8-4 .2501713. 刘莹, 13201712

1. 求下列曲面的面积.

(1) 平面$3x+2y+z=1$被椭圆柱面$2x^2+y^2=1$截下的部分;

(2) 锥面$z=\sqrt{x^2+y^2}$被柱面$z^2=2x$截下的部分;

(3) 双曲抛物面$z=xy$被柱面$x^2+y^2=R^2$所截下的部分;

(4) 圆柱面$x^2+y^2=R^2$被平面$x+z=0$, $x-z=0$ $(x>0,y>0)$所截部分;

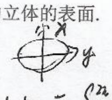
(5) 球面$x^2+y^2+z^2=3a^2$和抛物面$x^2+y^2=2az$ $(z\geq 0)$所围成区域的边界曲面;

(6) 锥面$z=\sqrt{x^2+y^2}$被柱面$\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$所截下的部分;

(7) 球面$x^2+y^2+z^2=R^2$夹在平面$z=\frac{R}{4}$与$z=\frac{R}{2}$之间的部分;

(8) 圆柱面$x^2+y^2=R^2$与$x^2+z^2=R^2$所围成的立体的表面.

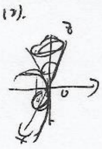
解: (1) $z=1-3x-2y$, 投影在xOy面上为$2x^2+y^2=1$

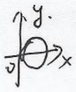


令$x=\frac{1}{\sqrt{2}}\rho\cos\theta$, $y=\rho\sin\theta$. 有$\rho=1$.

∴ $\frac{\partial z}{\partial x}=-3$, $\frac{\partial z}{\partial y}=-2$. ∴ $S=\iint_{D_{xy}}\sqrt{1+3^2+2^2}\,dxdy=\sqrt{14}\cdot\int_0^{2\pi}d\theta\cdot\int_0^1\frac{1}{\sqrt{2}}\rho d\rho=\sqrt{14}\times\frac{1}{\sqrt{2}}\times 2\pi\times\frac{1}{2}=\sqrt{7}\pi$

(2).





被截部分于xOy面投影为圆.

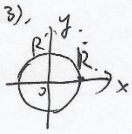
极坐标方程 $\rho=2\cos\theta$.

$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$.

$$S=\iint_{D_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy=\iint_{D_{xy}}\sqrt{2}\,dxdy=\sqrt{2}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}\rho d\rho$$

$$=\sqrt{2}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\cos^2\theta\times\frac{1}{2}\,d\theta=2\sqrt{2}\cdot\frac{1}{2}\cdot 4\cdot\int_0^{\frac{\pi}{2}}\cos^2\theta=\sqrt{2}\cdot 4\cdot\frac{1}{2}\times\frac{\pi}{2}=\sqrt{2}\pi$$

(3),



$\frac{\partial z}{\partial x}=y$, $\frac{\partial z}{\partial y}=x$. 投影如图所示为圆

极坐标方程 $\rho=R$.

$$\therefore S=\iint_{D_{xy}}\sqrt{1+x^2+y^2}\,dxdy=\int_0^{2\pi}d\theta\int_0^R\sqrt{1+\rho^2}\cdot\rho d\rho$$

$$=\int_0^{2\pi}d\theta\cdot\int_0^R\frac{1}{2}\cdot(1+\rho^2)^{\frac{1}{2}}\,d(1+\rho^2)$$

$$=\int_0^{2\pi}\frac{1}{3}\cdot\left[(1+R)^{\frac{3}{2}}-1\right]d\theta=\frac{2}{3}\pi\left[(1+R)^{\frac{3}{2}}-1\right].$$

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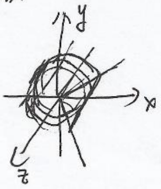
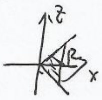
1.(4).





如图,阴影部分为被截部分在zOx面的投影.

$\therefore S=\iint\limits_{D_{xz}}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz$

$\frac{\partial y}{\partial x}=\frac{-x}{\sqrt{R^2-x^2}}$, $\frac{\partial y}{\partial z}=0$. $S=\int_0^R dx\int_{-x}^{x}\sqrt{\frac{R^2}{R^2-x^2}}\,dz$

$=R\cdot\int_0^R\frac{2x}{\sqrt{R^2-x^2}}dx=R\cdot\int_0^R-\frac{d(R^2-x^2)}{\sqrt{R^2-x^2}}=R\cdot\int_R^0\frac{d(R^2-x^2)}{\sqrt{R^2-x^2}}$

$=R\cdot 2\cdot(R^2-x^2)^{\frac{1}{2}}\Big|_R^0=2R^2-0=2R^2$

(5)

如图,球面和椭圆抛物面相交,交线为$\begin{cases}x^2+y^2=2a^2\\ z=a.\end{cases}$

曲面积分分为两部分 S_1 和 S_2.

由 $3a^2=x^2+y^2+z^2$, $\therefore\frac{\partial z}{\partial x}=-\frac{x}{z}$, $\frac{\partial z}{\partial y}=-\frac{y}{z}$. | 由 $2az=x^2+y^2$. $\frac{\partial z}{\partial x}=-\frac{x}{a}$, $\frac{\partial z}{\partial y}=-\frac{y}{a}$.

$\therefore S_1=\iint\limits_{D_{xy}}\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\,dx\,dy=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}a}\rho\sqrt{\frac{3a^2}{3a^2-\rho^2}}\,d\rho$

$=\int_0^{2\pi}d\theta\cdot\int_0^{\sqrt{2}a}\left(-\frac{1}{2}\right)\sqrt{3}\cdot a\cdot(3a^2-\rho^2)^{-\frac{1}{2}}\,d(3a^2-\rho^2)$

$=\int_0^{2\pi}\sqrt{3}a\cdot(3a^2-\rho^2)^{\frac{1}{2}}\Big|_{\sqrt{2}a}^{0}\,d\theta=2\pi\cdot\sqrt{3}a\cdot\left[(3a^2)^{\frac{1}{2}}-a^{\frac{1}{2}}\right]$

$=2\pi\cdot a^2\cdot(3-\sqrt{3})$

$S_2=\iint\limits_{D_{xy}}\sqrt{1+\frac{x^2+y^2}{a^2}}\,dx\,dy=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}a}\frac{1}{a}\rho\cdot\sqrt{a^2+\rho^2}\,d\rho$

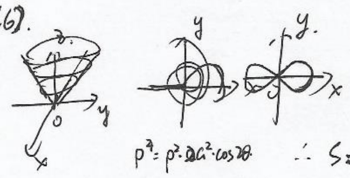
$=\int_0^{2\pi}d\theta\cdot\int_0^{\sqrt{2}a}\frac{1}{2}\cdot\frac{1}{a}\cdot(a^2+\rho^2)^{\frac{1}{2}}\,d(a^2+\rho^2)=\cancel{\int_0^{2\pi}d\theta\int_0^{\sqrt{2}a}\frac{1}{3}}\cdot\frac{1}{a}$

$=\int_0^{2\pi}\frac{1}{3}\cdot\frac{1}{a}\cdot(a^2+\rho^2)^{\frac{3}{2}}\Big|_0^{\sqrt{2}a}\,d\theta=2\pi\cdot\frac{1}{a}\cdot\frac{1}{3}\cdot(3\sqrt{3}-1)\cdot a^3=$

$=2\pi\cdot a^2\cdot\left(\sqrt{3}-\frac{1}{3}\right)$

$\therefore S_1+S_2=2\pi\cdot a^2\cdot\left(3-\sqrt{3}+\sqrt{3}-\frac{1}{3}\right)=\frac{16}{3}\pi a^2.$

1. (6).



如图，投影为双纽线，由对称性可只算第一象限。

$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$，$\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$

$\rho^4=\rho^2\cdot a^2\cdot\cos 2\theta$

$\rho^2=a^2\cdot\cos 2\theta$

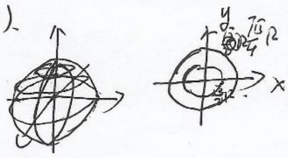
$\rho=\sqrt{a^2\cos 2\theta}$

$\therefore S=\iint\limits_D\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}\,dxdy=4\sqrt{2}\cdot\int_0^{\frac{\pi}{4}}d\theta\int_0^{a\sqrt{\cos 2\theta}}\rho d\rho$

$\cancel{=4\sqrt{2}\int_0^{a\sqrt{\cos 2\theta}}}=4\sqrt{2}\cdot\int_0^{\frac{\pi}{4}}\frac{1}{2}a^2\cos 2\theta\,d\theta$

$=\sqrt{2}a^2\sin 2\theta\Big|_0^{\frac{\pi}{4}}=\sqrt{2}a^2$

(7).



设 $x^2+y^2=a^2$，$\frac{\partial z}{\partial x}=-\frac{x}{z}$，$\frac{\partial z}{\partial y}=-\frac{y}{z}$

$S=\iint\limits_D\sqrt{1+\frac{x^2+y^2}{z^2}}\,dxdy=\int_0^{2\pi}d\theta\int_0^a\frac{R}{\sqrt{R^2-\rho^2}}\cdot\rho d\rho$

$=(-\frac{1}{2})\int_0^{2\pi}d\theta\int_0^a R\cdot(R^2-\rho^2)^{-\frac{1}{2}}d(R^2-\rho^2)$

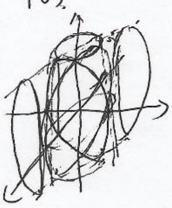
$=2\pi\cdot R\cdot(-1)\cdot\sqrt{R^2-\rho^2}\Big|_0^a$

$=2\pi R\cdot(R-\sqrt{R^2-a^2})$

$\therefore$ 当 $a^2=\frac{15}{16}R^2$，$S_1=\frac{3}{4}R^2\cdot 2\pi$，当 $a^2=\frac{3}{4}R^2$，$S_2=\frac{1}{2}R^2\cdot 2\pi$

$\therefore S=S_1-S_2=\frac{1}{4}R^2\cdot 2\pi=\frac{\pi}{2}R^2$

(8)



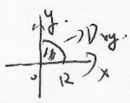
$z=\sqrt{R^2-x^2}$，$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{R^2-x^2}}$，$\frac{\partial z}{\partial y}=0$，由对称可知

可分为4个曲面，先计算一因至八卦限一个曲面的面积

$S'=4\cdot\iint\limits_D\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,dxdy=4\cdot\int_0^{\frac{\pi}{2}}d\theta\int_0^R\rho\cdot\frac{R}{\sqrt{R^2-\rho^2\cos^2\theta}}d\theta$

$=4R\cdot\int_0^{\frac{\pi}{2}}\frac{1}{\cos^2\theta}\cdot\sqrt{R^2-\rho^2\cos^2\theta}\Big|_R^0 d\theta=4R^2\int_0^{\frac{\pi}{2}}\sec^2\theta-\tan\theta\sec\theta\,d\theta$

$=4R^2\cdot(\tan\theta-\sec\theta)\Big|_0^{\frac{\pi}{2}}=4R^2\left[\left(\frac{\sin\theta-1}{\cos\theta}\right)\Big|_{\theta\to\frac{\pi}{2}}+1\right]$



$\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}=0$ $S'=4R^2$，$S=4S'=16R^2$

2. 设一半圆形薄片：$x^2+y^2\le 2ax\,(y\ge 0)$，其上任一点的面密度 $\mu(x,y)=\sqrt{4a^2-x^2-y^2}$，求该薄片的质量.

解：由题

$\rho=2a\cos\theta,\ \theta\in[0,\frac{\pi}{2}]$.

$$\therefore m=\iint\limits_{D_{xy}}\mu\,dxdy=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\rho\cdot\sqrt{4a^2-\rho^2}\,d\rho$$

$$=\left(-\frac{1}{2}\right)\int_0^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\sqrt{4a^2-\rho^2}\,d(4a^2-\rho^2)$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{2}}(4a^2-\rho^2)^{\frac{3}{2}}\Big|_{2a\cos\theta}^{0}\,d\theta$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{2}}8a^3(1-\sin^3\theta)\,d\theta$$

$$=\frac{8}{3}a^3\cdot\left(\frac{\pi}{2}-\frac{2}{3}\times 1\right)=\frac{8}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

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3. 求下列物体的质量.

(1) 球体 $V: x^2+y^2+z^2 \le R^2$，其上任一点的密度与该点到球心的距离成正比;

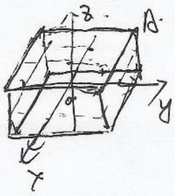
(2) 长方体: $|x| \le a$，$|y| \le a$，$|z| \le \frac{a}{8}(a>0)$，其上任一点的密度与该点到 z 轴的距离平方成正比，且在角上的密度为 1.

解(1): 由题 $\mu = kr$

$\therefore m = \iiint_V \mu \, dV = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_0^R r^2 \cdot kr \, dr$

$= 2\pi \times 2 \times \frac{1}{4} R^4 \cdot k = k\pi R^4$.

(2) 由题 $\mu = k(x^2+y^2)$，角点 $A(-a,-a,\frac{a}{8})$



$\mu_A = k \cdot 2a^2$，$k = \frac{1}{2a^2}$

$m = \iiint_V \mu \, dV = \int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-\frac{a}{8}}^{\frac{a}{8}} k(x^2+y^2) \, dz$

$= \int_{-a}^{a} dx \int_{-a}^{a} \frac{a}{4} k (x^2+y^2) \, dy$

$= \frac{a}{4} k \cdot \int_{-a}^{a} x^2 \cdot (2a) + \frac{1}{3} 2a^3 \, dx$

$= \frac{a}{4} k \cdot 2a \cdot \int_{-a}^{a} x^2 + \frac{1}{3} a^2 \, dx$

$= \frac{a^2 k}{2} \cdot \left(\frac{1}{3} 2a^3 + \frac{1}{3} a^2 \cdot 2a\right)$

$= a^2 k \cdot a^3 \cdot \frac{2}{3} = \frac{1}{3} a^3$

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4. 求下列平面薄片 D 的形心.

(1) D 由 $y=\sqrt{2x}$，$x=a\,(a>0)$，$y=0$ 围成;

(2) D 由心形线 $\rho=1+\cos\theta$ 围成;

(3) D 由双纽线 $\rho^2=2\cos 2\theta$ 的右边一支围成;

(4) $D: a\cos\theta\le\rho\le b\cos\theta\ (0<a<b)$;

(5) D 由摆线 $x=a(t-\sin t), y=a(1-\cos t)\ (0\le t\le 2\pi)$ 与 x 轴围成.

解：(1)

$\iint\limits_D \mu\, dxdy = \int_0^a dx\int_0^{\sqrt{2x}} dy = \frac{2\sqrt{2}}{3}a^{\frac{3}{2}}$

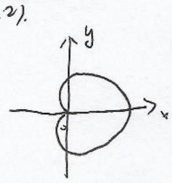
$\iint\limits_D x\,dxdy = \int_0^a x\,dx\int_0^{\sqrt{2x}} dy = \int_0^a \sqrt{2}\cdot x^{\frac{3}{2}}dx$

$= \sqrt{2}\cdot\frac{2}{5}a^{\frac{5}{2}}$.

$\iint\limits_D y\,dxdy = \int_0^a dx\int_0^{\sqrt{2x}} y\,dy = \int_0^a x\,dx = \frac{1}{2}a^2$.

$\therefore \bar{x}=\frac{3}{5}a$，$\bar{y}=\frac{\frac{1}{2}a^2}{\frac{2}{3}\sqrt{2}a^{\frac{3}{2}}}=\frac{3\sqrt{2a}}{8}$ $\therefore \left(\frac{3}{5}a, \frac{3}{8}\sqrt{2a}\right)$

(2).



由对称性知，$\bar{y}=0$

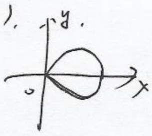
$\iint\limits_D dxdy = \int_0^{2\pi}d\theta\int_0^{1+\cos\theta}\rho\,d\rho = \int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta=\frac{3}{2}\pi$

$\iint\limits_D x\,dxdy = \int_0^{2\pi}d\theta\int_0^{1+\cos\theta}\rho\cos\theta\,d\rho=\frac{1}{3}\int_0^{2\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\cos\theta\,d\theta$

$=\frac{1}{3}\times 4\times\left(3\times\frac{1}{2}\times\frac{\pi}{2}+\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right)=\frac{5}{4}\pi$

$\therefore \bar{x}=\frac{5}{6}$ $\therefore$ 形心 $\left(\frac{5}{6}, 0\right)$

(3).



由对称性可知 $\bar{y}=0$

$\iint\limits_D dxdy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta\int_0^{\sqrt{2\cos 2\theta}}\rho\,d\rho = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}\rho^2\Big|_0^{\sqrt{2\cos 2\theta}}d\theta$

$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\theta\,d\theta=\frac{1}{2}\sin 2\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{1}{2}(1+1)=1$

$\iint\limits_D x\,dxdy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta\int_0^{\sqrt{2\cos 2\theta}}\rho^2\cos\theta\,d\rho=\frac{2}{3}\sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos\theta\cdot(\cos 2\theta)^{\frac{3}{2}}d\theta$

$=\frac{2}{3}\sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}d\sin\theta=\frac{2}{3}\sqrt{2}\int_{-\frac{\pi}{2}}^{0}\cdot\frac{1}{\sqrt{2}}\sin^4 t\ d\theta$

$=\frac{2}{3}\sqrt{2}\times 2\times\frac{1}{\sqrt{2}}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi}{4}$ $\therefore$ 形心 $\left(\frac{\pi}{4}, 0\right)$.

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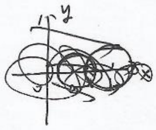
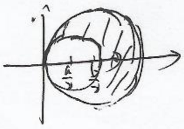
4、(4). 由对称性应有 $\bar{y}=0$





$$\iint_D dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{a\cos\theta}^{b\cos\theta} \rho d\rho$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a^2-b^2}{2} \cdot \cos^2\theta d\theta$$

$$= (a^2-b^2) \cdot \frac{\pi}{4}\ \frac{1}{2} \times \frac{\pi}{2} = (a^2-b^2) \cdot \frac{\pi}{4}$$

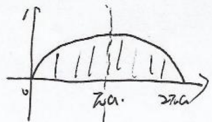
$$\iint_D x dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{a\cos\theta}^{b\cos\theta} \rho^2 \cos\theta d\theta.$$

$$= \frac{a^3-b^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta.$$

$$= \frac{a^3-b^3}{3} \cdot 2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = (a^3-b^3)\frac{\pi}{8}$$

$$\therefore \bar{x} = \frac{a^3-b^3}{a^2-b^2} \cdot \frac{1}{2} = \frac{a^2+ab+b^2}{2(a+b)}$$，∴形心 $\left(\frac{a^2+ab+b^2}{2(a+b)}, 0\right)$

(5)



由 $x=a(t-\sin t)$, $y=a(1-\cos t)$

有 $y=\frac{dx}{dt}$.

如图 由对称性知 $\bar{x}=\pi a$.

$$\iint_D dxdy = \int_0^{2\pi a} dx \int_0^{y(x)} dy = \int_0^{2\pi} a(1-\cos t)\, d(a(t-\sin t))$$

$$= \int_0^{2\pi} a^2(1-\cos t)^2 dt = 3\pi a^2$$

$$\iint_D y dxdy = \int_0^{2\pi a} dx \int_0^{y(x)} y dy.$$

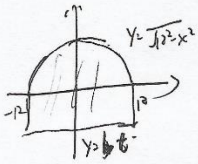
$$= \frac{1}{2}\int_0^{2\pi a} [y(x)]^2 dx = \frac{a^3}{2}\int_0^{2\pi} (1-\cos t)^3 dt.$$

$$= \frac{a^3}{2}\left(\cancel{2\pi+3\times\cos}\,(2\pi+3\times\frac{1}{2}\times\frac{\pi}{2})\times 4\right) = \frac{5a^3}{2}\pi.$$

$$\therefore \bar{y} = \frac{5}{2}a^3\pi / 3\pi a^2 = \frac{5}{6}a$$

∴ 形心 $(\pi a, \frac{5}{6}a)$

5. 质量均匀分布的薄片在 xOy 面上所占区域 D 是在半径为 R 的半圆的直径上拼接一个长为 $2R$ 的矩形，要使 D 的质心在圆心处，矩形的宽应为多少？



$\because$ 由对称性有 $\bar{x}=0$

现使 $\bar{y}=0$

$\bar{y}=\dfrac{\mu\cdot\iint y\,dxdy}{\mu S}$

$\therefore \iint_D y\,dxdy=0$

$\int_{-R}^{R}dx\int_{t}^{\sqrt{R^2-x^2}}y\,dy$

$=\int_{-R}^{R}\frac{1}{2}(R^2-t^2-x^2)\,dx=0$

$=\frac{1}{2}(R^2-t^2)\cdot 2R-\frac{1}{3}x^3\Big|_{-R}^{R}=0$

$\therefore (R^2-t^2)\cdot R=\frac{1}{3}\times\frac{1}{2}\times 2\times R^3$

$R^2-t^2=\frac{1}{3}R^2$

$t^2=\frac{2}{3}R^2$

$|t|=\sqrt{\frac{2}{3}}R$

第八章　重积分
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6. 设平面薄片D由抛物线$y=x^2$与直线$y=x$所围成，它在点(x,y)处的面密度$\mu(x,y)=x^2y$，求该薄片的质心.

解：

$$\iint_D \mu\, dxdy = \int_0^1 dx\int_{x^2}^{x} x^2y\, dy = \int_0^1 \frac{1}{2}(x^4-x^6)\,dx = \frac{1}{2}\times\left(\frac{1}{5}-\frac{1}{7}\right)=\frac{1}{35}$$

$$\iint_D x\mu\, dxdy = \int_0^1 x^3 dx\int_{x^2}^{x} y\, dy = \int_0^1 \frac{1}{2}\cdot(x^5-x^7)\,dx = \frac{1}{2}\cdot\left(\frac{1}{6}-\frac{1}{8}\right)=\frac{1}{48}$$

$$\iint_D y\cdot\mu\, dxdy = \int_0^1 x^2 dx\int_{x^2}^{x} y^2\, dy = \int_0^1 \frac{1}{3}(x^5-x^8)\,dx = \frac{1}{3}\times\left(\frac{1}{6}-\frac{1}{9}\right)=\frac{1}{54}$$

$$\therefore\ \bar{x}=\frac{\iint_D x\mu\, dxdy}{\iint_D \mu\, dxdy}=\frac{35}{48}$$

$$\bar{y}=\frac{\iint y\mu\, dxdy}{\iint \mu\, dxdy}=\frac{35}{54}$$

$\therefore$ 质心 $\left(\frac{35}{48},\frac{35}{54}\right)$

第八章 重积分
第四节 重积分的应用

7. 求下列物体 V 的形心.

(1) $V=\left\{(x,y,z)\middle| x^2+y^2\le 2z, x^2+y^2+z^2\le 3\right\}$;

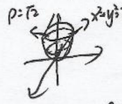
(2) $V=\left\{(x,y,z)\middle| 0\le z\le x^2+y^2, x\ge 0, y\ge 0, x+y\le 1\right\}$;

(3) $V=\left\{(x,y,z)\middle| x^2+y^2+z^2\ge 1, x^2+y^2+z^2\le 16, z\ge\sqrt{x^2+y^2}\right\}$;

(4) V 是由曲面 $x^2+z=1, y^2+z=1, z=0$ 所围成;

解：设该题的密度 $\rho=1$

(1) 由对称性知 $\bar{x}=0$, $\bar{y}=0$ 故求 $\bar{z}$



$$\iiint\limits_V dxdydz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\rho d\rho\int_{\frac{\rho^2}{2}}^{\sqrt{3-\rho^2}}dz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\left(\sqrt{3-\rho^2}-\frac{\rho^2}{2}\right)d\frac{1}{2}\rho^2=\int_0^{2\pi}\frac{1}{3}(3-\rho^2)\Big|_{\sqrt{2}}^{0}-\frac{1}{8}\rho^4\Big|_0^{\sqrt{2}}d\theta$$

$$=\int_0^{2\pi}\frac{6\sqrt{3}-5}{6}d\theta=2\pi\cdot\frac{6\sqrt{3}-5}{6}$$

$$\iiint\limits_V z\,dxdydz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\rho d\rho\int_{\frac{\rho^2}{2}}^{\sqrt{3-\rho^2}}z\,dz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\frac{1}{2}\left(3\rho-\rho^3-\frac{1}{4}\rho^5\right)d\rho$$

$$=\int_0^{2\pi}\frac{1}{2}\left(\frac{3}{2}\rho^2-\frac{1}{4}\rho^4-\frac{1}{24}\rho^6\right)\Big|_0^{\sqrt{2}}d\theta=\frac{5}{6}$$

$$\therefore \bar{z}=\frac{\frac{5}{6}}{\frac{6\sqrt{3}-5}{6}}=\frac{5(6\sqrt{3}+5)}{83}\quad\therefore(\bar{x},\bar{y},\bar{z})=\left(0,0,\frac{5(6\sqrt{3}+5)}{83}\right).$$

(2)

由对称性可知 $\bar{x}=\bar{y}$,

$$\iiint\limits_V dxdydz=\int_0^1dx\int_0^{1-x}dy\int_0^{x^2+y^2}dz=\int_0^1dx\int_0^{1-x}x^2+y^2dy=\int_0^1x^2(1-x)+\frac{1}{3}(1-x)^3dx=\frac{1}{6}$$

$$\iiint\limits_V x\,dxdydz=\int_0^1dx\int_0^{1-x}dy\int_0^{x^2+y^2}x\,dz=\int_0^1x^3(1-x)+\frac{1}{3}(1-x)^3x\,dx=\frac{1}{15}$$

$$\therefore\bar{y}=\bar{x}=\frac{\frac{1}{15}}{\frac{1}{6}}=\frac{2}{5}.\quad\iiint\limits_V z\,dxdydz=\int_0^1dx\int_0^{1-x}dy\int_0^{x^2+y^2}z\,dz=\int_0^1dx\int_0^{1-x}\frac{1}{2}(x^2+y^2)^2dy$$

$$=\int_0^1\frac{1}{2}x^4(1-x)+\frac{1}{3}x^2(1-x)^3+\frac{1}{8}(1-x)^5dx=\frac{7}{180},\quad\bar{z}=\frac{\frac{7}{180}}{\frac{1}{6}}=\frac{7}{30}$$

$$\therefore(\bar{x},\bar{y},\bar{z})=\left(\frac{2}{5},\frac{2}{5},\frac{7}{30}\right)$$

(3).

$$\iiint\limits_V dxdydz=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\,d\varphi\int_1^4r^2dr=2\pi\times\left(1-\frac{\sqrt{2}}{2}\right)\times\frac{1}{3}\times 63=21(2-\sqrt{2})$$

$$\iiint\limits_V z\,dxdydz=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\cos\varphi\,d\varphi\int_1^4r^3dr=2\pi\left[\int_0^{\frac{\pi}{4}}\sin\varphi\,d\sin\varphi\right]\frac{1}{4}(2^8-1)$$

$$=2\cdot\frac{1}{8}\cdot 255\cdot\pi$$

$$\therefore\bar{z}=\frac{255\cdot\pi(2+\sqrt{2})}{21\times2\times8}=\frac{85(2+\sqrt{2})}{112}$$

由对称性知, $\bar{x}=\bar{y}=0$. $\therefore(\bar{x},\bar{y},\bar{z})=\left(0,0,\frac{85}{112}(2+\sqrt{2})\right)$.

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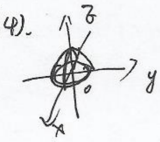
(4).



由题有

$x^2+y^2+2z=2$

$z=(1-\frac{\rho^2}{2})$

于xoy面内有

$\rho=\sqrt{2}$

(图).

$$\iiint\limits_V dxdydz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\rho d\rho\int_0^{1-\frac{\rho^2}{2}}dz$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\rho-\frac{\rho^3}{2}d\rho$$

$$=\int_0^{2\pi}d\theta\cdot\left(\frac{1}{2}\rho^2-\frac{1}{8}\rho^4\right)\Big|_0^{\sqrt{2}}=\pi$$

$$\iiint\limits_V z\,dxdydz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\rho d\rho\cdot\int_0^{1-\frac{\rho^2}{2}}z\,dz$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\frac{1}{2}\left(1-\frac{\rho^2}{2}\right)^2d\frac{\rho^2}{2}$$

$$=\int_0^{2\pi}\frac{1}{2}\cdot\left(-\frac{1}{3}\right)\cdot\left(1-\frac{\rho^2}{2}\right)^3\Big|_0^{\sqrt{2}}d\theta$$

$$=\int_0^{2\pi}\frac{1}{6}d\theta=\frac{\pi}{3}$$

$\therefore \bar{z}=\dfrac{\frac{\pi}{3}}{\pi}=\dfrac{1}{3}$

由对称性知 $\bar{x}=\bar{y}=0$

$\therefore (\bar{x},\bar{y},\bar{z})=(0,0,\frac{1}{3})$

8. 求下列物体V的质心.

(1) $V:\sqrt{x^2+y^2}\le z\le H$，其上任一点$(x,y,z)$处的密度$\mu(x,y,z)=1+x^2+y^2$;

(2) $V:x^2+y^2+z^2\le 2az\ (a>0)$，其上任一点的密度与该点到原点的距离成反比.

解：(1) $\iiint_V \mu\,dx\,dy\,dz=\int_0^{2\pi}d\theta\int_0^H\rho\,d\rho\int_\rho^H\mu\,dz=\int_0^{2\pi}d\theta\int_0^H\rho\,d\rho\int_\rho^H(1+\rho^2)\,dz$

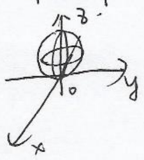
$=\int_0^{2\pi}d\theta\int_0^H H(\rho+\rho^3)-(\rho^2+\rho^4)\,d\rho=2\pi\cdot\frac{10+3H^2}{60}\cdot H^3$

$\iiint_V z\mu\,dx\,dy\,dz=\int_0^{2\pi}d\theta\int_0^H\rho\,d\rho\int_\rho^H(1+\rho^2)z\,dz$

$=\int_0^{2\pi}d\theta\int_0^H H^2(\rho+\rho^3)-(\rho^3+\rho^5)\,d\rho=2\pi\cdot\frac{3+H^2}{24}\cdot H^4$

$\therefore\bar z=\frac{5H(3+H^2)}{2(3H^2+10)}$ 由对称性可知$\bar x=\bar y=0$ $\therefore\left(0,0,\frac{5H(3+H^2)}{2(3H^2+10)}\right)$

(2) $r=2a\cos\varphi$ 由题$\mu=\frac{k}{r}$



$\iiint_V \mu\,dx\,dy\,dz=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{2a\cos\varphi}kr\,dr$

$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}2ka^2\cos^2\varphi\,d(-\cos\varphi)$

$=\int_0^{2\pi}d\theta\cdot\frac{2}{3}ka^2=2\pi\cdot\frac{2}{3}ka^2$

$\iiint_V z\mu\,dx\,dy\,dz=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{2a\cos\varphi}kr^2\cos\varphi\,dr$

$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\frac{8k}{3}a^3\cdot\cos^4\varphi\,d(-\cos\varphi)$

$=\int_0^{2\pi}\frac{8}{3}ka^3\cdot\frac{1}{5}\,d\theta=2\pi\cdot\frac{1}{5}\times\frac{8}{3}ka^3$

$\therefore\bar z=\frac{4}{5}a$. 由对称性知$\bar x=\bar y=0$

$\therefore\left(0,0,\frac{4}{5}a\right)$

第八章 重积分
第四节 重积分的应用

9. 求下列均匀薄片 D 或均匀物体 V 对指定直线或点的转动惯量. 设密度 $\mu=1$

(1) $D=\{(x,y)|0\le x\le a,0\le y\le b\}$，求 I_x,I_y,I_0；

(2) D 由抛物线 $y^2=\frac{9}{2}x$ 与直线 $x=2$ 围成，求 I_x,I_y；

(3) $D=\left\{(x,y)\middle|\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$，求 I_y；

(4) D 由抛物线 $y=x^2$ 与直线 $y=1$ 围成，求 D 对直线 $y=-1$ 的转动惯量；

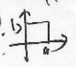
(5) D 由直线 $y=x,y=2x,y=1$ 围成，求 I_0；

(6) V 是底半径为 R，高为 H 的圆柱体，求 V 对其每一条母线的转动惯量；

(7) V 由曲面 $z=x^2+y^2$ 和平面 $z=0,|x|=a,|y|=a$ 围成，求 I_z；

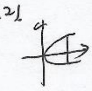
→与答案不一样

(8) $V=\{(x,y,z)|x^2+y^2+z^2\le 2,x^2+y^2\ge z^2\}$，求 I_z. ← $\mu=1$.

(1). 

$I_x=\int_0^a x^2dx\int_0^b dy=\frac13 a^3b\mu,\ I_y=\int_0^a dx\int_0^b y^2dy=\frac13 ab^3\mu$

$I_0=\int_0^a dx\int_0^b (x^2+y^2)dy=\int_0^a x^2b+\frac13 b^3dx=\frac13 ab(a^2+b^2)\cdot\mu.$

(2) 

$I_x=\int_0^2 dx\int_{-\sqrt{\frac92 x}}^{\sqrt{\frac92 x}}\mu y^2dy=\int_0^2\mu\frac{9}{\sqrt2}\cdot x^{\frac32}dx=\mu\frac{9}{\sqrt2}\times\frac25\times 2^{\frac52}=\frac{72}{5}\mu.$

$I_y=\int_0^2 x^2dx\int_{-\sqrt{\frac92 x}}^{\sqrt{\frac92 x}}\mu dy=\int_0^2\mu 3\times2\times\sqrt2\cdot x^{\frac52}dx=\mu\cdot 6\sqrt2\cdot\frac27\cdot 2^{\frac72}=\frac{96}{7}\mu.$

(3). 

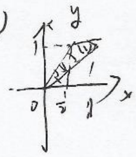
设 $x=\rho a\cos\theta,\ y=\rho b\cos\theta$. $I_y=\iint_D\mu x^2dxdy=ab\mu\int_0^{2\pi}d\theta\int_0^1\rho^3 a^2\cos^2\theta d\rho$

$=\mu\cdot a^3b\times4\times\frac12\times\frac{\pi}{2}\times\frac14=\frac14\pi a^3b\mu.$

(4). 

$I=\iint_D(y+1)^2dxdy=\mu\int_{-1}^1dx\int_{x^2}^1 y^2+2y+1\,dy=\int_{-1}^1\frac{\mu}{3}(1-x^6)+2-x^4-x^2\,dx$

$=\mu\left(\frac{14}{3}-2\int_0^1(\frac13x^6+x^4+x^2)dx\right)=\left[\frac{14}{3}-2\times(\frac{1}{21}+\frac15+\frac{7}{21})\right]\cdot\mu=\frac{368}{105}\mu.$

(5) 

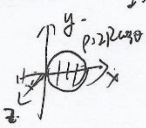
$I=\iint_D\mu(x^2+y^2)dxdy=\mu\cdot\int_0^1dy\int_{\frac12 y}^{y}(x^2+y^2)dx=\mu\int_0^1\frac13\times\frac78y^3+\frac12y^3dy$

$=\mu\int_0^1\frac{19}{24}y^3dy=\mu\cdot\frac{19}{96}=\frac{19}{96}\mu.$

第八章 重积分

第四节 重积分的应用

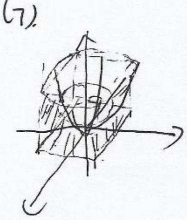
9. (6).



$$I=\mu H\cdot\iint_D(x^2+y^2)\,dxdy=\mu H\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2R\cos\theta}\rho^2\cdot\rho\,d\rho$$

$$=\mu H\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4R^4\cos^4\theta\,d\theta=\mu H\times2\times4\times\frac{3}{4}\times\frac{R^4}{2}\times\frac{\pi}{2}=\frac{3}{2}R^4H\mu.$$

(7).



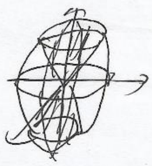
如图. 该几何体为旋转抛物面 $z=x^2+y^2$ 外侧面与面 $|x|=a, |y|=a$ 所围成

$$\therefore I=\mu\iiint_V z^2\,dxdydz=\mu\cdot\int_{-a}^{a}dx\int_{-a}^{a}dy\int_0^{x^2+y^2}(x^2+y^2)\,dz.$$

$$=4\mu\int_0^a dx\int_0^a(x^2+y^2)^2dy=4\mu\int_0^a x^4a+2x^2\cdot\frac{1}{3}a^3+\frac{1}{5}a^5\,dx$$

$$=4\mu\cdot a^6\cdot\left(\frac{1}{5}+\frac{2}{9}+\frac{1}{5}\right)=\frac{112}{45}a^6\mu.$$

(8).



所求为空白部分转动惯量.

$$I_z=\mu\int_0^{2\pi}d\theta\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\sin\varphi\,d\varphi\int_0^{\sqrt{2}}r^2\cdot r^2\sin^2\varphi\,dr$$

$$=\mu\int_0^{2\pi}d\theta\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\sin^3\varphi\,d\varphi\int_0^{\sqrt{2}}r^4\,dr$$

$$=\mu2\pi\times\frac{4}{5}\sqrt{2}\cdot\left(-\cos x+\frac{\cos^3x}{3}\right)\Big|_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}$$

$$=\mu2\pi\cdot\frac{4}{5}\sqrt{2}\times\frac{5}{6}\sqrt{2}=\frac{8}{3}\pi\mu=\frac{8}{3}\mu\pi$$

10. 设一薄片 D 由 $y=e^x, y=0, x=0, x=2$ 所围成，其面密度为 $\mu(x,y)=xy$，求 I_x, I_y；

D 如图.

$$I_y=\iint_D x^2\mu\,dxdy$$

$$=\int_0^2 dx\int_0^{e^x} x^3y\,dy=\frac{1}{2}\int_0^2 x^3 e^{2x}dx$$

$$=\frac{1}{2}\cdot e^{2x}\cdot\left[\frac{1}{2}x^3-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right]\Big|_0^2$$

$$=\frac{1}{16}(17e^4+3)$$

$$I_x=\iint_D y^2\mu\,dxdy=\int_0^2 dx\int_0^{e^x} xy^3dy$$

$$=\frac{1}{4}\int_0^2 x\cdot e^{4x}dx=\frac{1}{4}e^{4x}\left(\frac{1}{4}x-\frac{1}{16}\right)\Big|_0^2$$

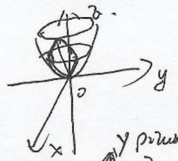
$$=\frac{1}{16}\cdot\left(e^8\cdot\frac{7}{4}+\frac{1}{4}\right)=\frac{7e^8+1}{64}$$

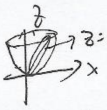
第八章 重积分

第四节 重积分的应用

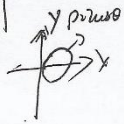
11. 设V是由曲面$z=x^2+y^2$和平面$z=2x$所围成的物体，其上任一点的密度等于该点到面xOz距离的平方，求I_z.

解：由题





$z=2\rho\cos\theta$，设$\mu=y^2$.



$$\therefore I_z=\iiint_V \mu\cdot y^2\,dxdydz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}\rho d\rho\cdot\int_{\rho^2}^{2\rho\cos\theta}\rho^2\cdot\rho^2\sin^2\theta\,dz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}(2\rho^6\sin^2\theta\cos\theta-\rho^7\sin^2\theta)\,d\rho$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{256}{7}\sin^2\theta\cos^8\theta-32\sin^2\theta\cos^8\theta\right)d\theta$$

$$=\frac{32}{7}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2(1-\cos^2\theta)\cos^8\theta\,d\theta$$

$$=\frac{64}{7}\int_0^{\frac{\pi}{2}}\cos^8\theta-\cos^{10}\theta\,d\theta$$

$$=\frac{64}{7}\cdot\left(\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right)\cdot\left(1-\frac{9}{10}\right)$$

$$=\frac{\pi}{8}.$$

第八章 重积分
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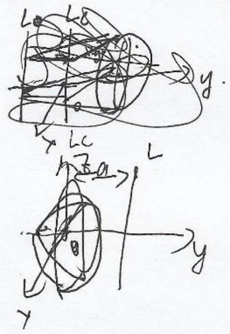
12. 设物体对直线 L 的转动惯量为 I_L，对通过质心 C 且平行 L 的直线 L_C 的转动惯量为 I_C，I_C 与 L 的距离为 a，试证：$I_L = L_C + Ma^2$，其中 M 为物体的质量，这一公式称为平行轴定理.



如图以物体的质心为原点，构建直角坐标系，L_c 为 z 轴，L 为 $y=a$，

∴ $I_L = \iiint_V (x^2+(y-a)^2)\cdot\mu(x,y,z)\,dV$

$= \iiint_V (x^2+y^2+a^2-2ay)\mu(x,y,z)\,dV$

$= a^2\iiint_V \mu\,dV + \iiint_V (x^2+y^2)\mu(x,y,z)\,dV - 2a\iiint_V y\mu\,dV$

∵ $\bar{y} = \dfrac{\iiint_V y\cdot\mu(x,y,z)\,dV}{\iiint_V \mu\,dV} = 0$

∴ $\iiint_V y\cdot\mu\,dV = 0$，$M = \iiint_V \mu\,dV$

∴ $I_L = \iiint_V \mu(x^2+y^2)\,dV + a^2\cdot\iiint_V \mu\,dV$

$= I_c + a^2\cdot M$

∴ $I_L = I_c + Ma^2$

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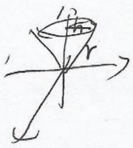
13. 求高为h，半顶角为α的均匀直圆锥体对位于其顶点的一单位质点的引力.

解:



如图, 由对称性知, $F_x=F_y=0$

故求 F_z，m为单位质量, μ为密度.

$$F_z=\iiint_V \frac{(z-0)}{r^3}\cdot Gm\mu\, dV$$

$$=Gm\mu\int_0^{2\pi}d\theta\int_0^{\alpha}\sin\varphi\, d\varphi\int_0^{\frac{h}{\cos\alpha}} r\cdot\cos\alpha\cdot\frac{1}{r^3}\cdot r^2 dr$$

$$=Gm\mu\int_0^{2\pi}d\theta\int_0^{\alpha}\sin\varphi\cdot\cos\varphi\cdot\frac{h}{\cos\varphi}\, d\varphi$$

$$=Gm\mu\cdot 2\pi\cdot h\,(-\cos\varphi)\Big|_0^{\alpha}=2\pi Gm\mu h(1-\cos\alpha)$$

$$\therefore F=(F_x, F_y, F_z)$$

$$=(0,0,2\pi Gm\mu h(1-\cos\alpha))$$

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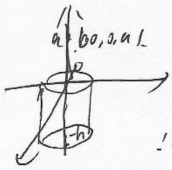
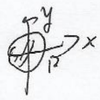
14. 设均匀物体 $V=\{(x,y,z)\mid x^2+y^2\le R^2,-h\le z\le 0\}$，求 V 对位于点 $(0,0,a)$ $(a>0)$ 处质量为 m 的质点的引力.



解：由对称性可知，x轴，y轴方向分力相互抵消，$F_x=F_y=0$

$$\therefore F_z=\iiint_V G\cdot\mu\cdot m\cdot\frac{z-a}{r^3}\,dV$$



$$=G\mu\cdot m\cdot\int_{-h}^{0}(z-a)\,dz\iint_D\frac{dx\,dy}{(x^2+y^2+(a-z)^2)^{\frac{3}{2}}}$$

$$=G\mu m\int_{-h}^{0}(z-a)\int_0^{2\pi}d\theta\int_0^{R}\frac{\rho\,d\rho}{[\rho^2+(a-z)^2]^{\frac{3}{2}}}$$

$$=G\mu m\cdot 2\pi\cdot\int_{-h}^{0}(z-a)\cdot\left(\frac{1}{a-z}-\frac{1}{\sqrt{R^2+(a-z)^2}}\right)dz$$

$$=2\pi G\mu\cdot m\left[-h+\sqrt{R^2+(a+h)^2}-\sqrt{R^2+a^2}\right]$$

$$\therefore F=\left(0,0,2\pi G\mu m\left(\sqrt{R^2+(a+h)^2}-\sqrt{R^2+a^2}-h\right)\right)$$

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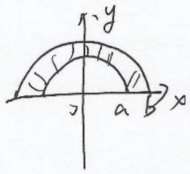
15. 设半圆环薄片 $D: a^2 \le x^2 + y^2 \le b^2\ (y \ge 0)$ 的面密度 $\mu(x,y) = y$，求 D 对位于原点处质量为 m 的质点的引力.



由对称性知 在x轴方向上. $F_x = 0$

故求 F_y，μ 为密度

$$F_y = \iint_D \frac{(y-0)}{\rho^3} \cdot Gm\mu \, dV$$

$$= Gm \cdot \int_0^{\pi} d\theta \int_a^b \rho \cdot \frac{\rho^2 \cancel{\cos} \sin^2\theta}{\rho^3} d\rho$$

$$= Gm \cdot \int_0^{\pi} \sin^2\theta \, d\theta \cdot [(b-a)]$$

$$= (b-a)Gm \cdot 2 \times \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{2} Gm(b-a)$$

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16. 设V是由曲面$x^2+y^2=4$, $x^2+y^2=9$和平面$z=0$, $z=4$围成的均匀物体，求V对位于原点的质量为m的质点的引力.

解：由对称性知 $F_x=F_y=0$

故求F_z

$$F_z=\iiint\limits_V \frac{Gm\mu\cdot(z-0)}{r^3}\,dV$$

$$=Gm\mu\cdot\int_0^4 z\,dz\int_0^{2\pi}d\theta\int_2^3\frac{\rho\,d\rho}{(\sqrt{\rho^2+z^2})^3}$$

$$=Gm\mu\int_0^4 z\,dz\cdot\int_0^{2\pi}\frac{1}{2}\cdot(-2)\cdot\frac{1}{\sqrt{\rho^2+z^2}}\Big|_2^3\,d\theta$$

$$=2\pi\cdot Gm\mu\cdot\int_0^4\left(\frac{1}{\sqrt{z^2+4}}-\frac{1}{\sqrt{z^2+9}}\right)d\frac{1}{2}z^2$$

$$=2\pi\cdot Gm\mu\cdot\left[(\sqrt{z^2+4})\Big|_0^4-(\sqrt{z^2+9})\Big|_0^4\right]$$

$$=2\pi Gm\mu\cdot[2\sqrt{5}-2-(5-3)]=4\pi Gm\mu\cdot(\sqrt{5}-2)$$

$\therefore F=(0,0,4\pi Gm\mu(\sqrt{5}-2))$

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