1. 求下列极限.

(1) $\lim\limits_{r\to 0}\dfrac{1}{\pi r^2}\iint\limits_{x^2+y^2\le r^2} e^{x^4-y^4}\cos\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y$;

(2) $\lim\limits_{t\to 0}\dfrac{1}{\pi t^4}\iiint\limits_{x^2+y^2+z^2\le t^2} f\left(\sqrt{x^2+y^2+z^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 $f(u)$ 有连续导数;

(3) $\lim\limits_{t\to +\infty}\dfrac{1}{t^4}\iiint\limits_{x^2+y^2+z^2\le t^2} \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$;

(4) $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\dfrac{i}{n^3}\cos\dfrac{i\cdot j}{n^2}$.

解: (1) 由积分中值定理.

$$\lim_{r\to 0}\cdot\frac{1}{\pi r^2}\cdot\iint_D f(x,y)\,ds=\lim_{r\to 0}\frac{1}{\pi t^2}f(\xi,\eta)\cdot\pi t^2=\lim_{t\to 0}f(\xi,\eta)=f(0,0)$$

$$=e^{0-0}\cdot\cos 0=1$$

(2). $$\lim_{t\to 0}\cdot\frac{1}{\pi t^4}\iiint_V f(r)\,dv=\lim_{t\to 0}\cdot\frac{\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{t}r^2f(r)\,dr}{\pi t^4}=\lim_{t\to 0}\cdot\frac{4\pi\int_0^{t}r^2f(r)\,dr}{\pi t^4}$$

$$=\lim_{t\to 0}\frac{4\cdot t^2f(t)}{4t^3}=\lim_{t\to 0}\frac{f(t)}{t}=\begin{cases}f'(0), & f(0)=0\\ \infty, & f(0)\neq 0\end{cases}$$

(3). $$\lim_{t\to +\infty}\frac{1}{t^4}\iiint_V r\,dv=\lim_{t\to +\infty}\frac{1}{t^4}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{t}r\cdot r^2\,dr$$

$$=\lim_{t\to +\infty}\cdot\frac{4\pi\times\frac{1}{4}t^4}{t^4}=\pi.$$

(4) $$\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i}{n^3}\cos\frac{i\cdot j}{n^2}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\sum_{j=1}^{n}\frac{1}{n^2}\cdot\cos\frac{i\cdot j}{n^2}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sum_{j=1}^{n}\frac{i}{n}\left(\cos\frac{i}{n}\cdot\frac{j}{n}\right)\cdot\frac{1}{n}$$

$$=\int_0^1dx\int_0^1x\cdot\cos(xy)\,dy=\int_0^1\sin(xy)\Big|_0^1dx$$

$$=\int_0^1\sin x\,dx=(-\cos x)\Big|_0^1=1-\cos 1$$

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2. 计算下列二重积分.

(1) $\iint\limits_D xy\ln\left(1+x^2+y^2\right)\mathrm{d}x\mathrm{d}y$，其中 D 由 $y=x^3, y=1, x=-1$ 围成;

(2) $\iint\limits_D \dfrac{y\mathrm{d}x\mathrm{d}y}{\left(1+x^2+y^2\right)^{\frac{3}{2}}}$，其中 $D: 0\le x\le 1, 0\le y\le 1$;

(3) $\iint\limits_D \left|\sin(x-y)\right|\mathrm{d}\sigma$，其中 $D: 0\le x\le y\le 2\pi$;

(4) $\int_{-1}^{1}\mathrm{d}x\int_{-1}^{x} x\sqrt{1-x^2+y^2}\,\mathrm{d}y$;

(5) $\iint\limits_{\substack{0\le x\le 2\\0\le y\le 2}} [x+y]\mathrm{d}x\mathrm{d}y$，其中 $[x+y]$ 表示不超过 $x+y$ 的最大整数;

(6) $\int_1^2\mathrm{d}x\int_{\frac{1}{x}}^{2} y e^{xy}\mathrm{d}y$;

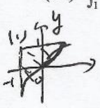
(7) $\iint\limits_D y\left(1+xe^{\frac{1}{2}\left(x^2+y^2\right)}\right)\mathrm{d}x\mathrm{d}y$，其中 D 是由直线 $y=x, y=-1, x=1$ 围成的区域;

(8) $\iint\limits_D e^{\max\left\{x^2,y^2\right\}}\mathrm{d}x\mathrm{d}y$，其中 $D=\left\{(x,y)\,\middle|\,0\le x\le 1, 0\le y\le 1\right\}$;

(9) $\iint\limits_{\substack{0\le x\le 1\\0\le y\le 1}} \left|x^2+y^2-1\right|\mathrm{d}\sigma$;

(10) $\int_1^2\mathrm{d}x\int_{\sqrt{x}}^{x}\sin\dfrac{\pi x}{2y}\mathrm{d}y+\int_2^4\mathrm{d}x\int_{\sqrt{x}}^{2}\sin\dfrac{\pi x}{2y}\mathrm{d}y$.

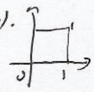
(1)



$\iint\limits_D xy\ln(1+x^2+y^2)dxdy=\int_{-1}^{1}dy\int_{-1}^{\sqrt[3]{y}} xy\ln(1+x^2+y^2)dx=\int_{-1}^{1}\frac{1}{2}y\cdot\left((1+y^{\frac{2}{3}}+y^2)\ln(1+y^{\frac{2}{3}}+y^2)-(2+y^2)\ln(2+y^2)\right)dy$

$=\int_{-1}^{1}F(y)dy$，由于 $F(-y)=-F(y)$ ∴ $\int_{-1}^{1}F(y)=0$. ∴ $\iint\limits_D f(x,y)dxdy=0$

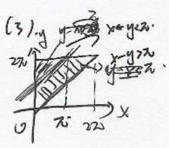
(2).



$\iint\limits_D\frac{ydxdy}{(1+x^2+y^2)^{\frac{3}{2}}}=\int_0^1dx\int_0^1\frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}dy=\int_0^1-\frac{1}{\sqrt{1+x^2+y^2}}\Big|_0^1dy=\int_0^1\frac{1}{\sqrt{1+x^2}}-\frac{1}{\sqrt{2+x^2}}dx$

$=\left(\ln(x+\sqrt{1+x^2})-\ln(x+\sqrt{2+x^2})\right)\Big|_0^1=\ln\frac{2+\sqrt{2}}{1+\sqrt{3}}$

(3).



$\iint\limits_D|\sin(x-y)|dxdy=\int_0^{\pi}dx\int_x^{x+\pi}\sin(x-y)dy+\int_0^{\pi}dx\int_{x+\pi}^{2\pi}\sin(y-x)dy$

$+\int_{\pi}^{2\pi}dx\int_y^{2\pi}\sin(x-y)dy$

$=\int_0^{\pi}(\cos x+1)dx+\int_0^{\pi}2dx+\int_{\pi}^{2\pi}(1-\cos x)dx$

$=4\pi$

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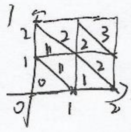
2. (4).

$$\int_{-1}^{1}dx\int_{-1}^{x}x\sqrt{1-x^2+y^2}\,dy=\int_{-1}^{1}dy\int_{y}^{1}x\cdot\sqrt{1-x^2+y^2}\,dx=\int_{-1}^{1}\left(-\frac{1}{3}\right)\cdot(1+y^2-x^2)^{\frac{3}{2}}\Big|_{y}^{1}dy$$

$$=\int_{-1}^{1}\frac{1}{3}(1-y^3)\,dy=\frac{1}{3}\times\left(2-\frac{2}{4}\right)=\frac{1}{2}$$

(5)



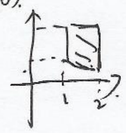
如图.将.D区域分划.数字代表f(x)

$$\iint_D f(x,y)\,dxdy=\int_0^1dx\int_0^{1-x}0\,dy+\int_0^1dx\int_{1-x}^{2-x}1\,dy+\int_0^1dx\int_{2-x}^{2}2\,dy$$

$$+\int_1^2dx\int_0^{2-x}1\,dy+\int_1^2dx\int_{2-x}^{3-x}2\,dy+\int_1^2dx\int_{3-x}^{2}3\,dy$$

$$=0+\frac{3}{2}\times1+\frac{3}{2}\times2+\frac{1}{2}\times3=6.$$

(6).



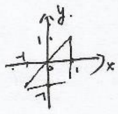
$$\iint_D \int_1^2dx\int_{\frac{1}{x}}^{2}ye^{xy}\,dy=\int_{\frac{1}{2}}^{1}dy\int_{\frac{1}{y}}^{2}y\cdot e^{xy}\,dx+\int_1^2dy\int_1^2y\cdot e^{xy}\,dx$$

$$=\int_{\frac{1}{2}}^{1}e^{xy}\Big|_{\frac{1}{y}}^{2}dy+\int_1^2e^{xy}\Big|_1^2dy$$

$$=\int_{\frac{1}{2}}^{1}(e^{2y}-e)\,dy+\int_1^2e^{2y}-e^{y}\,dy=\frac{1}{2}e^2-e+\frac{1}{2}e^4-\frac{1}{2}e^2-(e^2-e)$$

$$=\frac{1}{2}e^4-e^2$$

(7).

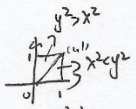


$$\iint_D y\left(1+x\cdot e^{\frac{1}{2}(x^2+y^2)}\right)dxdy=\cancel{\iint_D y\,dy}$$

$$=\int_{-1}^{1}dx\int_{-1}^{x}\left(1+x\cdot e^{\frac{1}{2}(x^2+y^2)}\right)d\frac{1}{2}y^2=\int_{-1}^{1}\frac{1}{2}(x^2-1)+x\cdot e^{\frac{1}{2}(x^2+y^2)}\Big|_{-1}^{x}dx$$

$$=\int_{-1}^{1}\left[\frac{1}{2}(x^2-1)+x\cdot e^{x^2}-x\cdot e^{\frac{1}{2}(x^2+1)}\right]dx=\int_{-1}^{1}\frac{1}{2}(x^2-1)dx=\left(\frac{1}{3}x^3-x\right)\Big|_{-1}^{1}=\frac{-2}{3}$$

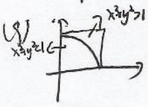
(8).



$$\iint_D e^{\max\{x^2,y^2\}}ds=\int_0^1dx\int_x^1e^{y^2}dy+\int_0^1dx\int_0^xe^{x^2}dy=\int_0^1dy\int_0^ye^{y^2}dx+\int_0^1dx\int_0^xe^{x^2}dy$$

$$=\int_0^1e^{y^2}d\left(\frac{1}{2}y^2\right)+\int_0^1e^{x^2}d\frac{1}{2}(x^2)=e-1$$

(9)

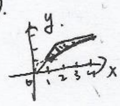


$$\iint_D|x^2+y^2-1|\,d\sigma=\int_0^1dx\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)dy+\int_0^1dx\int_{\sqrt{1-x^2}}^{1}(x^2+y^2-1)dy$$

$$=\int_0^1\frac{2}{3}(1-x^2)^{\frac{3}{2}}dx+\int_0^1\frac{2}{3}(1-x^2)^{\frac{3}{2}}dx+\int_0^1\left(x^2-\frac{2}{3}\right)dx=\frac{4}{3}\int_0^1(1-x^2)^{\frac{3}{2}}dx+\left(-\frac{1}{3}\right)$$

$$=\frac{4}{3}\cdot\int_0^{\frac{\pi}{2}}\cos^4t\,dt-\frac{1}{3}=\frac{4}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}-\frac{1}{3}=\frac{\pi}{4}-\frac{1}{3}$$

(10).



$$\int_1^2dx\int_{\sqrt{x}}^{x}\sin\frac{\pi}{2}\cdot\frac{x}{y}\,dy+\int_2^4dx\int_{\sqrt{x}}^{2}\sin\frac{\pi}{2}\frac{x}{y}\,dy=\int_1^2dy\int_y^{y^2}\sin\frac{\pi}{2}\cdot\frac{x}{y}\,dx$$

$$=-\frac{2}{\pi}\int_1^2y\cos\frac{\pi x}{2y}\Big|_y^{y^2}dy=-\frac{2}{\pi}\int_1^2y\cdot\cos\frac{\pi}{2}y\,dy=-\frac{2^2}{\pi^2}\left[\left(y\cdot\sin\frac{\pi}{2}y\right)\Big|_1^2-\int_1^2\sin\frac{\pi}{2}y\,dy\right]$$

$$=\left(\frac{2}{\pi}\right)^2\left(1+\frac{2}{\pi}\right)=\frac{4}{\pi^3}(\pi+2)$$

3. 计算下列三重积分.

(1) $\iiint\limits_V \frac{1}{\sqrt{(x-a)^2+y^2+z^2}}\mathrm{d}V$，其中$V: x^2+y^2+z^2 \le R^2, a>R$；

(2) $\iiint\limits_V \frac{z\ln\left(x^2+y^2+z^2+1\right)}{x^2+y^2+z^2+1}\mathrm{d}V$，其中$V: x^2+y^2+z^2 \le 1$；

(3) $\iiint\limits_V y\sqrt{1-x^2}\,\mathrm{d}V$，其中$V$是由曲面$y=-\sqrt{1-x^2-z^2}$，$x^2+z^2=1$，$y=1$所围成的区域；

(4) $\iiint\limits_V z\,\mathrm{d}V$，其中$V$是由曲面$z=\sqrt{4-x^2-y^2}$与$z=\frac{1}{3}\left(x^2+y^2\right)$所围成的区域；

(5) $\iiint\limits_V (x+y+z)^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V:(x-1)^2+(y-1)^2+(z-1)^2 \le R^2$；

(6) $\int_{-1}^{1}\mathrm{d}x\int_0^{\sqrt{1-x^2}}\mathrm{d}y\int_1^{1+\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{x^2+y^2+z^2}}\mathrm{d}z$；

(7) $\int_0^1\mathrm{d}x\int_x^1\mathrm{d}y\int_y^1 y\sqrt{1+z^4}\,\mathrm{d}z$.

(1) $\iiint\limits_V \frac{1}{\sqrt{(x-a)^2+y^2+z^2}}\,dV = \int_0^{2\pi}d\theta\int_{-R}^{R}dx\int_0^{\sqrt{R^2-x^2}}\frac{\rho\,d\rho}{\sqrt{(x-a)^2+\rho^2}} = \int_0^{2\pi}d\theta\int_{-R}^{R}\left(\sqrt{a^2+R^2-2ax}-(a-x)\right)dx$

$= 2\pi\cdot\left[\left(-\frac{1}{2a}\int_{-R}^{R}\sqrt{a^2+R^2-2ax}\,d(-2ax)\right)-2aR\right].$

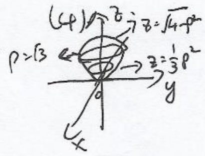
$= 2\pi\cdot\left[\frac{1}{3a}\left((a+R)^3-(a-R)^3\right)-2aR\right] = 2\pi\cdot\frac{2}{3a}R^3 = \frac{4}{3}\pi R^3$

(2). $\iiint\limits_V \frac{z\ln(x^2+y^2+z^2+1)}{x^2+y^2+z^2+1}\,dV = \int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^1\frac{r^3\cos\varphi\ln(r^2+1)}{r^2+1}\,dr = \int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^1\frac{r^3\ln(r^2+1)}{r^2+1}\,dr$

$= 0.$

(3) $\iiint\limits_V y\cdot\sqrt{1-x^2}\,dV = \int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dz\int_{-\sqrt{1-x^2-z^2}}^{1}y\cdot\sqrt{1-x^2}\,dy$

$= \int_{-1}^{1}\sqrt{1-x^2}\,dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{2}(x^2+z^2)\,dz = \int_{-1}^{1}\frac{1}{3}+\frac{1}{3}x^2-\frac{2}{3}x^4\,dx = \frac{28}{45}.$

(4) 

$\iiint\limits_V z\,dV = \int_0^{2\pi}d\theta\int_0^{\sqrt{3}}\rho\,d\rho\int_{\frac{1}{3}\rho^2}^{\sqrt{4-\rho^2}}z\,dz.$

$= \int_0^{2\pi}\frac{1}{2}d\theta\int_0^{\sqrt{3}}\rho\cdot z^2\Big|_{\frac{1}{3}\rho^2}^{\sqrt{4-\rho^2}}d\rho = \int_0^{2\pi}\frac{1}{2}d\theta\int_0^{\sqrt{3}}(4\rho-\rho^3-\frac{1}{9}\rho^5)\,d\rho$

$= \int_0^{2\pi}\frac{1}{2}d\theta\times\left(2\rho^2-\frac{1}{4}\rho^4-\frac{1}{9}\cdot\frac{1}{6}\rho^6\right)\Big|_0^{\sqrt{3}} = 2\pi\times\frac{1}{2}\times\left(6-\frac{9}{4}-\frac{1}{2}\right)$

$= \frac{13}{4}\pi.$

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3. (5) 令 $x'=x-1$，$y'=y-1$，$z'=z-1$，$|J|=1$，$x'^2+y'^2+z'^2\le R^2$.

$\therefore \iiint\limits_V (x+y+z)^2\,dx\,dy\,dz=\iiint\limits_V (x'+y'+z'+3)^2\,dx'dy'dz'$，设 $A=x'+y'+z'$

$=\iiint\limits_V A^2\,dV+6\iiint\limits_V A\,dV+9\iiint\limits_V dV$

由对称性知. $\iiint\limits_V A\,dV=0$. $\left(\cancel{\iiint x'\,dx\,dy\,dz=\int_{-R}^{R} y\,dy\int_{z_2(y)}^{z_1(y)} dz}\right)$

$\because \iiint x'\,dx'dy'dz'=\int_{-R}^{R}dy'\int_{z_2(y')}^{z_1(y')}dz\int_{-\sqrt{R^2-y^2-z^2}}^{\sqrt{R^2-y^2-z^2}}x'\,dx'$

$=0$

同理. $\iiint\limits_V A\,dV=0$

$9\iiint\limits_V dV=9\cdot\int_0^{2\pi}d\theta\cdot\int_0^{\pi}\sin\varphi\int_0^R r^2\,dr=9\times\frac{4}{3}\pi R^3=12\pi R^3$

$\iiint\limits_V (x'+y'+z')^2\,dV=\iiint\limits_V (x'^2+y'^2+z'^2)\,dV$

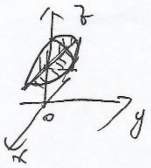
$\because$ 下面证明 $\iiint\limits_V x'y'\,dV=0$.

$\iiint\limits_V x'y'\,dV=\int_{-R}^{R}dz'\int_{x_2(z')}^{x_1(z')}x'\,dx\int_{-\sqrt{R^2-x^2-z^2}}^{\sqrt{R^2-x^2-z^2}}y\,dy=0$

$\therefore \iiint\limits_V A^2\,dV=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\int_0^R r^4\,dr=2\pi\times2\times\frac{1}{5}\cdot R^5=\frac{4}{5}\pi R^5$

$\therefore \iiint\limits_V (x+y+z)^2=\frac{4}{5}\pi R^5+12\pi R^3$

3 (6). 如图:



$$\int_{-1}^{1}dx\int_{0}^{\sqrt{1-x^2}}dy\int_{1}^{1+\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{x^2+y^2+z^2}}dz$$

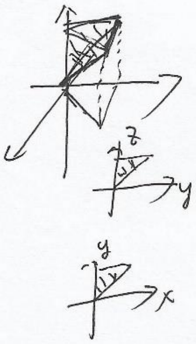
$$=\int_{0}^{\pi}d\theta\int_{1}^{2}dz\int_{0}^{\sqrt{1-(z-1)^2}}\frac{\rho d\rho}{\sqrt{z^2+\rho^2}}=\int_{0}^{\pi}d\theta\int_{1}^{2}\sqrt{z^2+\rho^2}\Big|_{0}^{\sqrt{1-(z-1)^2}}dz$$

$$=\int_{0}^{\pi}d\theta\cdot\int_{1}^{2}\sqrt{2z}-z\,dz=\pi\times\left(\frac{2\sqrt{2}}{3}\cdot z^{\frac{3}{2}}\Big|_{1}^{2}-\frac{1}{2}z^2\Big|_{1}^{2}\right)=\pi\left(\frac{2\sqrt{2}}{3}(2\sqrt{2}-1)-\frac{3}{2}\right)$$

$$=\pi\cdot\left(\frac{8}{3}-\frac{2}{3}\sqrt{2}-\frac{3}{2}\right)=\frac{7}{6}\pi-\frac{2}{3}\sqrt{2}\pi$$

(7).

如图:积分区域如阴影所示



$$\int_{0}^{1}dx\int_{x}^{1}dy\int_{y}^{1}y\sqrt{1+z^4}\,dz$$

$$=\int_{0}^{1}dz\int_{0}^{z}dy\int_{0}^{y}y\cdot\sqrt{1+z^4}\,dx$$

$$=\int_{0}^{1}\sqrt{1+z^4}\int_{0}^{z}y\,dy\int_{0}^{y}dx=\int_{0}^{1}\sqrt{1+z^4}\int_{0}^{z}y^2dy$$

$$=\int_{0}^{1}\frac{z^3}{3}\sqrt{1+z^4}\,dz=\frac{1}{12}\int_{0}^{1}\sqrt{1+z^4}\,dz^4$$

$$=\frac{1}{12}\cdot\frac{2}{3}\cdot(1+z^4)^{\frac{3}{2}}\Big|_{0}^{1}$$

$$=\frac{1}{18}\cdot(2\sqrt{2}-1)$$

4. 设 $F(x,y)$ 在 $D=\{(x,y)|a\le x\le b,c\le y\le d\}$ 上有二阶连续导数，且 $\frac{\partial^2 F}{\partial x\partial y}=f(x,y)$，证明
$\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y=F(b,d)-F(b,c)-F(a,d)+F(a,c)$.

解：设 $\frac{\partial F}{\partial x}=G(x,y)=\int f(x,y)dy$.

则有 $F=\int G(x,y)dx$

$$\therefore \iint\limits_D f(x,y)dx\,dy=\int_a^b dx\int_c^d f(x,y)dy$$

$$=\int_a^b G(x,d)-G(x,c)\ dx$$

$$=F(b,d)-F(a,d)-(G(b,c)-G(a,c))$$

$$=F(b,d)-F(b,c)-F(a,d)+F(a,c)$$

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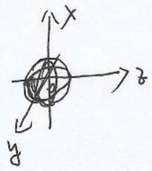
5. 设 $f(x)$ 在 $[-1,1]$ 上连续，证明 $\iiint\limits_V f(x)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\pi\int_{-1}^{1}f(x)\left(1-x^2\right)\mathrm{d}x$，

其中 $V: x^2+y^2+z^2\le 1$.



解：采用柱坐标系

令 $y^2+z^2=\rho^2$.

$\rho^2+x^2=1 \Rightarrow \rho=\sqrt{1-x^2}$

$\therefore \iiint\limits_V f(x)\,dxdydz$

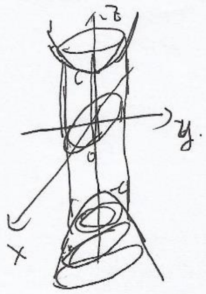
$$= \int_{-1}^{1}dx\int_0^{2\pi}d\theta\int_0^{\sqrt{1-x^2}}\rho f(x)\,d\rho$$

$$= \int_{-1}^{1}f(x)\,dx\int_0^{2\pi}d\theta\int_0^{\sqrt{1-x^2}}\rho\,d\rho$$

$$= 2\pi\cdot\int_{-1}^{1}f(x)\cdot\frac{1}{2}(1-x^2)\,dx$$

$$= \pi\cdot\int_{-1}^{1}f(x)(1-x^2)\,dx$$

6. 求由曲面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ 与 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 所围成的立体的体积.



解：令. $x=a\rho\cdot\cos\theta$,

$y=b\cdot\rho\cos\theta$,

图中所示为柱面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

与椭圆双曲面所围成图形

$$\therefore \iiint_V dV=\int_0^{2\pi}d\theta\int_0^1 ab\rho d\rho\int_{-c\sqrt{\rho^2+1}}^{c\sqrt{\rho^2+1}}dz$$

$$=\int_0^{2\pi}d\theta\int_0^1 ab(2c\sqrt{\rho^2+1})\cdot d\tfrac{1}{2}\rho^2$$

$$=\int_0^{2\pi}d\theta\int_0^1 abc\sqrt{\rho^2+1}\,d(\rho^2+1)$$

$$=abc\cdot(2\pi)\cdot\frac{2}{3}(1+\rho^2)^{\frac{3}{2}}\Big|_0^1$$

$$=\frac{4}{3}abc\pi\cdot(2\sqrt{2}-1)$$

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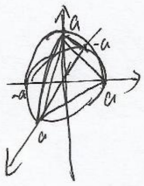
7. 求曲面 $az = a^2 - x^2 - y^2$ 与平面 $x + y + z = a\ (a > 0)$ 以及三个坐标面所围成立体的体积.



解：所求几何体

可由.椭圆抛物面 $az = a^2 - x^2 - y^2$ 在第一卦限的部分 减去三棱. $x+y+z=a$, $(x\geq 0, y\geq 0, z\geq 0)$ 的体积

$$V_{三棱锥} = \frac{1}{3} \times \frac{1}{2} \times a^2 \cdot a = \frac{1}{6}a^3$$

$$V_{椭抛面} = \cancel{\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{a} \rho\, d\rho \int_0^{a-\frac{\rho^2}{a}} dz$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{a} \left(a\rho - \frac{1}{a}\rho^3\right) d\rho$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}a\rho^2 - \frac{1}{4a}\cdot\rho^4\right)\Big|_0^{a} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}a^3 \, d\theta = \frac{\pi}{8}a^3$$

$$\therefore V = \frac{\pi}{8}a^3 - \frac{1}{6}a^3 = \frac{a^3}{3}\left(\frac{\pi}{4} - \frac{1}{3}\right)$$

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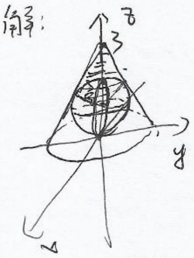
8. 求锥面 $z=3-\sqrt{3\left(x^2+y^2\right)}$ 与球面 $z=1+\sqrt{1-x^2-y^2}$ 所围成立体的体积.

解:



解：由题 $\begin{cases} z=3-\sqrt{3(x^2+y^2)} \\ z=1+\sqrt{1-x^2-y^2} \end{cases}$

可得 $x^2+y^2=\frac{3}{4}$.

且锥与球相切于 $\begin{cases} x^2+y^2=\frac{3}{4} \\ z=\frac{3}{2} \end{cases}$

$$\therefore V=\int_0^{2\pi}d\theta\int_0^{\frac{\sqrt{3}}{2}}\rho\,d\rho\cdot\int_{1+\sqrt{1-\rho^2}}^{3-\sqrt{3}\rho}dz$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\sqrt{3}}{2}}(2\rho-\sqrt{3}\rho^2-\rho\sqrt{1-\rho^2})d\rho$$

$$=2\pi\times\left(\rho^2-\frac{\sqrt{3}}{3}\rho^3+\frac{1}{3}(1-\rho^2)^{\frac{3}{2}}\right)\Big|_0^{\frac{\sqrt{3}}{2}}$$

$$=2\pi\times\left(\frac{3}{4}-\frac{3}{8}-\frac{1}{3}\times\frac{7}{8}\right)=\frac{\pi}{6}.$$

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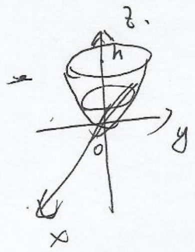
9. 设一形状为抛物面 $z=x^2+y^2$ 的容器已盛有 $8\pi\ \text{cm}^3$ 的液体，现又倒入 $120\pi\ \text{cm}^3$ 的液体，问液面比原来升高多少？



设高为h的体积为V 有 $h=\rho^2 \Rightarrow \rho=\sqrt{h}$

$\therefore V=\int_0^{2\pi}d\theta\int_0^{\sqrt{h}}\rho d\rho\int_0^{\rho^2}dz$

$=\int_0^{2\pi}d\theta\int_0^{\sqrt{h}}\rho^3 d\rho$

$=\int_0^{2\pi}\frac{1}{4}\rho^4\Big|_0^{\sqrt{h}}d\theta=\frac{\pi}{2}h^2$

$\therefore \frac{\pi}{2}\cdot h_1^2=8\pi$

$\frac{\pi}{2}\cdot h_2^2=120\pi+8\pi$

$\therefore h_1=4\text{cm}$

$h_2=16\text{cm}$

$\therefore \Delta h=12\text{cm}$

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10. 求抛物面 $z=1+x^2+y^2$ 的一个切平面，使得它与该抛物面与圆柱面 $(x-1)^2+y^2=1$ 围成的体积最小，试写出切平面方程，并求出最小体积.

解：

$\frac{\partial z}{\partial x}=2x,\ \frac{\partial z}{\partial y}=2y.$

$\therefore$ 切面法向量 $(2x_0,2y_0,-1)$

$\therefore$ 切面方程

$2x_0x+2y_0y-(x_0^2+y_0^2)+1$

$$\therefore V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}\rho d\rho\int_{2x_0x+2y_0y-(x_0^2+y_0^2)+1}^{1+x^2+y^2}dz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}\rho\cdot(\rho^2-2x_0\rho\cos\theta-2y_0\rho\sin\theta+(z_0-1))d\rho$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\cdot\left(\rho^3-\frac{2}{3}x_0\rho^3\sin\theta-\frac{2}{3}y_0\rho^3\sin\theta+\frac{1}{2}(z_0-1)\rho^2\right)\Big|_0^{2\cos\theta}$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\cos^4\theta-\frac{16}{3}x_0\cos^4\theta-\frac{2}{3}y_0\cos^3\theta\sin\theta+\frac{1}{2}(z_0-1)\cos^2\theta\,)d\theta$$

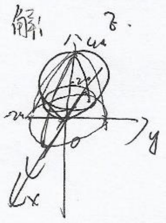
$$=(\frac{3}{2}-2x_0)\pi+(z_0-1)\pi=(\frac{1}{2}-2x_0+z_0)\pi$$

$$\therefore\begin{cases}2x_0=z_0\\ z_0=1+x_0^2+y_0^2\end{cases}\Rightarrow\begin{cases}x_0=1\\ z_0=2\end{cases}$$

$\therefore$ 平面方程 $z=2x$，$V=\frac{\pi}{2}$

11. 曲面 $x^2+y^2+az=4a^2$ 将球体 $x^2+y^2+z^2\le 4az$ 分成两个部分，试求两部分体积之比.

解：



$V_{球}=\frac{4}{3}\pi R^3=\frac{32}{3}\pi a^3$ 由题 $\rho^2+az=4a^2$

$\rho^2+(z-2a)^2=4a^2$.

$V_1=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}a}\rho d\rho\int_{2a-\sqrt{4a^2-\rho^2}}^{4a-\frac{\rho^2}{a}}dz$

$=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}a}\rho\cdot(2a-\frac{\rho^2}{a}+\sqrt{4a^2-\rho^2})\,d\rho$

$=\int_0^{2\pi}d\theta\int_0^{3a^2}\frac{1}{2}(2a-\frac{u}{a}+\sqrt{4a^2-u})\,du$. 令 $u=\rho^2$.

$=2\pi\times\frac{1}{2}(2a\times 3a^2-\frac{1}{2}\cdot\frac{1}{a}\cdot 9a^3+\frac{2}{3}(4a^2-u)^{\frac{3}{2}}\Big|_0^{3a^2})$

$=\pi\cdot\frac{37}{6}a^3=\frac{37}{6}\pi a^3$

$\therefore V_2=V_{球}-V_1=\frac{27}{6}\pi a^3$

$\therefore\frac{V_1}{V_2}=\frac{37}{27}$. V_1 为曲面 $x^2+y^2+az=4a^2$ 与球所围成部分

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12. 一个火山的形状可以用曲面 $z = he^{-\frac{\sqrt{x^2+y^2}}{4h}}$ $(h>0)$ 来表示，在一次火山爆发后，有体积为 V 的熔岩粘附在山上，使它具有和原来一样的形状，求火山高度变化的百分比.

解：设 V_h 为喷发前的体积，h_1 为喷发后的高度.

$\therefore V = V_1 - V_h$，求 $\frac{h_1-h}{h}$

$\therefore V_h = \iint_D he^{-\frac{\sqrt{x^2+y^2}}{4h}}dxdy$

$= \int_0^{2\pi}d\theta\int_0^{+\infty}h\cdot e^{-\frac{\rho}{4h}}\rho d\rho$

$= 2\pi h\cdot(-4h)\cdot[\rho\cdot e^{-\frac{\rho}{4h}}\Big|_0^{+\infty} - \int_0^{+\infty}e^{-\frac{\rho}{4h}}d\rho]$

$= 8\pi h^2(-4h)e^{-\frac{\rho}{4h}}\Big|_0^{+\infty} = 32\pi h^3$

$\therefore V_1 = 32\pi h_1^3 \quad V = 32\pi(h_1^3 - h^3)$

$h_1 = (h^3 + \frac{V}{32\pi})^{\frac{1}{3}}$

$\therefore \frac{h_1-h}{h} = \frac{1}{h}(\frac{V}{32\pi}+h^3)^{\frac{1}{3}} - 1$

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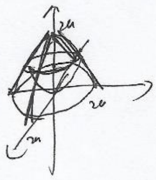
13. 求由曲面$x^2+y^2=az$, $z=2a-\sqrt{x^2+y^2}$ $(a>0)$所围成的立体的表面积.

解：



由图知，锥面$z_2=2a-\sqrt{x^2+y^2}$与$x^2+y^2=az_1$的交线$\begin{cases}x^2+y^2=a^2\\ z=a\end{cases}$

$$\frac{\partial z_1}{\partial x}=\frac{2x}{a},\quad \frac{\partial z_1}{\partial y}=\frac{2y}{a}.$$

$$\frac{\partial z_2}{\partial x}=\frac{-x}{\sqrt{x^2+y^2}},\quad \frac{\partial z_2}{\partial y}=\frac{-y}{\sqrt{x^2+y^2}}.$$

$$S_1=\iint_D\sqrt{1+\left(\frac{\partial z_1}{\partial x}\right)^2+\left(\frac{\partial z_1}{\partial y}\right)^2}\,dxdy=\iint_D\sqrt{1+\frac{4(x^2+y^2)}{a^2}}\,d\sigma$$

$$=\int_0^{2\pi}d\theta\int_0^a\rho\cdot\sqrt{1+\frac{4\rho^2}{a^2}}\,d\rho=\int_0^{2\pi}\frac{1}{8a}\cdot\frac{1}{3}(a^2+\rho^2)\Big|_0^a d\theta$$

$$=\frac{a^2}{6}\pi\cdot(5\sqrt{5}-1)$$

$$S_2=\iint_D\overset{d\sigma}{\sqrt{1+\left(\frac{\partial z_2}{\partial x}\right)^2+\left(\frac{\partial z_2}{\partial y}\right)^2}}=\iint_D\sqrt{2}\,dxdy$$

$$=\int_0^{2\pi}d\theta\int_0^a\sqrt{2}\rho\,d\rho=\sqrt{2}\cdot a^2.$$

$$\therefore S=S_1+S_2=\frac{1}{6}\pi a^2(6\sqrt{2}+5\sqrt{5}-1)$$

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14. 设半径为r的球的球心在半径为a的定球面上，试求r的值，使得半径为r的球的表面位于定球内部的那一部分的面积取最大值.

解：如图

先推导得球冠表面积公式，不含底面.

球半径为r，球冠高为h.

$dS = 2\pi \cdot r \cdot \cos\theta \, dh$

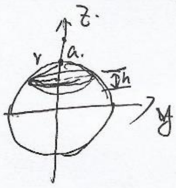
$dh = r \cdot d\theta$

∴ $dS = 2\pi r^2 \cos\theta \, d\theta$

∴ $S = \int_0^{\frac{\pi}{2}} 2\pi r^2 \cos\theta \, d\theta = 2\pi r^2 (1-\sin\theta)$

∵ $h = r(1-\sin\theta)$ ∴ $S = 2\pi r h$.

∴ $S = 2\pi r \cdot h$.



∴ 由勾股定理

$r^2 - (r-h)^2 = a^2 - (a-(r-h))^2$

∴ $h = \frac{2ar - r^2}{2a}$

∴ $S = 2\pi \cdot r \cdot \frac{2ar-r^2}{2a} = \frac{\pi}{a} \cdot (2ar^2 - r^3)$

∴ $S' = \frac{\pi}{a}(4ar - 3r^2) = 0$

$r = \frac{4}{3}a$ 或 $r = 0$

经验证：$r = \frac{4}{3}a$ 时 S 取极大值

∴ $r = \frac{4}{3}a$

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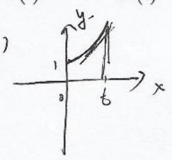
15. 曲线 $y=\dfrac{e^{x}+e^{-x}}{2}$ 与直线 $x=0, x=t\ (t>0)$ 及 $y=0$ 围成一曲边梯形，该曲边梯形绕 x 轴旋转一周所得旋转体的体积为 $v(t)$，侧面积为 $S(t)$，在 $x=t$ 处的底面积为 $F(t)$，

求(1) $\dfrac{S(t)}{V(t)}$；(2) $\lim\limits_{t\to+\infty}\dfrac{S(t)}{F(t)}$.

(1)



$\cancel{S(t)=\int_0^{2\pi}d\theta\int_0^{t}y\cdot\frac{dx}{\ }}$

$S(t)=2\pi\cdot\int_0^{t}y\cdot\sqrt{1+y'^2}\,dx$

$=2\pi\cdot\int_0^{t}\frac{e^x+e^{-x}}{2}\cdot\sqrt{1+\frac{e^{2x}-2+e^{-2x}}{4}}\,dx$

$=2\pi\cdot\int_0^{t}\left(\frac{e^x+e^{-x}}{2}\right)^2dx$

$V(t)=\pi\cdot\int_0^{t}y^2dx=\pi\cdot\int_0^{t}\left(\frac{e^x+e^{-x}}{2}\right)^2dx$

$\therefore\ \frac{S(t)}{V(t)}=2.$

(2). $F(t)=\pi\cdot y^2=\pi\cdot\left(\frac{e^t+e^{-t}}{2}\right)^2$

$\therefore\ \lim\limits_{t\to+\infty}\frac{S(t)}{F(t)}=\dfrac{2\pi\int_0^{t}\left(\frac{e^x+e^{-x}}{2}\right)^2dx}{\pi\cdot\left(\frac{e^t+e^{-t}}{2}\right)^2}$

$=\lim\limits_{t\to+\infty}\dfrac{2\left(\frac{e^t+e^{-t}}{2}\right)^2}{2\left(\frac{e^t+e^{-t}}{2}\right)\left(\frac{e^t-e^{-t}}{2}\right)}=\lim\limits_{t\to+\infty}\dfrac{e^t+e^{-t}}{e^t-e^{-t}}$

$=1$

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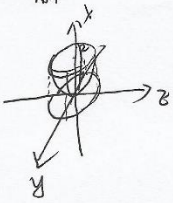
16. 求由曲面 $y^2+2z^2=4x$ 与平面 $x=2$ 所围成质量均匀分布立体的质心.

解



令 $y=\rho\cdot\cos\theta$.

$z=\frac{1}{\sqrt{2}}\rho\sin\theta$. ，$\mu$为密度.

$$\therefore M=\mu\cdot V=\mu\cdot\iiint dV=$$

$$=\mu\cdot\int_0^{2\pi}d\theta\int_0^{2\sqrt{2}}\frac{1}{\sqrt{2}}\rho\,d\rho\int_{\frac{\rho^2}{4}}^{2}dx$$

$$=\mu\cdot 2\pi\,\frac{1}{\sqrt{2}}\int_0^{2\sqrt{2}}\rho\cdot\left(2-\frac{\rho^2}{4}\right)d\rho$$

$$=\mu\cdot 2\pi\cdot\frac{1}{\sqrt{2}}\int_0^{8}\frac{1}{2}\left(2-\frac{u}{4}\right)du,\ \text{令}u=\rho^2.$$

$$=\mu\cdot 2\pi\cdot\frac{1}{\sqrt{2}}\cdot 4$$

$$\text{又}\iiint\mu x\,dV$$

$$=\mu\cdot\int_0^{2\pi}d\theta\int_0^{2\sqrt{2}}\frac{1}{\sqrt{2}}\rho\,d\rho\int_{\frac{\rho^2}{4}}^{2}x\,dx$$

$$=2\pi\mu\frac{1}{\sqrt{2}}\int_0^{2\sqrt{2}}\frac{1}{2}\left(4-\frac{\rho^4}{16}\right)\rho\,d\rho$$

$$=2\pi\cdot\frac{\mu}{\sqrt{2}}\int_0^{8}\frac{1}{4}\left(4-\frac{u^2}{16}\right)du,\ \text{令}u=\rho^2.$$

$$=2\pi\cdot\frac{\mu}{\sqrt{2}}\cdot\frac{16}{3}$$

$\therefore\ \bar{x}=\frac{4}{3}$. 由对称性可知有 $\bar{y}=\bar{z}=0$

$\therefore(\bar{x},\bar{y},\bar{z})=\left(\frac{4}{3},0,0\right)$

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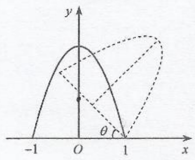
17. 欲设计一个装置，其中需要一平面薄片，如图，该薄片由 x 轴及抛物线 $y=a(1-x^2)$ 围成，密度为常数，对此薄片的要求是，当它以 $(1,0)$ 为支点向右方倾斜时，只要 θ 不超过 $45°$，则该薄片不会向右翻倒，问什么样的 a 值能保证达到此要求？



解：由对称性可知未翻转前，

质心的坐标为 $(0,\bar{y})$ 由题知 $0<\bar{y}$

当 $\bar{y}<1$ 时，翻转至 $45°$ 后不会倒

$$\bar{y}=\frac{\iint\limits_D y\,dxdy}{\iint\limits_D dxdy}$$

$$=\frac{\int_{-1}^{1}dx\int_0^{a(1-x^2)}y\,dy}{\int_{-1}^{1}dx\int_0^{a(1-x^2)}dy}=\frac{\int_{-1}^{1}\frac{1}{2}a^2(1-x^2)^2dx}{\int_{-1}^{1}a(1-x^2)dx}$$

$$=\frac{a^2\int_{-1}^{1}\frac{1}{2}(1-2x^2+x^4)dx}{a\cdot\int_{-1}^{1}(1-x^2)dx}=a\cdot\frac{(x-\frac{2}{3}x^3+\frac{1}{5}x^5)\big|_0^1}{(x-\frac{1}{3}x^3)\big|_0^1}$$

$$=a\cdot\frac{\frac{8}{15}}{\frac{4}{3}}=\frac{8}{15}\times\frac{3}{4}\times a=\frac{2}{5}a$$

$\because 0<\bar{y}<1$

$\therefore 0<a<\frac{5}{2}$

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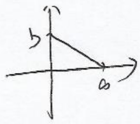
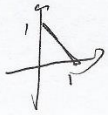
18. 设 D 是由直线 $\frac{x}{a}+\frac{y}{b}=1\,(a>0,b>0)$ 与坐标轴所围成的均匀薄片，求 I_x, I_O.



设面密度为 ρ

设 $u=\frac{x}{a}$, $v=\frac{y}{b}$

$\therefore J=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}=ab$



$I_x=\mu\iint\limits_D y^2\,dxdy=\iint\limits_{D'} b^2v^2(ab)\,dudv$

$I_x=\mu ab^3\int_0^1 du\int_0^{1-u} v^2dv$

$I_x=\mu ab^3\int_0^1 \frac{1}{3}(1-u)^3du$

$I_x=\frac{\mu}{3}ab^3\int_0^1 t^3dt=\frac{1}{12}\mu ab^3$

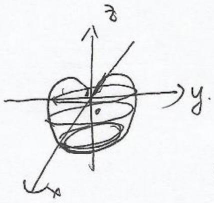
$I_O=\mu\iint(x^2+y^2)\,dxdy=I_y+I_x$

$I_y=\mu\iint\limits_D x^2\,dxdy=\frac{1}{12}\mu a^3b$

$\therefore I_O=\frac{1}{12}\mu ab(a^2+b^2)$

19. 求由曲面$r=1-\cos\varphi$所围成的苹果形均匀立体关于z轴的转动惯量.

解:



$$I_z=\left[\iiint_V (x^2+y^2)\,dV\right]\cdot\mu$$

$$=\mu\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1-\cos\varphi} r^2\cdot r^2\sin\varphi\,dr$$

$$=\mu\int_0^{2\pi}d\theta\cdot\int_0^{\pi}\sin^3\varphi\cdot\frac{1}{5}(1-\cos\varphi)^5\,d\varphi$$

令$u=\cos\varphi$

$$=2\pi\cdot\frac{1}{5}\int_{-1}^{1}(1-u^2)(1-u)^5\,du$$

$$=2\pi\frac{\mu}{5}\int_{-1}^{1}(1+u)(1-u)^6\,du$$

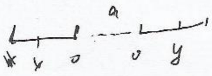
令$1-u=t$, $\therefore 0\le t\le 2$.

$$\therefore I_z=\left[2\pi\cdot\frac{1}{5}\int_0^{2}(2-t)\cdot t^6\,dt\right]\cdot\mu.$$

$$=\frac{2}{5}\pi\cdot\left(2\times\frac{2^7}{7}-\frac{2^8}{8}\right)=\frac{64}{35}\mu\cdot\pi.$$

20. 有两根质量均匀分布的细杆，长度都是l，质量都是M，若两根细杆位于同一直线上，近端相距为a，求它们相互的引力.

解：$\mu=\frac{M}{l}$，μ为线密度.



$\therefore dF=\frac{G\mu dx\cdot\mu dy}{(a+x+y)^2}$

$\therefore F=G\mu^2\int_0^l dx\int_0^l\frac{dy}{(a+x+y)^2}$

$=G\mu^2\int_0^l\left(\frac{1}{a+x}-\frac{1}{a+l+x}\right)dx$

$=G\mu^2\cdot[\ln(a+x)-\ln(a+l+x)]\big|_0^l$

$=G\mu^2\cdot[\ln(a+l)-\ln a-\ln(a+2l)+\ln(a+l)]$

$=G\mu^2\cdot\ln\frac{(a+l)^2}{a(a+2l)}=G\cdot\frac{M^2}{l^2}\cdot\ln\frac{(a+l)^2}{a(a+2l)}$

第八章　重积分
第六节　综合例题