8-3 ~~3050410~~ 05111713. 刘[illegible]. 13201712

1. 将三重积分 $I=\iiint\limits_V f(x,y,z)\mathrm{d}V$ 化成累次积分.

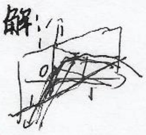
(1) V 是由双曲抛物面 $z=xy$，平面 $x+y-1=0$ 及 $z=0$ 围成的区域;

(2) V 是由曲面 $z=x^2+y^2$ 与平面 $z=1$ 围成的区域;

(3) V 是由曲面 $z=x^2+2y^2$ 与 $z=2-x^2$ 围成的闭区域;

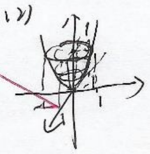
(4) V 是由柱面 $y=1-x^2$ 与平面 $z=0$ 及 $z=y$ 围成的区域.

解:



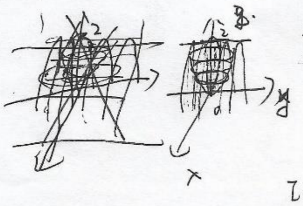
(1) $I=\iiint\limits_V f(x,y,z)dV=\int_0^1 dx\int_0^{1-x}dy\int_0^{xy}f(x,y,z)dV$

(2)



$I=\iiint\limits_V f(x,y,z)dV=\int_{-1}^1 dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_{x^2+y^2}^{1}f(x,y,z)dz$

(3)



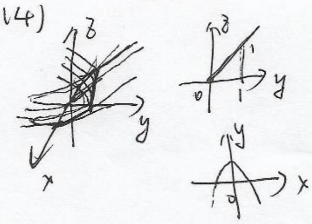
~~$I=\iiint\limits_V f(x,y,z)dV=\int_{-\sqrt{2}}^{\sqrt{2}}dx\int$~~

令 $z_1=x^2+2y^2=z_2=2-x^2$，有 $x^2+y^2=1$

~~$z_2-z_1=2-2(x^2+y^2)$~~

$I=\iiint\limits_V f(x,y,z)dV=\int_{-1}^1 dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_{x^2+2y^2}^{2-x^2}f(x,y,z)dz$

(4)



$I=\iiint\limits_V f(x,y,z)dV=\int_{-1}^1 dx\int_0^{1-x^2}dy\int_0^{y}f(x,y,z)dz.$

第八章　重积分
第三节　三重积分

2. 计算下列三重积分

(1) $\iiint\limits_V \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1+x+y+z)^3}$，其中$V$是平面$x+y+z=1$与三坐标面所围成的区域;

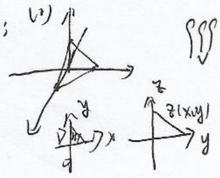
(2) $\iiint\limits_V xz\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由平面$z=0, z=y, y=1$以及$y=x^2$柱面所围成的区域;

(3) $\iiint\limits_V xy^2z^3\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由曲面$z=xy$与平面$y=x, x=1$和$z=0$所围成区域;

(4) $\iiint\limits_V e^x\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由平面$x=0, y=1, z=0, y=x$以及$x+y-z=0$所围成的区域;

(5) $\iiint\limits_V y\cos(x+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由柱面$y=\sqrt{x}$及平面$y=0, z=0, x+z=\frac{\pi}{2}$所围成的区域.

解: (1)



$$\iiint \frac{dxdydz}{(1+x+y+z)^3} = \int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}dz$$

$$= \int_0^1 dx\int_0^{1-x} \left(-\frac{1}{2}\right)\frac{1}{(1+x+y+z)^2}\Big|_0^{1-x-y} dy$$

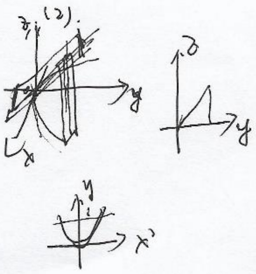
$$= \int_0^1 dx\int_0^{1-x} \left(-\frac{1}{2}\right)\left(\frac{1}{4}-\frac{1}{(1+x+y+z)^2}\right)dy$$

$$= \int_0^1 \left(-\frac{1}{2}\right)\left[\left(\frac{1}{4}\right)\cdot(1-x) + \left(\frac{1}{2}-\frac{1}{1+x}\right)\right]dx$$

$$\cancel{= -\frac{1}{8}x+\frac{1}{2}\ln(1+x)\Big|_0^1 - \frac{1}{2}\times\frac{1}{2}}$$

$$= -\frac{1}{8}\times\frac{1}{2}+\frac{1}{2}\ln(1+x) - \frac{1}{2}\times\frac{1}{2} = \frac{1}{2}\left(\ln 2-\frac{5}{8}\right)$$

(2)



$$\iiint\limits_V xz\,dxdydz = \int_{-1}^1 dx\int_0^{x^2} dy\int_0^y f(x,y,z)dz$$

$$= \int_{-1}^1 dx\int_0^{x^2} dy\int_0^y x\cdot z\,dz$$

$$= \int_{-1}^1 dx\int_0^{x^2} x\cdot\frac{1}{2}y^2 dy$$

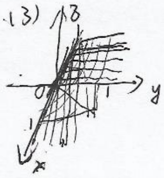
$$= \int_{-1}^1 x\cdot\frac{1}{6}x^6 dx = \int_{-1}^1 \frac{1}{6}x^7 dx = 0$$

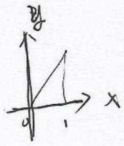
第八章　重积分

第三节　三重积分

2.

(3)





$$\iiint_V xy^2z^3\,dx\,dy\,dz$$

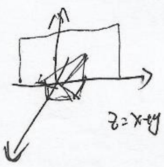
$$=\int_0^1 dx\int_0^x dy\int_0^{xy} xy^2z^3\,dz$$

$$=\int_0^1 dx\int_0^x \frac{1}{4}x^5y^2\cdot y^4\,dy=\int_0^1 dx\int_0^x \frac{1}{4}x^5y^6\,dy$$

$$\cancel{=\int_0^1 \frac{1}{28}x^{12}dx=\frac{1}{28}\cdot\frac{1}{13}\int_0^1 x^{13}d}$$

$$=\int_0^1 \frac{1}{28}\cdot\frac{1}{13}\cdot x^{12}dx=\frac{1}{364}\cdot x^{13}\Big|_0^1=\frac{1}{364}$$

(4).



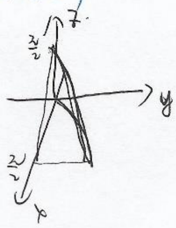
$$\iiint_V f(x,y,z)\,dV=\int_0^1 dx\int_x^1 dy\int_0^{x+y} e^x\,dz$$

$$=\int_0^1 dx\int_x^1 e^x(x+y)\,dy$$

$$=\int_0^1 e^x\cdot x+\frac{1}{2}e^x-\frac{3}{2}e^x\cdot x^2\,dx$$

$$=1+\frac{1}{2}e-\frac{3}{2}[e-2]=\frac{7}{2}-e$$

~~3)~~ (8)



$$\iiint_V f(x,y,z)\,dV=\int_0^{\frac{\pi}{2}} dx\int_0^{\sqrt{x}} dy\int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,dz$$

$$=\int_0^{\frac{\pi}{2}} dx\int_0^{\sqrt{x}} y\cdot\sin(x+z)\Big|_0^{\frac{\pi}{2}}dy$$

$$=\int_0^{\frac{\pi}{2}} dx\int_0^{\sqrt{x}} y(1-\sin x)\,dy=\int_0^{\frac{\pi}{2}}\frac{1}{2}y^2\Big|_0^{\sqrt{x}}\cdot(1-\sin x)\,dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{2}x\cdot(1-\sin x)\,dx=\frac{1}{2}\left[\frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}}-(\sin x-x\cdot\cos x)\Big|_0^{\frac{\pi}{2}}\right]$$

$$=\frac{1}{2}\left[\frac{\pi^2}{8}-(1-0)\right]=\frac{\pi^2-8}{16}$$

3. 在柱坐标系中计算下列三重积分.

(1) $\iiint\limits_V (x^2+y^2)\mathrm{d}V$，其中$V$是由曲面$x^2+y^2=2z$与平面$z=2$所围成的闭区域;

(2) $\iiint\limits_V \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由曲面$z=9-x^2-y^2$与平面$z=0$所围成的闭区域;

(3) $\iiint\limits_V z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由上半球面$x^2+y^2+z^2=4\ (z\geq 0)$与抛物面$z=\frac{1}{3}(x^2+y^2)$所围成的闭区域;

(4) $\iiint\limits_V x^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由曲面$z=2\sqrt{x^2+y^2}$，$x^2+y^2=1$与平面$z=0$所围成的闭区域;

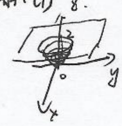
(5) $\iiint\limits_V (x+y)\mathrm{d}V$，其中$V$是介于两柱面$x^2+y^2=1$和$x^2+y^2=4$之间的被平面$z=0$和$z=x+2$所截下的部分;

(6) $\iiint\limits_V z\,\mathrm{d}V$，其中$V$是由曲面$z=x^2+y^2$与平面$z=2y$所围成的闭区域;

(7) $\iiint\limits_V z^2\mathrm{d}V$，其中$V: x^2+y^2+z^2\leq a^2, x^2+y^2\leq ax\ (a>0)$;

(8) $\iiint\limits_V y^2\mathrm{d}V$，其中$V$是由曲面$z=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$与平面$z=0$所围成的闭区域.

解：(1)



$\iiint\limits_V (x^2+y^2)dv = \iint\limits_D \rho d\theta d\rho \int_{\frac{1}{2}\rho^2}^{2} \rho^2 dz.$

$D=\{x,y \mid x^2+y^2\leq 4\}$ 即 $\rho\leq 2$

$\therefore \iiint\limits_V (x^2+y^2)dv = \int_0^{2\pi} d\theta \int_0^2 \rho d\rho \int_{\frac{1}{2}\rho^2}^{2} \rho^2 dz = \int_0^{2\pi} d\theta \int_0^2 (2\rho^3-\frac{1}{2}\rho^5)d\rho$

$= \int_0^{2\pi} (\frac{1}{2}\rho^4-\frac{1}{12}\rho^6)\Big|_0^2 d\theta = \int_0^{2\pi}\frac{8}{3}d\theta = \frac{16}{3}\pi$

(2)

$\iiint\limits_V \sqrt{x^2+y^2}\,dv = \iint\limits_D \rho d\theta d\rho \int_0^{9-\rho^2} \rho dz$，其中$D\{\rho \mid 0\leq\rho\leq 3\}$

$\therefore \iiint\limits_V \sqrt{x^2+y^2}\,dv = \int_0^{2\pi} d\theta \int_0^3 \rho d\rho \int_0^{9-\rho^2} \rho dz = \int_0^{2\pi} d\theta \int_0^3 \rho^2(9-\rho^2)d\rho$

$= \int_0^{2\pi} (3\rho^3-\frac{1}{5}\rho^5)\Big|_0^3 d\theta = \int_0^{2\pi} (3^4-\frac{1}{5}\cdot 3^5)\cdot d\theta$

$= \int_0^{2\pi} 3^4\cdot\frac{2}{5}d\theta = 3^4\times\frac{2}{5}\times 2\pi = \frac{81\times 4\pi}{5} = \frac{324}{5}\pi$

第八章 重积分

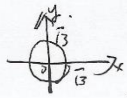
第三节 三重积分

3.

(3).

$$\iiint_V z\,dx\,dy\,dz = \iint_D \rho\,d\rho\,d\theta \int_{z_1(\rho,\theta)}^{z_2(\rho,\theta)} z\,dz.$$

$z_1=\frac{1}{3}\rho^2$, $z_2=\sqrt{4-\rho^2}$, [illegible]如图. (令 $\begin{cases} z^2=4-x^2-y^2 \\ 3z=x^2+y^2 \end{cases} \Rightarrow x^2+y^2=3 \Rightarrow \rho=\sqrt{3}$)



$$\iiint_V z\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\rho\,d\rho\int_{\frac{1}{3}\rho^2}^{\sqrt{4-\rho^2}} z\,dz.$$

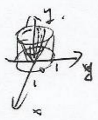
$$=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}\left(4\rho-\rho^3-\frac{1}{9}\rho^5\right)d\rho$$

$$=\frac{1}{2}\int_0^{2\pi}\left(2\rho^2-\frac{1}{4}\rho^4-\frac{1}{9}\times\frac{1}{6}\times\rho^6\right)\Big|_0^{\sqrt{3}}d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(6-\frac{9}{4}-\frac{1}{2}\right)d\theta=\frac{1}{2}\int_0^{2\pi}\frac{13}{4}d\theta$$

$$=\frac{13}{4}\times\frac{1}{2}\times 2\pi=\frac{13}{4}\pi$$

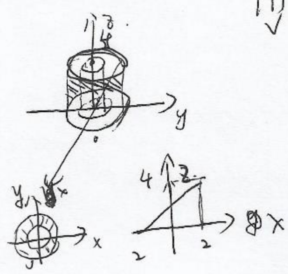
(4). $$\iiint_V x^2\,dV=\iint_D \rho\,d\theta\,d\rho\int_0^{2\rho}x^2\,dz=\int_0^{2\pi}d\theta\int_0^1\rho\,d\rho\int_0^{2\rho}\rho^2\cos^2\theta\,d\rho$$



$$=\int_0^{2\pi}d\theta\int_0^1 2\rho^4\cos^2\theta\,d\rho=\int_0^{2\pi}\frac{2}{5}\rho^5\cos^2\theta\Big|_0^1 d\theta$$

$$=\int_0^{2\pi}\frac{2}{5}\cos^2\theta\,d\theta=\frac{2}{5}\int_0^{2\pi}\frac{\cos 2\theta+1}{2}d\theta=\int_0^{2\pi}\frac{1}{5}d\theta=\frac{2}{5}\pi$$

(5).



$$\iiint_V (x+y)\,dV=\int_0^{2\pi}d\theta\int_1^2\rho\,d\rho\int_0^{\rho\cos\theta+2}\rho(\cos\theta+\sin\theta)\,dz$$

$$=\int_0^{2\pi}d\theta\cdot\int_1^2\rho(\cos\theta+\sin\theta)(\rho\cos\theta+2)\Big|_1^2 d\rho$$

$$=\int_0^{2\pi}d\theta\int_1^2\rho^3(\cos^2\theta+\sin\theta\cos\theta)+2\rho^2(\cos\theta+\sin\theta)\,d\rho$$

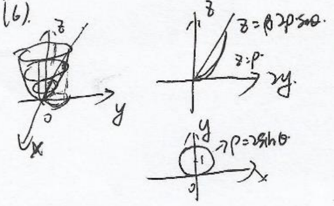
$$=\int_0^{2\pi}\frac{1}{4}\rho^4\Big|_1^2\left(\frac{\cos 2\theta+1}{2}+\frac{1}{2}\sin 2\theta\right)+\frac{2}{3}\rho^3\Big|_1^2(\cos\theta+\sin\theta)\,d\theta$$

$$=\int_0^{2\pi}\frac{15}{2\times 4}(\cos 2\theta+\sin 2\theta)+\frac{1}{2}\times\frac{15}{4}+\frac{14}{3}(\cos\theta+\sin\theta)\,d\theta$$

$$=\int_0^{2\pi}\frac{1}{2}\times\frac{15}{4}d\theta=\frac{15}{4}\pi$$

3.

(6)



$$\iiint z\,dV=\int_0^{\pi}d\theta\int_0^{2\sin\theta}\rho\,d\rho\int_{\rho^2}^{2\rho\sin\theta}z\,dz$$

$$=\int_0^{\pi}d\theta\int_0^{2\sin\theta}\rho\cdot\left(\frac{1}{2}z^2\Big|_{\rho^2}^{2\rho\sin\theta}\right)d\rho$$

$$=\int_0^{\pi}d\theta\int_0^{2\sin\theta}\frac{1}{2}(4\rho^3\sin^2\theta-\rho^5)\,d\rho$$

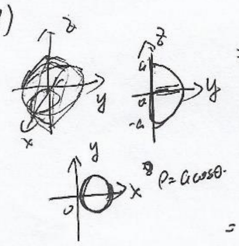
$$=\frac{1}{2}\int_0^{\pi}d\theta\int_0^{2\sin\theta}4\rho^3\sin^2\theta-\rho^5\,d\rho$$

$$=\frac{1}{2}\int_0^{\pi}\left(\rho^4\sin^2\theta-\frac{1}{6}\cdot\rho^6\right)\Big|_0^{2\sin\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\left(16\sin^6\theta-\frac{2^6}{6}\sin^6\theta\right)d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\frac{32}{6}\sin^6\theta\,d\theta=\int_0^{\frac{\pi}{2}}\frac{32}{6}\sin^6\theta\,d\theta=\frac{32}{6}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=\frac{5}{6}\pi$$

(7)



$$z=\sqrt{a^2-\rho^2}.$$

$$\iiint z^2\,dV=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\rho\,d\rho\int_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}}z^2\,dz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\rho\cdot\frac{1}{3}z^3\Big|_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}}d\rho=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\frac{2}{3}(a^2-\rho^2)^{\frac{3}{2}}\rho\,d\rho$$

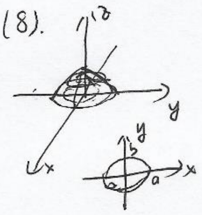
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\left(-\frac{1}{3}\right)(a^2-\rho^2)^{\frac{3}{2}}d(a^2-\rho^2)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-\frac{1}{3}\right)\cdot\frac{2}{5}(a^2-\rho^2)^{\frac{5}{2}}\Big|_0^{a\cos\theta}d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{2}{15}(a^5-a^5\cdot\sin^5\theta)\,d\theta=\frac{2}{15}\left[a^5\left(\frac{\pi}{2}+\frac{\pi}{2}\right)-2\int_0^{\frac{\pi}{2}}\sin^5\theta\,d\theta\right]$$

$$=\frac{2}{15}a^5\left[\pi-\frac{4}{5}\times\frac{2}{3}\times1\times2\right]=\frac{2}{15}a^5\left(\pi-\frac{16}{15}\right)$$

3.

(8).



设 $x = a\cdot\rho\cdot\cos\theta$

$y = b\cdot\rho\sin\theta$

$\therefore \rho^2 = 1 \Rightarrow \rho = 1$

$dxdydz = ab\rho d\rho d\theta$

$$\therefore \iiint\limits_V y^2 dV = \int_0^{2\pi} d\theta \int_0^1 ab\rho d\rho \int_0^{\sqrt{1-\rho^2}} \rho^2 \sin^2\theta\, dz$$

$$= ab^3 \int_0^{2\pi} d\theta \cdot \int_0^1 \sin^2\theta \cdot \rho^3 \sqrt{1-\rho^2}\, d\rho$$

$$\int_0^1 \rho^3 \sqrt{1-\rho^2}\, d\rho = -\int_0^1 ((1-\rho^2)-1)\sqrt{1-\rho^2}\, d\tfrac{1}{2}\rho^2$$

$$= \frac{1}{2}\int_0^1 ((1-\rho^2)-1)\cdot\sqrt{1-\rho^2}\, d(1-\rho^2)$$

$$= \frac{-1}{2}\int_0^1 (u-1)\cdot\sqrt{u}\, du = \frac{1}{2}\int_1^0 u^{\frac{3}{2}} - u^{\frac{1}{2}}\, du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}\cdot u^{\frac{3}{2}}\right)\Big|_1^0 = \frac{2}{15}$$

$$\therefore \iiint\limits_V y^2 dV = \frac{2}{15}ab^3 \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \frac{2}{15}ab^3 \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{2}{15}ab^3 \int_0^{2\pi} \frac{1}{2}\, d\theta$$

$$= \frac{2}{15}ab^3\cdot\pi$$

4. 在球坐标系中计算下列三重积分.

(1) $\iiint\limits_V (x^2+y^2+z^2)\mathrm{d}V$，其中$V$是由球面$x^2+y^2+z^2=1$所围成的闭区域;

(2) $\iiint\limits_V y^2\mathrm{d}V$，其中$V: x^2+y^2+z^2\le a^2, x^2+y^2+z^2\le b^2\ (0\le a\le b)$;

(3) $\iiint\limits_V (x^2+y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由曲面$z=\sqrt{x^2+y^2}$和$z=\sqrt{1-x^2-y^2}$所围成的闭区域;

(4) $\iiint\limits_V z\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由$x^2+y^2+(z-a)^2\le a^2$和$x^2+y^2\le z^2$所确定的区域;

(5) $\iiint\limits_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.

解：(1)

$f(x,y,z)=r^2$.

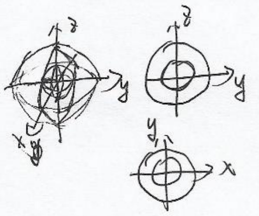
$x^2+y^2+z^2=r^2=1 \Rightarrow r=1$

$\therefore \iiint\limits_V (x^2+y^2+z^2)\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^1 r^2\cdot r^2\,\mathrm{d}r$

$=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\frac{1}{5}r^5\Big|_0^1\mathrm{d}\varphi=\frac{1}{5}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi$

$=\frac{1}{5}\int_0^{2\pi}2\,\mathrm{d}\theta=\frac{4}{5}\pi.$

(2). $y^2=r^2\sin^2\varphi\sin^2\theta$



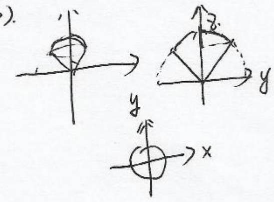
$\iiint\limits_V y^2\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_a^b r^2\sin^2\varphi\sin^2\theta\cdot r^2\,\mathrm{d}r.$

$=\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta\cdot\int_0^{\pi}\sin^3\varphi\,\mathrm{d}\varphi\cdot\int_a^b r^4\,\mathrm{d}r.$

$=4\times\frac{1}{2}\times\frac{\pi}{2}\cdot 2\times\frac{2}{3}\times 1\times\frac{1}{5}\times(b^5-a^5)$

$=\pi\cdot\frac{4}{3}\cdot\frac{1}{5}\cdot(b^5-a^5)=\frac{4}{15}\pi(b^5-a^5)$

(3).



$\iiint\limits_V (x^2+y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\sin\varphi\,\mathrm{d}\varphi\int_0^1 r^2\sin^2\varphi\, r^2\,\mathrm{d}r$

$=2\pi\cdot\int_0^{\frac{\pi}{4}}\sin^3\varphi\,\mathrm{d}\varphi\cdot\int_0^1 r^4\,\mathrm{d}r$

$=2\pi\times\frac{1}{5}\times\int_0^{\frac{\pi}{4}}(\cos^2\varphi-1)\,\mathrm{d}\cos\varphi$

$=\frac{2}{5}\pi\times\left(\frac{1}{3}u^3-u\right)\Big|_1^{\frac{\sqrt{2}}{2}}=\frac{2}{5}\pi\times\frac{8-5\sqrt{2}}{12}=\frac{8-5\sqrt{2}}{30}\pi.$

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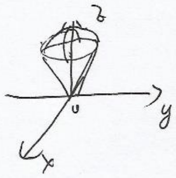
(4). 由 $x^2+y^2+(z-a)^2\le a^2 \Rightarrow r\le 2a\cos\varphi$



$$\iiint_V z\,dV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\, d\varphi\int_0^{2a\cos\varphi} r\cos\varphi\cdot r^2\, dr$$

$$=\cancel{\oint}\, 2\pi\cdot\int_0^{\frac{\pi}{4}}\cos\varphi\left(\frac{1}{4}r^4\right)\Big|_0^{2a\cos\varphi} d(-\cos\varphi)$$

$$=2\pi\cdot\int_0^{\frac{\pi}{4}}\cos^4\varphi\cdot 4a^4\cdot(-1)\cdot d\cos\varphi$$

$$=8\pi a^4\int_{\frac{1}{\sqrt{2}}}^{1} u^5\, du=8\pi a^4\cdot\frac{1}{6}\left(1^6-\left(\frac{1}{\sqrt{2}}\right)^6\right)$$

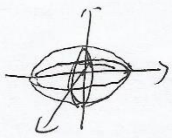
$$=8\pi a^4\times\frac{1}{6}\times\frac{7}{8}=\frac{7}{6}\pi a^4$$

(5). 利用广义坐标系，$x=ar\cdot\cos\theta\sin\varphi$，$y=b\cdot r\cdot\sin\theta\cdot\sin\varphi$，$z=c\cdot r\cdot\sin\varphi$

∴ $dV=abc\cdot r^2\sin\varphi\, dr\, d\theta\, d\varphi$

∴ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^2$，且 $r\le 1$

$$\therefore \iiint_V\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\, d\varphi\int_0^1 abc\cdot r^2\cdot r^2\, dr$$



$$=abc\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\, d\varphi\int_0^1 r^4\, dr$$

$$=abc\cdot 2\pi\cdot 2\cdot\frac{1}{5}=\frac{4}{5}\pi abc$$

5. 选用适当的坐标系计算下列三重积分. $r \le \pi \cdot \cos\varphi$

(1) $\iiint\limits_V \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$，其中$V$是由球面$x^2+y^2+z^2=z$所围成的闭区域;

(2) $\iiint\limits_V \sin z\,\mathrm{d}V$，其中$V$是由曲面$z=\sqrt{x^2+y^2}$与平面$z=\pi$所围成的闭区域;

(3) $\iiint\limits_V \frac{1}{1+x^2+y^2}\,\mathrm{d}V$，其中$V$是由曲面$x^2+y^2=z^2$与平面$z=1$所围成的区域;

(4) $\iiint\limits_V z\left(x^2+y^2\right)\mathrm{d}V$，其中$V: z\ge\sqrt{x^2+y^2}, 1\le x^2+y^2+z^2\le 4$;

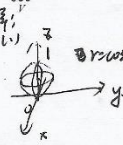
(5) $\iiint\limits_V \frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}V$，其中$V$是由曲面$z=\sqrt{x^2+y^2}$和平面$z=1$所围成的闭区域;

(6) $\iiint\limits_V z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中$V$是由曲面$z=1+\sqrt{1-x^2-y^2}$与平面$z=1$所围成的闭区域;

(7) $\iiint\limits_V z^2\,\mathrm{d}V$，其中$V$是球体$x^2+y^2+z^2\le R^2$与$x^2+y^2+z^2\le 2Rz$的公共部分.

解:

(1)



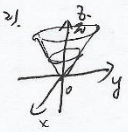
$r=\cos\varphi$

$$\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\, d\varphi\int_0^{\cos\varphi}\sqrt{r^2}\cdot r^2\,dr$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\frac{1}{4}r^4\Big|_0^{\cos\varphi}d(-\cos\varphi)=\int_0^{2\pi}d\theta\int_0^1\frac{1}{4}u^4\,du$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{4}\times\frac{1}{5}\times 1\,d\theta=\frac{1}{20}\times 2\pi=\frac{\pi}{10}$$

(2)



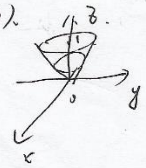
$$\iiint\limits_V \sin z\,dV=\int_0^{2\pi}d\theta\int_0^{\pi}\rho\, d\rho\int_\rho^{\pi}\sin z\,dz$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi}\rho(\cos\rho+1)\,d\rho$$

$$=\int_0^{2\pi}\left(\rho\cdot\sin\rho+\cos\rho+\frac{1}{2}\rho^2\right)\Big|_0^{\pi}d\theta$$

$$=\int_0^{2\pi}\left(\frac{1}{2}\pi^2-2\pi\right)d\theta=2\pi\cdot\left(\frac{1}{2}\pi^2-2\right)=\pi^3-4\pi$$

(3)



$$\iiint\limits_V \frac{1}{1+x^2+y^2}\,dV=\int_0^{2\pi}d\theta\int_0^1\rho\, d\rho\int_\rho^1\frac{1}{1+\rho^2}\,dz$$

$$=\int_0^{2\pi}d\theta\int_0^1\frac{\rho(1-\rho)}{1+\rho^2}\,d\rho=\int_0^{2\pi}d\theta\int_0^1\left(-1+\frac{1}{\rho^2+1}+\frac{\rho}{\rho^2+1}\right)d\rho$$

~~$=2\pi\cdot(-x+\arctan x+\frac{1}{2}$~~

$$=2\pi\cdot\left(-\rho+\arctan\rho+\frac{1}{2}\ln(1+\rho^2)\right)\Big|_0^1=2\pi\cdot\frac{2\ln 2+\pi-4}{4}$$

$$=\frac{\pi}{2}\cdot(2\ln 2+\pi-4)=\pi\cdot\left(\ln 2-2+\frac{\pi}{2}\right).$$

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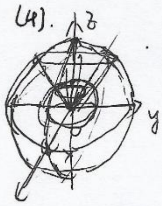
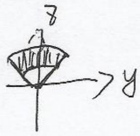
5.

(4).



$z=r,\ 1\le r\le 2$



$$\iiint_V z(x^2+y^2)\,dV$$

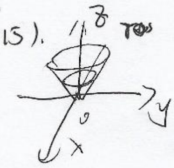
$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\,d\varphi\int_1^2 r^3\cos\varphi(r^2\sin^2\varphi)\,dr$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin^3\varphi\,d\sin\varphi\int_1^2 r^5\,dr$$

$$=2\pi\cdot\frac{1}{4}\sin^4\varphi\Big|_0^{\frac{\pi}{4}}\cdot\frac{1}{6}r^6\Big|_1^2$$

$$=2\pi\cdot\frac{1}{4}\times\frac{1}{4}\times\frac{1}{6}\times 63=\frac{21}{16}\pi$$

(5).

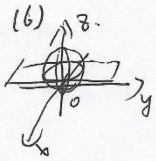


$$\iiint_V\frac{1}{\sqrt{x^2+y^2+z^2}}\,dV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\,d\varphi\int_0^{\frac{1}{\cos\varphi}}r\,dr$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\cdot\frac{1}{2}\frac{1}{\cos^2\varphi}\,d\theta=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\sec\theta$$

$$=\frac{1}{2}\times 2\pi\times(\sqrt{2}-1)=(\sqrt{2}-1)\cdot\pi$$

(6)



$$\iiint_V z\,dx\,dy\,dz=\int_0^{2\pi}d\theta\int_0^1\rho\,d\rho\int_1^{1+\sqrt{1-\rho^2}}z\,dz$$

$$=\int_0^{2\pi}d\theta\int_0^1\frac{1}{2}(1-\rho^2+2\cdot\sqrt{1-\rho^2})\,d\frac{1}{2}\rho^2$$

$$=\int_0^{2\pi}d\theta\,\frac{1}{4}\int_0^1(u+2\sqrt{u})\,du=\int_0^{2\pi}\frac{1}{4}\left(\frac{1}{2}u^2+\frac{4}{3}u^{\frac{3}{2}}\right)\Big|_0^1 d\theta$$

$$=\frac{11}{6}\times\frac{1}{4}\times 2\pi=\frac{11}{12}\pi$$

5.

(7) $\iiint\limits_V z^2 dV = \int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{3}}{2}R} \rho\, dr \int_{R-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} z^2 dz$

$= \frac{2}{3}\pi \int_0^{\frac{\sqrt{3}}{2}R} [(R^2-\rho^2)^{\frac{3}{2}} - (R-\sqrt{R^2-\rho^2})^3] dr$

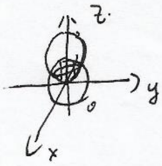
$= \frac{\pi}{3} \int_0^{\frac{\sqrt{3}}{2}R} [R^3 - 3R^2(R^2-\rho^2)^{\frac{1}{2}} + 3R(R^2-\rho^2) - 2(R^2-\rho^2)^{\frac{3}{2}}] d(R^2-\rho^2)$

$= \frac{\pi}{3} [R^3(R^2-\rho^2) - 2R^2(R^2-\rho^2)^{\frac{3}{2}} + \frac{3}{2}R(R^2-\rho^2)^2 - \frac{4}{5}(R^2-\rho^2)^{\frac{5}{2}}]\Big|_0^{\frac{\sqrt{3}}{2}a}$

$= \frac{\pi}{3} R^5 (\frac{1}{4} - \frac{1}{4} + \frac{3}{32} - \frac{1}{40} - 1 + 2 - \frac{3}{2} - \frac{4}{5})$

$= \frac{59}{480}\pi$

5(7)



$$\iiint z^2 dV = \cancel{\int_0^{\frac{\sqrt{3}}{2}R} \rho d}$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{3}}{2}R} \rho d\rho \cdot \int_{R-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} z^2 dz$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{3}}{2}R} \rho \left(\frac{1}{3}(R^2-\rho^2)^{\frac{3}{2}} - \frac{1}{3}(R-\sqrt{R^2-\rho^2})^{\frac{3}{2}}\right) d\rho$$

$$= \int_0^{2\pi} \left(-\frac{1}{3}\right) d\theta \cdot \frac{1}{2} \int_0^{\frac{\sqrt{3}}{2}R} (R^2-\rho^2)^{\frac{3}{2}} - (R-\sqrt{R^2-\rho^2})\, d(R^2-\rho^2)$$

令 $u = R^2-\rho^2$.

$$= \int_0^{2\pi} \frac{1}{3} d\theta \frac{1}{2} \int_{\frac{1}{4}R^2}^{R^2} (2u^{\frac{3}{2}} - R^3 + 3R^2\sqrt{u} - 3R \cdot u)\, du$$

$$= \frac{2}{3} \times \frac{1}{2} \times \pi \times \left(\frac{4}{5}u^{\frac{5}{2}} - R^3 \cdot u + 2R^2 \cdot u^{\frac{3}{2}} - \frac{3}{2}Ru^2\right)\Big|_{\frac{1}{4}R^2}^{R^2}$$

$$= \frac{\pi}{3} \times \left(\frac{4}{5} \times \frac{31}{32} - \frac{3}{4} + \frac{7}{4} - \frac{45}{32}\right) = \frac{\pi}{3} \times \left(\frac{31}{40} - \frac{13}{32}\right) = \frac{59}{480}\pi \cdot R^5.$$

关于坐标系的选择，有2个方面需要考虑，

第一点，被积函数，直角坐标，柱坐标，球坐标，这三种形式的方程，何者更简便，何者积分计算可能更方便。

第二点，包围区域V的表达形式，更简单，就采用哪种方法。

注：以上只是选择的经验，并不绝对正确，方法正确，3种方法都可以计算，如5(7)柱坐标与球坐标的计算量差不多，不存在绝对的简便优势。

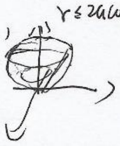
6. 求下列立体V的体积.

(1) $V: x^2+y^2+z^2\le 2az\ (a>0), x^2+y^2\le z^2$;

(2) V 由曲面 $z=\sqrt{x^2+y^2}$ 和 $z=x^2+y^2$ 所围成;

(3) V 由曲面 $z=\sqrt{5-x^2-y^2}$ 和 $x^2+y^2=4z$ 所围成;

(4) V 由曲面 $x^2+y^2=2ax, az=x^2+y^2\ (a>0)$ 及平面 $z=0$ 所围成.

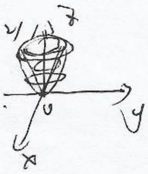
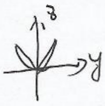
(1) 

$r\le 2a\cos\varphi$

$$V=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi d\varphi\int_0^{2a\cos\varphi}r^2dr$$

$$=2\pi\cdot\int_0^{\frac{\pi}{4}}\frac{8}{3}a^3\cos\varphi\, d(-\cos\varphi)=2\pi\times\frac{8}{3}\times a^3\times\frac{1}{4}(1-\frac{1}{4})$$

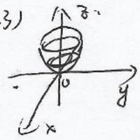
$$=\pi a^3$$

(2)  

$$V=\int_0^{2\pi}d\theta\int_0^1\rho d\rho\int_{\rho^2}^{\rho}dz$$

$$=\int_0^{2\pi}d\theta\cdot\int_0^1\rho^2-\rho^3d\rho$$

$$=2\pi\times(\frac{1}{3}-\frac{1}{4})=\frac{\pi}{6}$$

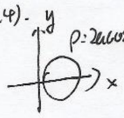
(3) 

$$V=\int_0^{2\pi}d\theta\int_0^{2}\rho d\rho\int_{\frac{1}{4}\rho^2}^{\sqrt{5-\rho^2}}dz$$

$$=\int_0^{2\pi}d\theta\int_0^2\rho\cdot\sqrt{5-\rho^2}-\frac{1}{4}\rho^3d\rho$$

$$=\int_0^{2\pi}(-\frac{1}{3})(5-\rho^2)\Big|_0^2-\frac{1}{16}\rho^4\Big|_0^2d\theta=\int_0^{2\pi}\frac{1}{3}5\sqrt{5}-\frac{1}{3}-1\,d\theta$$

$$=\frac{2}{3}\pi(5\sqrt{5}-4)$$

(4). 

$\rho=2a\cos\theta$.

$az=\rho^2 \Rightarrow z=\frac{\rho^2}{a}$

$$V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\rho d\rho\int_0^{\frac{\rho^2}{a}}dz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\frac{\rho^3}{a}d\rho=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}a^3\cdot\rho^4\Big|_0^{2a\cos\theta}d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4a^3\cos^4\theta d\theta=4a^3\cdot2\cdot\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta=4a^3\cdot2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=\frac{3}{2}a^3\pi$$

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