习题 7-6　　05011713 刘璧. 13201716072.

1. $\cos\alpha = -\frac{\sqrt{2}}{2}$

$\cos\beta = \frac{\sqrt{2}}{2}$

$\frac{\partial z}{\partial x} = 12x^2+y$, $\frac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=2}} = 12+2 = 14$

$\frac{\partial z}{\partial y} = x+3y^2$, $\frac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=2}} = 1+3\times4 = 13$

$\therefore \frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cdot\cos\alpha + \frac{\partial z}{\partial y}\cdot\cos\beta$

$= -7\sqrt{2}+\frac{13}{2}\sqrt{2} = -\frac{\sqrt{2}}{2}$

2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{2x}{a^2}+\frac{2y\cdot y'}{b^2}=0$

$y' = -\frac{b^2}{a^2}\cdot\frac{x}{y}$

$y'\Big|_{\substack{x=\frac{a}{\sqrt{2}}\\y=\frac{b}{\sqrt{2}}}} = -\frac{b^2}{a^2}\cdot\frac{a}{b} = -\frac{b}{a}$

内法向正切值 $\tan\theta_1 = \frac{-1}{y'} = \frac{a}{b}$

$\therefore \cos\alpha = \frac{-b}{\sqrt{a^2+b^2}}$, $\cos\beta = \frac{-a}{\sqrt{a^2+b^2}}$

$\frac{\partial z}{\partial x} = -\frac{2x}{a^2}$

$\frac{\partial z}{\partial y} = -\frac{2y}{a^2}$

$\frac{\partial z}{\partial x}\Big|_{\substack{x=\frac{a}{\sqrt{2}}\\y=\frac{b}{\sqrt{2}}}} = -\frac{\sqrt{2}}{a}$

$\frac{\partial z}{\partial x}\Big|_{\substack{x=\frac{a}{\sqrt{2}}\\y=\frac{b}{\sqrt{2}}}} = -\frac{\sqrt{2}}{b}$

$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cdot\cos\alpha + \frac{\partial z}{\partial y}\cdot\cos\beta$

$= \frac{\sqrt{2}}{ab}\cdot\sqrt{a^2+b^2}$

4. $\overrightarrow{AB} = \{2,0\}$, $\vec{e_1} = \{1,0\}$

$\overrightarrow{AC} = \{0,4\}$, $\vec{e_2} = \{0,1\}$

$\therefore \frac{\partial z}{\partial l_1} = \left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)\cdot\vec{e_1} = \frac{\partial z}{\partial x} = 3$

$\frac{\partial z}{\partial l_2} = \left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)\cdot\vec{e_2} = \frac{\partial z}{\partial y} = 26$

$\because \overrightarrow{AD} = \{5,12\}$, $\vec{e_3} = \left\{\frac{5}{13},\frac{12}{13}\right\}$

$\frac{\partial z}{\partial l_3} = \frac{\partial z}{\partial x}\cdot\cos\alpha + \frac{\partial z}{\partial y}\cdot\cos\beta = \frac{15}{13}+\frac{312}{13}$

$= \frac{327}{13}$

3. $\vec{l} = \{4,3,12\}$

$\vec{e} = \frac{\vec{l}}{|\vec{l}|} = \left\{\frac{4}{13},\frac{3}{13},\frac{12}{13}\right\}$

$\therefore \frac{\partial u}{\partial x} = yz$, $\frac{\partial u}{\partial x}\Big|_{\substack{y=1\\z=2}} = 2$

$\frac{\partial u}{\partial y} = x\cdot z$, $\frac{\partial u}{\partial y}\Big|_{\substack{x=5\\z=2}} = 10$

$\frac{\partial u}{\partial z} = x\cdot y$, $\frac{\partial u}{\partial z}\Big|_{\substack{x=5\\y=1}} = 5$

$\therefore \frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\cdot\frac{4}{13} + \frac{\partial u}{\partial y}\cdot\frac{3}{13} + \frac{\partial u}{\partial z}\cdot\frac{12}{13}$

$= \frac{8}{13}+\frac{30}{13}+\frac{60}{13}$

$= \frac{98}{13}$

5. $\frac{\partial f}{\partial x} = 2x+y+3$

$\frac{\partial f}{\partial y} = 4y+x-2$

$\frac{\partial f}{\partial z} = 6z-6$

$\therefore$ 在点 $(0,0,0)$ 处

$\frac{\partial f}{\partial x} = 3$, $\frac{\partial f}{\partial y} = -2$, $\frac{\partial f}{\partial z} = -6$

$\therefore \max|\mathrm{grad}\, f(0,0,0)| = \sqrt{9+4+36} = \sqrt{49} = 7$

$\mathrm{grad}\, f(0,0,0) = \{3,-2,-6\}$

在点 $(1,1,1)$

$\frac{\partial f}{\partial x} = 6$　$\frac{\partial f}{\partial y} = 3$　$\frac{\partial f}{\partial z} = 0$

$\therefore \mathrm{grad}\, f(1,1,1) = (6,3,0)$

$\max|\mathrm{grad}\, f(1,1,1)| = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

6. $\frac{\partial u}{\partial x}=\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$

$\frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2+z^2)^2}$

$\frac{\partial u}{\partial z}=\frac{-2xz}{(x^2+y^2+z^2)^2}$.

∴在 $(1,2,2)$ 处

$\frac{\partial u}{\partial x}=\frac{7}{9^2}$，$\frac{\partial u}{\partial y}=-\frac{4}{9^2}$，$\frac{\partial u}{\partial z}=-\frac{4}{9^2}$

$\operatorname{grad} f(1,2,2)=(\frac{7}{9^2},-\frac{4}{9^2},-\frac{4}{9^2})$.

设 $l_1 \parallel \operatorname{grad} f(1,2,2)$，$l_1=(7,-4,-4)$ 与 $\operatorname{grad} f(1,2,2)$ 同向.

在 $(-3,1,0)$

~~$\frac{\partial u}{\partial x}=(\frac{-8}{100},\frac{6}{100},0)$~~

$\frac{\partial u}{\partial x}=\frac{-8}{100}$，$\frac{\partial u}{\partial y}=\frac{6}{100}$，$\frac{\partial u}{\partial z}=0$

∴ $\operatorname{grad} f(-3,1,0)=(-\frac{8}{100},\frac{6}{100},0)$

设 $l_2=(-4,3,0)$，l_2 与 $\operatorname{grad} f(-3,1,0)$ 同向

∴设夹角为 θ.

$\cos\theta=\cos\langle l_1,l_2\rangle=\frac{-40}{|9|\cdot|5|}=-\frac{8}{9}$.

∴ $\theta=\arccos(-\frac{8}{9})$

7. 由题得度 t

$t=\frac{k}{\sqrt{x^2+y^2+z^2}}$，$k$ 为常数.

问题等价于证明 $\vec{g}(\frac{\partial t}{\partial x},\frac{\partial t}{\partial y},\frac{\partial t}{\partial z})$ 与 $(-x,-y,-z)$ 同向.

∴ $\frac{\partial t}{\partial x}=\frac{-kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$\frac{\partial t}{\partial y}=\frac{-ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$\frac{\partial t}{\partial y}=\frac{-kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$.

在任意点 (x_0,y_0,z_0)，指向圆原点的方向向量 $\vec{e}=(-x_0,-y_0,-z_0)$

$\frac{\partial t}{\partial x}=\frac{-kx_0}{(x_0^2+y_0^2+z_0^2)^{\frac{3}{2}}}$；$\frac{\partial t}{\partial y}=\frac{-ky_0}{(x_0^2+y_0^2+z_0^2)^{\frac{3}{2}}}$；$\frac{\partial t}{\partial z}=\frac{-kz_0}{(x_0^2+y_0^2+z_0^2)^{\frac{3}{2}}}$.

设夹角为 θ.

$\cos\theta=\frac{(\frac{\partial t}{\partial x},\frac{\partial t}{\partial y},\frac{\partial t}{\partial z})\cdot(-x_0,-y_0,-z_0)}{|(\frac{\partial t}{\partial x},\frac{\partial t}{\partial y},\frac{\partial t}{\partial z})|\cdot|(-x_0,-y_0,-z_0)|}=1$

∴ $\theta=0$

∴ $\vec{g}$ 与 $\vec{e}$ 同向，证毕.

8. $\frac{\partial v}{\partial x}=-4x$，$\frac{\partial v}{\partial y}=16$

∴设 $\vec{g}=(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y})$

在点 $(1,-2)$ 处

$\vec{g}=(-4,16)$.

(1) 设 $\vec{e}$ 为方向向量.

当 $\vec{e}$ 与 $\vec{g}$ 同向时，电压升高最快

$\vec{e}=(-1,4)$

(2) 当 $\vec{e}$ 与 $\vec{g}$ 反向时，电压下降最快

$\vec{e}=(1,-4)$

(3) $\operatorname{grad} V(1,-2)=(-4,16)$

∴ $\max|\operatorname{grad} V(1,-2)|=4\cdot\sqrt{17}$

$\min|\operatorname{grad} V(1,-2)|=-\max|\operatorname{grad} V(1,-2)|=-4\sqrt{17}$

∴上升的速率为 $4\sqrt{17}$，下降的速率为 $4\sqrt{17}$.

(4). $\vec{e}=(m,n)$，$\vec{e}$ 与 $\vec{g}$ 垂直时电压变化最慢为0，$\vec{e}\cdot\vec{g}=-m+4n=0$，$m=4n$.

∴ $\vec{e}=(4,1)$ 或 $\vec{e}=(-4,-1)$

9. $\frac{\partial u}{\partial x}=-\frac{2x}{a^2}$

$\frac{\partial u}{\partial y}=-\frac{2y}{b^2}$

$\frac{\partial u}{\partial z}=\frac{2z}{c^2}$.

在点 (a,b,c) 处.

$\frac{\partial u}{\partial x}=-\frac{2}{a}$，$\frac{\partial u}{\partial y}=-\frac{2}{b}$，$\frac{\partial u}{\partial z}=\frac{2}{c}$

设 $\vec{g}=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})=(-\frac{2}{a},-\frac{2}{b},\frac{2}{c})$

∴设 $\vec{e}=(l,m,n)$

$\vec{e}$ 与 $\vec{g}$ 同向时，$\vec{e_1}=(-\frac{1}{a},-\frac{1}{b},\frac{1}{c})$

$\vec{e}$ 与 $\vec{g}$ 反向时，$\vec{e_2}=(\frac{1}{a},\frac{1}{b},-\frac{1}{c})$

沿 $\vec{e_1}$ 增大得最快，沿 $\vec{e_2}$ 减小的最快.

由 $\vec{g}$ 为法向量，可确定一个平面.

$\frac{1}{a}x+\frac{1}{b}y-\frac{z}{c}=0$.

任何与此面平行或在平面内的向量，沿这些向量，变化率为0.

10、

$$\frac{\partial z}{\partial x}=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x=\frac{x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{y}{x^2+y^2}$$

$$\therefore \operatorname{grad} z(x,y)=\frac{x}{x^2+y^2}\cdot\vec{i}+\frac{y}{x^2+y^2}\cdot\vec{j}$$

设常数 t.

$$t=\frac{1}{2}\cdot\ln(x^2+y^2)$$

$$0=\frac{2x+2y\cdot y'}{x^2+y^2}\Rightarrow y'=-\frac{x}{y}$$

设切线方向向量 $\vec{e}=(-y,x)=-y\cdot\vec{i}+x\cdot\vec{j}$

$$\therefore \vec{e}\cdot\operatorname{grad} z(x,y)=\frac{-xy}{x^2+y^2}+\frac{xy}{x^2+y^2}=0$$

11.

$$\frac{\partial u}{\partial x}=\frac{1}{r}\cdot\frac{a-x}{((x-a)^2+(y-b)^2+(z-c)^2)^{\frac{1}{2}}}=r\cdot\frac{a-x}{r^3}$$

$$=\frac{1}{r^2}\cdot(a-x)$$

$$\frac{\partial u}{\partial y}=\frac{1}{r^2}(b-y)$$

$$\frac{\partial u}{\partial z}=\frac{1}{r^2}(c-z)$$

$$\therefore \operatorname{grad} u=\frac{1}{r^2}(a-x,b-y,c-z)$$

$$|\operatorname{grad} u|=\frac{1}{r^2}\cdot\sqrt{(a-x)^2+(b-y)^2+(c-z)^2}=\frac{1}{r}$$

故当 $r=1$ 时，$|\operatorname{grad} u|=1$

习题 7-7.

1、$t=\frac{\pi}{3}$ 时.

$$x_0=\frac{3}{4}a,\ y_0=\frac{\sqrt{3}}{4}b,\ z_0=\frac{1}{4}c$$

$$x'(t)=a\cdot\sin 2t,\quad x'(t)|_{t=\frac{1}{3}\pi}=\frac{\sqrt{3}}{2}a.$$

$$y'(t)=b\cdot\cos 2t,\quad y'(t)|_{t=\frac{1}{3}\pi}=-\frac{1}{2}b$$

$$z'(t)=-c\cdot\sin 2t,\quad z'(t)|_{t=\frac{1}{3}\pi}=-\frac{\sqrt{3}}{2}c.$$

$\therefore$ 切线方程

为

$$\frac{x-\frac{3}{4}a}{\sqrt{3}a}=\frac{y-\frac{\sqrt{3}}{4}b}{-b}=\frac{z-\frac{1}{4}c}{-\sqrt{3}c}$$

方向向量 $(\frac{\sqrt{3}}{2}a,-\frac{1}{2}b,-\frac{\sqrt{3}}{2}c)$

2、

$$x'(t)=1$$

$$y'(t)=2t$$

$$z'(t)=3t^2$$

$\therefore$ 切线方向向量 $\vec{s}=\vec{i}+2t\vec{j}+3t^2\cdot\vec{k}$

该平面法向量.

$$\vec{n}=(1,2,1)=\vec{i}+2\vec{j}+\vec{k}$$

$\therefore$ 由题有

$$\vec{n}\cdot\vec{s}=0=1+2\cdot 2t+3t^2=0$$

$$(3t+1)(t+1)=0$$

$$\therefore t=-\frac{1}{3},\ t=-1$$

$$\therefore x_1=-\frac{1}{3},\ y_1=\frac{1}{9},\ z_1=-\frac{1}{27}$$

$$x_2=-1,\ y_2=1,\ z_2=-1$$

$\therefore$ 点 $(-\frac{1}{3},\frac{1}{9},-\frac{1}{27})$

与点 $(-1,1,-1)$

为所求点.