1. 画出下列积分的积分区域并计算积分.

(1) $\iint\limits_D \sqrt{x+y}\,\mathrm{d}x\mathrm{d}y$，$D: 0\le x\le 1, 0\le y\le 3$;

(2) $\iint\limits_D \frac{2y}{1+x}\mathrm{d}x\mathrm{d}y$，$D$是由直线$y=x-1$与两坐标轴所围成的区域;

(3) $\iint\limits_D y\mathrm{e}^x\mathrm{d}x\mathrm{d}y$，$D$是顶点为$(0,0),(2,4)$和$(6,0)$的三角形区域;

(4) $\iint\limits_D \mathrm{e}^{x+y}\mathrm{d}x\mathrm{d}y$，$D$是由$|x|+|y|\le 1$所确定的区域;

(5) $\iint\limits_D (y^2-x)\mathrm{d}x\mathrm{d}y$，$D$是由抛物线$x=y^2$和$x=3-2y^2$所围成的区域;

(6) $\iint\limits_D \sin\frac{x}{y}\mathrm{d}x\mathrm{d}y$，$D$是由直线$y=x, y=2$和曲线$x=y^3$所围成的区域;

(7) $\iint\limits_D (x^2+y^2-x)\mathrm{d}x\mathrm{d}y$，$D$是由直线$y=2, y=x$及$y=2x$所围成的区域;

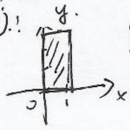
(8) $\iint\limits_D x^2\mathrm{e}^{-y^2}\mathrm{d}x\mathrm{d}y$，$D$是由直线$x=0, y=1$及$y=x$所围成的区域;

(9) $\iint\limits_D \sin y^2\mathrm{d}x\mathrm{d}y$，$D$是由直线$x=0, y=1$及$y=x$所围成的区域;

(10) $\iint\limits_D \frac{x^2}{y^2}\mathrm{d}x\mathrm{d}y$，$D$是由$y=2, y=x, xy=1$所围成的区域.

解：

(1).



$\iint\limits_D \sqrt{x+y}\,dxdy$

$=\int_0^1 dx\int_0^3 \sqrt{x+y}\,dy$

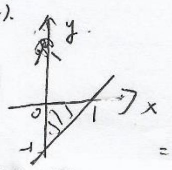
$=\int_0^1 \left(\frac{2}{3}(x+3)^{\frac{3}{2}}-\frac{2}{3}x^{\frac{3}{2}}\right)dx$

$=\frac{2}{3}\cdot\frac{2}{5}\left[(x+3)^{\frac{5}{2}}\Big|_0^1 - x^{\frac{5}{2}}\Big|_0^1\right]$

$=\frac{4}{15}\left[4^{\frac{5}{2}}-3^{\frac{5}{2}}-1^{\frac{5}{2}}\right]$

$=\frac{4}{15}\left[31-9\sqrt{3}\right]$.

(2).



$\iint\limits_D \frac{2y}{1+x}dxdy$

$=\int_0^1 dx\int_{x-1}^0 \frac{2y}{1+x}dy$

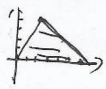
$=\int_0^1 \left(\frac{y^2}{1+x}\right)\Big|_{x-1}^0 dx$

$=\int_0^1 -\frac{(x-1)^2}{1+x}dx=-\int_0^1\left(x-3+\frac{4}{x+1}\right)dx$

原式$=-\left(\frac{1}{2}x^2-3x+4\ln(1+x)\right)\Big|_0^1$

$=-\left(-\frac{5}{2}+4\ln2\right)=\frac{5}{2}-4\ln2$.

(3).



$\iint\limits_D ye^x dxdy$

$=\int_0^4 dy\int_{\frac{1}{2}y}^{-y+6} ye^x dy$

$=\int_0^4\left(y\cdot e^{6-y}-y\cdot e^{\frac{1}{2}y}\right)dy$

$=\int_0^4 y\cdot e^{6-y}dy-\int_0^4 y\cdot e^{\frac{1}{2}y}dy$

$=-\left[e^{6-y}\cdot y\Big|_0^4-\int_0^4 e^{6-y}dy\right]-2\left[e^{\frac{1}{2}y}\cdot y\Big|_0^4-\int_0^4 e^{\frac{1}{2}y}dy\right]$

$=-\left[4e^2+e^2-e^6\right]-2\left[4e^2-2e^2+2\right]$

$=e^6-5e^2-4e^2-4$

$=e^6-9e^2-4$.

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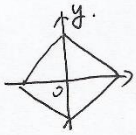
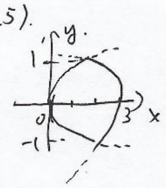
(4).



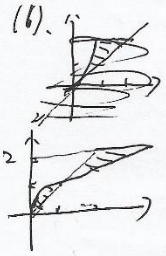
$$\iint_D e^{x+y}dxdy$$
$$=\int_{-1}^{0}dx\int_{-(x+1)}^{x+1}e^{x+y}dy+\int_0^1 dx\int_{x-1}^{-(x-1)}e^{x+y}dy$$
$$=\int_{-1}^{0}(e^{2x+1}-\tfrac{1}{e})dx+\int_0^1(e-e^{2x-1})dx$$
$$=\tfrac{1}{2}(e-\tfrac{1}{e})-\tfrac{1}{e}+e-\tfrac{1}{2}(e-\tfrac{1}{e})$$
$$=e-\tfrac{1}{e}$$

(5).



$$\iint_D (y^2-x)dxdy$$
$$=\int_{-1}^{1}dy\int_{y^2}^{3-2y^2}(y^2-x)dx$$
$$=\int_{-1}^{1}-\tfrac{9}{2}y^4+9y^2-\tfrac{9}{2}dy$$
$$=2\times(-\tfrac{9}{2}\times\tfrac{1}{5}y^5\big|_0^1+3y^3\big|_0^1-\tfrac{9}{2}y\big|_0^1)$$
$$=2\times(-0.9+3-4.5)=-4.8$$

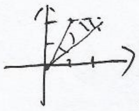
(6).



$$\iint_D \sin\tfrac{x}{y}dxdy$$
$$=\int_0^1 dy\int_{y^3}^{y}\sin\tfrac{x}{y}dx+\int_1^2 dy\int_y^{y^3}\sin\tfrac{x}{y}dx$$
$$=\int_0^1(-y\cos 1+y\cos y^2)dy+\int_1^2((-y)\cdot\cos y^2+\cos 1\,y)dy$$
$$=-\tfrac{1}{2}y^2\cos 1\big|_0^1+\tfrac{1}{2}\sin y^2\big|_0^1+-\tfrac{1}{2}\sin y^2\big|_1^2+\tfrac{1}{2}y^2\cos 1\big|_1^2$$
$$=-\tfrac{1}{2}\cos 1+\tfrac{1}{2}\sin 1-\tfrac{1}{2}\sin 4+\tfrac{1}{2}\sin 1+2\cos 1-\tfrac{1}{2}\cos 1$$
$$=\cos 1+\sin 1-\tfrac{1}{2}\sin 4$$

(7).

$$\iint_D (x^2+y^2-x)dxdy$$



$$= \int_0^2 dy \int_{\frac{y}{2}}^{y} (x^2+y^2-x)dx$$

$$= \int_0^2 \left(\frac{1}{3}x^3+y^2x-\frac{1}{2}x^2\right)\Big|_{\frac{y}{2}}^{y} dy$$

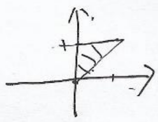
$$= \int_0^2 \left(\frac{19}{24}y^3-\frac{3}{8}y^2\right)dy$$

$$= \left[\frac{19}{24}\times\frac{1}{4}\times y^4-\frac{3}{8}\times\frac{1}{3}\times y^3\right]\Big|_0^2$$

$$= \frac{19}{24}\times\frac{1}{4}\times 16-\frac{3}{8}\times\frac{1}{3}\times 8 = \frac{52}{24} = \frac{13}{6}$$

(8)

$$\iint_D x^2\cdot e^{-y^2}dxdy$$



$$= \int_0^1 dy \int_0^y x^2\cdot e^{-y^2}\cdot dx$$

$$= \frac{1}{3}\int_0^1 y^3\cdot e^{-y^2}dy = \frac{1}{12}\int_0^1 e^{-y^2}dy^4$$

令 $y^2=u$. 原式 $= \frac{1}{12}\int_0^1 e^{-u}du^2 = \frac{-1}{6}\cdot\int_0^1 u\cdot de^{-u}$

$$= -\frac{1}{6}\left[e^{-u}\cdot u\Big|_0^1+e^{-u}\Big|_0^1\right] = -\frac{1}{6}\left[e^{-1}+e^{-1}-e^0\right]$$

$$= \frac{1}{6}-\frac{1}{3e}$$

(9). $\iint_D \sin y^2 dxdy$

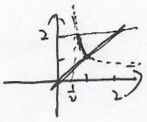
$$= \int_0^1 dy \int_0^y \sin y^2\cdot dx$$

$$= \int_0^1 y\cdot\sin y^2 dy = \frac{1}{2}\int_0^1 \sin y^2 dy^2$$

$$= \frac{1}{2}(-\cos y^2)\Big|_0^1 = \frac{1}{2}-\frac{1}{2}\cos 1$$

$$= \frac{1}{2}(1-\cos 1)$$

(10). $\iint\limits_{D} \frac{x^2}{y^2} dx dy$,



$= \int_1^2 dy \int_{\frac{1}{y}}^{y} \frac{x^2}{y^2} dx$

$= \int_1^2 (\frac{1}{3} \frac{x^3}{y^2} \Big|_{\frac{1}{y}}^{y}) dy.$

$= \frac{1}{3} \int_1^2 (y - \frac{1}{y^5}) dy.$

$= \frac{1}{3} \times \frac{1}{2} y^2 \Big|_1^2 + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{y^4} \Big|_1^2.$

$= \frac{1}{3} \times \frac{1}{2} \times 3 + \frac{1}{3} \times \frac{1}{4} \times (\frac{1}{2^4} - 1)$

$= \frac{1}{2} - \frac{5}{64} = \frac{27}{64}.$

2. 改变下列积分的积分次序.

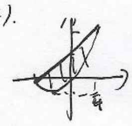
(1) $\int_{-1}^{1}\mathrm{d}x\int_{x^2+x}^{x+1}f(x,y)\mathrm{d}y$;

(2) $\int_{0}^{a}\mathrm{d}x\int_{a-x}^{\sqrt{a^2-x^2}}f(x,y)\mathrm{d}y$;

(3) $\int_{0}^{1}\mathrm{d}y\int_{y}^{2-y}f(x,y)\mathrm{d}x$

(4) $\int_{0}^{\frac{a}{2}}\mathrm{d}y\int_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}}f(x,y)\mathrm{d}x+\int_{\frac{a}{2}}^{a}\mathrm{d}y\int_{0}^{\sqrt{a^2-y^2}}f(x,y)\mathrm{d}x\ (a>0)$.

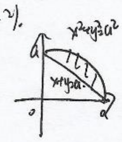
(1).



$$\int_{-1}^{1}dx\int_{x^2+x}^{x+1}f(x,y)dy$$

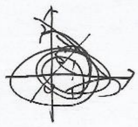
$$=\int_{-\frac{1}{4}}^{0}dy\int_{\frac{-\sqrt{1+4y}-1}{2}}^{\frac{\sqrt{1+4y}-1}{2}}f(x,y)dx+\int_{0}^{2}dy\int_{y-1}^{\frac{\sqrt{1+4y}-1}{2}}f(x,y)dx$$

(2).



$$\int_{0}^{a}dx\int_{a-x}^{\sqrt{a^2-x^2}}f(x,y)dy=\int_{0}^{a}dy\int_{a-y}^{\sqrt{a^2-y^2}}f(x,y)dx$$

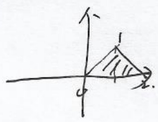
(3)



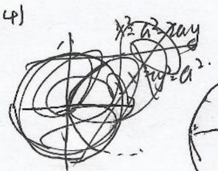
$$\int_{0}^{1}dy\int_{y}^{2-y}f(x,y)dx$$

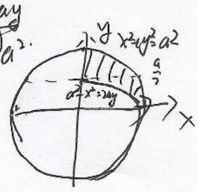
$$=\int_{0}^{1}dx\int_{0}^{x}f(x,y)dy$$

$$+\int_{1}^{2}dx\int_{0}^{2-x}f(x,y)dy.$$



(4)





$$\int_{0}^{\frac{a}{2}}dy\int_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}}f(x,y)+\int_{\frac{a}{2}}^{a}dy\int_{0}^{\sqrt{a^2-y^2}}f(x,y)dx$$

$$=\int_{0}^{a}dx\int_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}}f(x,y)dy.$$

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3. 计算下列积分.

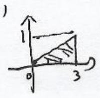
(1) $\int_0^1 dy \int_{3y}^3 e^{x^2} dx$;

(2) $\int_0^1 dy \int_{\sqrt{y}}^1 \sqrt{x^3+1} dx$;

(3) $\int_0^1 dx \int_{x^2}^1 x^3 \sin y^3 dy$;

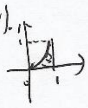
(4) $\int_0^1 dy \int_{\arcsin y}^{\frac{\pi}{2}} \cos x \cdot \sqrt{1+\cos^2 x} dx$.

(1)



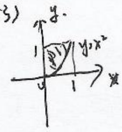
$$\int_0^1 dy \int_{3y}^3 e^{x^2} dx = \int_0^3 dx \int_0^{\frac{1}{3}x} e^{x^2} dy = \int_0^3 \frac{1}{3}x \cdot e^{x^2} dx = \frac{1}{6}\int_0^3 e^{x^2} dx^2 = \frac{1}{6}(e^9-1)$$

(2)



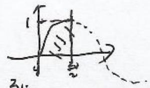
$$\int_0^1 dy \int_{\sqrt{y}}^1 \sqrt{x^3+1} dx = \int_0^1 dx \int_0^{x^2} \sqrt{x^3+1} dy = \int_0^1 x^2 \cdot \sqrt{x^3+1} dx = \frac{1}{3}\int_0^1 \sqrt{x^3+1} d(x^3+1) = \frac{1}{3} \cdot \frac{2}{3} \cdot (x^3+1)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{9} \cdot (2\sqrt{2}-1)$$

(3)



$$\int_0^1 dx \int_{x^2}^1 x^3 \sin y^3 dy = \int_0^1 dy \int_0^{\sqrt{y}} x^3 \sin y^3 dx = \int_0^1 \frac{1}{4}y^2 \sin y^3 dy = \frac{1}{12}\int_0^1 \sin y^3 dy^3 = \frac{1}{12}(-\cos y^3)\Big|_0^1 = \frac{1}{12}(1-\cos 1)$$

(4)



$$\int_0^1 dy \int_{\arcsin y}^{\frac{\pi}{2}} \cos x \cdot \sqrt{1+\cos^2 x} dx = \int_0^{\frac{\pi}{2}} dx \int_0^{\sin x} \cos x \sqrt{1+\cos^2 x} dy = \int_0^{\frac{\pi}{2}} \sin x \cos x \cdot \sqrt{1+\cos^2 x} dx$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos 2x+3}{2}} d\frac{\cos 2x+3}{2} = -\frac{1}{2} \times \frac{2}{3} \cdot \left(\frac{\cos 2x+3}{2}\right)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}} = -\frac{1}{3} \times \left[\left(\frac{2}{2}\right)^{\frac{3}{2}} - \left(\frac{4}{2}\right)^{\frac{3}{2}}\right] = \frac{1}{3} \times (2\sqrt{2}-1)$$

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4. 求下列立体V的体积.

(1) V由三坐标面，平面$x=4$, $y=4$及抛物面$z=x^2+y^2+1$所围成;

(2) V由曲面$z=x^2+2y^2$和$z=6-2x^2-y^2$所围成;

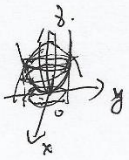
(3) V由平面$z=0$, $y=x$，曲面$x=y^2-y$和$z=3x^2+y^2$所围成.

(1)

$$V=\iint_D f(x,y)d\sigma=\int_0^4 dx\int_0^4 x^2+y^2+1\,dy=\int_0^4\left(4x^2+4+\frac{64}{3}\right)dx$$

$$=\left(\frac{4}{3}x^3+4x+\frac{64}{3}x\right)\Big|_0^4=\frac{64}{3}\times4\times2+4\times4=186\frac{2}{3}$$

(2). $f(x,y)=z_2-z_1=6-3x^2-3y^2=3(2-x^2-y^2)$



$\therefore \iint_D f(x,y)d\sigma$. D为$\{(x,y)\mid x^2+y^2\le2\}$

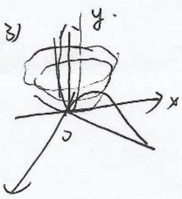
故用极坐标代换. $x=\rho\cos\theta$, $y=\rho\sin\theta$. $x^2+y^2=\rho^2$.

$$\iint_D f(x,y)d\sigma=\int_0^{2\pi}d\theta\int_0^{\sqrt2}f(\rho\cos\theta,\rho\sin\theta)\cdot\rho d\rho$$

$$=3\int_0^{2\pi}\frac12\left(2\rho^2-\frac12\rho^4\right)\Big|_0^{\sqrt2}d\theta=\frac32\int_0^{2\pi}2d\theta$$

$$=\frac32\times4\pi=6\pi.$$

(3)



$$\iint_D f(x,y)d\sigma=\int_0^2 dy\int_{y^2-y}^{y}3x^2+y^2dx$$

$$=\int_0^2\left(x^3+y^2\cdot x\right)\Big|_{y^2-y}^{y}dy$$

$$=\int_0^2\left(-y^6+3y^5-4y^4+4y^3\right)dy$$

$$=\left(-\frac17y^7+\frac36y^6-\frac45y^5+y^4\right)\Big|_0^2=\left(-\frac{y^7}{7}+\frac{y^6}{2}-\frac45y^5+y^4\right)\Big|_0^2$$

$$=-\frac17 2^7+\frac{2^6}{2}-\frac45 2^5+16=\frac{144}{35}.$$

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5. 利用二次积分证明二重积分的性质(7).

(对称性质)

设区域 D 关于 y 轴对称，函数 $f(x,y)$ 在 D 上可积，如果 $f(x,y)$ 关于 x 是奇函数，即满足 $f(-x,y)=-f(x,y)$，则 $\iint\limits_D f(x,y)\mathrm{d}\sigma=0$；如果 $f(x,y)$ 关于 x 是偶函数，即满足 $f(-x,y)=f(x,y)$，并设 D_1 是 D 的右边一半区域，则 $\iint\limits_D f(x,y)\mathrm{d}\sigma=2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma$.

同样，设区域 D 关于 x 轴对称，函数 $f(x,y)$ 在 D 上可积，如果 $f(x,y)$ 关于 y 是奇函数，即满足 $f(x,-y)=-f(x,y)$，则 $\iint\limits_D f(x,y)\mathrm{d}\sigma=0$；如果 $f(x,y)$ 关于 y 是偶函数，即满足 $f(x,-y)=f(x,y)$，并设 D_1 是 D 的上边一半区域，则 $\iint\limits_D f(x,y)\mathrm{d}\sigma=2\iint\limits_{D_1} f(x,y)\mathrm{d}\sigma$.

解：

(1) $\iint\limits_D f(x,y)d\sigma = \int_{x_0}^{x_0} dx \int_{-y(x)}^{y(x)} f(x,y)dy = \int_{x_0}^{x_0} F(x)dx$

$\because f(-x,y)=-f(x,y)$. 故 $F(x)$ 为偶函数.

$\therefore \int_{-x_0}^{x_0} F(x)dx = 0$. 即 $\iint\limits_D f(x,y)=0$

(2) $\iint\limits_D f(x,y)d\sigma = \int_{x_1}^{x_0} dx \int_{y(x)}^{y(x)} f(x,y)dy = \int_{x_0}^{x_0} F(x)dx$

$\because f(-x,y)=f(x,y)$，$\therefore f(x,y)$ 为偶函数，故 $F(x)$ 为奇函数

$\therefore \int_{-x_0}^{x_0} F(x)dx = 2\int_0^{x_0} F(x)dx$ 即 $\iint\limits_D f(x,y)d\sigma = 2\iint\limits_{D_1} f(x,y)d\sigma$

同理可证关于 y 的奇偶性的结论.

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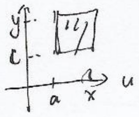
6. 设 $f(x,y)$ 在矩形区域 $D: a\le x\le b$, $c\le y\le d$ 上连续，$g(x,y)=\int_a^x \mathrm{d}u\int_c^y f(u,v)\mathrm{d}v$，证明 $g''_{xy}(x,y)=g''_{yx}(x,y)=f(x,y)$.



$g(x,y)=\int_a^x F(u,v)du$

$=G(x,v)-G(a,v)$

$\therefore g'_x=G'(x,v)=F(x,v)$

$F(x,v)=\int_c^y f(x,v)dv$

$=F(x,y)-F(x,c)$

$\cancel{f(x,y)=F'(x,y)}$

$\therefore g'_{xy}=F'(x,v)=F'(x,y)=f'(x,y)$

$\because g(x,y)=\int_a^x du\int_c^y f(u,v)dv$

$=\int_c^y dv\int_a^x f(u,v)du$

$\therefore g'_{xy}=g'_{yx}=f(x,y)$.

阅

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7. 利用极坐标计算下列二重积分.

(1) $\iint\limits_D \ln\left(1+x^2+y^2\right)\mathrm{d}x\mathrm{d}y$，其中 D 是圆 $x^2+y^2=1$ 与坐标轴在第一象限所围成的区域;

(2) $\iint\limits_D \arctan\frac{y}{x}\mathrm{d}x\mathrm{d}y$，其中 D 是曲线 $x^2+y^2=4, x^2+y^2=1$ 及直线 $y=0, y=x$ 在第一象限所围成的区域;

(3) $\iint\limits_D \left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y$，其中 $D=\left\{(x,y)\middle|2x\le x^2+y^2\le 4x\right\}$;

(4) $\iint\limits_D (x+y)\mathrm{d}x\mathrm{d}y$，其中 D 是由曲线 $x^2+y^2=x+y$ 所围成的区域;

(5) $\iint\limits_D \frac{\mathrm{d}x\mathrm{d}y}{\left(a^2+x^2+y^2\right)^{\frac{3}{2}}}$，其中 $D: 0\le x\le a, 0\le y\le a$.

解：(1) $\rho^2 = x^2+y^2$.

$$\therefore \iint\limits_D \ln(1+x^2+y^2)\,dxdy = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \ln(1+\rho^2)\cdot\rho\, d\rho$$

$$= \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^1 \frac{1}{2}\ln(1+\rho^2)\, d(1+\rho^2) = \int_0^{\frac{\pi}{2}} \left[(1+\rho^2)\cdot\ln(1+\rho^2)\Big|_0^1 - \int_0^1 1\,d\rho\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(2\ln 2 - 1)\, d\theta = \frac{1}{2}(2\ln 2-1)\cdot\theta\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}(2\ln 2-1).$$

(2).

$$\iint\limits_D f(x,y)\,dxdy = \int_0^{\frac{\pi}{4}} d\theta \int_1^2 \theta\cdot\rho\cdot d\rho = \int_0^{\frac{\pi}{4}} \frac{3}{2}\theta\, d\theta$$

$$= \frac{3}{2}\times\frac{1}{2}\cdot\theta^2\Big|_0^{\frac{\pi}{4}} = \frac{3}{4}\times\frac{\pi^2}{16} = \frac{3}{64}\pi^2.$$

(3).

$$\therefore \iint\limits_D f(x,y)\,dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{2\cos\theta}^{4\cos\theta} \rho^2\cdot\rho\, d\rho = 2\cdot\int_0^{\frac{\pi}{2}} \frac{1}{4}\rho^4\Big|_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= 2\cdot 60\cdot \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 2\times 60\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{45}{2}\pi.$$

(4) $$\iint\limits_D (x+y)\,dxdy = \int_{-\frac{1}{4}\pi}^{\frac{3}{4}\pi} d\theta \int_0^{\cos\theta+\sin\theta} (\rho\cos\theta+\rho\sin\theta)\cdot\rho\, d\rho$$

$\rho = \cos\theta+\sin\theta$.

$$= \int_{-\frac{1}{4}\pi}^{\frac{3}{4}\pi} d\theta \cdot \int_0^{\cos\theta+\sin\theta} (\cos\theta+\sin\theta)\rho^2\, d\rho$$

$$= \int_{-\frac{1}{4}\pi}^{\frac{3}{4}\pi} \frac{1}{3}(\cos\theta+\sin\theta)^4\cdot d\theta.$$

$$= \frac{1}{3}\int_{-\frac{1}{4}\pi}^{\frac{3}{4}\pi} 4\sin^4\left(\theta+\frac{\pi}{4}\right)d\theta = 2\cdot\frac{4}{3}\int_0^{\frac{\pi}{2}} \sin^4 u\, du.$$

$$= 2\times\frac{4}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{2}$$

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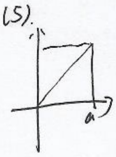
(5).



$$\iint_D \frac{dxdy}{(a^2+x^2+y^2)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{a}{\cos\theta}} \frac{\rho}{(a^2+\rho^2)^{\frac{3}{2}}} d\rho + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{\frac{a}{\sin\theta}} \frac{\rho}{(a^2+\rho^2)^{\frac{3}{2}}} d\rho$$

$$= \int_0^{\frac{\pi}{4}} (-1)\cdot \frac{1}{\sqrt{a^2+\rho^2}}\Big|_0^{\frac{a}{\cos\theta}} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (-1)\cdot \frac{1}{\sqrt{a^2+\rho^2}}\Big|_0^{\frac{a}{\sin\theta}} d\theta$$

$$= \frac{1}{a}\left[\int_0^{\frac{\pi}{4}} \left(1-\frac{\cos\theta}{\sqrt{1+\cos^2\theta}}\right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1-\frac{\sin\theta}{\sqrt{1+\sin^2\theta}}\right) d\theta\right].$$

$$= \frac{1}{a}\left[\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{d\sin\theta}{\sqrt{2-\sin^2\theta}} + \frac{\pi}{4} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\cos\theta}{\sqrt{2-\cos^2\theta}}\right]$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \left(\arcsin\frac{\sin\theta}{\sqrt{2}}\right)\Big|_0^{\frac{\pi}{4}} + \left(\arcsin\frac{\cos\theta}{\sqrt{2}}\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right].$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \frac{1}{6}\pi - \frac{1}{6}\pi\right] = \frac{\pi}{6a}$$

8. 将下列积分化成极坐标系中的累次积分并计算积分的值.

$y^2 = 2x - x^2$

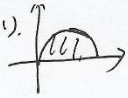
(1) $\int_0^2 dx \int_0^{\sqrt{2x-x^2}} (x^2+y^2) dy$;

(2) $\int_0^1 dx \int_{x^2}^{x} \frac{1}{\sqrt{x^2+y^2}} dy$;

(3) $\int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} \frac{1}{\left(x^2+y^2\right)^{\frac{3}{2}}} dy$;

(4) $\int_1^2 dx \int_0^x \frac{y\sqrt{x^2+y^2}}{x} dy$.

(1).



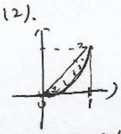
$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} (x^2+y^2) dy$

$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho^3 d\rho$

$= \int_0^{\frac{\pi}{2}} 4\cos^4\theta \, d\theta = 4 \cdot \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta$

$= 4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{3}{4}\pi$

(2).



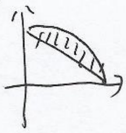
$\rho = \sec\theta \cdot \tan\theta$

$\int_0^1 dx \int_{x^2}^{x} \frac{1}{\sqrt{x^2+y^2}} dy$

$= \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sec\theta \cdot \tan\theta} \frac{\rho}{\sqrt{\rho^2}} d\rho$

$= \int_0^{\frac{\pi}{4}} \rho \Big|_0^{\sec\theta \cdot \tan\theta} d\theta = \sec\theta \Big|_0^{\frac{\pi}{4}} = \sqrt{2} - 1$

(3)

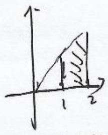


$\int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} \frac{1}{(x^2+y^2)^{\frac{3}{2}}} dy = \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \frac{\rho}{(\rho^2)^{\frac{3}{2}}} d\rho$

$= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{\rho}\right) \Big|_{\frac{1}{\cos\theta+\sin\theta}}^{1} d\theta = \int_0^{\frac{\pi}{2}} [(\cos\theta + \sin\theta) - 1] d\theta$

$= 2 - \frac{\pi}{2}$

(4).



$\rho \cdot \cos\theta = 1 \Rightarrow \rho = \frac{1}{\cos\theta}$

$\rho \cdot \cos\theta = 2 \Rightarrow \rho = \frac{2}{\cos\theta}$

$\int_1^2 dx \int_0^x \frac{y\sqrt{x^2+y^2}}{x} dy$

$= \int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} \frac{\rho \cdot \sin\theta \cdot \sqrt{\rho^2}}{\rho \cdot \cos\theta} \cdot \rho \cdot d\rho =$

$= \int_0^{\frac{\pi}{4}} d\theta \cdot \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} \rho^2 \tan\theta \, d\rho$

$= \int_0^{\frac{\pi}{4}} \frac{1}{3}\rho^3 \Big|_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} d\theta = \int_0^{\frac{\pi}{4}} \frac{7}{3} \frac{\sin\theta}{\cos^4\theta} d\theta$

$= -\frac{7}{3} \int_0^{\frac{\pi}{4}} \frac{d\cos\theta}{\cos^4\theta} = -\frac{7}{3} \int_1^{\frac{\sqrt{2}}{2}} \frac{du}{u^4} = \frac{7}{9} \int_1^{\frac{1}{\sqrt{2}}} d\frac{1}{u^3}$

$= \frac{7}{9} \frac{1}{u^3} \Big|_1^{\frac{\sqrt{2}}{2}} = \frac{7}{9}(2\sqrt{2}-1) = \frac{14\sqrt{2}-7}{9}$

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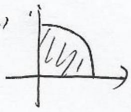
9. 计算下列积分.

(1) $\iint\limits_D \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\mathrm{d}\sigma$，其中 D 是由圆 $x^2+y^2=1$ 及坐标轴在第一象限所围成的区域;

(2) $\iint\limits_D y\mathrm{d}x\mathrm{d}y$，其中 D 是由直线 $x=-2, y=0, y=2$ 及曲线 $x=-\sqrt{2y-y^2}$ 所围成的区域;

(3) $\iint\limits_D \sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}\mathrm{d}x\mathrm{d}y$，其中 $D:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1, x\ge 0, y\ge 0\ (a>0, b>0)$.

(1)



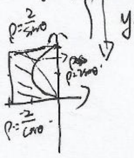
$$\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}} = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \sqrt{\frac{1-\rho^2}{1+\rho^2}}\cdot\rho\cdot d\rho = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^1 \sqrt{\frac{1-\rho^2}{1+\rho^2}}\,d\rho^2$$

令 $u=\rho^2$，原式 $=\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^1 \sqrt{\frac{1-u}{1+u}}\,du = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^1 \frac{1-u}{\sqrt{1-u^2}}\,du$

$$\cancel{=\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \arcsin u\Big|_0^1 + \sqrt{1-u^2}\Big|_0^1\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-1\right) d\theta$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{\pi-2}{2} = \frac{\pi}{8}(\pi-2)$$

(2)



$$\iint\limits_D y\,dxdy = \int_{\frac{3}{4}\pi}^{\pi} d\theta \int_{2\sin\theta}^{\frac{-2}{\cos\theta}} \rho^2\sin\theta\, d\rho + \int_{\frac{\pi}{2}}^{\frac{3}{4}\pi} d\theta \int_{2\sin\theta}^{\frac{2}{\sin\theta}} \rho^2 \sin\theta\, d\rho$$

$$= \int_{\frac{\pi}{2}}^{\frac{3}{4}\pi} \frac{1}{3}\rho^3\sin\theta\Big|_{2\sin\theta}^{\frac{2}{\sin\theta}}\cdot d\theta + \int_{\frac{3}{4}\pi}^{\pi} \frac{1}{3}\rho^3\sin\theta\Big|_{2\sin\theta}^{\frac{-2}{\cos\theta}}\cdot d\theta$$

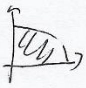
$$= \int_{\frac{\pi}{2}}^{\frac{3}{4}\pi} \frac{8}{3}\csc^2\theta\, d\theta + \int_{\frac{3}{4}\pi}^{\pi} \frac{-8}{3}\tan\theta\cdot\sec^2\theta\, d\theta - \int_{\frac{\pi}{2}}^{\pi} \frac{8}{3}\cdot\sin^4\theta\, d\theta$$

$$= \frac{8}{3}(-\cot\theta)\Big|_{\frac{\pi}{2}}^{\frac{3}{4}\pi} + \frac{-8}{3}\times\frac{1}{2}\times\tan^2\theta\Big|_{\frac{3}{4}\pi}^{\pi} - \frac{8}{3}\int_0^{\frac{\pi}{2}}\sin^4\theta\, d\theta$$

$$= \frac{8}{3}\times 1 + \frac{4}{3} - \frac{8}{3}\times\frac{3}{4}\times\frac{1}{3}\times\frac{\pi}{2} = 4-\frac{\pi}{2}$$

(3) 令 $x=a\cdot\rho\cos\theta$，$y=b\cdot\rho\cdot\sin\theta$

$\therefore\ dxdy = ab\cdot\rho d\rho\, d\theta$



$$\therefore \iint\limits_D f(x,y)\,dxdy = ab\cdot\int_0^{\frac{\pi}{2}} d\theta \int_0^1 \rho\sqrt{1-\rho^2}\,d\rho = ab\int_0^{\frac{\pi}{2}} \frac{-1}{3}(1-\rho^2)^{\frac{3}{2}}\Big|_0^1 d\theta$$

$$= ab\int_0^{\frac{\pi}{2}} \frac{1}{3}\, d\theta = \frac{\pi}{6}ab$$

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10. 求下列立体V的体积.

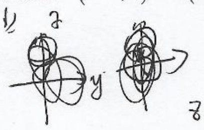
(1) V是球体$x^2+y^2+z^2\le R^2$与$x^2+y^2+z^2\le 2Rz$的公共部分;

(2) V由柱面$x^2+y^2=y$和平面$6x+4y+z=12$, $z=0$所围成;

(3) V由锥面$z=\sqrt{x^2+y^2}$和半球面$z=\sqrt{1-x^2-y^2}$所围成;

(4) V由柱面$V\left(x^2+y^2\right)^2=2\left(x^2-y^2\right)$以及抛物面$z=x^2+y^2$和$xOy$面所围成.

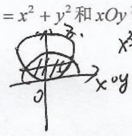
(1)



$z_1=R-\sqrt{R^2-x^2-y^2}$

$z_2=\sqrt{R^2-x^2-y^2}$

$z=2\sqrt{R^2-x^2-y^2}-R$.



$x^2+y^2=\frac{3}{4}R^2$

$\therefore V=\int_0^{2\pi}d\theta\int_0^{\frac{\sqrt3}{2}R}\left(2\sqrt{R^2-\rho^2}-R\right)\rho d\rho$

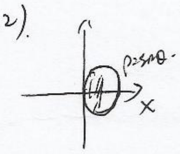
$=\int_0^{2\pi}\left(-\frac{2}{3}(R^2-\rho^2)^{\frac{3}{2}}\Big|_0^{\frac{\sqrt3}{2}R}-\frac{1}{2}R\cdot\rho^2\Big|_0^{\frac{\sqrt3}{2}R}\right)d\theta$

$=\int_0^{2\pi}\frac{2}{3}\times\frac{7}{8}R^3-\frac{9}{24}R^3d\theta=\int_0^{2\pi}\frac{5}{24}R^3d\theta$

$=\frac{5}{12}\pi R^3$

(2).

$V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\sin\theta}(12-6\rho\cdot\cos\theta-4\rho\sin\theta)\rho d\rho$



$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\sin\theta}(12\rho-6\rho^2\cos\theta-4\rho^2\sin\theta)d\rho$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(6\sin^2\theta-2\sin^3\theta\cos\theta-\frac{4}{3}\sin^4\theta)d\theta$

$=2\int_0^{\frac{\pi}{2}}(6\sin^2\theta-\frac{4}{3}\sin^4\theta)d\theta-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\sin^3\theta d\sin\theta$.

$=12\cdot\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta-\frac{8}{3}\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta$

$=12\times\frac{1}{2}\times\frac{\pi}{2}-\frac{8}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=(6-1)\times\frac{\pi}{2}=\frac{5}{2}\pi$

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