8-5、8-6、2017713 刘楚 13271072

阅

1. 作适当的变换计算下列积分.

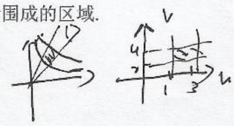
(1) $\iint\limits_D x^2y^2\mathrm{d}x\mathrm{d}y$，其中 $D$ 是由曲线 $xy=2, xy=4, y=x, y=3x$ 在第一象限所围成的区域;

(2) $\iint\limits_D \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\mathrm{d}x\mathrm{d}y$，其中 $D=\left\{(x,y)\middle|\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$;

(3) $\iint\limits_D \cos\frac{x-y}{x+y}\mathrm{d}x\mathrm{d}y$，其中是 $D$ 由 $x+y=1, x=0, y=0$ 所围成的区域;

(4) $\iint\limits_D xy\mathrm{d}x\mathrm{d}y$，其中是 $D$ 由 $y^2=x, y^2=4x, x^2=y, x^2=4y$ 所围成的区域.

(1) 设 $u=\frac{y}{x}, v=xy \Rightarrow x=\sqrt{\frac{v}{u}}, y=\sqrt{uv}$



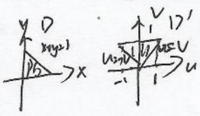
$\therefore J=\begin{vmatrix} x_u' & x_v' \\ y_u' & y_v' \end{vmatrix}=\begin{vmatrix} -\frac{1}{2u}\cdot\sqrt{\frac{v}{u}} & \frac{1}{2}\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\cdot\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}=-\frac{1}{2u}$

$\therefore \iint\limits_D x^2y^2\mathrm{d}x\mathrm{d}y=\iint\limits_{D'} v^2|J|\mathrm{d}u\mathrm{d}v=\int_1^3\frac{1}{2u}\mathrm{d}u\int_2^4 v^2\mathrm{d}v=\frac{1}{2}\ln u\Big|_1^3\cdot\frac{1}{3}v^3\Big|_2^4=\frac{28}{3}\ln 3$

(2) 设 $x=a\cdot\rho\cdot\cos\theta, y=b\rho\sin\theta$. $\therefore J=\begin{vmatrix} x_\rho' & x_\theta' \\ y_\rho' & y_\theta' \end{vmatrix}=\begin{vmatrix} a\cos\theta, & a\rho(-\sin\theta) \\ b\sin\theta & b\rho\cos\theta \end{vmatrix}=ab\rho$

$\therefore \iint \frac{x^2}{a^2}+\frac{y^2}{b^2}\mathrm{d}x\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho^2\cdot ab\rho\mathrm{d}\rho=ab\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho^3\mathrm{d}\rho=\frac{\pi}{2}ab$

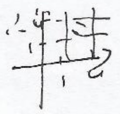
(3) 设 $x-y=u, x+y=v \Rightarrow x=\frac{v+u}{2}, y=\frac{v-u}{2}, J=\begin{vmatrix} x_u' & x_v' \\ y_u' & y_v' \end{vmatrix}=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}=\frac{1}{2}$



$\iint\limits_D \cos\frac{x-y}{x+y}\mathrm{d}x\mathrm{d}y=\iint\limits_{D'}\frac{1}{2}\cos\frac{u}{v}\mathrm{d}u\mathrm{d}v=\int_0^1\mathrm{d}v\int_{-v}^{v}\cos\frac{u}{v}\mathrm{d}u\cdot\frac{1}{2}$

$=\frac{1}{2}\cdot\int_0^1 v\cdot\sin\frac{u}{v}\Big|_{-v}^{v}\mathrm{d}v=\int_0^1 v\sin 1\mathrm{d}v=\frac{1}{2}\sin 1$

(4). 设 $\frac{y^2}{x}=u, \frac{x^2}{y}=v \Rightarrow x=(uv^2)^{\frac{1}{3}}, y=(u^2v)^{\frac{1}{3}}, J=\begin{vmatrix} \frac{v^{\frac{2}{3}}}{3}u^{-\frac{2}{3}} & \frac{2}{3}u^{\frac{1}{3}}v^{-\frac{1}{3}} \\ \frac{2}{3}v^{\frac{1}{3}}u^{-\frac{1}{3}} & \frac{u^{\frac{2}{3}}}{3}v^{-\frac{2}{3}} \end{vmatrix}=-\frac{1}{3}$



$\iint\limits_D xy\mathrm{d}x\mathrm{d}y=\left|\frac{1}{3}\right|\iint\limits_{D'} uv\mathrm{d}u\mathrm{d}v=\frac{1}{3}\int_1^4 u\mathrm{d}u\int_1^4 v\mathrm{d}v$

$=\frac{1}{3}\times\frac{15}{2}\times\frac{15}{2}=\frac{75}{4}$

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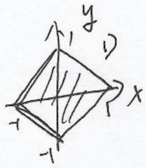
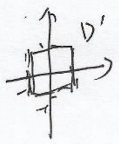
2. 证明 $\iint\limits_D f(x+y)\,\mathrm{d}x\mathrm{d}y=\int_{-1}^{1} f(u)\,\mathrm{d}u$，其中 $D=\{(x,y)\,|\,|x|+|y|\le 1\}$.

解：令 $\begin{cases} u=x+y \\ v=x-y \end{cases} \Rightarrow \begin{cases} x=\frac{v+u}{2} \\ y=\frac{v-u}{2}. \end{cases} \quad \therefore J=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}=\frac{1}{2}$

$\therefore$




$\therefore \iint\limits_D f(x+y)\,dxdy$

$=\iint\limits_{D'} f(u)\cdot\frac{1}{2}\,dudv$

$=\frac{1}{2}\cdot\int_{-1}^{1} f(u)\,du \int_{-1}^{1} dv$

$=\int_{-1}^{1} f(u)\,du.$

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3. 计算 $\iiint\limits_V xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 $V=\left\{(x,y,z)\left|\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1, x\ge 0, y\ge 0, z\ge 0\right.\right\}$.

解：令. $x=a r\cos\theta\sin\varphi$

$y=b r\cdot\sin\theta\sin\varphi$

$z=c\cdot r\cos\varphi$

$$\therefore J=\begin{vmatrix} a\cos\theta\sin\varphi & a(-r)\sin\theta\sin\varphi & a r\cos\theta\cos\varphi \\ b\sin\theta\sin\varphi & b r\cos\theta\sin\varphi & b r\sin\theta\cos\varphi \\ c\cos\varphi & 0 & c(-r\sin\varphi) \end{vmatrix}$$

$$=abc(-r^2)\cdot\sin\varphi$$

$$\therefore \iiint\limits_V xyz\,dV=\iiint\limits_{V'} abc\cdot r^3\sin^2\varphi\cdot\cos\varphi\sin\theta\cos\theta\cdot|J|\cdot dV$$

$$=(abc)^2\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_0^{\frac{\pi}{2}}\sin^3\varphi\cos\varphi\,d\varphi\int_0^1 r^5\,dr$$

$$=a^2b^2c^2\cdot\int_0^{\frac{\pi}{2}}\sin\theta\,d\sin\theta\int_0^{\frac{\pi}{2}}\sin^3\varphi\,d\sin\varphi\int_0^1 r^5\,dr$$

$$=a^2b^2c^2\times\frac{1}{2}\times\frac{1}{4}\times\frac{1}{6}=\frac{a^2b^2c^2}{48}$$

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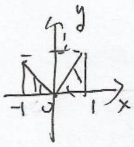
4. 求下列极限.

(1) $\lim\limits_{y\to 0}\int_{-1}^{1}\sqrt{x^2+y^2}\,\mathrm{d}x$;

(2) $\lim\limits_{x\to 0}\int_{0}^{2}y^2\cos(xy)\,\mathrm{d}y$.

(1) $\lim\limits_{y\to 0}\int_{-1}^{1}\sqrt{x^2+y^2}\,dx=\int_{-1}^{1}\lim\limits_{y\to 0}\sqrt{x^2+y^2}\,dx=\int_{-1}^{1}|x|\,dx=1$

如图.




(2).

$\lim\limits_{y\to 0}\int_{0}^{2}y^2\cos(xy)\,dy=\int_{0}^{2}\lim\limits_{x\to 0}y^2\cos(xy)\,dy$

$=\int_{0}^{2}y^2\,dy=\frac{1}{3}y^3\Big|_0^2=\frac{8}{3}$

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5. 求下列函数的导数.

(1) $F(x)=\int_x^{x^2} e^{-xy^2}\,dy$;

(2) $F(x)=\int_0^x \frac{\ln(1+xy)}{y}\,dy$.

解

(1) $F'(x)=\int_x^{x^2}(e^{-xy^2})'_x\,dy+(x^2)'\cdot e^{-x\cdot x^4}-(x)'\cdot e^{-x\cdot x^2}$

$=2x\cdot e^{-x^5}-e^{-x^3}-\int_x^{x^2}y^2\cdot e^{-xy^2}\,dy$

(2).

$F'(x)=\int_0^x\cdot\left(\frac{\ln(1+xy)}{y}\right)'_x\,dy+(x)'\cdot\frac{\ln(1+x^2)}{x}$

$=\int_0^x\cdot\frac{1}{1+xy}\,dy+\frac{1}{x}\cdot\ln(1+x^2)$

$=\frac{1}{x}\cdot\ln(1+xy)\Big|_0^x+\frac{1}{x}\ln(1+x^2)=\frac{2}{x}\ln(1+x^2)$

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6. 计算下列积分.

(1) $\int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\mathrm{d}x\ (0<a<b)$;

(2) $\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}\mathrm{d}x\ (a>0,b>0)$.

解：(1) $\int_0^1 \sin(\ln\frac{1}{x})\frac{x^b-x^a}{\ln x}dx = \int_0^1 dx\int_a^b \sin(\ln\frac{1}{x})\cdot x^y dy$

$=\int_a^b dy\int_0^1 \sin(\ln\frac{1}{x})\cdot x^y dx$，令 $\ln\frac{1}{x}=u$，则 $x=e^{-u}$

原式 $=\int_a^b dy\int_0^{+\infty}\sin u\cdot e^{-u(y+1)}du$，对 $\int_0^{+\infty}\sin u\cdot e^{-u(y+1)}du=A$ 讨论.

由分部积分法得 $A=-\frac{1}{y+1}[\sin u\cdot e^{-u(y+1)}\Big|_0^{+\infty}-\int_0^{+\infty}e^{-u(y+1)}\cos u du]$

$=-\frac{1}{y+1}[0+\frac{1}{y+1}\int_0^{+\infty}\cos u d e^{-u(y+1)}]=-\frac{1}{(y+1)^2}[\cos u\cdot e^{-u(y+1)}\Big|_0^{+\infty}+A]$.

$\therefore A=-\frac{1}{(y+1)^2}\cdot[A-1]$. $A=\frac{1}{(y+1)^2+1}$

原式 $=\int_a^b\frac{1}{(y+1)^2+1}dy=\arctan(y+1)\Big|_a^b=\arctan(b+1)-\arctan(a+1)$

(2). $\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}dx=\int_0^{+\infty}dx\int_a^b e^{-xy}dy$

$=\int_a^b dy\int_0^{+\infty}e^{-xy}dx=\int_a^b -\frac{1}{y}\cdot(e^{-xy})\Big|_0^{+\infty}dy$

$=\int_a^b\frac{1}{-y}\cdot(0-1)dy=\ln y\Big|_a^b=\ln b-\ln a=\ln\frac{b}{a}$

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