Two Essays on Theory

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A concealed arithmetical exercise of the soul, unaware of its counting—Leibniz's description of music, 17 April 1712

First Essay: The Rationalization of a Harmonically Irrational Set of Pitches

For more than two thousand years it has been a well-known fact that two pitches that have a simple frequency relationship between them (such as 1:2, where one frequency is twice as high as the other) form a harmonic interval (an octave in this example). However, it is undisputable that a given interval with a complex numerical relationship in the direct vicinity of another, more harmonic interval, falls into the pull of the stronger one, as it were. It thus operates as an approximation (for instance: an interval with the frequency relationship of 100:199 is only 0.7% smaller than an octave and is therefore heard as an octave); this "bending into place" is the topic of this text.

Harmonicity

How can we measure the *harmonicity* of an interval? Differently put, how can we express the degree of simplicity of a numerical relationship? The relationships 1:2 (octave), 2:3 (perfect fifth), 3:4 (perfect fourth), 4:5 (major third), or 5:6 (minor third) are obviously simpler than 8:9 (major tone) or 9:10 (minor tone) . . . and likewise these are simpler than 15:16 (minor second) or 32:45 (augmented fourth) or 45:64 (diminished fifth). To all appearances, this is equally true of the corresponding intervals given here in brackets. However, a comparison between the major sixth (3:5) and major third (4:5) does not clearly end in favor of the sixth, even though 3 is smaller.

In the previously stated list I have deliberately left out the intervals 6:7 and 7:8, as well as 10:11, 11:12, 12:13, and 14:15. None of the classical intervals contain these numbers. The numbers 8 and 9 have been preferred since antiquity to the smaller 7, the numbers 15 and 16 to 11, 13, and 14. (12, for instance, is to be found in the minor tenth 5:12.) It is striking that while the historically preferred numbers are based on the prime numbers 2, 3, and 5, the others contain higher prime factors of 7, 11, and 13. Therefore, not only the smallness of the numbers of a relationship is relevant for the construction of a harmonic interval, but also their *divisibility.*

To measure harmonicity, it would be of use to have at one's disposal a coefficient for natural numbers that would combine these features. This thought led me in 1978 to develop the *indigestibility* function $\xi(N)$, shown in Eq. 1. A practical aspect of this function is that $\xi(ab) = \xi(a) + \xi(b)$, as with logarithms.

$$\xi(N) = 2\sum_{r=1}^{\infty}\left\{\frac{n_r(p_r - 1)^2}{p_r}\right\}$$

where

$$N = \prod_{r=1}^{\infty} p_r^{n_r};$$

p is a prime; and
n is a natural number. (1)

Table 1 shows the indigestibility values for the numbers 1 to 16 in tabular form. If we arrange these 16 numbers in the order of increasing indigestibility, we obtain the sequence 1, 2, 4, 3, 8, 6, 16, 12, 9, 5, 10, 15, 7, 14, 11, 13. . . . The last four are the aforementioned "outcasts."

From the inversion of the sum of the indigestibility values of the numbers P and Q (mutually prime), a function of harmonicity for any interval $P:Q$ can be derived: the more indigestible P and Q, the less har-

Translated from German by Henning Lohner, and approved by the author.

Table 1. Indigestibility values for the integers 1 to 16

N	$\xi(N)$
1	0,000000
2	1,000000
3	2,666667
4	2,000000
5	6,400000
6	3,666667
7	10,285714
8	3,000000
9	5,333333
10	7,400000
11	18,181818
12	4,666667
13	22,153846
14	11,285714
15	9,066667
16	4,000000

Table 2. All intervals that are more harmonic than 0.06 within the octave

Interval Size (Cents)	*Interval as Product of Powers of Primes:* 2	3	5	7	11	13	*Frequency Ratio*	*Harmonicity Value*
0,000	0	0	0	0	0	0	1:1	Infinity
111,731	4	−1	−1	0	0	0	15:16	−0,076531
182,404	1	−2	1	0	0	0	9:10	0,078534
203,910	−3	2	0	0	0	0	8:9	0,120000
231,174	3	0	0	−1	0	0	7:8	−0,075269
266,871	−1	−1	0	1	0	0	6:7	0,071672
294,135	5	−3	0	0	0	0	27:32	−0,076923
315,641	1	1	−1	0	0	0	5:6	−0,099338
386,314	−2	0	1	0	0	0	4:5	0,119048
407,820	−6	4	0	0	0	0	64:81	0,060000
435,084	0	2	0	−1	0	0	7:9	−0,064024
498,045	2	−1	0	0	0	0	3:4	−0,214286
519,551	−2	3	−1	0	0	0	20:27	−0,060976
701,955	−1	1	0	0	0	0	2:3	0,272727
764,916	1	−2	0	1	0	0	9:14	0,060172
813,686	3	0	−1	0	0	0	5:8	−0,106383
884,359	0	−1	1	0	0	0	3:5	0,110294
905,865	−4	3	0	0	0	0	16:27	0,083333
933,129	2	1	0	−1	0	0	7:12	−0,066879
968,826	−2	0	0	1	0	0	4:7	0,081395
996,090	4	−2	0	0	0	0	9:16	−0,107143
1017,596	0	2	−1	0	0	0	5:9	−0,085227
1088,269	−3	1	1	0	0	0	8:15	0,082873
1200,000	1	0	0	0	0	0	1:2	1,000000

monic the interval. The formula can be glanced at in Eq. 2. If P and Q are mutually prime, then $(\mathrm{hcf}_{P,Q})$ is zero. (hcf means "highest common factor.")

$$h(P,Q) = \frac{\mathrm{sgn}[\xi(P) - \xi(Q)]}{\xi(P) + \xi(Q) - 2\xi(\mathrm{hcf}_{P,Q})} \tag{2}$$

where

$\mathrm{sgn}(x) = -1$ when x is negative, otherwise $\mathrm{sgn}(x) = +1$;
$\mathrm{hcf}_{a,b}$ is the highest common factor of a and b; and
$\xi(x)$ is the indigestibility of x.

To return to the issue of harmonicity, Table 2 shows a listing of all intervals within an octave, with harmonicity being not less than 0.06. The occasional minus signs indicate a polarization of the interval to the higher note; this pitch then functions as the root. In Table 3 one can see the resolution of the numbers of an intervallic relationship into the powers of their prime factors. For the imposed margin of 0.06 *minimum harmonicity,* the *sequence of maximum absolute powers* is 6, 4, 1, 1, for the prime numbers 2, 3, 5, 7. Larger prime numbers do not occur and therefore they all have the power 0. The total number of admissible intervals within this octave is 24. Table 3 shows the sequences of maximum powers and total quantity of intervals for other minimum harmonicities; the range is again arbitrarily an octave.

As is to be expected, raising the minimum harmonicity reduces the interval density (the value 0.1065 allows only a justly-intoned Mixolydian scale!), and vice versa. There is a direct correspondence between the maximum powers sequences shown in Table 3 and their equivalent minimum harmonicities expressed in Eq. 3.

Table 3. Resolution of the numbers of an intervallic relationship into the powers of their prime factors

Harmonicity Minimum	*Largest Powers of Prime Numbers* 2	3	5	7	11	13	*Number of Intervals in an Octave*
0,10	4	2	1	0	0	0	9
0,09	4	2	1	0	0	0	10
0,08	4	3	1	1	0	0	14
0,07	5	3	1	1	0	0	19
0,06	6	4	1	1	0	0	24
0,05	7	4	2	1	0	0	38
0,04	9	6	2	1	1	0	76
0,03	12	8	3	2	1	1	211
0,02	19	11	5	3	2	1	?
0,01	37	23	11	7	4	3	?

$$N(p) = \left[\frac{\omega + 1/h}{1 + (\log_e(256)/\log_e(27))}\right], \qquad (3)$$

where $p = 2$, otherwise

$$N(p) = \left[\frac{\omega + 1/h}{\xi(p) + (\log_e(p)/\log_e(2))}\right]$$

where

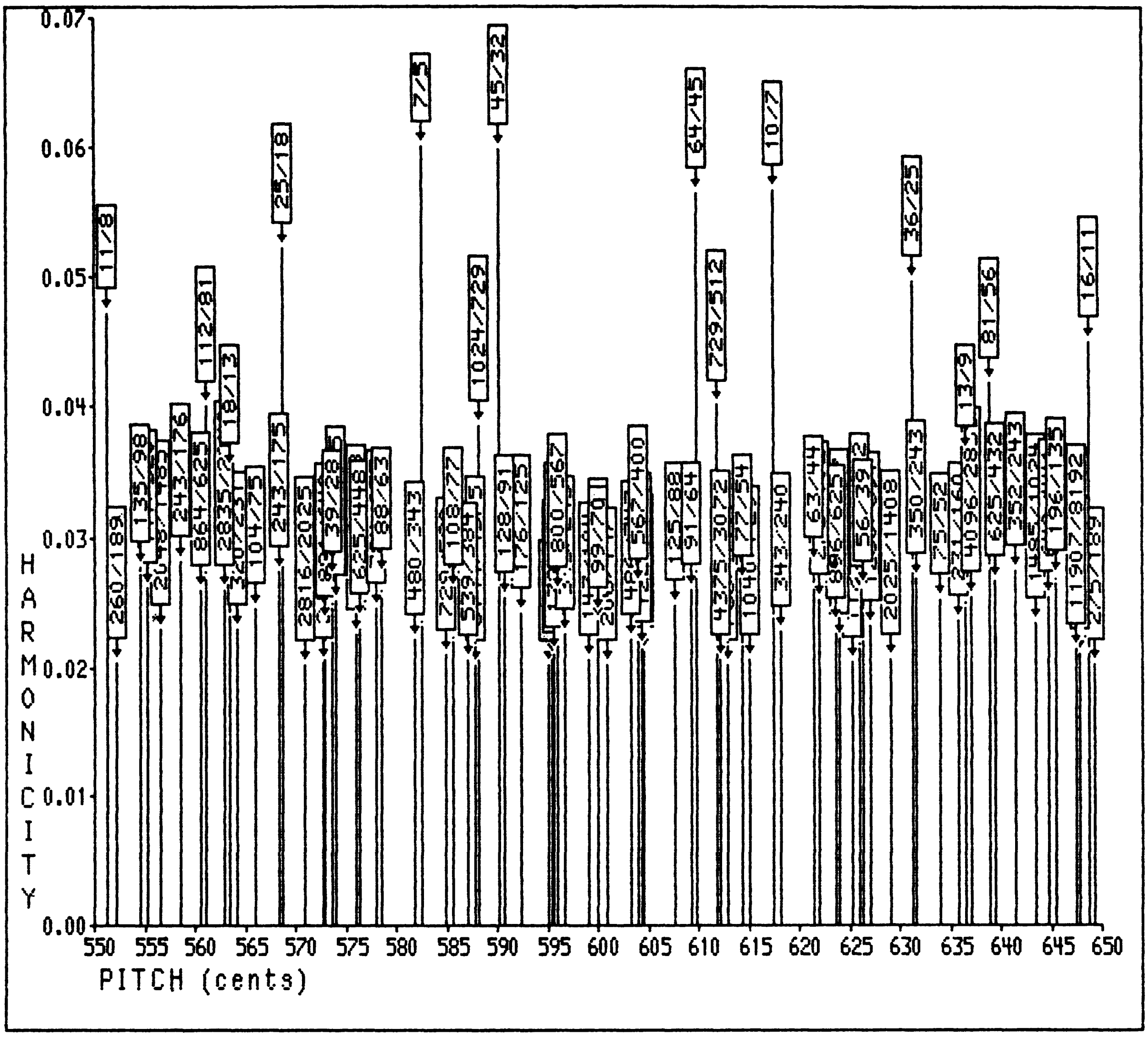
p is the prime number, the maximum power of which is desired;
h is the minimum harmonicity;
ω is the pitch range in octaves;
$\xi(x)$ is the indigestibility of x; and
$[x]$ is the integer part of x.

Because of the extremely high powers represented in the last two lines of Table 3, the corresponding interval quantities could not be determined. The powers sequence itself was estimated via Eq. 3. One has to consider that a maximum powers sequence includes intervals, the harmonicities of which may lie below the minimum suggested by Eq. 3. For instance, the sequence $-4, 0, 1, 2, -1$ for the interval 176:245 lies within the values corresponding to a minimum harmonicity of 0.03, but this interval nevertheless has a harmonicity of 0.020345. The maximum powers sequence guarantees merely that all intervals that are more harmonic than a given minimum value can be expressed by the sequence. A harmonicity minimum of 0.03 and a corresponding maximum powers sequence 12, 8, 3, 2, 1, 1, results in as many as 3,964 different intervals within one octave (!), of which only 211 are truly more harmonic than 0.03. Within these proceedings, however, intervals that lie below the given minimum harmonicity are not taken into consideration. The interval listings to be used contain no intervals that are less harmonic than the indicated harmonicity minimum.

The maximum powers sequence of the traditionally accepted interval tunings as mentioned by Western (and incidentally also Indian) theoreticians is 9, 6, 2, 0, 0, . . . , whereas interval construction in ancient Greece (Pythagoras and others) proceeded from the series 9, 6, 0, ... Interestingly, both series look as if they were hesitant approaches to the sequence 9, 6, 2, 1, 1, 0, 0, . . . , based on a minimum harmonicity of 0.04. Indeed, theoreticians have—for understandable reasons—been reluctant to introduce larger prime numbers into their calculation systems. (The exception is Harry Partch's exuberant employment of the numbers 11 and 13, although in my opinion, he was not able to use them convincingly in his compositions.) On the other hand, it is also true that interval measurers have let themselves be guided by an almost unlimited numerological joy, instead of examining whether their calculations have practical application. Frequently, theory and practice remain apart because appropriate means of testing are not available.

To sum up, one can say: larger prime numbers are harmonicity inhibitors. Knowing this, theoreticians have often been reluctant to consider them at all. To explain certain intervals, monstrous conglomerations of small prime numbers have often been installed in situations where smaller products of larger prime numbers may have offered the more elegant solution. An example is the tritone, which was represented in Europe until a few centuries ago as an augmented fourth of the ratio 512:729 (sequence of powers: $-9, -6, 0, 0, \ldots$ = six fifths minus three octaves). In comparison, theoreticians today usually allocate the ratio 32:45 (sequence $-5, 3, 1, 0, 0, \ldots$ = two fifths plus a third, minus one octave) to this interval.

Fig. 1. Graphic representation of all intervals more harmonic than 0.02 within the pitch frequencies of 550 and 600 cents.



Tuning Tolerance

Let us look closely at the tritone—more exactly: the half step between 550 and 650 cents. Using a minimum harmonicity of 0.02 as basis, we find no less than 81 intervals here, represented graphically in Fig. 1. Among these, the most harmonic are 5 : 7 and 32 : 45 and, slightly weaker, their inversions 7 : 10 and 45 : 64, all intervals that can be taken as tritones. The next question is, which one of these and other neighboring variants is recognized and understood when an interval of 600 cents is intoned. This question cannot be answered without an exact inspection of the musical context in which the interval occurs.

An interval of 600 cents is, musically speaking, without any intervallic significance in itself, just as the syllables "damp light" could just as well mean "damn plight," depending on the semantic context. If, taking C as a base, pitches of 3, 5, and 6 half steps above are sounded, it will not be difficult to recognize them as E-flat, F, and G-flat. If, however, notes appear in the sequence of 3, 7, and 6 half steps above C, then this would suggest the series C, E-flat, G, F-sharp. The tunings of the four notes of the first series would be 1:1, 6:5, 4:3, 64:45, those of the second, 1:1, 6:5, 3:2, 45:32. In the second case, the last note (F-sharp) sounds a little bit sharper than in the first case (G-flat), even if the series is played on a well-tempered piano. (Actually, 64:45 is higher than 45:32!) Skeptics are advised to play the series of notes C, D, F, E (Mozart) and C-sharp, B-sharp, E, D-sharp (J. S. Bach) on a piano to compare the impression produced by the third C–E and the diminished fourth B-sharp–E, respectively. The degree of tension definitely differs between the two. This difference results from a more or less subconscious bending of the notes to an optimal tuning within the mind's ear of the listener.

I have developed a method to find the most convincing tunings of an initially neutral series of pitches (indicated in cents). It proceeds in the following manner. First, all admissable tuning alternatives of the notes must be known. Their choice depends on two main factors: the harmonicity minimum and the tuning tolerance. In harmonically more complex music (such as that of the late Romantic era) the harmonicity minimum would have to be fixed at a lower level than that of harmonically simpler music. With less fastidious interpreters (such as children) the *tuning tolerance* would have to be fixed at a higher level than with more articulate interpreters. From a chosen harmonicity minimum one now calculates a maximum powers sequence according to Eq. 3. This is used to generate a listing of intervals (such as the ones in Table 2 and Fig. 1). Of course it is possible to choose the maximum powers sequence at random, as for instance 0, 0, 0, 5, 3, 0, . . . , a tuning based solely on the 7 and 11. However, such a series is very difficult to support compositionally (cf. Partch) and may under certain conditions sound like an out-of-tune rendition of another tuning generated from smaller prime numbers.

Concerning the tuning tolerance, one sets a Gaussian-like bell over the pitch to be tuned. This bell, which has a width (variance) proportional to the indicated tolerance, dampens—increasingly upwards and downwards in pitch—the harmonicity values as shown in Fig. 2. Thus all intervals that are far away from the central area or that are simply too weak fail as tuning alternatives. I preferred this method to a nonweighted frequency window, which seems less differentiated. The nominal tolerance is the value at which the damping factor is 20. Figure 3 shows this damping procedure in the attempt to tune a major scale (indicated in cents). The nominal tolerance was fixed at 50 cents, which is half the smallest distance between two neighboring steps of the scale.

The next step is to decide how many of the best tuning alternatives should be taken into consideration for each pitch to be tuned. Then the sum of the harmonicities of all possible intervallic links between the degrees of the scale is determined for each constellation of tuning alternatives. The constellation that results in the largest harmonicity sum is chosen. If the number of tunable pitches is, for instance, 8, and the number of their corresponding tuning alternatives 3, then the total number of tuning constellations is $3^8 = 6{,}561$. For 8 pitches there are 28 intervallic connections, so all in all $6{,}561 \times 28 = 183{,}708$ harmonicity values must be added to find the optimum tuning—clearly a job for the computer! When programming this problem, I found it to be more efficient not to add the harmonicities, but to add their inversions, whereby the smallest sum is the "winner." I call the square of the number of pitches divided by this value the *specific harmonicity* of the tuning constellation.

Tuning a major scale within one octave, we obtain, given a harmonicity minimum of 0.04 and a nominal tolerance of 50 cents, the most, second most, and third most harmonic alternative tunings for each pitch, as shown in Table 4.

After checking all 28 intervallic connections 6,561 times, the program decides upon the following tuning combination:

$$1:1 \quad 8:9 \quad 4:5 \quad 3:4 \quad 2:3 \quad 3:5 \quad 8:15 \quad 1:2.$$

The specific harmonicity of this tuning is 0.2575. Under the same circumstances, the computer allo-

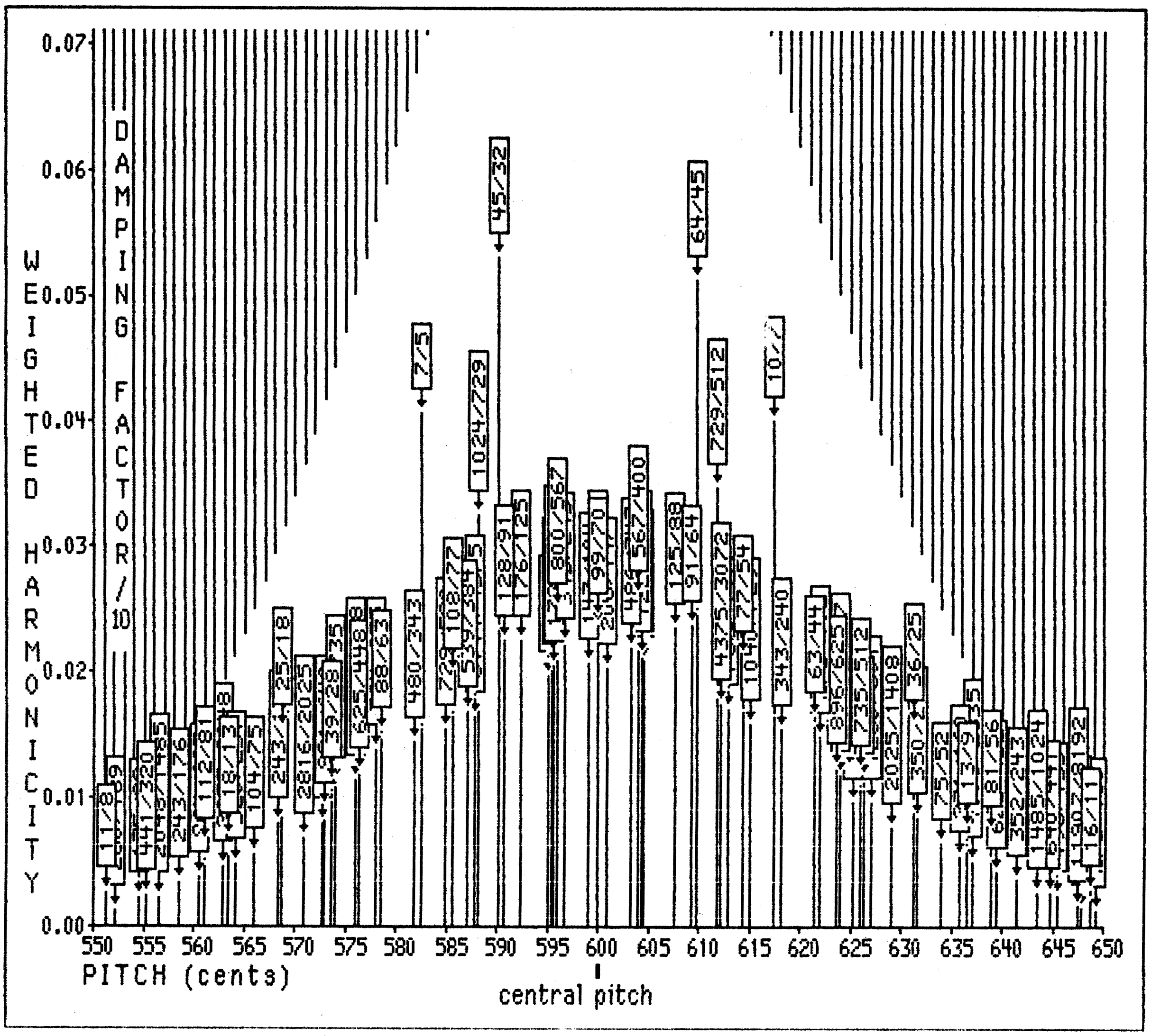
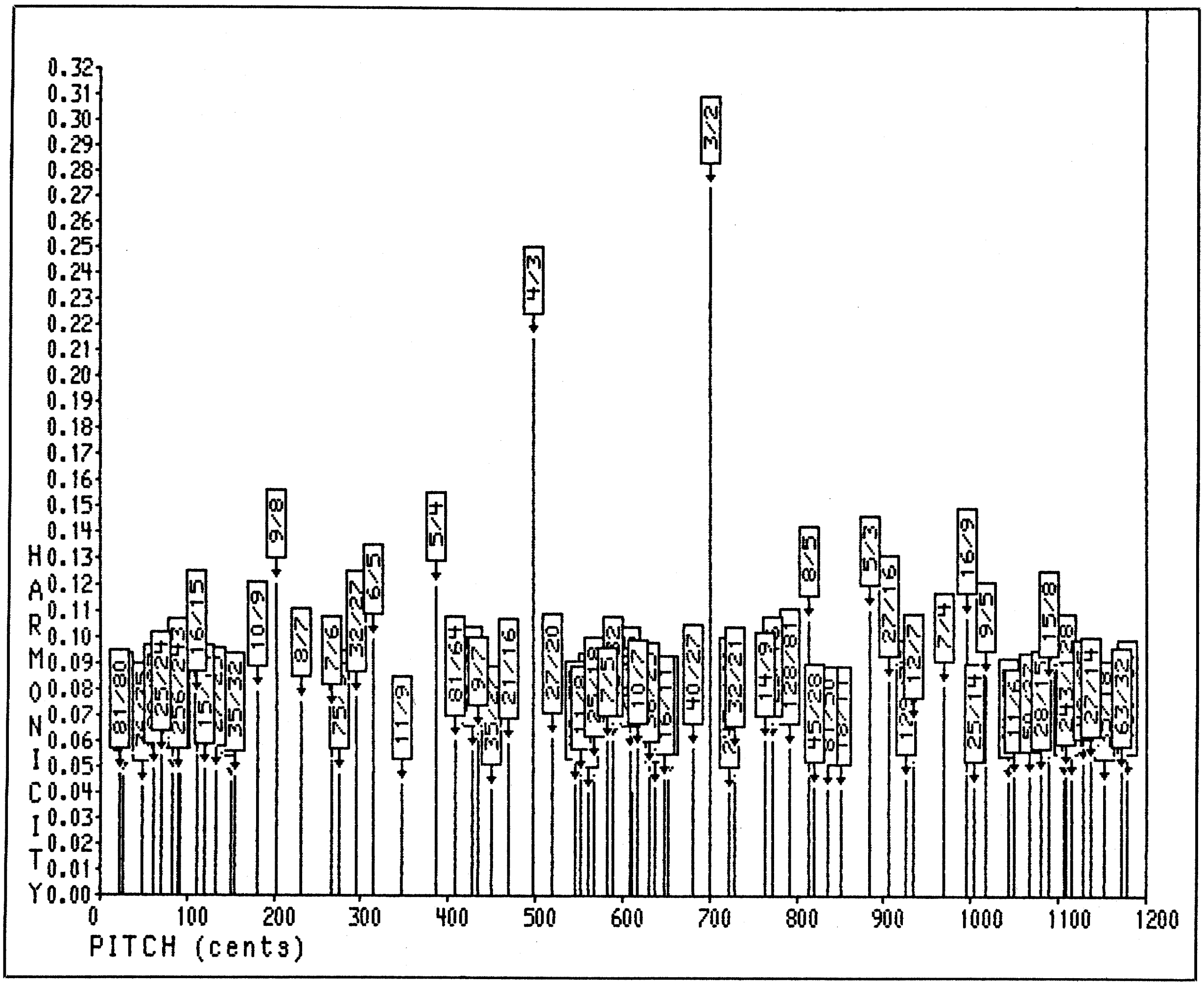


Fig. 2. The harmonicities shown in Fig. 1 are damped by an attenuation curve.

Table 4. Alternative tunings for each pitch

Pitches (Cents)	*0*	*200*	*400*	*500*	*700*	*900*	*1100*	*1200*
1. Alternative:	1:1	8:9	4:5	3:4	2:3	3:5	8:15	1:2
2. Alternative:	80:81	9:10	64:81	20:27	27:40	16:27	128:243	81:160
3. Alternative:	63:64	25:28	25:32	16:21	160:243	75:128	135:256	32:63

Fig. 3. Original (left) and damped (right) harmonicities (minimum value 0.04) for the tuning of a major scale (nominal tolerance 50 cents).



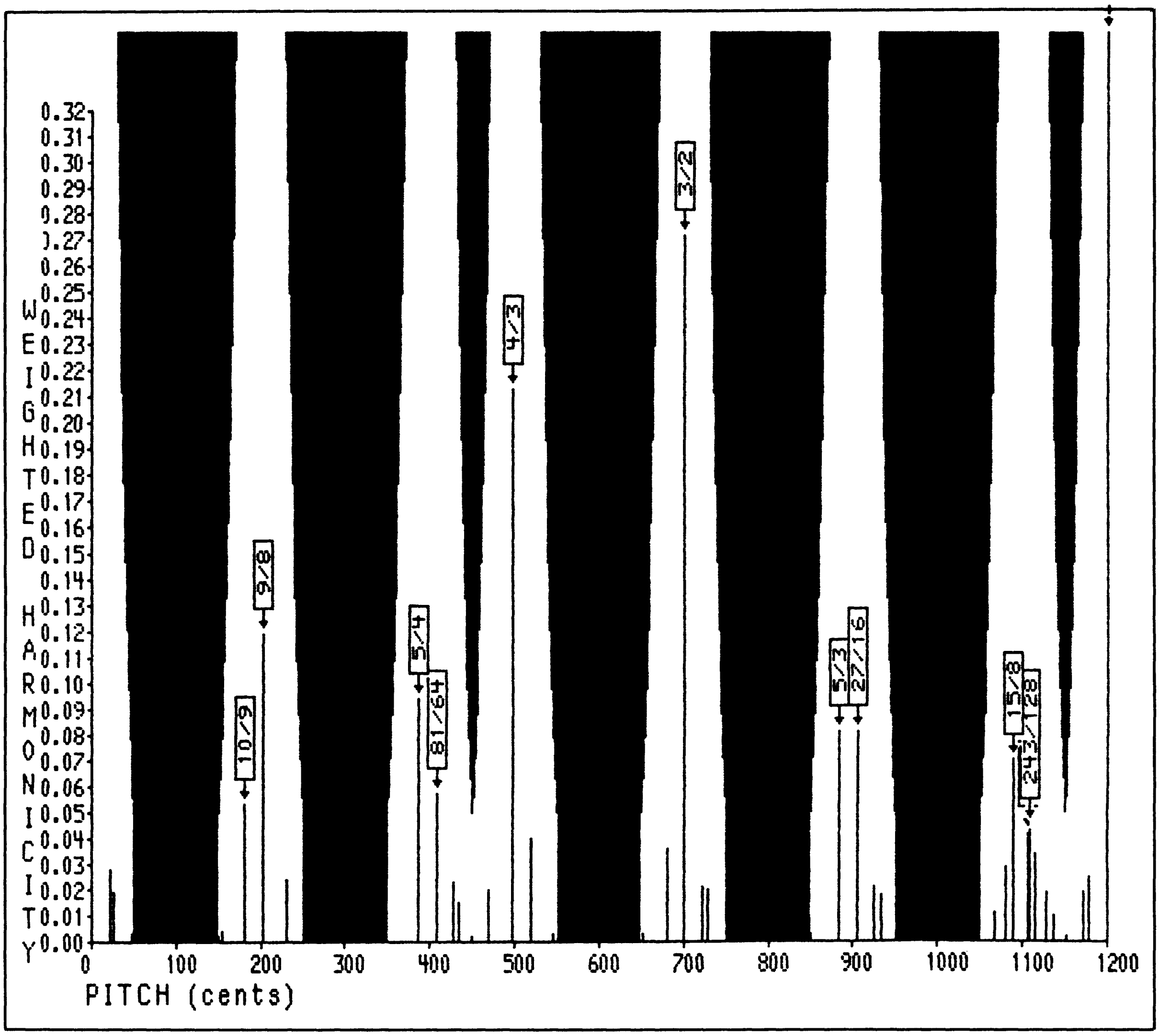

0.32
0.31
0.30
0.29
0.28
0.27
0.26
0.25
0.24
0.23
0.22
0.21
0.20
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
WEIGHTED HARMONICITY
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
PITCH (cents)
10/9
9/8
5/4
81/64
4/3
3/2
5/3
27/16
15/8
243/128

Table 5. Alternative tunings for a minor scale with a minimum harmonicity of 0.06

Pitches (Cents):	*0*	*150*	*300*	*500*	*700*	*850*	*1000*	*1200*	*Specific Harmonicity*
MH = 0.06 NT = 50ct	1:1	15:16	5:6	3:4	2:3	5:8	9:16	1:2	0.2575
MH = 0.06 NT = 20ct	1:1	——	5:6	3:4	2:3	——	9:16	1:2	——
MH = 0.06 NT = 13ct	1:1	——	27:32	3:4	2:3	——	9:16	1:2	——
MH = 0.04 NT = 50ct	1:1	25:27	5:6	20:27	2:3	50:81	5:9	1:2	0.1880
MH = 0.04 NT = 20ct	1:1	25:27	5:6	20:27	2:3	50:81	5:9	1:2	0.1880
MH = 0.04 NT = 13ct	1:1	11:12	27:32	3:4	2:3	11:18	9:16	1:2	0.1475

cates the following tuning constellation to the so-called melodic minor scale:

1:1 8:9 5:6 3:4 2:3 5:8 5:9 1:2.

Here, the specific harmonicity is 0.2556. The pentatonic scale (for instance: C-D-E-G-A-c, with the small letter meaning the next higher octave) can be fixed as follows, with the highest specific harmonicity at 0.2843:

1:1 9:10 4:5 2:3 3:5 1:2.

(The well-known Pythagorean tuning of this scale, which is made up solely of fifths, has a specific harmonicity of 0.2792 and is therefore placed just behind the one chosen here.) These three results remain the same at a minimum harmonicity of 0.06.

The tuning of the wholetone scale can be—at a harmonicity minimum of 0.04—either:

1:1 9:10 4:5 45:64 5:8 9:16 1:2

or

1:1 8:9 4:5 32:45 5:8 5:9 1:2.

Both tunings have the same specific harmonicity, namely 0.1882. The difference between the two lies in a movable link of two natural thirds. Taking 1:1 = C as a base, this link is found between the notes G-flat, B-flat, and D (last one through d) in the first case, whereas in the second case this link stretches from B-flat (through d) to D to F-sharp. A second link between A-flat, c, C, E is common to both cases. Up to a harmonicity minimum of 0.056 both tunings remain the same. From 0.06 upwards, however, no more rationalization is possible because then there are no more admissible tuning possibilities for the fourth step (tritone) of the scale within the range of 520 to 700 cents (Table 2).

All of the previously mentioned findings were also confirmed for a tuning tolerance of 20 cents.

An interesting case is a minor scale in which the second and sixth steps have been lowered by a quarter tone. Table 5 shows the results of diverse tests. (The abbreviations are: MH = minimum harmonicity and NT = nominal tolerance.)

Given a harmonicity minimum of 0.06, there are only very few intervals at our disposal. At a nominal tolerance of 32 cents or less, the lowered second and sixth steps are eliminated totally—at this tolerance level there are no more tuning alternatives for them. Given a minimum harmonicity of 0.04, things look better because interval density is now three times as high. The prime number 7 is not available—however, 11 is, although only within a very narrow tolerance in the last example.

All results shown up to now are based on the choice of three alternatives per pitch. A reexamination of this quarter-tone minor scale just described at a harmonicity minimum of 0.04—but this time with two alternatives—produces the tunings shown in Table 6.

Here the prime number 11 is allowed entry already at a 50 cent tolerance. This time its most threatening competitors, the intervals 25:27 and

Table 6. Alternative tunings for a quarter-tone minor scale with a minimum harmonicity of 0.04

Pitches (Cents):		*0*	*150*	*300*	*500*	*700*	*850*	*1000*	*1200*	*Specific Harmonicity*
MH = 0.04	NT = 50ct	1:1	11:12	5:6	3:4	2:3	11:18	9:16	1:2	0.1529
MH = 0.04	NT = 20ct	1:1	11:12	5:6	3:4	2:3	11:18	9:16	1:2	0.1529
MH = 0.04	NT = 13ct	1:1	11:12	27:32	3:4	2:3	11:18	9:16	1:2	0.1475

50:81, are omitted from this selection, a fate that they were previously able to avoid at the third starting position of tuning alternatives.

Figure 4 displays the numerical proportions as well as the harmonicities for 78 interval connections between the 13 steps of a twelve-tone tempered octave. This optimum tuning is the result of the following prerequisites: harmonicity minimum: 0.04, nominal tolerance: 30 cents, and number of alternatives: two (three instead of two would lead to 1,594,323 rationalization propositions!). The interval connections that have harmonicities larger than 0.115 are drawn as lines in the included diagram. This method of rationalization is illustrated for the last time by the 14 steps of an octave that has been divided into 13 equal intervals. Given the same harmonicity minimum (0.04), the same nominal tolerance (30 cents), and the same number of alternatives (two), a network of tunings shown in Fig. 5 results. In Fig. 5 all interval connections more harmonic than 0.07 are displayed; the steps therein have been numbered.

Summary

If one has a certain number of possible tunings available for each pitch, they can be extracted from a listing of all possible available intervals within the range of the examined pitches. These tunings can be derived according to a given harmonicity minimum and according to the selective action of a given nominal tolerance. The alternative tunings can be checked for their collective harmonicity (expressed as *specific harmonicity* of the constellation) within all constellations. The constellation with the largest specific harmonicity is the optimum rationalization of the set of pitches.

Postscript

The method described here is also available in my own real-time implementation. With this realization one can play on a keyboard tuned or tempered to one's own will, so that values in cents are conveyed to the computer, which sends them via a pitch-wheel controllable synthesizer to the loudspeakers. The pitch set to be rationalized is thereby a variable buffer; its contents vary with each played note. At the present moment, rationalizations with buffers of five notes and two alternative tunings per note are quite possible in real time: the time span between playing and hearing is approximately one tenth of a second—an acceleration of this is planned for the near future.

Unfortunately, the contents of the buffer tend to fluctuate microtonally, a curious phenomenon which can be well exemplified through the played notes C, A, G. At the time of the input of A, there is no doubt as to its tuning: it receives a frequency 5/6ths of the frequency of C. When G is entered, the computer establishes that the most harmonic constellation for C, A, and G entails a 27:32 between C and A, and a 8:9 between A and G. G is therefore intoned a major tone beneath A, and C, if it still sounds, is abruptly lowered by a syntonic comma. Even if C is already silent, G sounds astonishingly low. To avoid this effect, I had to accept tunings of second, third, or n-best specific harmonicity of the buffer contents (given the smallest possible n), insofar as they prevent such fluctuations.

Fig. 4. Optimized tuning network of the 13 degrees of a twelve-tone tempered scale.

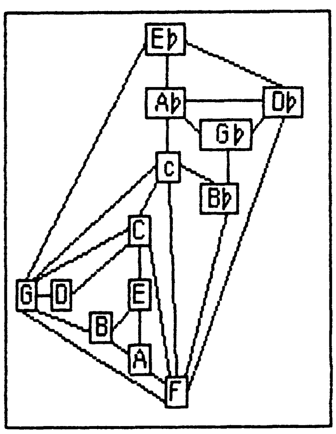
Minimum Harmonicity : 0,04 **Nominal Tolerance : 30 cents** Tuning alternatives : 2

Given size:	**0**	**100**	**200**	**300**	**400**	**500**	**600**	**700**	**800**	**900**	**1000**	**1100**	**1200** cents
Tuning:	1/1	16/15	9/8	6/5	5/4	4/3	45/32	3/2	8/5	5/3	16/9	15/8	2/1
Deviation:	0	+12	+4	+16	-14	-2	-10	+2	+14	-16	-4	-12	0 cents

Tuning-network [1000 x Harmonicity of each intervals in square brackets] :

pitch2	pitch3	pitch4	pitch5	pitch6	pitch7	pitch8	pitch9	pitch10	pitch11	pitch12	pitch13	against
16/15 [-77]	9/8 [120]	6/5 [-99]	5/4 [119]	4/3 [-214]	45/32 [60]	3/2 [273]	8/5 [-106]	5/3 [110]	16/9 [-107]	15/8 [83]	2/1 [1000]	pitch1
	135/128 [47]	9/8 [120]	75/64 [47]	5/4 [119]	675/512 [34]	45/32 [60]	3/2 [273]	25/16 [60]	5/3 [110]	225/128 [40]	15/8 [83]	pitch2
		16/15 [-77]	10/9 [79]	32/27 [-77]	5/4 [119]	4/3 [-214]	64/45 [-56]	40/27 [57]	128/81 [-57]	5/3 [110]	16/9 [-107]	pitch3
			25/24 [54]	10/9 [79]	75/64 [47]	5/4 [119]	4/3 [-214]	25/18 [52]	40/27 [57]	25/16 [60]	5/3 [110]	pitch4
				16/15 [-77]	9/8 [120]	6/5 [-99]	32/25 [-56]	4/3 [-214]	64/45 [-56]	3/2 [273]	8/5 [-106]	pitch5
					135/128 [47]	9/8 [120]	6/5 [-99]	5/4 [119]	4/3 [-214]	45/32 [60]	3/2 [273]	pitch6
						16/15 [-77]	256/225 [-38]	32/27 [-77]	512/405 [-38]	4/3 [-214]	64/45 [-56]	pitch7
							16/15 [-77]	10/9 [79]	32/27 [-77]	5/4 [119]	4/3 [-214]	pitch8
								25/24 [54]	10/9 [79]	75/64 [47]	5/4 [119]	pitch9
									16/15 [-77]	9/8 [120]	6/5 [-99]	pitch10
										135/128 [47]	9/8 [120]	pitch11
											16/15 [-77]	pitch12

Network of intervals more harmonic than 0.115



Specific harmonicity: 0.1734

Fig. 5. Optimized tuning network of the 14 degrees of a thirteen-tone tempered octave.

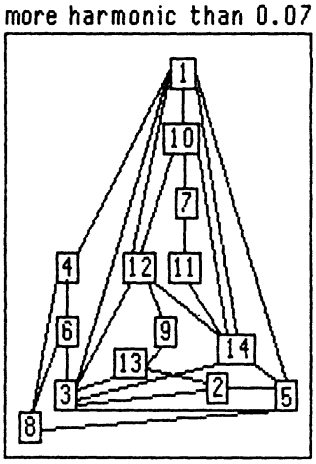
Minimum harmonicity : 0,04 Nominal tolerance : 30 cents Tuning alternatives : 2

Given size:	0	92	185	277	369	462	554	646	738	831	923	1015	1108	1200 cents
Tuning:	1/1	135/128	9/8	7/6	5/4	21/16	48/35	81/56	243/160	8/5	12/7	9/5	243/128	2/1
Deviation:	0	+2	+19	-10	+17	+9	-7	-7	-14	-17	+10	+3	+2	0 cents

Tuning network [1000 x harmonicity of each interval in square brackets] :

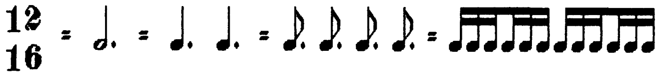
pitch2	pitch3	pitch4	pitch5	pitch6	pitch7	pitch8	pitch9	pitch10	pitch11	pitch12	pitch13	pitch14	against
135/128 [47]	9/8 [120]	7/6 [72]	5/4 [119]	21/16 [59]	48/35 [-43]	81/56 [-42]	243/160 [40]	8/5 [-106]	12/7 [-67]	9/5 [-85]	243/128 [49]	2/1 [1000]	pitch1
	16/15 [-77]	448/405 [-30]	32/27 [-77]	56/45 [40]	2048/1575 [-25]	48/35 [-43]	36/25 [-50]	1024/675 [-32]	512/315 [-32]	128/75 [-45]	9/5 [-85]	256/135 [-45]	pitch2
		28/27 [49]	10/9 [79]	7/6 [72]	128/105 [-38]	9/7 [-64]	27/20 [-61]	64/45 [-56]	32/21 [-56]	8/5 [-106]	27/16 [83]	16/9 [-107]	pitch3
			15/14 [-49]	9/8 [120]	288/245 [-27]	243/196 [-28]	729/560 [-27]	48/35 [-43]	72/49 [-35]	54/35 [-39]	729/448 [-31]	12/7 [-67]	pitch4
				21/20 [47]	192/175 [-31]	81/70 [-35]	243/200 [-34]	32/25 [-56]	48/35 [-43]	36/25 [-50]	243/160 [40]	8/5 [-106]	pitch5
					256/245 [-29]	54/49 [-34]	81/70 [-35]	128/105 [-38]	64/49 [-38]	48/35 [-43]	81/56 [-42]	32/21 [-56]	pitch6
						135/128 [47]	567/512 [33]	7/6 [72]	5/4 [119]	21/16 [59]	2835/2048 [26]	35/24 [45]	pitch7
							21/20 [47]	448/405 [-30]	32/27 [-77]	56/45 [40]	21/16 [59]	112/81 [40]	pitch8
								256/243 [-47]	640/567 [-29]	32/27 [-77]	5/4 [119]	320/243 [-39]	pitch9
									15/14 [-49]	9/8 [120]	1215/1024 [34]	5/4 [119]	pitch10
										21/20 [47]	567/512 [33]	7/6 [72]	pitch11
											135/128 [47]	10/9 [79]	pitch12
												256/243 [-47]	pitch13

Network of intervals more harmonic than 0.07



Specific harmonicity: 0.1025

Fig. 6. Stratification of a 12/16 bar.



Second Essay: Is Harmony a Special Case of Polymetricity?

If the rhythmic organization of music is not exactly totally random, it shows a more or less heavy leaning towards a meter. The meter is typically characterized by an inner hierarchy, with which this text is concerned.

Stratification of Meter

The *stratification* of a meter can be well presented through the example of a 12/16 beat pattern in Fig. 6.

One can say that a 12/16 bar shows a stratification of 2 × 2 × 3; if one investigates this type of bar more in depth, continuing to 32nds, 64ths, and so forth, the indicated geometrical series could be extended by a constant binary division: . . . × 2 × 2 × 2 × 2. . . . The place holder of the last number of the series that is not 2 determines the *order* of the meter—therefore 12/16 is a meter of third order, 6/8 (= 2 × 3) of second order, 3/4 (= 3(× 2 × 2 . . .)) of first order, and 4/4 of zero order. Independent of the order of meter, their singular, successive pulses show diverse metrical relevance (or *indispensability,* as I call it) on every level of stratification. On its first level, a 12/16 bar reveals two beats in the form of dotted quarter-notes, a rhythmically stronger one followed by a rhythmically weaker one. The succeeding levels lack a sufficiently differentiated verbal description. The evaluation of indispensability already on the second level of stratification with four dotted eighth-notes makes a numerical description necessary.

In 1978 I devised a formula that could calculate a convincing value of indispensability for each pulse on any level of any metric order. The values, according to this formula, for the six eighth-notes of a 3/4 bar, are: 5-0-3-1-4-2, and those for a 6/8 bar: 5-0-2-4-1-3. Both series of numbers show the strongest accent on the first eighth-note. The second strongest in the 6/8 bar lies on the fourth eighth-note, and in 3/4 it is on the fifth eighth-note. Here are the indispensabilities for the twelve sixteenth-notes of the meters 3/4, 6/8, and 12/16:

3/4	(3 × 2 × 2):	11	0	6	3	9	1	7	4	10	2	8	5
6/8	(2 × 3 × 2):	11	0	6	2	8	4	10	1	7	3	9	5
12/16	(2 × 2 × 3):	11	0	4	8	2	6	10	1	5	9	3	7.

The values for the first and second levels are also contained herein: to expose them, just subtract the difference between the pulse quantity on the displayed and desired levels from the indispensability value and keep only the positive numbers. For instance for 3/4 (number of pulses on the third level: 12):

1. level (pulse quantity 3—subtract 9 (= 12 − 3)): 2---0---1---
2. level (pulse quantity 6—subtract 6 (= 12 − 6)): 5-0-3-1-4-2-.

One notices on all levels that the indispensability of the first pulse is always one less than the number of pulses; that of the second is always zero.

The relevance of the indispensability values can be illustrated via a *thinning process:* if all "dispensable" attacks on the initially saturated *x*-level of a bar are taken away one by one, the impression of the type of meter always remains well maintained. Compare, for instance, the second levels of two of the previously listed bars, 3/4 and 6/8 (Fig. 7).

Indispensability values are a means toward a better understanding and more conscious handling of the most diverse meters, for example when these should be made more vivid through dynamic accentuation or stochastic distribution. (A statistical analysis of 27 Franconian dance pieces revealed striking parallels between the real frequency of occurrence and calculated indispensability of the single eighth-notes.) I based the complete rhythmic structure of my piece *Çoğluotobüsişletmesi* on this concept.

The indispensability formula, which at first was only intended for successive subdivisions of a bar by the numbers 2 and 3, has now been expanded to encompass any other prime-number divisors. Prerequisite for each of these divisors is a series of *fundamental indispensabilities* for a first-order bar having a pulse quantity equivalent to the corresponding prime number, such as for 2 the series 1-0, for 3 the series 2-0-1, for 5 the series 4-0-3-1-2, etc.

Given a bar with the stratification $q_1 \times q_2 \times q_3 \times \ldots \times q_z$, the indispensability for the nth pulse is given by Eq. 4.

$$\psi_z(n) = \sum_{r=0}^{z}\left\{\prod_{i=0}^{z-r-1} q_i \Psi_{q_{z-r}}\left(1 + \left[1+\frac{(n-2) \bmod \prod_{j=1}^{z}}{\prod_{k=0}^{r} q_{z+1-k}}\right] \bmod q_{z-r}\right)\right\} \quad (4)$$

$$\boxed{q_0 = q_{z+1} = 1}$$

where

n is the position in the bar of the pulse in question;
q_j is the stratification divisor on level j;
z is the depth of the stratification (number of levels);
$\Psi_h(x)$ is the indispensability of the xth pulse of a bar of first order with the prime stratification h; and
$a \bmod b$ is the remainder of $(a + mb)/b$; m must be large enough to make the remainder positive.

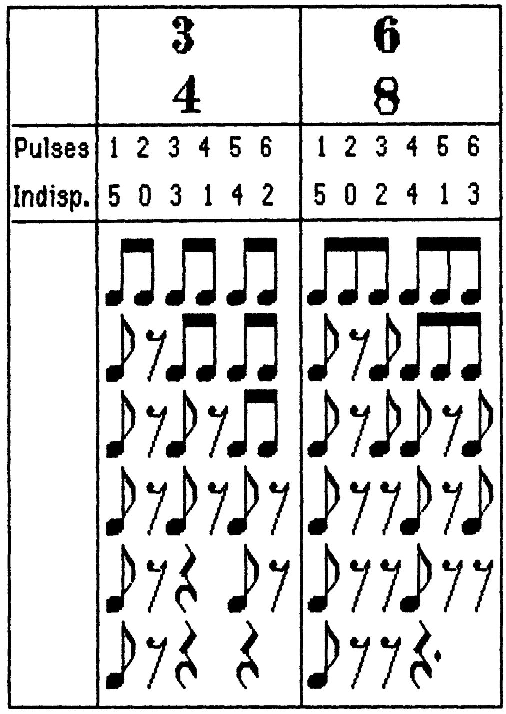
The fundamental indispensabilities for prime numbers can be set up according to one's own judgment or according to Eq. 5. The basis for their measurement here is the series of indispensabilities for a bar level with one pulse less, in which the upper divisors are the largest ones. For example, to find the fundamental indispensabilities for a cycle of 23 pulses, one must at first calculate the values of 22 pulses according to Eq. 4, whereby 22 is displayed as 11×2. The values for 11 pulses are based on those for 10, which are to be understood as 5×2. . . .

$$\Psi_h(n) = \left\{\psi_{h-1}(n') + \omega\left(\left[\frac{\psi_{h-1}(n')}{[h/4]}\right]\right)\right\} \omega(h-n-1) + [h/4]\{1-\omega(h-n-1)\} \quad (5)$$

when h is larger than 3, else:

$$\Psi_h(n) = (h + n - 2) \bmod h,$$

Fig. 7. *Thinning table for the second level of 3/4 and 6/8.*



where

$\omega(x) = 0$ if $x = 0$ else $\omega(x) = 1$ and $n' = n - 1 + \omega(h - n)$;
n is the position in the bar of the pulse in question;
$\psi_k(x)$ is the indispensability of the xth pulse on the kth level of the bar; and
$[x]$ is, as in previous formulae, the integer part of x.

Metrical Affinities

In search of a method of determining affinity between two meters of different stratification and speed, I was able to use indispensability measurement satisfactorily. For instance a $2 \times 2 \times 3$ meter of bar-tempo MM20 is to be compared to a 3×5 meter of bar-tempo MM16. Pulse-tempos are MM240 in both cases; also, five bars of the first meter equal four bars of the second (both cycles have 60 pulses). One should juxtapose—pulse for pulse—a series of indispensabilities for $2 \times 2 \times 3$ pulses five times in

Table 7. A comparison of indispensability measurements for two meters

Current Pulse:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
2 × 2 × 3 Pulses:	1	2	3	4	5	6	7	8	9	10	11	12	1	2	3	4	5	6	7	8
Indispensability:	11	0	4	8	2	6	10	1	5	9	3	7	11	0	4	8	2	6	10	1
3 × 5 Pulses:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	1	2	3	4	5
Indispensability:	14	0	9	3	6	12	1	10	4	7	13	2	11	5	8	14	0	9	3	6
Current Pulse:	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*
2 × 2 × 3 Pulses:	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12	1	2	3	4
Indispensability:	5	9	3	7	11	0	4	8	2	6	10	1	5	9	3	7	11	0	4	8
3 × 5 Pulses:	6	7	8	9	10	11	12	13	14	15	1	2	3	4	5	6	7	8	9	10
Indispensability:	12	1	10	4	7	13	2	11	5	8	14	0	9	3	6	12	1	10	4	7
Current Pulse:	*41*	*42*	*43*	*44*	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*	*53*	*54*	*55*	*56*	*57*	*58*	*59*	*60*
2 × 2 × 3 Pulses:	5	6	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12
Indispensability:	2	6	10	1	5	9	3	7	11	0	4	8	2	6	10	1	5	9	3	7
3 × 5 Pulses:	11	12	13	14	15	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Indispensability:	13	2	11	5	8	14	0	9	3	6	12	1	10	4	7	13	2	11	5	8

a row against a series of 3 × 5 repeated four times in a row, as is shown in Table 7.

If both meters don't have equally fast pulses, their stratifications have to be prolonged by further divisors until this equivalence is achieved on a deeper level. For example: a 2 × 5 meter at bar-tempo MM50 (pulse tempo MM500) and a 3 × 2 meter at MM60 (pulse tempo MM360) are expanded to 2 × 5 × 3 × 3 × 2 and 3 × 2 × 5 × 5 respectively because the pulse tempo of both meters is now MM9000. Here, too, the higher levels are reserved for the larger divisors (i.e., the extension of the 2 × 5 meter is × 3 × 3 × 2, not × 2 × 3 × 3).

The best method which I was able to find for the determination of metrical affinities was to multiply the *relative indispensability* (nominal value divided by largest value, i.e., by pulse number minus one) of each pulse of one of the meters with that of each pulse of the other, then adding all products. Concurrences of stronger pulses increase for the total sum even more if the products of the relative indispensability are squared before their addition. It becomes apparent that the mean product square (let us call it *MPS*) is larger for more related meters (such as for 2 × 3 and 3 × 2 in equal bar tempo: 0.3245) as for less-related ones (such as for 2 × 2 × 3/MM20 and 3 × 5/MM16: 0.1573). It so happens that with this procedure the MPS of the least-related meters tends to 1/9 (the MPS of all pairs of numbers not larger than N is $N^4/9 + N^3/3 + 13N^2/36 + N/6 + 1/36$), therefore 9MPS − 1 in the mentioned case tends towards zero. The most convincing and convenient scaling in half of the negative reciprocal of the natural logarithm of (9MPS − 1)/3.5 (3.5 is the largest possible 9MPS − 1 between two 2-pulse meters of equal tempo). This coefficient of metrical affinity is defined in Eq. 6.

Preliminary conventions:

M is the metric affinity of meters 1 and 2;
m is the number-label of the individual meters (= 1 or 2);
u_m is the original stratification depth (number of levels) of meter m;
z_m is the extended stratification depth of meter m;
v_m is the bar tempo of meter m;
q_{m_j} is the stratification divisor on the jth level of meter m;

Table 8. Metrical affinities of stratifications up to the third order. The relative bar tempo of each meter is indicated before the stratification.

Meter 1	*Meter 2*	*Affinity*
1 2×2×2	1 2×2×2	0.46382
1 2×2×2	1 2×2×3	0.24638
1 2×2×2	1 2×3×2	0.18956
1 2×2×2	1 3×2×2	0.16747
1 2×2×3	1 2×2×2	0.24638
1 2×2×3	1 2×2×3	0.41454
1 2×2×3	1 2×3×2	0.26958
1 2×2×3	1 3×2×2	0.20797
1 Z×3×Z	1 Z×Z×Z	0.18956
1 2×3×2	1 2×2×3	0.26958
1 2×3×2	1 2×3×2	0.41454
1 2×3×2	1 3×2×2	0.36421
1 3×2×2	1 2×2×2	0.16747
1 3×2×2	1 2×2×3	0.20797
1 3×2×2	1 2×3×2	0.36421
1 3×2×2	1 3×2×2	0.41454
1 2×2×2	2 2×2×2	0.42381
1 2×2×2	2 2×2×3	0.39233
1 2×2×2	2 2×3×2	0.25708
1 2×2×2	2 3×2×2	0.19808
1 2×2×3	2 2×2×2	0.18485
1 2×2×3	2 2×2×3	0.25708
1 2×2×3	2 2×3×2	0.39233
1 2×2×3	2 3×2×2	0.34635
1 2×3×2	2 2×2×2	0.16281
1 2×3×2	2 2×2×3	0.19808
1 2×3×2	2 2×3×2	0.34635
1 2×3×2	2 3×2×2	0.39233
1 3×2×2	2 2×2×2	0.15474
1 3×2×2	2 2×2×3	0.17378
1 3×2×2	2 2×3×2	0.32603
1 3×2×2	2 3×2×2	0.37737
1 2×2×2	3 2×2×2	0.17633
1 2×2×2	3 2×2×3	0.19094
1 2×2×2	3 2×3×2	0.13830
1 2×2×2	3 3×2×2	0.11984
1 2×2×3	3 2×2×2	0.33779
1 2×2×3	3 2×2×3	0.35407

Meter 1	*Meter 2*	*Affinity*
1 2×2×3	3 2×3×2	0.20399
1 2×2×3	3 3×2×2	0.15166
1 2×3×2	3 2×2×2	0.39582
1 2×3×2	3 2×2×3	0.37889
1 2×3×2	3 2×3×2	0.24842
1 2×3×2	3 3×2×2	0.18120
1 3×2×2	3 2×2×2	0.41227
1 3×2×2	3 2×2×3	0.38555
1 3×2×2	3 2×3×2	0.25318
1 3×2×2	3 3×2×2	0.19492
2 2×2×2	1 2×2×2	0.42381
2 2×2×2	1 2×2×3	0.18485
2 2×2×2	1 2×3×2	0.16281
2 2×2×2	1 3×2×2	0.15474
2 2×2×3	1 2×2×2	0.39233
2 2×2×3	1 2×2×3	0.25708
2 2×2×3	1 2×3×2	0.19808
2 2×2×3	1 3×2×2	0.17378
2 2×3×2	1 2×2×2	0.25708
2 2×3×2	1 2×2×3	0.39233
2 2×3×2	1 2×3×2	0.34635
2 2×3×2	1 3×2×2	0.32603
2 3×2×2	1 2×2×2	0.19808
2 3×2×2	1 2×2×3	0.34635
2 3×2×2	1 2×3×2	0.39233
2 3×2×2	1 3×2×2	0.37737
Z Z×Z×Z	3 Z×Z×Z	0.16149
2 2×2×2	3 2×2×3	0.13297
2 2×2×2	3 2×3×2	0.11317
2 2×2×2	3 3×2×2	0.10609
2 2×2×3	3 2×2×2	0.18416
2 2×2×3	3 2×2×3	0.19395
2 2×2×3	3 2×3×2	0.14147
2 2×2×3	3 3×2×2	0.12338
2 2×3×2	3 2×2×2	0.35240
2 2×3×2	3 2×2×3	0.23679
2 2×3×2	3 2×3×2	0.17158

Meter 1	*Meter 2*	*Affinity*
2 2×3×2	3 3×2×2	0.15084
2 3×2×2	3 2×2×2	0.41575
2 3×2×2	3 2×2×3	0.24132
2 3×2×2	3 2×3×2	0.18509
2 3×2×2	3 3×2×2	0.16105
3 2×2×2	1 2×2×2	0.17633
3 2×2×2	1 2×2×3	0.33779
3 2×2×2	1 2×3×2	0.39582
3 2×2×2	1 3×2×2	0.41227
3 2×2×3	1 2×2×2	0.19094
3 2×2×3	1 2×2×3	0.35407
3 2×2×3	1 2×3×2	0.37899
3 2×2×3	1 3×2×2	0.38555
3 2×3×2	1 2×2×2	0.13830
3 2×3×2	1 2×2×3	0.20399
3 2×3×2	1 2×3×2	0.24842
3 2×3×2	1 3×2×2	0.25318
3 3×2×2	1 2×2×2	0.11984
3 3×2×2	1 2×2×3	0.15166
3 3×2×2	1 2×3×2	0.18120
3 3×2×2	1 3×2×2	0.19492
3 2×2×2	2 2×2×2	0.16149
3 2×2×2	2 2×2×3	0.18416
3 2×2×2	2 2×3×2	0.35240
3 2×2×2	2 3×2×2	0.41575
3 2×2×3	2 2×2×2	0.13297
3 2×2×3	2 2×2×3	0.19395
3 2×2×3	2 2×3×2	0.23679
3 2×2×3	2 3×2×2	0.24132
3 2×3×2	2 2×2×2	0.11317
3 2×3×2	2 2×2×3	0.14147
3 2×3×2	2 2×3×2	0.17158
3 2×3×2	2 3×2×2	0.18509
3 3×2×2	2 2×2×2	0.10609
3 3×2×2	2 2×2×3	0.12338
3 3×2×2	2 2×3×2	0.15084
3 3×2×2	2 3×2×2	0.16105

Ω_j is the number of pulses in the stratification on level j;
Ω_0 is the number of pulses in the entire nonrepeating cycle of both meters; and
$\mathrm{lcm}(a,b)$ is the lowest common multiple of a and b.

These relationships now hold:

$$\Omega_{u_m} = \prod_{i=1}^{u_m} q_{m_i}$$

$$\Omega_{z_m} = \mathrm{lcm}(v_1\Omega_{u_1},\ v_2\Omega_{u_2})$$

$$\Omega_0 = \mathrm{lcm}(\Omega_{z_1},\ \Omega_{z_2})$$

$$M: -\frac{1}{2\log_e\left[\dfrac{18\sum_{n=1}^{\Omega_0}\left[\prod_{i=1}^{2}\left\{\psi_{z_i}(1+(n-1)\bmod \Omega_{z_i})\right\}\right]^2-2}{7\,\Omega_0\prod_{i=1}^{2}(\Omega_{z_i}-1)^2}\right]} \qquad (6)$$

Fig. 8. Graphic comparison between metric affinities and the harmonicity of the 32 intervals displayed in Table 9.

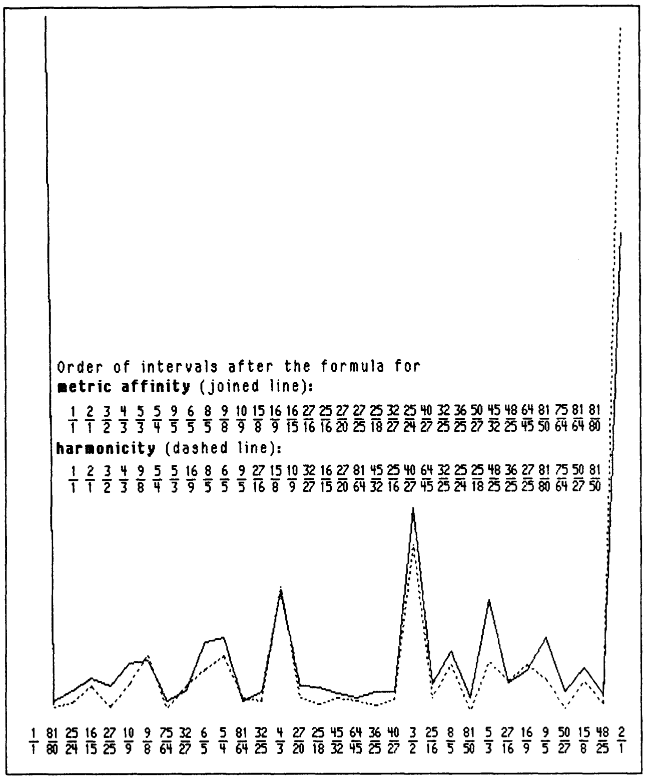


Table 9. Comparison between metrical affinity and harmonicity of 32 ratios

Ratio	Metric Affinity	Harmonicity
1. 1/1	undefined	infinity
2. 81/80	0.055380	0.047468
3. 25/24	0.070050	0.054152
4. 16/15	0.087120	0.076531
5. 27/25	0.076070	0.048077
6. 10/9	0.108890	0.078534
7. 9/8	0.113040	0.120000
8. 75/64	0.056880	0.046584
9. 32/27	0.071380	0.076923
10. 6/5	0.137830	0.099338
11. 5/4	0.143700	0.119048
12. 81/64	0.056820	0.060000
13. 32/25	0.069200	0.056180
14. 4/3	0.207970	0.214286
15. 27/20	0.077500	0.060976
16. 25/18	0.074240	0.052265
17. 45/32	0.066680	0.059761
18. 64/45	0.060820	0.056391
19. 36/25	0.068340	0.049669
20. 40/27	0.069520	9.057471
21. 3/2	0.324470	0.272727
22. 25/16	0.080010	0.059524
23. 8/5	0.124420	0.106383
24. 81/50	0.059610	0.040872
25. 5/3	0.196130	0.110294
26. 27/16	0.080470	0.083333
27. 16/9	0.098370	0.107143
28. 9/5	0.142840	0.085227
29. 50/27	0.068100	0.045872
30. 15/8	0.102090	0.082873
31. 48/25	0.065620	0.051370
32. 2/1	0.705070	1.000000

Table 8 shows the calculated metrical affinities for the four metrical stratifications up to the third order. Their pulse quantities do not exceed 12. The tempo relationships are given in terms of the combinations of whole numbers from 1 to 3.

Metrical Affinity and Harmonicity

If one were to consider audible pitch as an extremely rapid series of pulses with frequencies specified as tempo indications, intervallic harmonicity would be a kind of "micrometrical affinity!" Table 9 shows the metrical affinity of 32 different pairs of bars of equal bar tempo. For comparison, the harmonicities of the corresponding pitch intervals are indicated. The parallels are evident in the graphic display in Fig. 8.