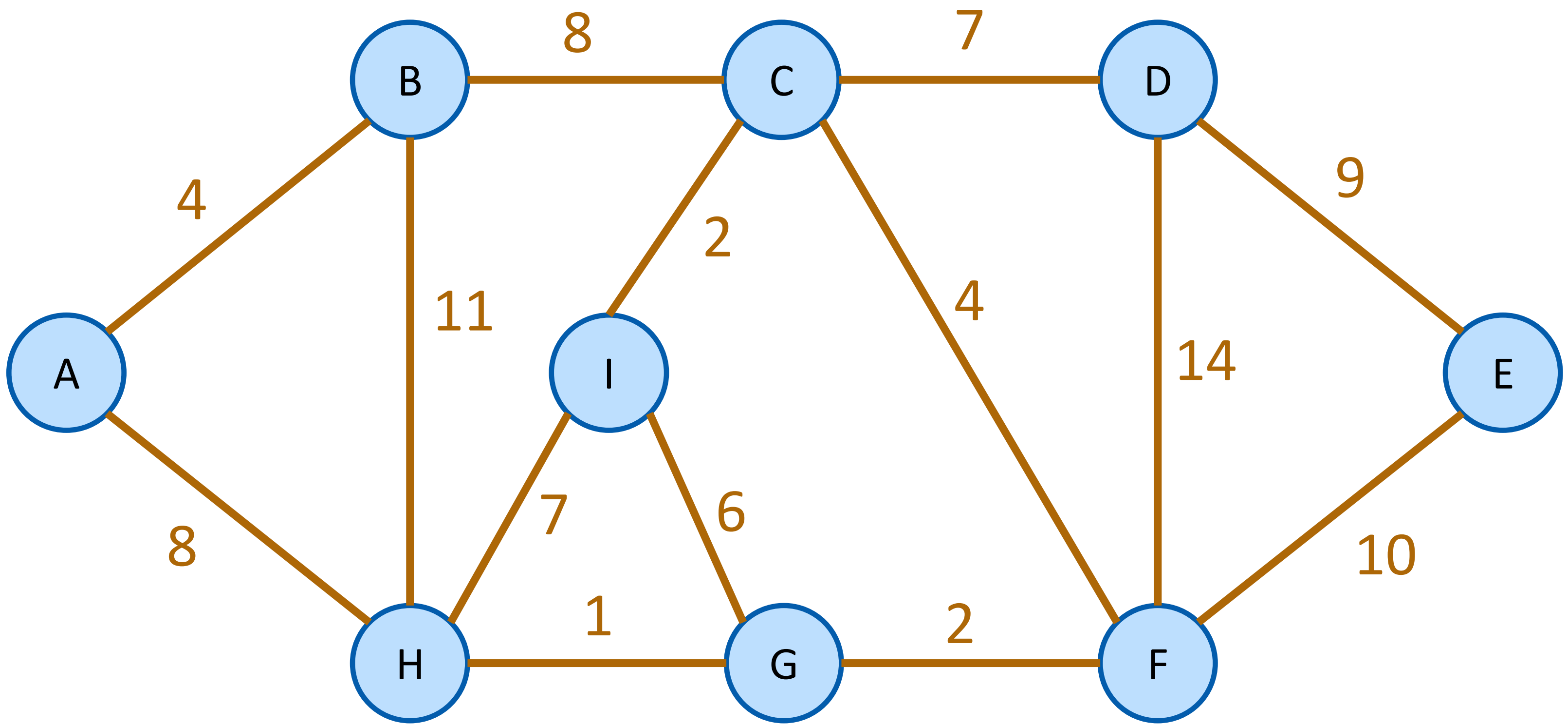


This is a spanning tree with cost 67.

A **spanning tree** is a **tree** that connects all of the vertices.

Minimum Spanning Trees

- Say we have an undirected weighted graph



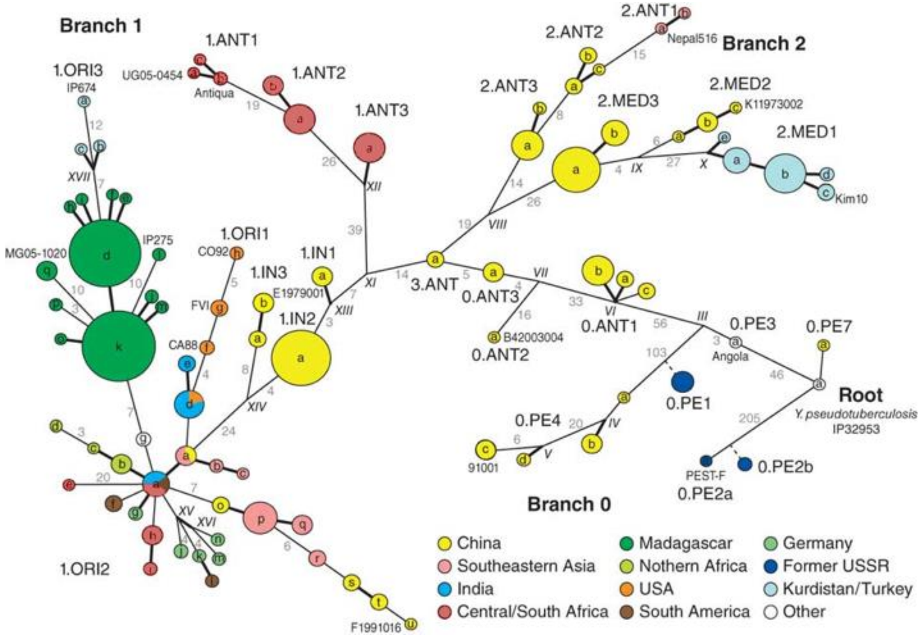
minimum

of minimum cost

A **spanning tree** is a **tree** that connects all of the vertices.

Why MSTs?

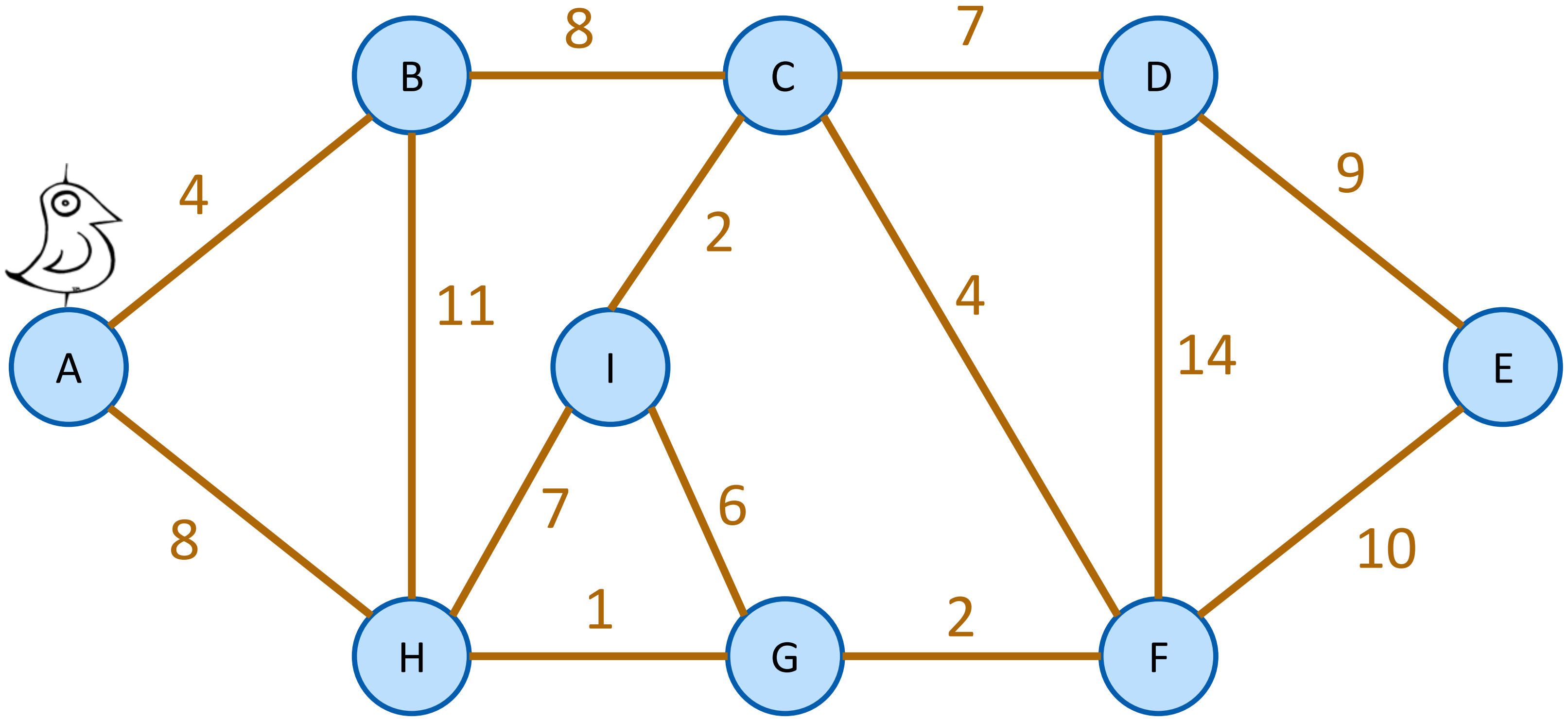
- Network design
  - Connecting cities with roads/electricity/telephone/…
- Cluster analysis
  - E.g., genetic distance
- Image processing
  - E.g., image segmentation
- Useful primitive
  - For other graph algs



How to find an MST

Idea:

Start growing a tree, greedily add the shortest edge we can to grow the tree.



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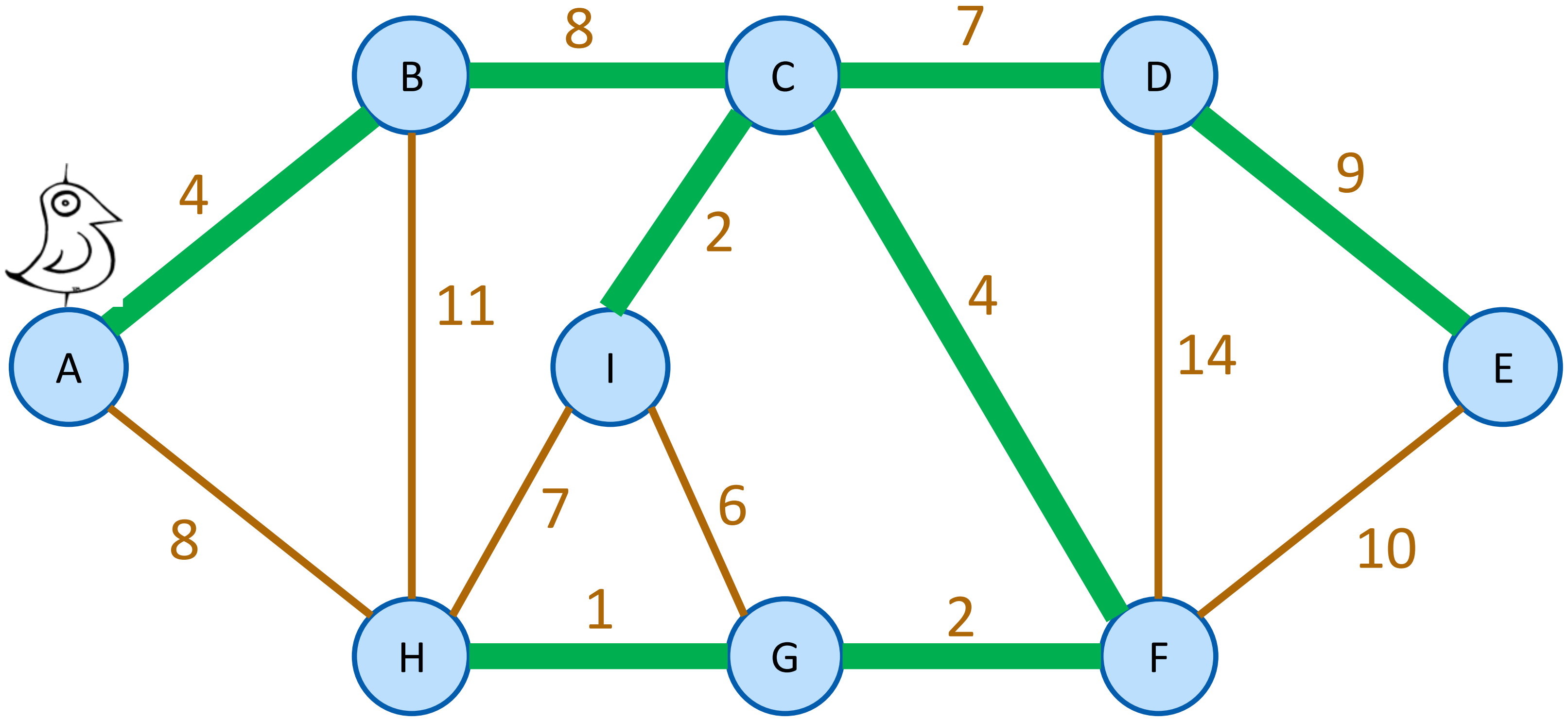


We've discovered Prim's algorithm!

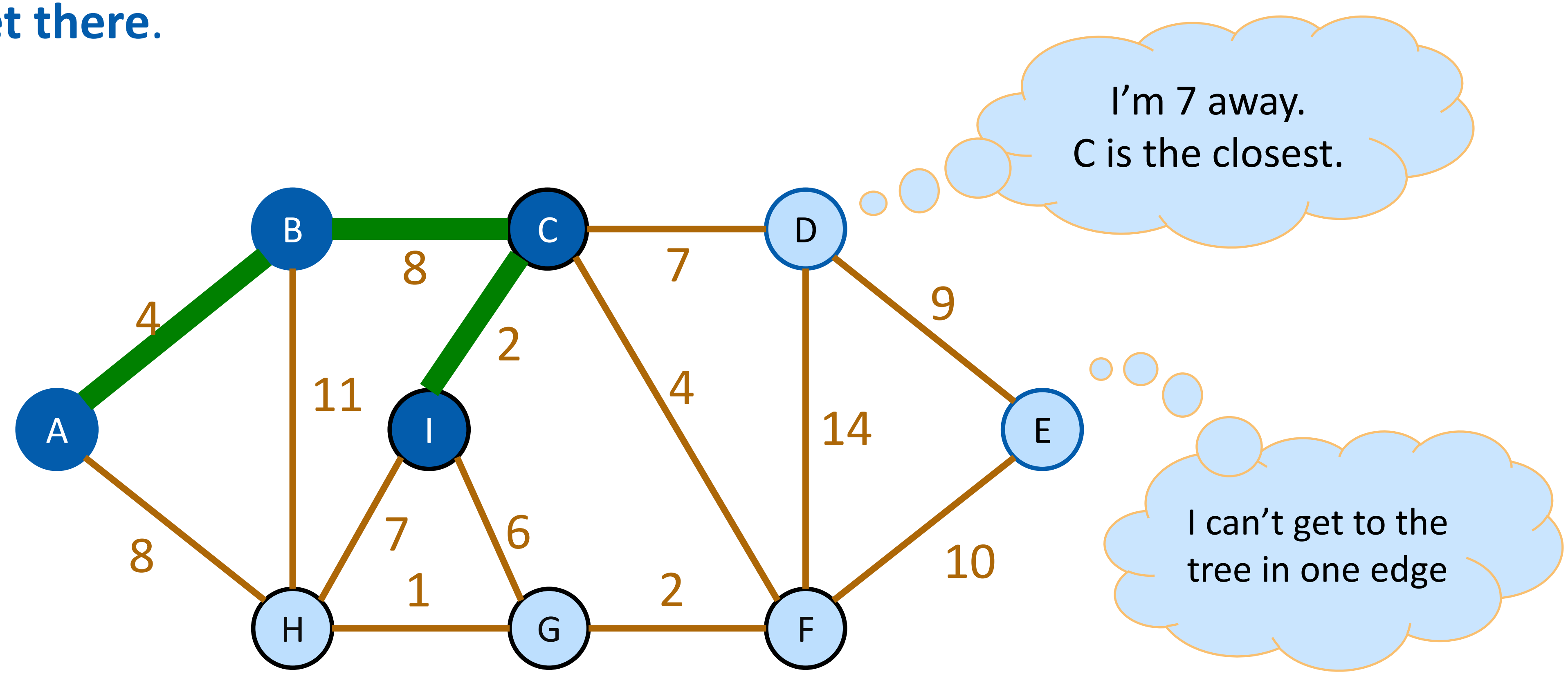
- slowPrim(G = (V,E), starting vertex s):
  - MST = {}
  - verticesVisited = { s }
  - **while** |verticesVisited| < |V|:
    - find the lightest edge {x,v} in E so that:
      - x is in verticesVisited
      - v is not in verticesVisited
    - add {x,v} to MST
    - add v to verticesVisited
  - **return** MST

Naively, the running time is O(nm):

- For each of ≤n-1 iterations of the while loop:
  - Go through all the edges.

Efficient Implementation

- Each vertex keeps:
  - the **(single-edge) distance** from itself to the **growing spanning tree**
  - **how to get there**.



Efficient Implementation

- Each vertex keeps:
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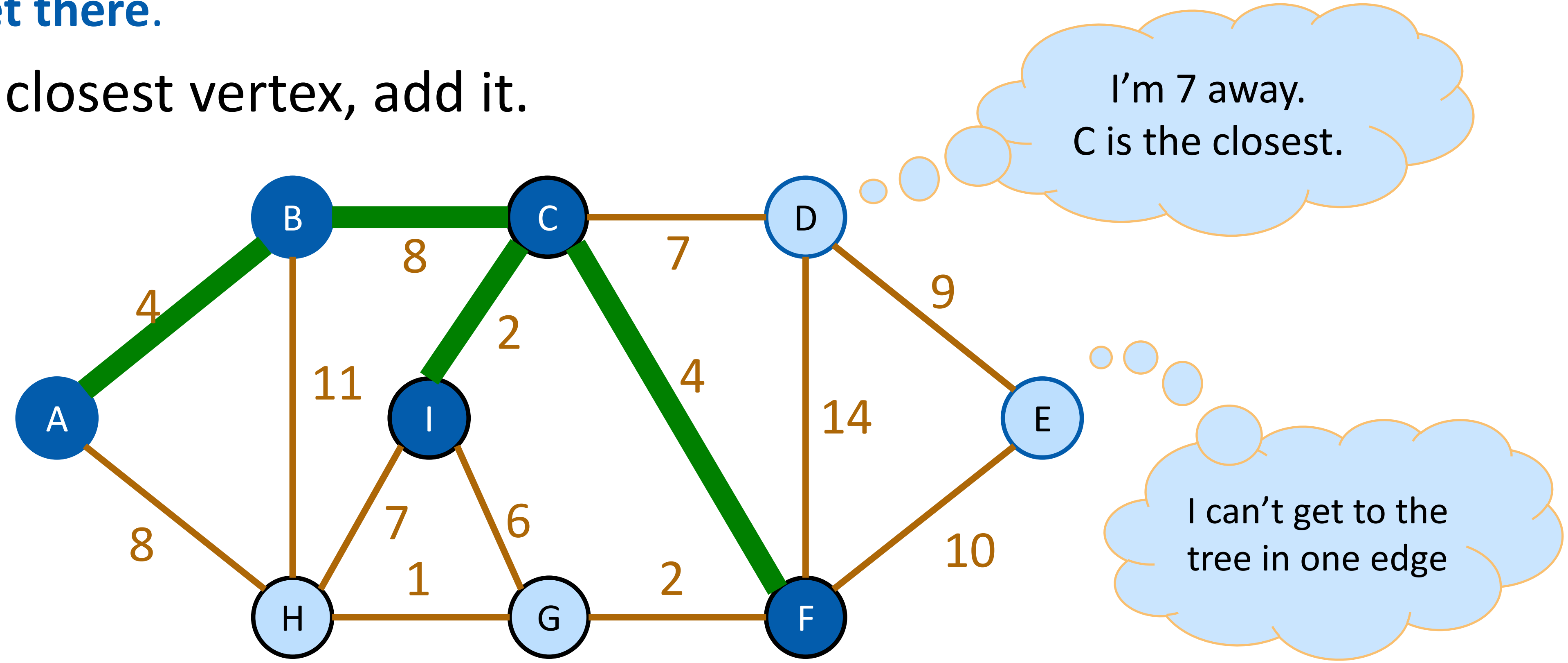
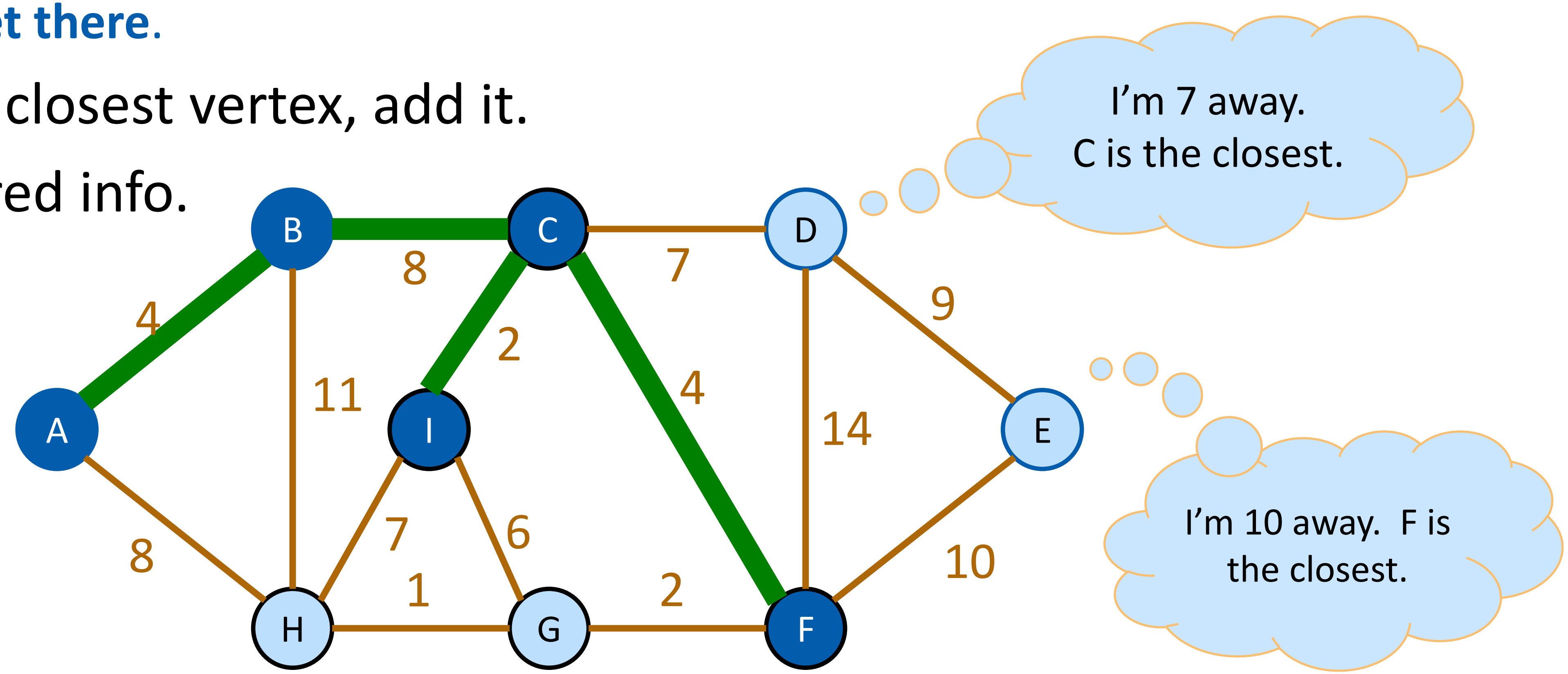
- Choose the closest vertex, add it.



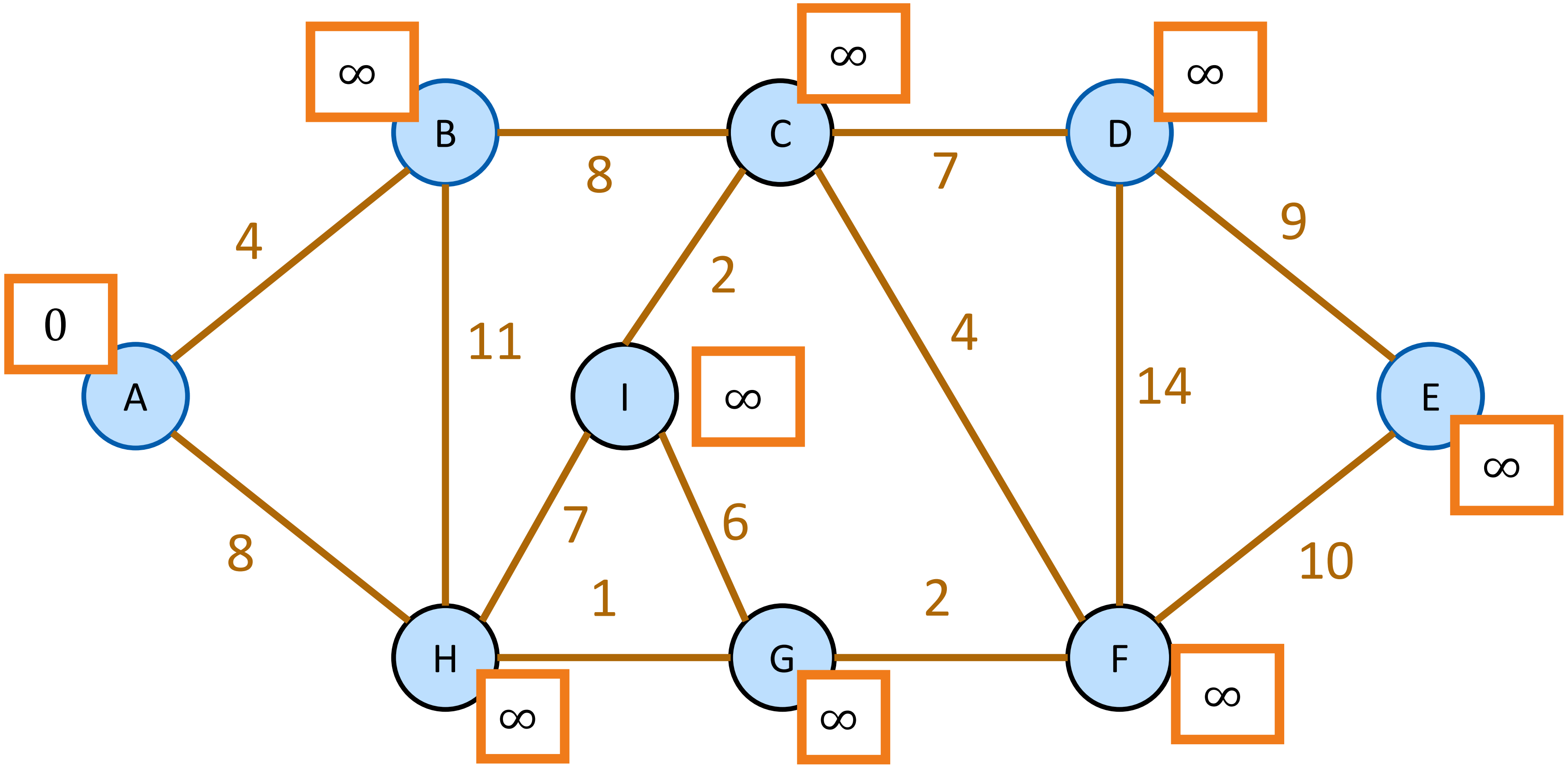
Efficient Implementation

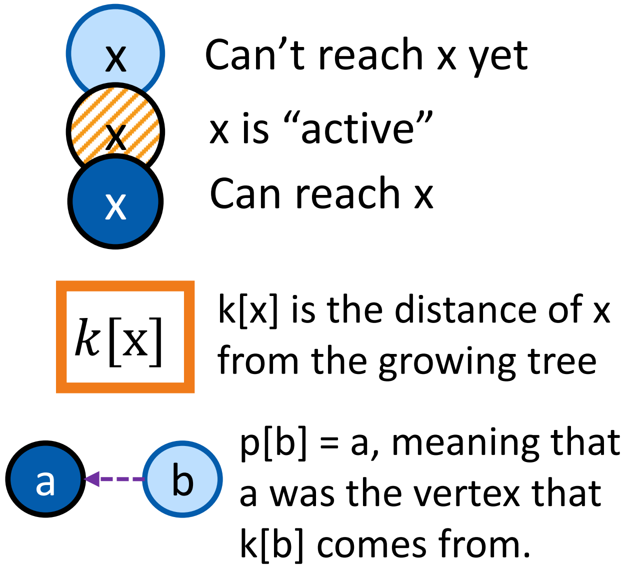
- Each vertex keeps:
  - the **(single-edge) distance** from itself to the **growing spanning tree**
  - **how to get there**.
- Choose the closest vertex, add it.
- Update stored info.



Efficient Implementation

Every vertex has a key and a parent





Until all the vertices are **reached:**

- Activate the **unreached** vertex u with the smallest key.
- **for each** of u’s unreached neighbors v:
  - k[v] = min(k[v], weight(u,v))
  - if k[v] updated, p[v] = u

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Efficient Implementation

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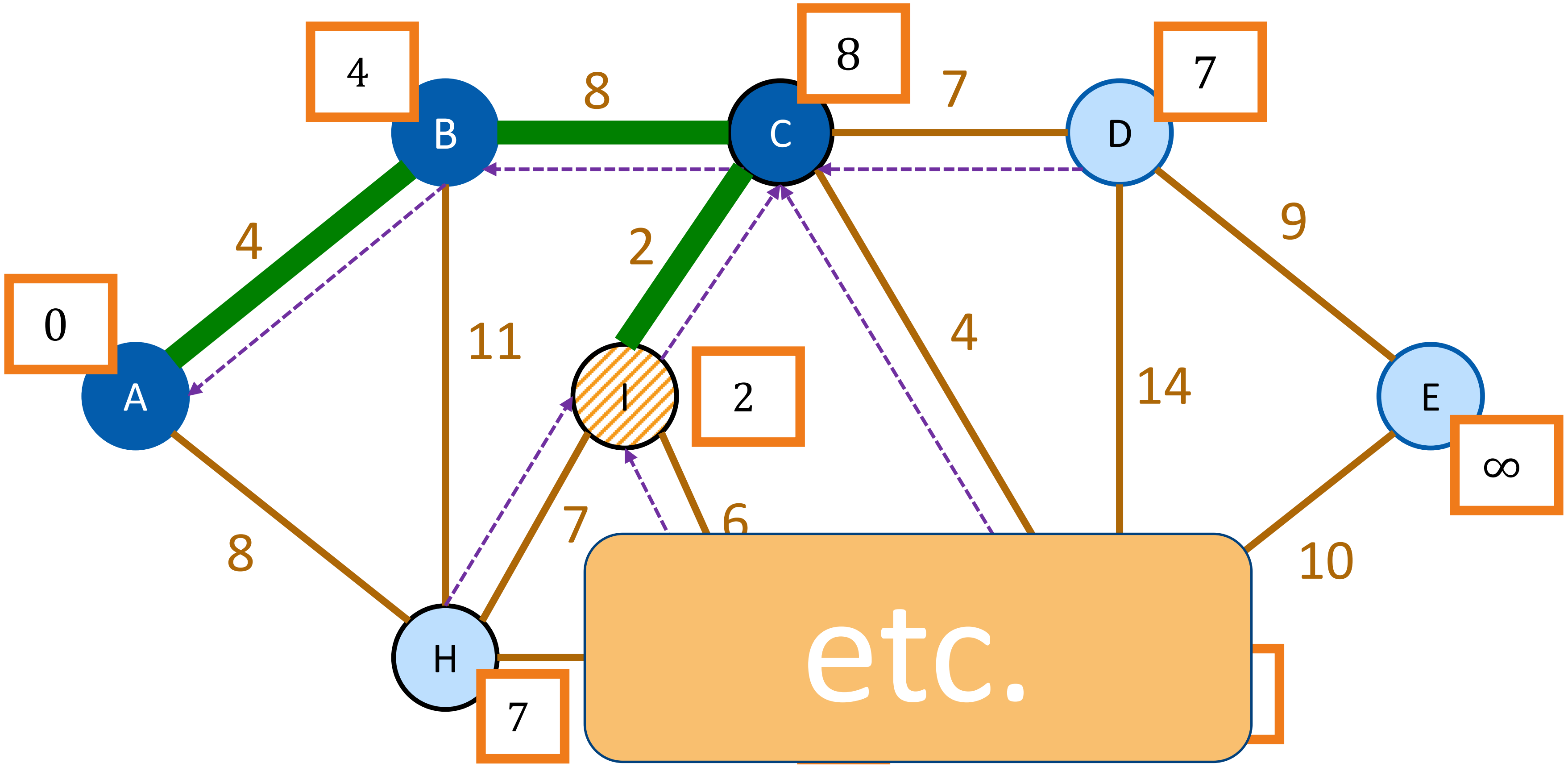

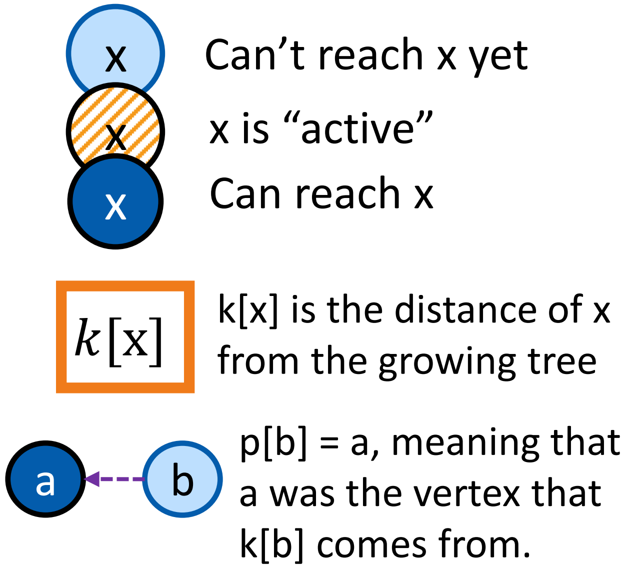

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Prim's Algorithm

- Very similar to Dijkstra's algorithm!
- **Differences:**
  1. Keep track of p[v] in order to return a tree at the end
     - But Dijkstra's can do that too, that's not a big difference.
  2. Instead of d[v] which we update by
     - d[v] = min(d[v], d[u] + w(u,v))

     we keep k[v] which we update by
     - k[v] = min(k[v], w(u,v))

Thing 2 is the main difference.

Kruskal's Algorithm

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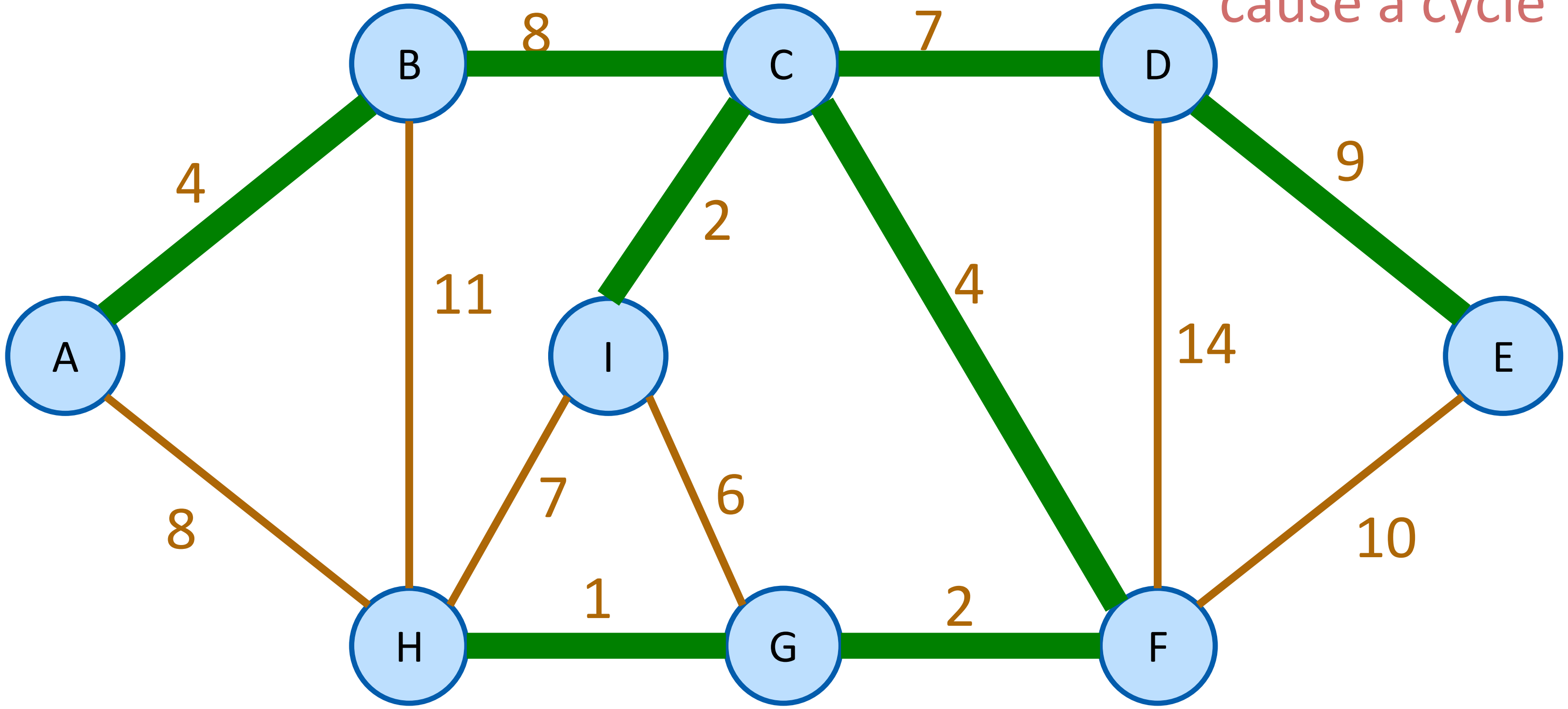
what if we just always take the cheapest edge?
whether or not it's connected to what we have so far?

That won't cause a cycle



Kruskal’s Algorithm

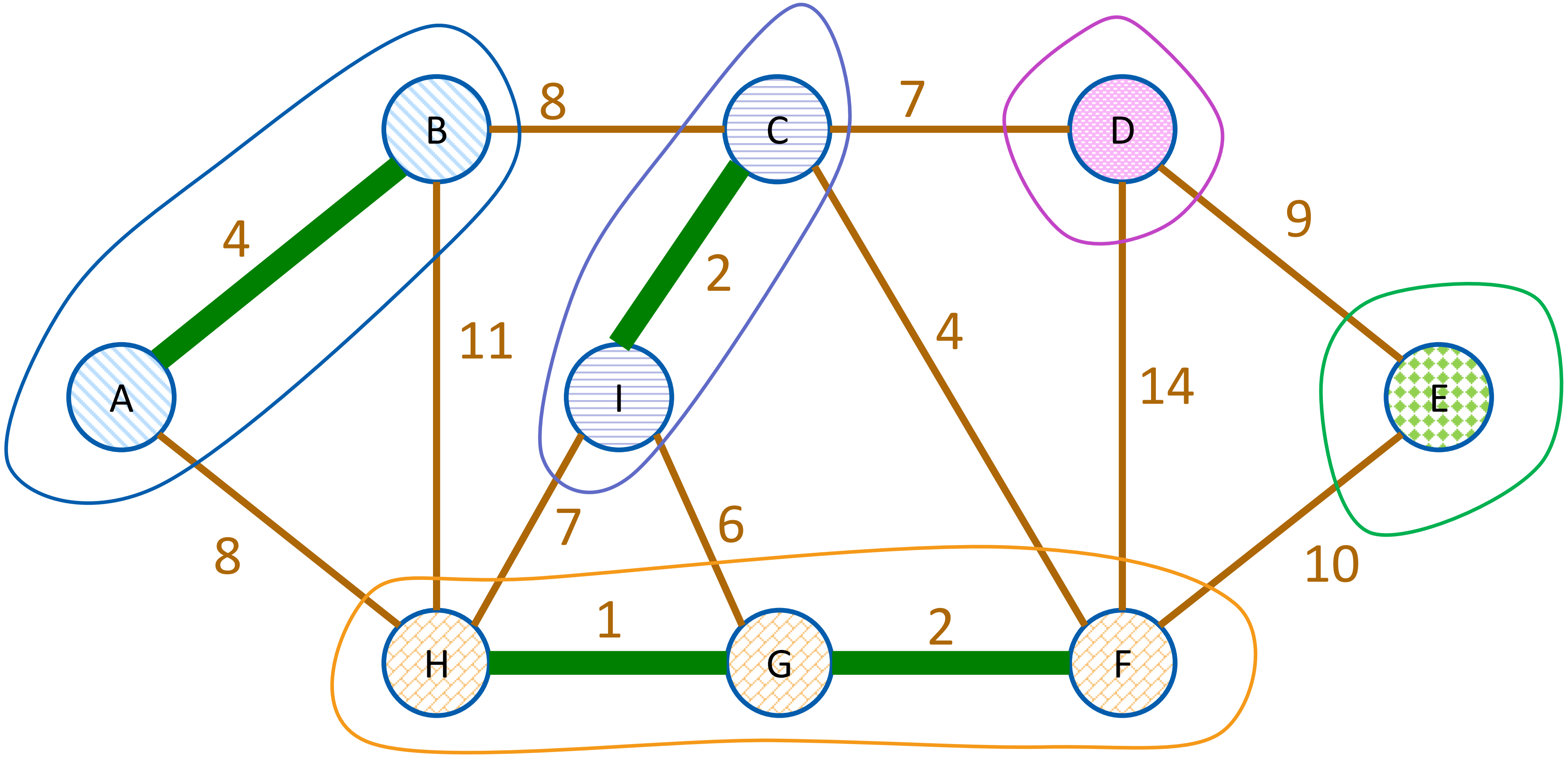
- **slowKruskal**(G = (V,E)):
  - Sort the edges in E by non-decreasing weight.
  - MST = {}
  - **for** e in E (in sorted order):
    - **if** adding e to MST won’t cause a cycle:
      - add e to MST.
  - **return** MST

m iterations through this loop

How do we check this?

Kruskal's Algorithm

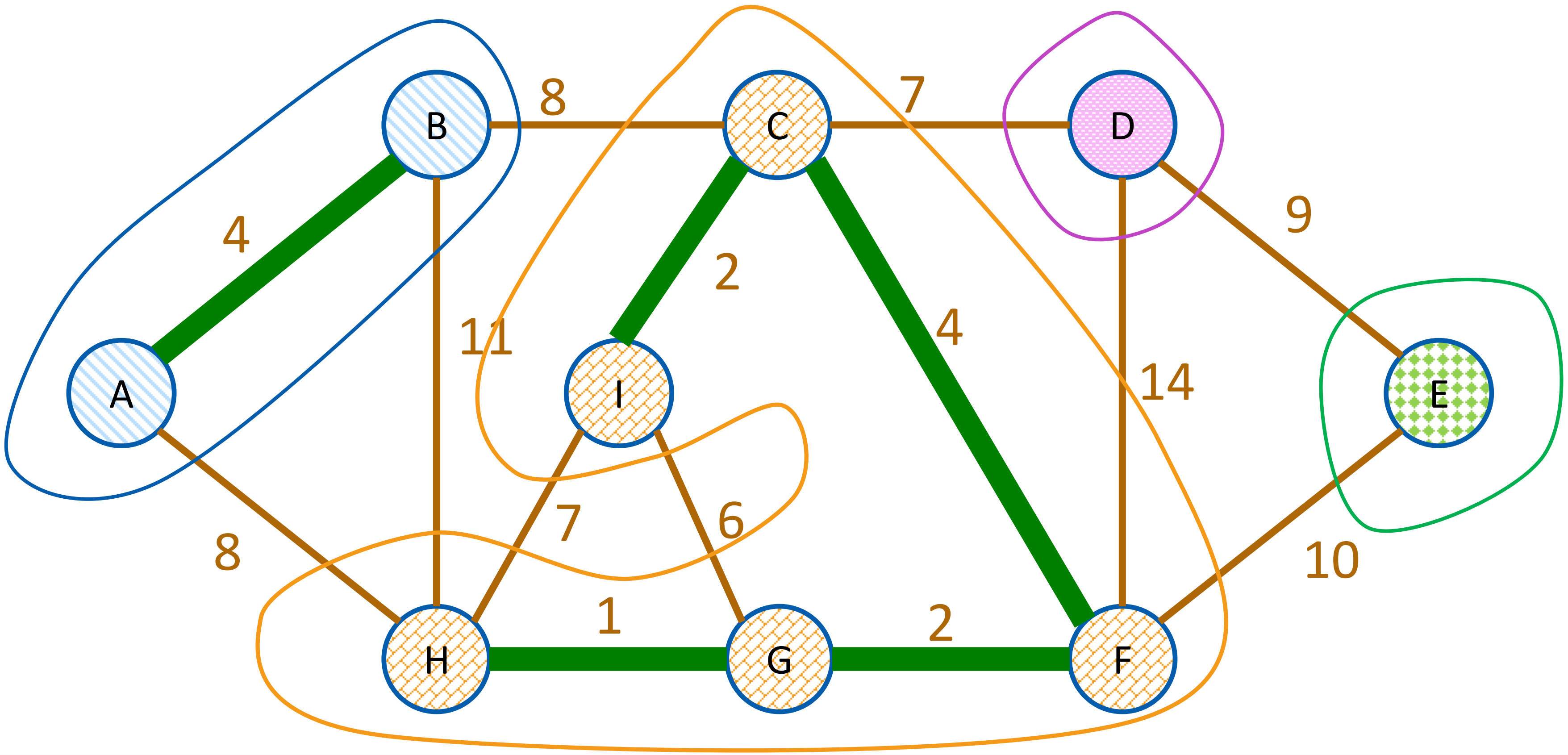
At each step of Kruskal's, we are maintaining a forest.



Kruskal's Algorithm

At each step of Kruskal's, we are maintaining a forest.

When we add an edge, we merge two trees:



Kruskal's Algorithm

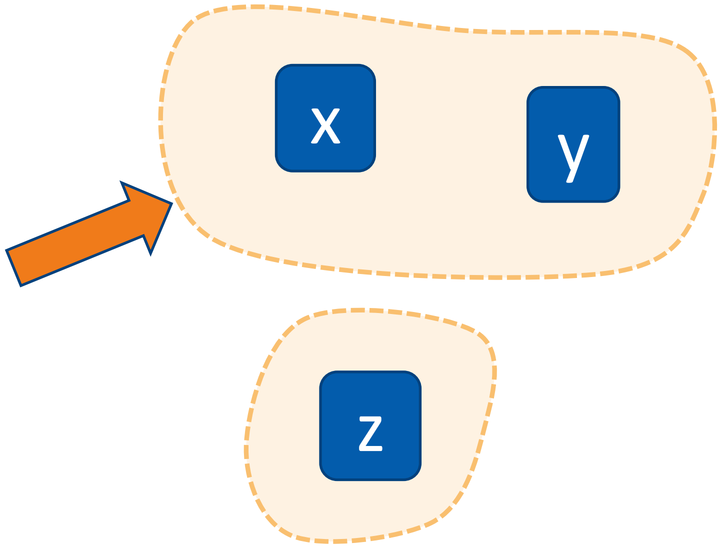
Union-find data structure

Implementation – Lab 13

- Used for storing collections of sets
- Supports:
  - **makeSet(u):** create a set {u}
  - **find(u):** return the set that u is in
  - **union(u,v):** merge the set that u is in with the set that v is in.

```
makeSet(x)
makeSet(y)
makeSet(z)

union(x,y)
find(x)
```



Kruskal's Algorithm

- **kruskal**(G = (V,E)):
  - Sort E by weight in non-decreasing order
  - MST = {} // initialize an empty tree
  - **for** v in V:
    - **makeSet**(v) // put each vertex in its own tree in the forest
  - **for** (u,v) in E: // go through the edges in sorted order
    - **if find**(u) != **find**(v): // if u and v are not in the same tree
      - add (u,v) to MST
      - **union**(u,v) // merge u's tree with v's tree
  - **return** MST

Kruskal's Algorithm

Running time

- Sorting the edges takes O(m log(n))
  - In practice, if the weights are small integers we can use radixSort and take time O(m)
- For the rest:
  - n calls to **makeSet**
    - put each vertex in its own set
  - 2m calls to **find**
    - for each edge, **find** its endpoints
  - n-1 calls to **union**
    - we will never add more than n-1 edges to the tree,
    - so we will never call **union** more than n-1 times.
- Total running time: **O(mlog(n))**

Complexity Classes

P and NP

- Definition: The class P consists of all decision problems that are solvable in polynomial time

- Definition: The class NP consists of all decision problems such that, for each yes-input, there exists a certificate that can be verified in polynomial time.
  – NP stands for "Nondeterministic Polynomial time".

- P = NP?

The End