[illegible] (i) 1

Thm. $(X, \mathcal{A}, \mu)$ measure sp. p, q: conjugate exponent. $\left(\frac{1}{p}+\frac{1}{q}=1\right)$.

let $f, g \in L^1(\mu)$, $f, g \geq 0$ non-neg. mble

then $\int_X f \cdot g \leq \left[\int_X |f|^p\right]\left[\int_X |g|^q\right]$ (Hölder ineq.)

and $\left[\int_X (f+g)^p\right]^{\frac{1}{p}} \leq \left[\int_X |f|^p\right]^{\frac{1}{p}} + \left[\int_X |g|^p\right]^{\frac{1}{p}}$.

(Minkowski ineq)

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pf. put $\left[\int_X |f|^p\right]^{\frac{1}{p}} =: A$, $\left[\int_X |g|^q\right]^{\frac{1}{q}} =: B$.

If $A=\infty$, then we are done.

If $A=0$, then $\int f^p = 0 \Rightarrow f^p = 0$ a.e.

$\Rightarrow f=0$ a.e. $\Rightarrow f \cdot g = 0$ a.e. $\Rightarrow \int_X f \cdot g = 0$.

So we will assume $0 < A, B < \infty$.

Set $F(x) := \frac{f(x)}{A}$, $\frac{g(x)}{B} =: G(x)$.

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Then $\int F^p = \frac{A^p}{A} = 1$, $\int_X G^q = \frac{B^q}{B^q} = 1$.

Since $F^p, G^q \neq 0, \infty$ a.e. so we may assume that $0 < F(x), G(x) < \infty$ "$\forall$" $x \in X$.

Hence for each $x \in X$, $\exists!\, s, t \in \mathbb{R}$ s.t.

$F(x) = \exp\left(\frac{s}{p}\right)$, $G(x) = \exp\left(\frac{t}{q}\right)$, $\left[\frac{1}{p}+\frac{1}{q}=1\right]$

$\Rightarrow e^{\frac{s}{p}} \cdot e^{\frac{t}{q}} \leq \frac{1}{p} e^s + \frac{1}{q} e^t$ by convexity of exp.

$\Leftrightarrow F(x) \cdot G(x) \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q$.

$\Rightarrow \int_X F \cdot G \leq \frac{1}{p}\int F^p + \frac{1}{q}\int G^q = \frac{1}{p}+\frac{1}{q} = 1$.

$\Leftrightarrow \frac{1}{A \cdot B}\int_X f \cdot g \leq 1 \Leftrightarrow \int_X f \cdot g \leq A \cdot B$.

$\therefore$ Hölder ineq.

Next, from $\frac{1}{p}+\frac{1}{q}=1 \Leftrightarrow p+q=pq$.

$\Leftrightarrow \begin{cases} (p-1)\cdot q = p \\ (q-1)\cdot p = q \end{cases}$

Denote $\left[\int_X |f|^p\right]^{\frac{1}{p}} = \|f\|_{L^p} \overset{\text{or}}{=} \|f\|_p$, L^p-norm.

Rmk. ① $p=1 \Rightarrow L^1(\mu)$: the collection of Lebesgue mble fns.

② X = A countable set

μ : counting measure.

$\ell^p(\mu) = \{f : A \to \mathbb{C}$ mble $| \int_A |f|^p \, d\mu < \infty\}$

& small ℓ^p-sp. $= \{(a_n)$ cplx-valued seq. $| \sum_{n=1}^{\infty} |a_n|^p < \infty\}$.

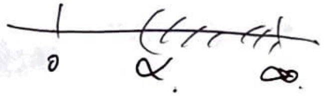
Def. let $g : X \to [0,\infty]$ mble, $(X, \mathcal{F}, \mu)$ given mea. sp.

Consider $g^{-1}(\alpha, \infty) = 0$.

$\mu(\quad)$

let $S := \{\alpha \in [0,\infty] | \mu(g^{-1}(\alpha,\infty]) \neq 0\}$.

measure zero



$(f+g)^p = (f+g)(f+g)^{p-1} = f(f+g)^{p-1} + g(f+g)^{p-1}$.

$\Rightarrow \int_X f(f+g)^{p-1} \underset{\text{Hölder}}{\leq} \left[\int_X f^p\right]^{\frac{1}{p}} \cdot \left[\int_X (f+g)^{(p-1)q}\right]^{\frac{1}{q}}$

$= \left[\int_X f^p\right]^{\frac{1}{p}} \left[\int_X (f+g)^p\right]^{\frac{1}{q}}$.

$\Rightarrow \int_X (f+g)^p = \int_X f(f+g)^{p-1} + \int_X g(f+g)^{p-1}$.

$\leq \left(\int_X (f+g)^p\right)^{\frac{1}{q}} \left\{\left(\int_X f^p\right)^{\frac{1}{p}} + \left(\int_X g^p\right)^{\frac{1}{p}}\right\}$.

$\Leftrightarrow \left(\int_X (f+g)^p\right)^{1-\frac{1}{q}} \leq \left(\int_X f^p\right)^{\frac{1}{p}} + \left(\int_X g^p\right)^{\frac{1}{p}}$.

$1 - \frac{1}{q} = \frac{1}{p}$. ☑

Def. Given a measure sp $(X, \mathcal{F}, \mu)$, for $1 \leq p < \infty$,

define $L^p(\mu) := \{f : X \to \mathbb{C}$ mble $| \|f\|_p < \infty\}$

& called a L^p-sp.

6.

$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$ (Minkowski).

$\|f \cdot g\|_{L^1} \le \|f\|_{L^p} \cdot \|g\|_{L^q}$. (Hölder)

Rmk. $\|f\|_{L^p} = 0 \Rightarrow f = 0$ "a.e".

Hence L^p has a "semi" norm. $\|\cdot\|_{L^p}$.

Consider $L^p(\mu)$, $1 \le p \le \infty$,

Give the (equi. rel) "$\sim$"

$f \sim f_0$ if $\underbrace{\|f - f_0\|_{L^p}}_{\lceil d(f, f_0) \rfloor} = 0$ as follows.

we may regard.

$$L^p(\mu) := \{ \underbrace{[f]}_{\text{equi. class}} \mid \underbrace{f : X \to \mathbb{C}}_{\text{msble}}, \ (\int_X |f|^p)^{\frac{1}{p}} < \infty \}.$$

we will say $\int_X |[f]|^p = \int_X |f|^p$.

(Convention).

7.

Define $\|g\|_{L^\infty} := \inf S$.

Rmk. ① If $S = \emptyset$ then $\inf S = \infty$.

② $\inf S \in S$.

$\lceil$ If $S \neq \emptyset$, $\beta = \inf S \in \mathbb{R}$.

Then $g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} \underbrace{g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right)}_{\text{measure zero}}$

$\Rightarrow \mu(g^{-1}(\beta, \infty]) = 0$.

$\Rightarrow \beta \in S$.

$\rfloor$.

Call $\|g\|_{L^\infty}$ essential bound of g.

(essential sup).

Define $L^\infty(\mu) := \{ f : X \to \mathbb{C}$ msble $\mid \|f\|_{L^\infty} < \infty \}$.

Thm.

$1 \le p \le \infty$, $f, g \in L^p(\mu)$

then $\forall \alpha \in \mathbb{C}$, $\|\alpha f\|_{L^p} = |\alpha| \|f\|_{L^p}$ and

"$(\int_X |\alpha f|^p)^{\frac{1}{p}}$.

Rmk. ① $\int_X |f\cdot g| \le \|f\|_{L^p} \cdot \|g\|_{L^q}$.

Equality holds iff $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$\alpha f^p = \beta g^q$ for all $x \in X$.

② If $|f| \le r$ a.e. then $\|f\|_{L^\infty} \le r$.

Pf: If $\|f\|_{L^\infty} > r$, $\mu(f^{-1}((r,\infty])) = 0$.

$r \in S$. ~~of~~ $\rfloor$.

Conversely if $\|f\|_{L^\infty} \le r$ then $|f| \le r$. a.e.

Thm. For $1 \le p \le \infty$, $(X, \mathcal{F}, \mu)$ measure sp.

$L^p(\mu)$ is a complete ~~metric~~ sp.

normed linear sp,

Called the Banach sp).

Rmk. If $p=2$, $L^2(\mu)$ is a complete

inner product sp. Called the Hilbert sp.