10、

$\frac{\partial z}{\partial x}=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x=\frac{x}{x^2+y^2}$

$\frac{\partial z}{\partial y}=\frac{y}{x^2+y^2}$

$\therefore \operatorname{grad} z(x,y)=\frac{x}{x^2+y^2}\cdot\vec{i}+\frac{y}{x^2+y^2}\cdot\vec{j}$

设常数 t.

$t=\frac{1}{2}\cdot\ln(x^2+y^2)$

$0=\frac{2x+2y\cdot y'}{x^2+y^2}\Rightarrow y'=-\frac{x}{y}$

设切线方向向量 $\vec{e}=(-y,x)=-y\cdot\vec{i}+x\cdot\vec{j}$

$\therefore \vec{e}\cdot\operatorname{grad} z(x,y)=\frac{-xy}{x^2+y^2}+\frac{xy}{x^2+y^2}=0$

11.

$\frac{\partial u}{\partial x}=\frac{1}{\frac{1}{r}}\cdot\frac{a-x}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{3}{2}}}=r\cdot\frac{a-x}{r^3}$

$=\frac{1}{r^2}\cdot(a-x)$

$\frac{\partial u}{\partial y}=\frac{1}{r^2}(b-y)$

$\frac{\partial u}{\partial z}=\frac{1}{r^2}(c-z)$

$\therefore \operatorname{grad} u=\frac{1}{r^2}(a-x,b-y,c-z)$.

$|\operatorname{grad} u|=\frac{1}{r^2}\cdot\sqrt{(a-x)^2+(b-y)^2+(c-z)^2}=\frac{1}{r}$

故当 $r=1$ 时，$|\operatorname{grad} u|=1$

习题 7-7.

1、$t=\frac{\pi}{3}$ 时.

$x_0=\frac{3}{4}a$，$y_0=\frac{\sqrt{3}}{4}b$，$z_0=\frac{1}{4}c$

$x'(t)=a\cdot\sin 2t$，$x'(t)|_{t=\frac{1}{3}\pi}=\frac{\sqrt{3}}{2}a$.

$y'(t)=b\cdot\cos 2t$，$y'(t)|_{t=\frac{1}{3}\pi}=-\frac{1}{2}b$

$z'(t)=-c\cdot\sin 2t$，$z'(t)|_{t=\frac{1}{3}\pi}=-\frac{\sqrt{3}}{2}c$.

$\therefore$ 切线方程

为

$$\frac{x-\frac{3}{4}a}{\sqrt{3}a}=\frac{y-\frac{\sqrt{3}}{4}b}{-b}=\frac{z-\frac{1}{4}c}{-\sqrt{3}c}$$

方向向量 $(\frac{\sqrt{3}}{2}a,-\frac{1}{2}b,-\frac{\sqrt{3}}{2}c)$

2、

$x'(t)=1$

$y'(t)=2t$

$z'(t)=3t^2$

$\therefore$ 切线方向向量 $\vec{s}=\vec{i}+2t\vec{j}+3t^2\cdot\vec{k}$

该平面法向量.

$\vec{n}=(1,2,1)=\vec{i}+2\vec{j}+\vec{k}$

$\therefore$ 由题有

$\vec{n}\cdot\vec{s}=0=1+2\cdot 2t+3t^2=0$

$(3t+1)(t+1)=0$

$\therefore t=-\frac{1}{3}$，$t=-1$

$\therefore x_1=-\frac{1}{3}$，$y_1=\frac{1}{9}$，$z_1=-\frac{1}{27}$

$x_2=-1$，$y_2=1$，$z_2=-1$

$\therefore$ 点 $(-\frac{1}{3},\frac{1}{9},-\frac{1}{27})$

与点 $(-1,1,-1)$

为所求点.

3. 对方程组求对x的导数

$$\begin{cases} 2x+2y\cdot\frac{dy}{dx}+2\cdot z\frac{dz}{dx}=0 \\ 1+\frac{dy}{dx}+\frac{dz}{dx}=0 \end{cases}$$

代入 $(1,-2,1)$

有 $\begin{cases} \frac{dy}{dx}=0 \\ \frac{dz}{dx}=-1 \end{cases}$ 切向量 $(1,0,-1)$

∴切线方程为

$$\begin{cases} x-1=1-z \\ y=-2 \end{cases}$$

法平面方程为

$$(x-1)+0(y+2)-(z-1)=0$$

$$(x-1)-(z-1)=0$$

$\therefore\ x-z=0$

5. $x'(t)=-a\sin t$

$y'(t)=a\cdot\cos t$

$z'(t)=b$

$\therefore \vec{s}=-a\cdot\sin t\,\vec{i}+a\cdot\cos t\cdot\vec{j}+b\cdot\vec{k}$

∴设夹角为 $\theta=\langle\vec{s},\vec{k}\rangle$

$|\vec{s}|=\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}=\sqrt{a^2+b^2}$

$\therefore \cos\theta=\frac{\vec{s}\cdot\vec{k}}{|\vec{s}||\vec{k}|}=\frac{b}{\sqrt{a^2+b^2}}$

$\theta=\arccos\frac{b}{\sqrt{a^2+b^2}}$

即该曲线切线与z轴成定角

4. $x'(t)=1$

$y'(t)=-2t$

$z'(t)=3t^2$

在 $P(1,-1,1)$ 处

$x'(t)|_{t=1}=1$

$y'(t)|_{t=1}=-2$

$z'(t)|_{t=1}=3$

∴方向向量 $\vec{s}=\vec{i}-2\vec{j}+3\vec{k}$, $\vec{e}=\frac{\vec{s}}{|\vec{s}|}=\left(\frac{1}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

$\therefore \frac{\partial u}{\partial x}=6z\cdot x-y$, $\frac{\partial u}{\partial x}\Big|_{\substack{x=1\\y=-1\\z=1}}=7$

$\frac{\partial u}{\partial y}=-x$ $\quad \frac{\partial u}{\partial y}=-1$

$\frac{\partial u}{\partial z}=3x^2+2z$ $\quad \frac{\partial u}{\partial z}=5$

∴方向导数

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma$$

$$=\frac{7+2+15}{\sqrt{14}}=\frac{24}{\sqrt{14}}$$

6. 求导后有

$$\begin{cases} 2x-\frac{dz}{dx}=0 \\ 3+2\frac{dy}{dx}+0=0 \end{cases}$$

代入 $P(1,-2,1)$

$\therefore \frac{dz}{dx}=2,\ \frac{dy}{dx}=-\frac{3}{2}$

∴切向量 $\vec{s}=(1,-\frac{3}{2},2)$

$\vec{a}=3\vec{i}-5\vec{j}+5\vec{k}$

$\vec{b}=\vec{i}+5\vec{k}$

$\therefore \vec{a}\times\vec{b}=5(-5\vec{i}-2\vec{j}+\vec{k})$

∴设 $\vec{n}=-5\vec{i}-2\vec{j}+\vec{k}$

$\therefore \vec{s}\cdot\vec{n}=-5+3+2=0$

$\therefore \vec{s}\perp\vec{n}$, 证毕

7. $F_x'(P_0)=y=1$

$F_y'(P_0)=x=2$

$F_z'(P_0)=e^z+1=2$

代入(2,1,0)有切平面法向量

为 $\vec{n}=(F_x'(P_0),F_y'(P_0),F_z'(P_0))$

$=(1,2,2)=\vec{i}+2\vec{j}+2\vec{k}$

∴平面方程 $(x-2)+2(y-1)+2z=0$

∴ $x+2y+2z=4$.

8. 由该切平面法向量 $\vec{n}=\vec{i}-\vec{j}+2\vec{k}=\{1,-1,2\}$

$F_x'=2x$, $F_y'=2y$, $F_z'=2z$.

∴ $\vec{n_0}=\{x,2y,[illegible]z\}$, 设切点 (x_0,y_0,z_0)

且有 $\vec{n_0}\parallel\vec{n}$

有 $\left\{\cancel{x_0^2+2y_0^2+z_0^2=1},\ \cancel{\frac{x_0}{1}=\frac{2y_0}{-1}=\frac{z_0}{2}}\right.$

$\begin{cases}x_0^2+2y_0^2+z_0^2=1\\ \frac{x_0}{1}=\frac{2y_0}{-1}=\frac{z_0}{2}\end{cases}$

有 $\begin{cases}x_0=\frac{2}{\sqrt{22}}\\ y_0=\frac{-1}{\sqrt{22}}\\ z_0=\frac{4}{\sqrt{22}}\end{cases}$ 或 $\begin{cases}x_0=\frac{-2}{\sqrt{22}}\\ y_0=\frac{1}{\sqrt{22}}\\ z_0=\frac{-4}{\sqrt{22}}\end{cases}$

∴平面方程

$\frac{2}{\sqrt{22}}x-\frac{2}{\sqrt{22}}y+\frac{4}{\sqrt{22}}z-1=0$

或 $\frac{2}{\sqrt{22}}x-\frac{2}{\sqrt{22}}y+\frac{4}{\sqrt{22}}z+1=0$

9. $\frac{\partial z}{\partial x}=4x$; $\frac{\partial z}{\partial x}\Big|_{x=1}=4$

$\frac{\partial z}{\partial y}=8y$; $\frac{\partial z}{\partial y}\Big|_{y=1}=8$

∴曲面在该点法向量

$\vec{n}=\left\{\frac{\partial z}{\partial x}\Big|_{P_0},\frac{\partial z}{\partial y}\Big|_{P_0},-1\right\}=\{4,8,-1\}$

∴平面方程 $4(x-1)+8(y-1)-(z-6)=0$

$4x-4+8y-8-z+6=0$

$4x+8y-z-6=0$

法线方程 $\frac{x-1}{4}=\frac{y-1}{8}=\frac{z-6}{-1}$

10. 由题知，该曲面

方程为 $3x^2+2y^2+3z^2=12$.

∴ $F_x'=6x$, $F_y'=4y$, $F_z'=6z$.

∴ $\vec{n}=(F_x'|_{P_0},F_y'|_{P_0},F_z'|_{P_0})$

$=\{0,4\sqrt{3},6\sqrt{2}\}$

∴ $\vec{n_0}=\frac{\vec{n}}{|\vec{n}|}=\left\{0,\frac{2}{\sqrt{10}},\frac{3}{\sqrt{15}}\right\}$

11. $F_x'=yz$, $F_y'=xz$, $F_z'=xy$.

∴在 $P_0(x_0,y_0,z_0)$ 处 $\vec{n}=\{y_0z_0,x_0z_0,x_0y_0\}$

∴该切平面 $y_0z_0\cdot x+x_0z_0\cdot y+x_0y_0\cdot z=3x_0y_0z_0$

∴截距 $x'=3x_0$, $y'=3y_0$, $z'=3z_0$

∴ $x'y'z'=27x_0y_0z_0$

∵P在曲面 $xyz=c^3$, ∴ $x'y'z'=27c^3$, 证毕.

12. $F_x'=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$, $F_y'=\frac{1}{2}\frac{1}{\sqrt{y}}$, $F_z'=\frac{1}{2}\cdot\frac{1}{\sqrt{z}}$

∴在 $P_0(x_0,y_0,z_0)$, $\vec{n}=\frac{1}{2}\left\{\frac{1}{\sqrt{x_0}},\frac{1}{\sqrt{y_0}},\frac{1}{\sqrt{z_0}}\right\}$.

∴切平面方程 $\frac{1}{\sqrt{x_0}}\cdot x+\frac{1}{\sqrt{y_0}}\cdot y+\frac{1}{\sqrt{z_0}}\cdot z=\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{a}$

∵ P_0 在曲面 $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$,

∴ $\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{a}$.

∴截距 $x'=\sqrt{a}\sqrt{x_0}$, $y'=\sqrt{a}\sqrt{y_0}$, $z'=\sqrt{a}\sqrt{z_0}$

∴ $x'+y'+z'=\sqrt{a}\cdot\sqrt{a}=a$.

证毕.

13. 由题 $\begin{cases}F(x,y,z)=xy-z^2\\ G(x,y,z)=x^2+y^2+z^2-9\end{cases}$

∴ $F_x'=y$, $F_y'=x$, $F_z'=-2z$

有 $G_x'=2x$, $G_y'=2y$, $G_z'=2z$.

∴在 $P(x_0,y_0,z_0)$ 处

$\vec{n_1}=\{y_0,x_0,-2z_0\}$

$\vec{n_2}=\{2x_0,2y_0,2z_0\}$

∴ $\vec{n_1}\cdot\vec{n_2}$

$=2x_0y_0+2x_0y_0-4z_0^2$

$=4(x_0y_0-z_0^2)$

∵ $P(x_0,y_0,z_0)$ 在 $F(x,y,z)$ 上

∴ $x_0y_0=z_0^2$

∴ $\vec{n_1}\cdot\vec{n_2}=0$

∴ $\vec{n_1}\perp\vec{n_2}$

14. $F_x'=6x$, $F_y'=2y$, $F_z'=2z$,

两直线方向向量 $\vec{s_1}=\{4,5,8\}$, $\vec{s_2}=\{1,1,1\}$

∴在 $P(x_0,y_0,z_0)$ 处，切平面法向量

$\vec{n}=\{6x_0,2y_0,2z_0\}$

∴ $\begin{cases}\vec{n}\cdot\vec{s_1}=0\\ \vec{n}\cdot\vec{s_2}=0\\ 3x_0^2+y_0^2+z_0^2=16\end{cases}\Rightarrow\begin{cases}x_0=z_0\\ y_0=-4z_0\\ 20z_0^2=16\end{cases}\Rightarrow\begin{cases}x_0=\sqrt{\frac{4}{5}}\\ y_0=-4\sqrt{\frac{4}{5}}\\ z_0=\sqrt{\frac{4}{5}}\end{cases}$ 或 $\begin{cases}x_0=-\sqrt{\frac{4}{5}}\\ y_0=4\sqrt{\frac{4}{5}}\\ z_0=-\sqrt{\frac{4}{5}}\end{cases}$

∴ $P\left(\sqrt{\frac{4}{5}},-4\sqrt{\frac{4}{5}},\sqrt{\frac{4}{5}}\right)$ 或 $\left(-\sqrt{\frac{4}{5}},4\sqrt{\frac{4}{5}},-\sqrt{\frac{4}{5}}\right)$

15. $F_x'=2x$, $F_y'=2y$, $F_z'=-1$

设 $P(x_0,y_0,z_0)$ 在曲线上.

∴法向量 $\vec{n}=(2x_0,2y_0,-1)$

法线方程 $\frac{x-x_0}{2x_0}=\frac{y-y_0}{2y_0}=\frac{z-z_0}{-1}$

无论令 $x=0$, $y=0$, 则 $z=\frac{1}{2}+z_0$. 无论 x_0, y_0 取任何值，法线都过 $(0,0,\frac{1}{2}+z_0)$

$\vec{n}\cdot\vec{k}=-1$

$\cos\langle\vec{n},\vec{k}\rangle=\frac{\vec{n}\cdot\vec{k}}{|\vec{n}||\vec{k}|}=\frac{-1}{\sqrt{4z_0+1}}$

$\theta=\langle\vec{n},\vec{k}\rangle=\arccos\left(\frac{-1}{\sqrt{4z_0+1}}\right)$

即 θ 为定角，故法线会形成一个锥面.