1. 利用高斯公式计算下列第二类曲面积分.

(1) $\oiint_S 3xy\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}z\mathrm{d}x - x^2y^4\mathrm{d}x\mathrm{d}y$，其中 S 是以点 $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$ 为顶点的四面体表面的外侧;

(2) $\oiint_S yz\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}z\mathrm{d}x + x^2y\mathrm{d}x\mathrm{d}y$，其中 S 是柱面 $x^2+y^2=9$ 与平面 $z=0$ 和 $z=y-3$ 所围成区域的边界曲面外侧;

(3) $\oiint_S 2xz\mathrm{d}y\mathrm{d}z + yz\mathrm{d}z\mathrm{d}x - z^2\mathrm{d}x\mathrm{d}y$，其中 S 是由锥面 $z=\sqrt{x^2+y^2}$ 与半球面 $z=\sqrt{2-x^2-y^2}$ 所围成区域边界曲面的外侧;

(4) $\oiint_S z^2\mathrm{d}y\mathrm{d}z$，其中 S 是椭球面 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ 的外侧;

(5) $\iint_S (x^2-yz)\mathrm{d}y\mathrm{d}z + (y^2-zx)\mathrm{d}z\mathrm{d}x + 2z\mathrm{d}x\mathrm{d}y$，其中 S 为锥面 $z=1-\sqrt{x^2+y^2}$ 被平面 $z=0$ 所截得的有限部分的上侧;

(6) $\iint_S x^3\mathrm{d}y\mathrm{d}z + 2xz^2\mathrm{d}z\mathrm{d}x + 3y^2z\mathrm{d}x\mathrm{d}y$，其中 S 是抛物面 $z=4-x^2-y^2$ 被平面 $z=0$ 所截得的有限部分的下侧;

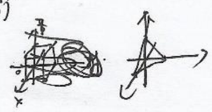
(7) $\iint_S 2(1-x^2)\mathrm{d}y\mathrm{d}z + 8xy\mathrm{d}z\mathrm{d}x - 4xz\mathrm{d}x\mathrm{d}y$，其中 S 是 xOy 面上曲线 $x=\mathrm{e}^y\ (0\le y\le a)$ 绕 x 轴旋转所成旋转曲面的凸的一侧;

(8) $\iint_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$，其中 $\boldsymbol{A}=\dfrac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}}$，$S$ 是半球面 $x^2+y^2+z^2=R^2\ (z\ge 0)$ 的下侧;

(9) $\iint_S xy\sqrt{1-x^2}\mathrm{d}y\mathrm{d}z + \mathrm{e}^x\sin y\,\mathrm{d}\cancel{z}\,\mathrm{d}y$，其中 S 为柱面 $x^2+z^2=1\ (0\le y\le 2)$ 的外侧.

解:

(1)



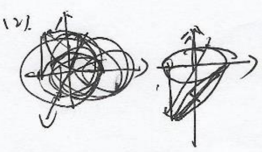
$\oiint_S 3xy\,dydz+y^2dzdx-x^2y^4dxdy$

$=\iiint_V 5y\,dV=\int_0^1dx\int_0^{1-x}5y\,dy\int_0^{1-x-y}dz$

$=\int_0^1dx\int_0^{1-x}5(1-x)y-5y^2\,dy=\int_0^1\frac{5}{2}(1-x)^3-\frac{5}{3}(1-x)^3\,dx$

$=\frac{5}{6}\times\frac{1}{4}=\frac{5}{24}$

(2)



$\oiint yz\,dydz+y^2dzdx+x^2y\,dxdy=\iint_{D_{xy}}2y\int_{y-3}^{0}dz$

$=\iint_{D_{xy}}2y(3-y)\,dxdy=\int_0^{2\pi}d\theta\int_0^3 2\rho^2\sin\theta(3-\rho\sin\theta)\,d\rho$

$=\int_0^{2\pi}6\times\frac{\rho^3}{3}\sin\theta\Big|_0^3-\frac{1}{2}\rho^4\sin^2\theta\Big|_0^3\,d\theta$

$=\int_0^{2\pi}-\frac{81}{2}\sin^2\theta\,d\theta=-\frac{81}{2}\times4\times\frac{1}{2}\times\frac{\pi}{2}=-\frac{81}{2}\pi$

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1\.

(3).

原式 $= \iiint\limits_V (2z+z-2z)\,dV$

$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi\, d\varphi \int_0^{\sqrt{2}} r\cdot\cos\varphi\cdot r^2 dr = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi\cos\varphi \int_0^{\sqrt{2}} r^3 dr$

$= 2\pi \times \frac{1}{4} \times 1 = \frac{\pi}{2}$

(4). $\frac{\partial X}{\partial x} = 0 \quad \therefore \oiint\limits_S z^2\,dydz = \iiint\limits_V 0\,dxdydz = 0$

(5). $\frac{\partial X}{\partial x} = 2x,\ \frac{\partial Y}{\partial y} = 2y,\ \frac{\partial Z}{\partial z} = 2.\ \therefore$ 原式 $= 2\iiint\limits_V (x+y+1)\,dV$

$= 2\cdot\int_0^1 dz\cdot\int_0^{2\pi} d\theta \int_0^{1-z} \rho(\rho\sin\theta + \rho\cos\theta + 1)\,d\rho$

$= 2\cdot\int_0^1 dz\cdot \int_0^{2\pi} \frac{1}{3}(1-z)^3(\sin\theta+\cos\theta) + \frac{1}{2}(1-z)^2\, d\theta$

$= 2\int_0^1 (1-z)^2\cdot\frac{1}{2}\cdot 2\pi\, dz = 2\pi\cdot\int_0^1 (1-z)^2 dz = \frac{2}{3}\pi.$

(6).

$\iint\limits_{S^-} x^3 dydz + 2xz^2 dzdx + 3y^2 z\, dxdy$

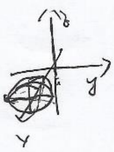
$= -\oiint\limits_{S'} x^3 dydz + 2xz^2 dzdx + 3y^2 z\, dxdy$，其中 S' 为 $\begin{cases} z=0 \\ x^2+y^2\le 4 \end{cases}$

原式 $= -\iiint\limits_V 3x^2+3y^2\,dV = -3\int_0^4 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{4-z}} \rho\cdot\rho^2 d\rho$

$= -3\cdot 2\pi\cdot \int_0^4 \frac{1}{4}(4-z)^2 dz = -3\times 2\pi\times\frac{1}{4}\times 4^3 \times \frac{1}{3}$

$= -32\pi.$

(7).



补充曲面 $x=e^a$ 与 $x=e^{\sqrt{y^2+z^2}}$ 相交部分为 S_1

$\therefore \iint\limits_S 2(1-x^2)\,dydz + 8xy\,dzdx - 4xz\,dxdy$

$= \oiint\limits_{S+S_1} (-4x+8x-4x)\,dV - \iint\limits_{y^2+z^2\le a^2} 2(1-e^{2a})\,dydz$

$= 0 + 2\cdot(e^{2a}-1)\cdot\iint\limits_{y^2+z^2\le a^2} dydz = 2(e^{2a}-1)\pi a^2.$

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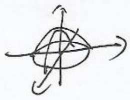
(8)，补充面 $S_1:\begin{cases} z=0 \\ x^2+y^2\le R^2 \end{cases}$



$\therefore \iint\limits_{S} A\cdot dS = -\oiint\limits_{S+S_1} A\cdot dS = \iiint\limits_{V}$.

$\frac{\partial X}{\partial x} = \frac{y^2+z^2}{(\sqrt{x^2+y^2+z^2})^3} = \frac{y^2+z^2}{r^3}$；$\frac{\partial Y}{\partial y} = \frac{x^2+z^2}{r^3}$；$\frac{\partial Z}{\partial z} = \frac{x^2+y^2}{r^3}$

$\therefore$ 上式 $\oiint\limits_{S+S_1} A\,dS = \iiint\limits_{V} \frac{2}{r}\,dV = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_0^{R} \frac{2}{r}\cdot r^2\, dr$

$= 2\pi\times1\times R^2 = 2\pi R^2$

$\therefore \iint\limits_{S} A\cdot dS = -2\pi R^2$

(9)，补面 $\begin{cases} S_1: y=0 \text{ 取左侧} \\ S_2: y=2 \text{ 取右侧} \end{cases}$

$\therefore \iint\limits_{S_1} P\,dy\,dz + R\,dx\,dy = \iint\limits_{S_2} P\,dy\,dz + R\,dx\,dy = 0$

$\therefore \iint\limits_{S} xy(\sqrt{1-x^2})\,dy\,dz + e^x \sin y\, dx\,dy = \oiint\limits_{S+S_1+S_2} P\,dy\,dz + R\,dx\,dy$

$= \iiint\limits_{V} \frac{1-2x^2}{\sqrt{1-x^2}}\,dV = \int_{-1}^{1} \frac{1-2x^2}{\sqrt{1-x^2}}\,dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz \int_0^2 dy$

$= 4\cdot\int_{-1}^{1} \frac{1-2x^2}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2}\cdot dx = 4\cdot\int_{-1}^{1} 1-2x^2\,dx$

$= 4\cdot2\cdot\int_0^1 1-2x^2\,dx = 8\cdot(1-\frac{2}{3}) = \frac{8}{3}$

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2. 求流速为$\boldsymbol{v}$的流体穿过曲面S外侧的流量，其中:

(1) $\boldsymbol{v}=x^3\boldsymbol{i}+y^3\boldsymbol{j}+z^3\boldsymbol{k}$，$S:x^2+y^2+z^2=a^2$;

(2) $\boldsymbol{v}=\left(x(y-z)\right),y(z-x),z(x-y)\big)$，$S:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

解: (1).

$$Q=\oiint_S v\cdot ds$$

$$\therefore \frac{\partial X}{\partial x}=3x^2;\ \frac{\partial Y}{\partial y}=3y^2;\ \frac{\partial Z}{\partial z}=3z^2.$$

$$\therefore Q=\iiint_V 3(x^2+y^2+z^2)\,dV=3\cdot\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{a}r^4dr$$

$$=3\times2\pi\times2\times\frac{a^5}{5}=\frac{12}{5}\pi\cdot a^5$$

(2).

$$Q=\oiint_S v\,ds$$

$$\frac{\partial X}{\partial x}=y-z,\quad \frac{\partial Y}{\partial y}=z-x,\quad \frac{\partial Z}{\partial z}=x-y$$

$$\therefore \frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0$$

$$\therefore Q=\oiint_S v\,ds=0$$

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3. 求下列向量场 A 的散度.

(1) $A=\left(x^2+yz\right)i+\left(y^2+xz\right)j+\left(z^2+xy\right)k$;

(2) $A=e^{xy}i+\cos(xy)j+\cos\left(xz^2\right)k$.

(1). $\operatorname{div} A=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$

$=2x+2y+2z$

(2). $\frac{\partial X}{\partial x}=y\cdot e^{xy}$

$\frac{\partial Y}{\partial y}=-x\cdot\sin(xy)$

$\frac{\partial Z}{\partial z}=-2z\cdot\sin(xz^2)$

$\therefore \operatorname{div} A=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=y\cdot e^{xy}-x\sin(xy)-2z\sin(xz^2)$

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