

Chapter 7 Symmetric Matrices and Quadratic Forms

Section 7.1: Diagonalization of Symmetric Matrices

Section 7.2: Quadratic Forms

Symmetric Matrices

- A **symmetric matrix** is a matrix A such that $A^T = A$.
- Such a matrix is necessarily square.
- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.

Symmetric Matrices

- **Theorem:** If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

An $n \times n$ matrix A is said to be **orthogonally diagonalizable** if there are an orthogonal matrix P (with $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1} \quad (1)$$

Such a diagonalization requires n linearly independent and orthonormal eigenvectors.

Symmetric Matrices

$$A = PDP^T = PDP^{-1} \quad (1)$$

- When is this possible?
- If A is orthogonally diagonalizable as in (1), then

$$A^T = (PDP^T)^T = P^{TT} D^T P^T = PDP^T = A$$

Thus A is symmetric.

Symmetric Matrices

- **Theorem:** An $n \times n$ matrix A is **orthogonally diagonalizable** if and only if A is symmetric matrix.

- **Example:** Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}, \text{ whose characteristic equation is}$$

$$0 = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2)$$

Symmetric Matrices

- **Solution:** The usual calculations produce bases for the eigenspaces:

$$\lambda = 7 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \lambda = -2 : \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

- Although \mathbf{v}_1 and \mathbf{v}_2 are linearly independent, they are not orthogonal.
- The projection of \mathbf{v}_2 onto \mathbf{v}_1 is $\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$.

Symmetric Matrices

- The component of \mathbf{v}_2 orthogonal to \mathbf{v}_1 is

$$\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

- Then $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$.
- (Note that \mathbf{z}_2 is linear combination of the eigenvectors \mathbf{v}_1 and \mathbf{v}_2 , so \mathbf{z}_2 is in the eigenspace).

Symmetric Matrices

- Normalize \mathbf{v}_1 and \mathbf{z}_2 to obtain the following orthonormal basis for the eigenspace for $\lambda = 7$:

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

- An orthonormal basis for the eigenspace for $\lambda = -2$ is

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

Hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

Symmetric Matrices

- Let

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Then P orthogonally diagonalizes A , and $A = PDP^{-1}$.

The Spectral Theorem

- The set of eigenvalues of a matrix A is sometimes called the **spectrum** of A , and the following description of the eigenvalues is called a **spectral theorem**.

The Spectral Theorem

Theorem: **The Spectral Theorem for Symmetric Matrices**

An $n \times n$ symmetric matrix A has the following properties:

- a. A has n real eigenvalues, counting multiplicities.
- b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
- c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
- d. A is orthogonally diagonalizable.

Spectral Decomposition

- Suppose $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ of A and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are in the diagonal matrix D .
- Then, since $P^{-1} = P^T$,

$$\begin{aligned} A = PDP^T &= [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \\ &= [\lambda_1 \mathbf{u}_1 \quad \cdots \quad \lambda_n \mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \end{aligned}$$

Spectral Decomposition

- Using the column-row expansion of a product, we can write

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \quad (2)$$

- This representation of A is called a **spectral decomposition** of A because it breaks up A into pieces determined by the spectrum (eigenvalues) of A .
- Each term in (2) is an $n \times n$ matrix of rank 1.

Spectral Decomposition

- **Example:** Construct a spectral decomposition of the matrix A that has the orthogonal diagonalization

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

- **Solution:** Denote the columns of P by \mathbf{u}_1 and \mathbf{u}_2 .

- Then
$$A = 8\mathbf{u}_1\mathbf{u}_1^T + 3\mathbf{u}_2\mathbf{u}_2^T$$

Spectral Decomposition

- To verify the decomposition of A , compute

$$\mathbf{u}_1 \mathbf{u}_1^T = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$\mathbf{u}_2 \mathbf{u}_2^T = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

and

$$8\mathbf{u}_1 \mathbf{u}_1^T + 3\mathbf{u}_2 \mathbf{u}_2^T = \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A$$

Section 7.1: Diagonalization of Symmetric Matrices

Section 7.2: Quadratic Forms

Quadratic Forms

- A **quadratic form** on \mathbb{R}^n is a function Q defined on \mathbb{R}^n whose value at a vector \mathbf{x} in \mathbb{R}^n can be computed by an expression of the form $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, where A is an $n \times n$ symmetric matrix.
- The matrix A is called the **matrix of the quadratic form**.

Quadratic Forms

- **Example:** Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $\mathbf{x}^T A \mathbf{x}$ for the following matrices.

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

Quadratic Forms

- **Solution:**

a. $\mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2$

b. Watch the two -2 entries from A :

$$\mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 2x_1x_2 - 2x_2x_1 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

Quadratic Forms

- The presence of $-4x_1x_2$ in the quadratic form in Example 1(b) is due to the -2 entries off the diagonal in the matrix A .
- In contrast, the quadratic form associated with the diagonal matrix A in Example 1(a) has no x_1x_2 *cross-product* term.

Change of Variable in a Quadratic Form

- If \mathbf{x} represents a variable vector in \mathbb{R}^n , then a **change of variable** is an equation of the form

$$\mathbf{x} = P\mathbf{y}, \text{ or equivalently, } \mathbf{y} = P^{-1}\mathbf{x} \quad (1)$$

where P is an invertible matrix and \mathbf{y} is a new variable vector in \mathbb{R}^n .

- Here \mathbf{y} is the coordinate vector of \mathbf{x} relative to the basis of \mathbb{R}^n determined by the columns of P .
- If the change of variable (1) is made in a quadratic form $\mathbf{x}^T A \mathbf{x}$, then

$$\mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T (P^T A P) \mathbf{y} \quad (2)$$

and the new matrix of the quadratic form is $P^T A P$.

Change of Variable in a Quadratic Form

- **Example:** Make a change of variable that transforms the quadratic form $Q(\mathbf{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$ into a quadratic form with no cross-product term.
- **Solution:** The matrix of the given quadratic form is

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

Change of Variable in a Quadratic Form

- The first step is to orthogonally diagonalize A .
- Its eigenvalues turn out to be $\lambda = 3$ and $\lambda = -7$.
- Associated unit eigenvectors are

$$\lambda = 3: \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}; \quad \lambda = -7: \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

- These vectors are automatically orthogonal (because they correspond to distinct eigenvalues) and so provide an orthonormal basis for \mathbb{R}^2 .

Change of Variable in a Quadratic Form

- Let

$$P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

- Then $A = PDP^{-1}$ and $D = P^{-1}AP = P^T AP$.
- A suitable change of variable is

$$\mathbf{x} = P\mathbf{y}, \text{ where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Change of Variable in a Quadratic Form

- Then

$$\begin{aligned}x_1^2 - 8x_1x_2 - 5x_2^2 &= \mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) \\&= \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T D \mathbf{y} \\&= 3y_1^2 - 7y_2^2\end{aligned}$$

- To illustrate the meaning of the equality of quadratic forms in Example, we can compute $Q(\mathbf{x})$ for $\mathbf{x} = (2, -2)$ using the new quadratic form.

Change of Variable in a Quadratic Form

- First, since $\mathbf{x} = P\mathbf{y}$,

$$\mathbf{y} = P^{-1}\mathbf{x} = P^T\mathbf{x}$$

so

$$\mathbf{y} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

- Hence

$$\begin{aligned} 3y_1^2 - 7y_2^2 &= 3(6/\sqrt{5})^2 - 7(-2/\sqrt{5})^2 = 3(36/5) - 7(4/5) \\ &= 80/5 = 16 \end{aligned}$$

- This is the value of $Q(\mathbf{x})$ when $\mathbf{x} = (2, -2)$.

The Principal Axis Theorem

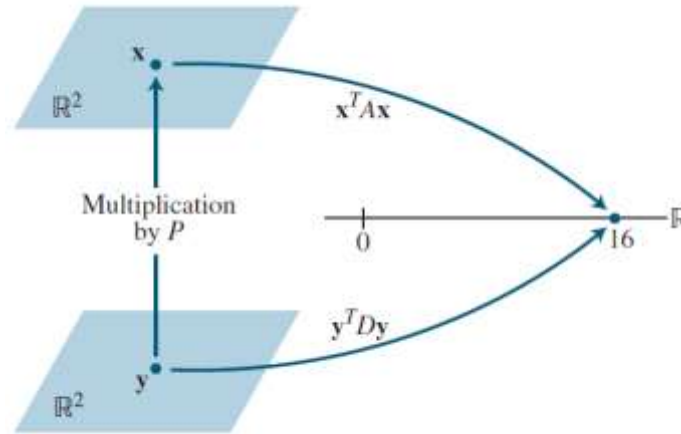


FIGURE 1 Change of variable in $\mathbf{x}^T A \mathbf{x}$.

- **Theorem:** Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A \mathbf{x}$ into a quadratic form $\mathbf{y}^T D \mathbf{y}$ with no cross-product term.

The Principal Axis Theorem

- The columns of P in theorem are called the **principal axes** of the quadratic form $\mathbf{x}^T A \mathbf{x}$.
- The vector \mathbf{y} is the coordinate vector of \mathbf{x} relative to the orthonormal basis of \mathbb{R}^n given by these principal axes.

Classifying Quadratic Forms

- **Definition:** A quadratic form Q is:
 - a. **positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,
 - b. **negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,
 - c. **indefinite** if $Q(\mathbf{x})$ assumes both positive and negative values.
- Also, Q is said to be **positive semidefinite** if $Q(\mathbf{x}) \geq 0$ for all \mathbf{x} , and **negative semidefinite** if $Q(\mathbf{x}) \leq 0$ for all \mathbf{x} .

Quadratic Forms and Eigenvalues

- **Theorem:** Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\mathbf{x}^T A \mathbf{x}$ is:
 - a. positive definite if and only if the eigenvalues of A are all positive,
 - b. negative definite if and only if the eigenvalues of A are all negative, or
 - c. indefinite if and only if A has both positive and negative eigenvalues.