On the other hand, b is a limit pt of $(x_n)_{n=1}^{\infty}$ [3].

[that is, $\forall$ nbd U of b, $(U - \{b\}) \cap \{x_n\}_{n=1}^{\infty} \neq \emptyset$].

Since U_b contains only finite only x_n's

$\cancel{\alpha}$. □

Exc.

① (X,τ), (Y,τ) Hausdorff sps,

$\Rightarrow (X \times Y, \tau_{prod})$: Hausdorff.

② $A \subseteq (X,\tau)$ Hausdorff

$\Rightarrow (A, \tau_{subsp})$: Hausdorff.

Continuity of a fn. on $\mathbb{R}$

Let $E \subseteq \mathbb{R}$, $p \in E$, $f : E \to \mathbb{R}$ be a fn.

We say f is conti. at p.

[$\lim_{x \to p} f(x) = f(p)$].

if $\forall \varepsilon > 0$. $\exists \delta > 0$. s.t. [If $|x-p| < \delta$ then

$|f(x) - f(p)| < \varepsilon$].

Hausdorff axiome, continuity of fn. ①

Recall that for a Hausdorff sp (X,τ) every finite set in X is closed. i.e, Hausdorff $(T_2) \Rightarrow T_1$

(called the T_1-axiom)

Prop. Given a Hausdorff sp. (X,τ),

let $(x_n)_{n=1}^{\infty}$ be a seq in X satisfying $x_n \to a$

and $x_n \to b$ as $n \to \infty$. $a, b \in X$.

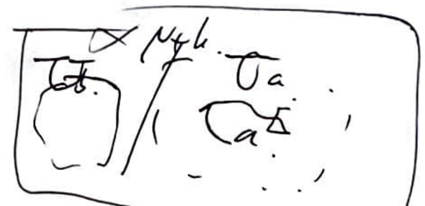
Then $a = b$. (Uniqueness of a limit).

pf. If $a \neq b$. $\exists$ nbd U_a and U_b s.t.

$U_a \cap U_b = \emptyset$.

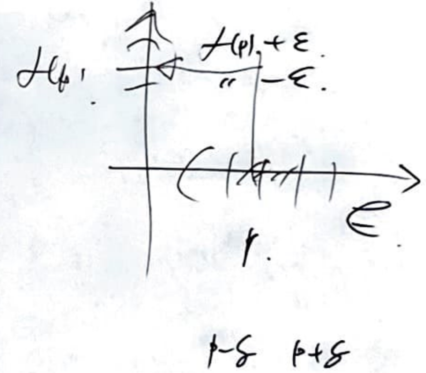
Since $x_n \to a$, as $n \to \infty$.

$\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $x_n \in U_a$.



$\Rightarrow U_b$ contains at most $x_1 \cdots x_{N_1 - 1}$.

(3)



$(*)$.

Rmk. $(*) \iff \forall\, \varepsilon>0,\ \exists\, \delta>0$ s.t.

If $x \in B_\delta(p)$. Then $f(x) \in B_\varepsilon(f(p))$.

$B_\delta(p) = \{x \in \mathbb{R} \mid |x-p| < \delta\}$.

$\iff \forall\, \varepsilon>0,\ \exists\, \delta>0$ s.t. $f(\underbrace{B_\delta(p)}_{\delta\text{-nbhd}}) \subseteq \underbrace{B_\varepsilon(f(p))}_{\varepsilon\text{-nbhd}}$

$\iff \forall$ nbd V of $f(p)$, $\exists$ nbd U of p s.t. $f(U) \subseteq V$.

$\iff \forall$ nbd V of $f(p)$, $\exists$ nbd U of p s.t. $U \subseteq f^{-1}(V)$.

(4)

(Recall that for $f: A \to B$ fn, $C \subseteq A$, $D \subseteq B$,

$f(C) \subseteq D \iff C \subseteq f^{-1}(D)$.)

$\iff \forall$ nbd U of $f(p)$, $f^{-1}(U)$ is a nbd of p.

~~If f is conti. for any $p \in \mathbb{R}$,~~

$\iff \forall$ open..

If $f: X \to Y$,

$\forall$ open U in Y, $f^{-1}(U)$ is open in X.
open set

Def. (of conti fn.)

Given top sps $(X, \tau_X), (Y, \tau_Y)$,

we say $f: X \to Y$ is conti if $\forall\, U \in \tau_Y$,

$f^{-1}(U) \in \tau_X$.

Prop. TFAE

Given (X, τ_X) (Y, τ_Y).

① f is conti. i.e. $\forall\, U \in \tau_Y$, $f^{-1}(U) \in \tau_X$.

6.

② ⇒ ①.

Take any open set U in Y.

Claim: $f^{-1}(U)$ is open in X.

By ②, $\forall p \in f^{-1}(U)$, $\exists$ nbd U_p of p s.t.

$f(U_p) \subseteq U \iff U_p \subseteq f^{-1}(U)$

$$\Rightarrow f^{-1}(U) = \bigcup_{p \in f^{-1}(U)} U_p.$$

$\ulcorner \supseteq$: $\forall p \in f^{-1}(U)$, $U_p \subseteq f^{-1}(U)$.

$\subseteq$: $\forall p \in f^{-1}(U)$, $p \in U_p \subseteq \bigcup_{p \in f^{-1}(U)} U_p$. $\lrcorner$.

In particular, $f^{-1}(U)$ is an union of open sets,

i.e. $f^{-1}(U)$ is an open in X. (✓).

② ⇒ ③. Take any $A \subseteq X$.

Claim: $f(\bar{A}) \subseteq \overline{f(A)}$.

let $f(p) \in f(\bar{A})$. $p \in \bar{A}$.

5

② $\forall$ nbd U of $f(p)$, $\exists$ nbd U of p. s.t.

$f(U) \subseteq U$. $\forall p \in X$.

③ $\forall A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.

④. $\forall$ closed F in Y. $f^{-1}(F)$ is closed in X.

pf. ① ⇔ ④.

Note that $\forall B \subseteq Y$, $f^{-1}(B^c) = X - f^{-1}(B)$.

Suppose ①.

Take any closed F in Y.

Then $\underset{\text{open in } Y}{f^{-1}(F^c)} = X - f^{-1}(F)$: open in X ... ④.

④ ⇒ ①. similar.

① ⇒ ②. $\forall p \in X$, let V be a nbd of $f(p)$.

$f(p) \in V \iff p \in f^{-1}(V)$.

Take $U = f^{-1}(V)$. — general fact.

Then $f(U) = f(f^{-1}(V)) \subseteq V$.

⑦

NTS: $f(p) \in \overline{f(A)} \iff \forall$ nbd of $f(p)$,

$V \cap f(A) \neq \emptyset$.

Take any nbd V of $f(p)$

By ②, $\exists$ nbd U of p s.t. $f(U) \subseteq V$.

Since $p \in \bar{A}$, $U \cap A \neq \emptyset$. $\Rightarrow \exists x \in U \cap A$.

$\Rightarrow f(x) \in f(U) \subseteq V$, $f(x) \in f(A)$.

$\Rightarrow f(x) \in V \cap f(A) \neq \emptyset$. (∪)

$$\begin{array}{ccc} ① & \iff & ④ \\ \Uparrow & & \Uparrow \\ ② & \Rightarrow & ③ \end{array}$$

③ $\Rightarrow$ ④. Take any closed F in Y.

F is closed iff $\bar{F} = F$.

Claim: $f^{-1}(F)$ is closed in X.

$\iff \overline{f^{-1}(F)} \overset{?}{=} f^{-1}(F)$.

$\supseteq$: clear.

$\subseteq$.

⑧

$\subseteq$: let $p \in \overline{f^{-1}(F)}$

By ③, letting $A = f^{-1}(F) \subseteq X$.

Then $f(\bar{A}) \subseteq \overline{f(A)}$.

$\iff f(\overline{f^{-1}(F)}) \subseteq \overline{f(f^{-1}(F))}$.

Thus $f(p) \in f(\overline{f^{-1}(F)}) \subseteq \overline{f(f^{-1}(F))}$.

$\subseteq \bar{F} = F$.

Thus $f(p) \in F$, $p \in f^{-1}(F)$ ☐