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A STOCHASTIC MAXIMUM PRINCIPLE FOR A MARKOV REGIME-SWITCHING JUMP-DIFFUSION MODEL AND ITS APPLICATION TO FINANCE*

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Abstract. This paper develops a sufficient stochastic maximum principle for a stochastic optimal control problem, where the state process is governed by a continuous-time Markov regime-switching jump-diffusion model. We also establish the relationship between the stochastic maximum principle and the dynamic programming principle in a Markovian case. Applications of the stochastic maximum principle to the mean-variance portfolio selection problem are discussed.

Key words. stochastic maximum principle, regime switching, jump-diffusion, dynamic programming, mean-variance portfolio selection

AMS subject classifications. 93E20, 91G80, 91G10

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1. Introduction.

The stochastic maximum principle is an important result in stochastic optimal control. It is a stochastic extension of the Pontryagin maximum principle which is used in the optimal control theory of deterministic dynamic systems. Like the Pontryagin maximum principle, the basic idea of the stochastic maximum principle is to derive a set of necessary and sufficient conditions that must be satisfied by any optimal control. Informally, the stochastic maximum principle states that any optimal control must satisfy a system of forward-backward stochastic differential equations (SDEs), called the optimality system, and to maximize a functional, called the Hamiltonian. Indeed, the converse is also true, giving the sufficient maximum principle. The merit of the stochastic maximum principle is to make the stochastic optimal problem, which is infinite-dimensional, more tractable. It leads to explicit solutions for the optimal controls in some cases. The stochastic maximum principle can be applied to situations where the state processes have random coefficients and state constraints are present. Bensoussan [1], Elliott [5], and Yong and Zhou [19] provide some theories and applications of the stochastic maximum principle.

There is interest in applying the stochastic maximum principle in finance. The first use of the stochastic maximum principle in finance is probably due to Cadenillas and Karatzas [3]. Some attention has been paid to applying the stochastic maximum principle to mean-variance portfolio selection problems (see, for example, Yong and Zhou [19] and Zhou and Yin [20]), where the problem was formulated as a stochastic linear-quadratic problem.

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Some work has been done on extending the stochastic maximum principle to SDEs involving jumps (see Øksendal and Sulem [14] and the references therein). In particular, a sufficient stochastic maximum principle was developed for SDEs with jumps. However, it appears that the sufficient stochastic maximum principle for Markov regime-switching jump-diffusion processes has not yet been published. A sufficient stochastic maximum principle provides a solid theoretical background to justify some of the existing approaches for the mean-variance portfolio selection in Markov regime-switching models.

In this paper, we develop a sufficient stochastic maximum principle for a stochastic optimal control problem where the state process is governed by a continuous-time Markov regime-switching jump-diffusion model. This extends the results of Framstad, Øksendal, and Sulem [9], which discusses a jump-diffusion case. In our case, the model is general enough to nest two important classes of models, namely, jump-diffusion models and Markov regime-switching diffusion models. For the sake of generality, we do not impose any Markovian assumption when we establish the sufficient stochastic maximum principle in the general setting. However, in the case of Markovian controls or feedback controls (i.e., the control $u(t)$ at time t is chosen as a measurable function of time t, the state $X(t-)$ and the regime state $\alpha(t-)$), we establish the relationship between the stochastic maximum principle and the dynamic programming principle. In particular, the adjoint processes are represented in terms of the derivatives of the classical solution of a Hamilton–Jacobi–Bellman equation. Applications of the stochastic maximum principle to the mean-variance portfolio selection problem are discussed. We also apply the sufficient maximum principle to discuss the mean-variance portfolio selection problem. Explicit solutions are obtained in some particular cases.

The paper is organized as follows. The next section presents the model dynamics and the optimal control problem. In section 3 we prove the sufficient stochastic maximum principle. Section 4 establishes the relationship between the sufficient stochastic maximum principle and the dynamical programming approach in the Markovian case. We apply the sufficient stochastic maximum principle to the mean-variance portfolio selection problem in section 5. The final section gives concluding remarks.

2. A Markov regime-switching jump-diffusion model and the control problem. In the rest of our paper, we shall adopt the following notation:

$\mathbb{R}$:	the set of real numbers;
$\mathbb{N}$:	the set of natural numbers;
$\mathrm{Diag}(y)$:	the diagonal matrix with the elements of y on the diagonal;
M^T:	the transpose of any vector or matrix M;
$\mathrm{tr}(M)$:	the trace of a square matrix M;
$\langle x, y\rangle$:	the inner product of $x, y \in \mathbb{R}^L$, that is, $\langle x, y\rangle := x^T y$;
$C^{k,l}(0,T] \times \mathbb{R}^L), k, l \in \mathbb{N}$:	the set of the functions $f(t,x)$ whose partial derivatives of orders $\leq k$ in the first variable t and of orders $\leq l$ in the second variable x are continuous on $[0,T] \times \mathbb{R}^L$.

We consider a continuous-time, finite-state, observable Markov chain $\{\alpha(t) | t \in [0,T]\}$ on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$, where $\mathbb{F} := \{\mathcal{F}(t) | t \in [0,T]\}$ is a right-continuous, $\mathcal{P}$-completed filtration to which all of the processes defined below, including the Markov chain, the Brownian motions, and the Poisson random measures, are adapted. Following the convention of Elliott, Aggoun, and Moore [7], we identify the state space of the chain with a finite state space $S := \{e_1, e_2 \ldots, e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$, and the jth component of e_i is the Kronecker delta δ_{ij} for each $i, j = 1, 2, \ldots, D$. The state space S is called a canonical state space and its use faciliates the mathematics. We suppose that the chain is homogeneous and irreducible.

To specify statistical or probabilistic properties of the chain α, we define the generator $\Lambda := [\lambda_{ij}]_{i,j=1,2,\ldots,D}$ of the chain under $\mathcal{P}$. This is also called the rate matrix, or the Q-matrix. Here, for each $i, j = 1, 2, \ldots, D$, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t. Note that $\lambda_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$, so $\lambda_{ii} \leq 0$. In what follows for each $i, j = 1, 2, \ldots, D$ with $i \neq j$, we suppose that $\lambda_{ij} > 0$, so $\lambda_{ii} < 0$.

Elliott, Aggoun, and Moore [7] obtained the following semimartingale dynamics for the chain α:

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^T \alpha(u) du + \mathbf{M}(t),$$

where $\{\mathbf{M}(t) | t \in [0, T]\}$ is an $\mathbb{R}^D$-valued, $(\mathbb{F}, \mathcal{P})$-martingale and y^T denotes the transpose of a matrix (or, in particular, a vector).

To model the controlled state process, we first need to introduce a set of Markov jump martingales associated with the chain α. Here we follow the results of Elliott [6] and Elliott, Aggoun, and Moore [7].

For each $i, j = 1, 2, \ldots, D$, with $i \neq j$, and $t \in [0, T]$, let $J^{ij}(t)$ be the number of jumps from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ up to time t. Then

$$\begin{aligned}
J^{ij}(t) &:= \sum_{0<s\leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s), e_j \rangle \\
&= \sum_{0<s\leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s) - \alpha(s-), e_j \rangle \\
&= \int_0^t \langle \alpha(s-), e_i \rangle \langle d\alpha(s), e_j \rangle \\
&= \int_0^t \langle \alpha(s-), e_i \rangle \left\langle \Lambda^T \alpha(s), e_j \right\rangle ds + \int_0^t \langle \alpha(s-), e_i \rangle \langle d\mathbf{M}(s), e_j \rangle \\
&= \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle ds + m_{ij}(t),
\end{aligned}$$

where $m_{ij} := \{m_{ij}(t) | t \in \mathcal{T}\}$ with $m_{ij}(t) := \int_0^t \langle \alpha(s-), e_i \rangle \langle d\mathbf{M}(s), e_j \rangle$ is an $(\mathbb{F}, \mathcal{P})$-martingale. The m_{ij}'s are called the basic martingales associated with the chain α.

Now, for each fixed $j = 1, 2, \ldots, D$, let $\Phi_j(t)$ be the number of jumps into state e_j up to time t. Then

$$\begin{aligned}
\Phi_j(t) &:= \sum_{i=1, i\neq j}^{D} J^{ij}(t) \\
&= \sum_{i=1, i\neq j}^{N} \lambda_{ij} \int_0^t \langle \alpha(s), e_i \rangle ds + \widetilde{\Phi}_j(t),
\end{aligned}$$

where $\widetilde{\Phi}_j(t) := \sum_{i=1, i\neq j}^{D} m_{ij}(t)$ and, for each $j = 1, 2, \ldots, D$, $\widetilde{\Phi}_j := \{\widetilde{\Phi}_j(t) | t \in \mathcal{T}\}$ is again an $(\mathbb{F}, \mathcal{P})$-martingale.

Write for each $j = 1, 2, \ldots, D$

$$\lambda_j(t) := \sum_{i=1, i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s), e_i \rangle \, ds. \tag{2.1}$$

Then, for each $j = 1, 2, \ldots, D$,

$$\widetilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t) \tag{2.2}$$

is an $(\mathbb{F}, \mathcal{P})$-martingale.

We now introduce a Markov regime-switching Poisson random measure. Let $\mathbb{R}^+ := [0, +\infty)$ be the time index set and $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ be a measurable space, where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field generated by the open subsets of $\mathbb{R}^+$.

Let $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and $\mathcal{B}_0$ the Borel σ-field generated by open subset O of $\mathbb{R}_0$, whose closure $\bar{O}$ does not contain the point 0. In what follows, let $L, M, N, K \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Suppose that $N_i(dt, dz), i = 1, 2, \ldots, M$, are independent Poisson random measures on $(\mathbb{R}^+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}_0)$ under $\mathcal{P}$. Assume that the Poisson random measure $N_i(dt, dz)$ has the following compensator:

$$n^i_\alpha(dt, dz) := \nu^i_{\alpha(t-)}(dz|t)\eta(dt) = \left\langle \alpha(t-), \nu^i(dz|t) \right\rangle \eta(dt), \tag{2.3}$$

where $\eta(dt)$ is a σ-finite measure on $\mathbb{R}^+$ and $\nu^i(dz|t) := (\nu^i_{e_1}(dz|t), \nu^i_{e_2}(dz|t), \ldots, \nu^i_{e_D}(dz|t))^T \in \mathbb{R}^D$ is a function of time t. Here we use the subscript α in $n^i_\alpha, i = 1, 2, \ldots, M$, to indicate the dependence of the probability law of the Poisson random measure on the Markov chain $\alpha(t)$. Indeed, for each $j = 1, 2, \ldots, D$, $\nu^i_{e_j}(dz|t)$ is the conditional Lévy density of jump sizes of the random measure $N_i(dt, dz)$ at time t when $\alpha(t-) = e_j$.

In what follows, we shall consider only the case where $\nu^i(dz|t)$ is a function of z, that is,

$$\nu^i(dz|t) := \nu^i(dz) = (\nu^i_{e_1}(dz), \nu^i_{e_2}(dz), \ldots, \nu^i_{e_D}(dz))^T.$$

Furthermore we assume that $\eta(dt) := dt$ and write

$$\widetilde{N}_\alpha(dt, dz) := (N_1(dt, dz) - n^1_\alpha(dt, dz), \ldots, N_M(dt, dz) - n^M_\alpha(dt, dz))^T. \tag{2.4}$$

We now introduce the state process $X := \{X(t)|t \in [0, T]\}$. Suppose that we are given a set $U \subset \mathbb{R}^K$ and a control process $u(t) = u(t, \omega) : \Omega \times [0, T] \to U$. We also require that $\{u(t, \omega)|t \in [0, T]\}$ is $\mathbb{F}$-predictable and has right limits. The state $X(t) := X^{(u)}(t)$ of the controlled Markov regime-switching jump-diffusion in $\mathbb{R}^L$ is modeled by

$$\begin{aligned} dX(t) &= b(t, X(t-), u(t), \alpha(t-))dt + \sigma(t, X(t-), u(t), \alpha(t-))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \eta(t, X(t-), u(t-), \alpha(t-), z)\widetilde{N}_\alpha(dt, dz) \\ &\quad + \gamma(t, X(t-), u(t-), \alpha(t-))d\widetilde{\Phi}(t). \end{aligned} \tag{2.5}$$

Here $b : [0, T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^L, \sigma : [0, T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^{L\times N}, \eta : [0, T] \times \mathbb{R}^L \times U \times S \times \mathbb{R}_0 \to \mathbb{R}^{L\times M}$, and $\gamma : [0, T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^{L\times D}$ are given continuous functions, $W(t) := (W_1(t), \ldots, W_N(t))^T$ is an N-dimensional standard Brownian motion, $\widetilde{N}_\alpha(dt, dz)$ is M-dimensional Markov regime-switching random measures defined by (2.4), $\widetilde{\Phi}(t) := (\widetilde{\Phi}_1(t), \ldots, \widetilde{\Phi}_D(t))^T$ with $\widetilde{\Phi}_j(t), j = 1, \ldots, D$, defined by (2.2). In what follows, we consider the process $\{X(t)|t \in [0, T]\}$ as the solution of (2.5) associated with the control process $\{u(t)|t \in [0, T]\}$.

The state process $\{X(t)|t \in [0,T]\}$ has two kinds of jumps. It entails some financial interpretation. The Brownian motion can be interpreted as random shocks in the price of a risky financial asset which are caused by market events or news having marginal impacts on the asset price. The Poisson random measure models large jumps in the asset price which are triggered by the emergence of sudden market events or news having extraordinary impacts on the asset price. These are those jumps in the well-known Merton jump-diffusion model (see Merton [12]). The jumps in the asset price described by the basic martingales of the Markov chain are attributed to transitions in economic conditions and are similar to those of Naik [13]. These transitions have impacts in both the coefficients in the asset price process and its jumps.

Consider a performance criterion defined for each $x \in \mathbb{R}^L, e_i \in S$ as

$$J(x, e_i, u) := E_{x,e_i}\left[\int_0^T f(t, X(t), u(t), \alpha(t))dt + h(X(T), \alpha(T))\right]. \tag{2.6}$$

Here E_{x,e_i} is the conditional expectation given $X(0) = x$ and $\alpha(0) = e_i$ under $\mathcal{P}$. We say that the control process $u(t)$ is admissible if the following two conditions are satisfied:

1. the SDE (2.5) for the state process $X(t)$ has a unique strong solution;
2. $E_{x,e_i}[\int_0^T |f(t, X(t), u(t), \alpha(t))|dt + |h(X(T), \alpha(T))|] < \infty$.

Write $\mathcal{A}$ for the set of admissible controls. The stochastic control problem is to find an optimal control $u^* \in \mathcal{A}$ such that

$$J(x, e_i, u^*) = \inf_{u\in\mathcal{A}} J(x, e_i, u). \tag{2.7}$$

Let $\mathcal{R}$ be the set of functions $r(t, z) : [0,T] \times \mathbb{R}_0 \to \mathbb{R}^{L\times M}$ and define the Hamiltonian $\mathcal{H} : [0,T] \times \mathbb{R}^L \times U \times S \times \mathbb{R}^L \times \mathbb{R}^{L\times N} \times \mathcal{R} \times \mathbb{R}^{L\times D} \to R$ by

$$\begin{aligned}\mathcal{H}(t, x, u, e_i, p, q, r, s) :=\ & f(t, x, u, e_i) + b^T(t, x, u, e_i)p + \mathrm{tr}(\sigma^T(t, x, u, e_i)q)\\ &+ \int_{\mathbb{R}_0} \sum_{m=1}^{M} \sum_{n=1}^{L} \eta_{nm}(t, x, u, e_i, z)r_{nm}(t, z)\nu^m_{e_i}(dz)\\ &+ \sum_{m=1}^{D} \sum_{n=1}^{L} \gamma_{nm}(t, x, u, e_i)s_{nm}(t)\lambda_{im}.\end{aligned} \tag{2.8}$$

We assume that the Hamiltonian $\mathcal{H}$ is differentiable with respect to x.

The adjoint equations corresponding to u and $X(t) := X^{(u)}(t)$ (i.e., the controlled state process associated with the control process u) for the unknown, adapted processes $\{p(t)|t \in [0,T]\}$, $\{q(t)|t \in [0,T]\}$, $\{r(t, z)|t \in \mathcal{T}, z \in \mathbb{R}_0\}$, and $\{s(t)|t \in [0,T]\}$, where $p(t) \in \mathbb{R}^L, q(t) \in \mathbb{R}^{L\times N}, r(t, z) \in \mathbb{R}^{L\times M}$, and $s(t) \in \mathbb{R}^{L\times D}$, are given by the following backward stochastic differential equation (BSDE):

$$\begin{cases} dp(t) = -\nabla_x\mathcal{H}(t, X(t-), u(t), \alpha(t-), p(t-), q(t), r(t, \cdot), s(t))dt\\ \qquad\qquad + q(t)dW(t) + \int_{\mathbb{R}_0} r(t, z)\widetilde{N}_\alpha(dt, dz) + s(t)d\widetilde{\Phi}(t),\\ p(T) = \nabla_x h(X(T), \alpha(T)),\end{cases} \tag{2.9}$$

where ∇_x is the gradient operator with respect to the variable at the position of $X(t)$. Note that the process $\{(p(t), q(t), r(t, z), s(t))|t \in [0,T], z \in \mathbb{R}_0\}$ is the solution of the above BSDE with jumps. For the existence and uniqueness of the solutions of BSDEs driven by Brownian motions, see Pardoux and Peng [15]. For the existence and uniqueness of solutions of BSDEs driven by Poisson jumps and Markov chains, see Tang and Li [18] and Cohen and Elliott [4], respectively.

3. Sufficient stochastic maximum principle. Here we state and prove a sufficient stochastic maximum principle.

THEOREM 3.1 (sufficient stochastic maximum principle). *Write*

$$\lambda(t) := (\lambda_1(t), \ldots, \lambda_D(t))^T, \quad \nu_\alpha(dz) := (\nu^1_{\alpha(t-)}(dz), \ldots, \nu^M_{\alpha(t-)}(dz))^T,$$

where $\lambda_j(t), j = 1, 2, \ldots, D$, are defined by (2.1) *and $\nu^i_{\alpha(t-)}(dz|t)$ is the conditional Lévy density of jump sizes of the random measure $N_i(dt, dz)$ at time t.*

Let $\hat{u} \in \mathcal{A}$ with a corresponding solution $\hat{X} := X^{(\hat{u})}$ and suppose there exists an adapted solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z), \hat{s}(t))$ of the corresponding adjoint equation (2.9) *such that for all $u \in \mathcal{A}$ we have*

$$E\bigg[\int_0^T (\hat{X}(t) - X(t))^T \bigg\{ \hat{q}(t)\hat{q}(t)^T + \int_{\mathbb{R}_0} \hat{r}(t,z)\mathrm{Diag}(\nu_\alpha(dz))\hat{r}(t,z)^T + \hat{s}(t)\mathrm{Diag}(\lambda(t))\hat{s}(t)^T \bigg\} \times (\hat{X}(t) - X(t))dt\bigg] < \infty$$

and

$$\begin{aligned} E\bigg[\int_0^T \hat{p}(t)^T \bigg\{ &(\sigma\sigma^T)(t, X(t-), u(t), \alpha(t-)) \\ &+ \gamma(t, X(t-), u(t-), \alpha(t-))\mathrm{Diag}(\lambda(t))\gamma(t, X(t-), u(t-), \alpha(t-))^T \\ &+ \int_{\mathbb{R}_0} \eta(t, X(t-), u(t-), \alpha(t-), z) \\ &\quad \mathrm{Diag}(\nu_\alpha(dz))\eta(t, X(t-), u(t-), \alpha(t-), z)^T \bigg\} \hat{p}(t)dt \bigg] < \infty. \end{aligned}$$

Furthermore, assume that the following three conditions hold:

1. *For almost all $t \in [0, T]$,*
$$\begin{aligned} &\mathcal{H}(t, \hat{X}(t-), \hat{u}(t), \alpha(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)) \\ &\quad = \inf_{u \in U} \mathcal{H}(t, \hat{X}(t-), u, \alpha(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)). \end{aligned}$$
2. *For each fixed pair $(t, e_i) \in [0, T] \times S$,*
$$\hat{\mathcal{H}}(x) := \inf_{u \in U} \mathcal{H}(t, x, u, e_i, \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t))$$
exists and is a convex function of x.
3. *$h(x, e_i)$ is a convex function of x for each $e_i \in S$.*

Then $\hat{u}$ is an optimal control process and $\hat{X}$ is the corresponding controlled state process.

To prove this theorem we first need the following lemma on the integration by parts formula for the jump processes.

LEMMA 3.2. *Suppose that $Y^{(j)}(t), j = 1, 2$, are processes defined by the forward SDEs,*

$$\begin{cases} dY^{(j)}(t) = b^{(j)}(t,\omega)dt + \sigma^{(j)}(t,\omega)dW(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \eta^{(j)}(t,z,\omega)\widetilde{N}_\alpha(dt,dz) + \gamma^{(j)}(t,\omega)d\widetilde{\Phi}(t), \\ Y^{(j)}(0) = y^{(j)}, \quad j = 1, 2, \end{cases} \tag{3.1}$$

where $b^{(j)}(t) \in \mathbb{R}^L, \sigma^{(j)}(t) \in \mathbb{R}^{L\times N}, \eta^{(j)}(t) := [\eta^{(j)}_{nm}(t)] \in \mathbb{R}^{L\times M}, \gamma^{(j)}(t) := [\gamma^{(j)}_{nm}(t)] \in$

$\mathbb{R}^{L\times D}$, $t \in [0,T]$, *are predictable processes such that the integrals in* (3.1) *exist. Then*

$$\begin{aligned}
&\left\langle Y^{(1)}(T), Y^{(2)}(T)\right\rangle \\
&\quad = \left\langle y^{(1)}, y^{(2)}\right\rangle + \int_0^T \left\langle Y^{(1)}(t-), dY^{(2)}(t)\right\rangle + \int_0^T \left\langle Y^{(2)}(t-), dY^{(1)}(t)\right\rangle \\
&\qquad + \int_0^T \operatorname{tr}[(\sigma^{(1)}(t,\omega))^T \sigma^{(2)}(t,\omega)]dt \\
&\qquad + \int_0^T \int_{\mathbb{R}_0} \sum_{m=1}^{M} \sum_{n=1}^{L} \eta_{nm}^{(1)}(t,z,\omega)\eta_{nm}^{(2)}(t,z,\omega)\nu_{\alpha(t-)}^m(dz)dt \\
&\qquad + \int_0^T \sum_{m=1}^{D} \sum_{n=1}^{L} \gamma_{nm}^{(1)}(t,\omega)\gamma_{nm}^{(2)}(t,\omega)\lambda_m(t)dt.
\end{aligned}$$

Here, as usual, $\langle x, y\rangle$ *is the inner product of* $x, y \in \mathbb{R}^L$*.*

Proof. Let $\widetilde{N}_\alpha^m(dt,dz)$ and $\widetilde{\Phi}_m(t)$ be the mth element of the vectors $\widetilde{N}_\alpha(dt,dz)$ and $\widetilde{\Phi}(t)$, respectively. For each $j = 1, 2$, we define the following terms:

$$\begin{aligned}
\eta_{nm}^{(j)}(t,z) \circ \widetilde{N}_\alpha^m(dt,dz) &:= \int_0^T \int_{\mathbb{R}_0} \eta_{nm}^{(j)}(t,z,\omega)\widetilde{N}_\alpha^m(dt,dz), n = 1,\dots,L, m = 1,\dots,M, \\
\gamma_{nm}^{(j)}(t,\omega) \circ \widetilde{\Phi}_m(t) &:= \int_0^T \gamma_{nm}^{(j)}(t,\omega)d\widetilde{\Phi}_m(t), n = 1,\dots,L, m = 1,\dots,D.
\end{aligned}$$

The result then follows by applying Itô's product rule to the semi-martingales (see, for example, Elliott [5, Corollary 12.22] or Protter [16, Chapter II]) and the fact that

$$\begin{aligned}
&\left[\eta_{nm}^{(1)}(t,z) \circ \widetilde{N}_\alpha^m(dt,dz), \eta_{nm}^{(2)}(t,z) \circ \widetilde{N}_\alpha^m(dt,dz)\right] \\
&\quad = \int_0^T \int_{\mathbb{R}_0} \eta_{nm}^{(1)}(t,z,\omega)\eta_{nm}^{(2)}(t,z,\omega)\nu_{\alpha(t-)}^m(dz)dt, \\
&\left[\gamma_{nm}^{(1)}(t,\omega) \circ \widetilde{\Phi}_m(t), \gamma_{nm}^{(2)}(t,\omega) \circ \widetilde{\Phi}_m(t)\right] \\
&\quad = \int_0^T \gamma_{nm}^{(1)}(t,\omega)\gamma_{nm}^{(2)}(t,\omega)\lambda_m(t)dt,
\end{aligned}$$

where $[X,Y]$ is the optional quadratic variation process of two semi-martingales X and Y. ∎

Proof of Theorem 3.1. To simplify our notation, we suppress the subscripts x and e_i and write E for E_{x,e_i}. For any $u \in \mathcal{A}$ and the corresponding state process X^u, consider

$$\begin{aligned}
J(x,e_i,u) - J(x,e_i,\hat{u}) = E\bigg[\int_0^T \Big(f(t,X(t),u(t),\alpha(t)) - f(t,\hat{X}(t),\hat{u}(t),\alpha(t))\Big)\,dt \\
+ h(X(T),\alpha(T)) - h(\hat{X}(T),\alpha(T))\bigg].
\end{aligned}$$

By the convexity of h in x, we have

$$\begin{aligned}
E[h(X(T),\alpha(T)) - h(\hat{X}(T),\alpha(T))] &\geq E(X(T)-\hat{X}(T))^T \nabla_x h(\hat{X}(T),\alpha(T)) \\
&= E(X(T)-\hat{X}(T))^T \hat{p}(T).
\end{aligned}$$

From (2.5), we have

$$
\begin{aligned}
&d(X(T)-\hat{X}(T))\\
&\quad=[b(t,X(t-),u(t),\alpha(t-))-b(t,\hat{X}(t-),\hat{u}(t),\alpha(t-))]dt\\
&\qquad+[\sigma(t,X(t-),u(t),\alpha(t-))-\sigma(t,\hat{X}(t-),\hat{u}(t),\alpha(t-))]dW(t)\\
&\qquad+\int_{\mathbb{R}_0}[\eta(t,X(t-),u(t-),\alpha(t-),z)-\eta(t,\hat{X}(t-),\hat{u}(t-),\alpha(t-),z)]\widetilde{N}_\alpha(dt,dz)\\
&\qquad+[\gamma(t,X(t-),u(t-),\alpha(t-))-\gamma(t,\hat{X}(t-),\hat{u}(t-),\alpha(t-))]d\widetilde{\Phi}(t).
\end{aligned}
$$

Consequently, using Lemma 3.2,

$$
\begin{aligned}
&E[(X(T)-\hat{X}(T))^T\hat{p}(T)]\\
&\quad=E\Bigg[\int_0^T(X(t-)-\hat{X}(t-))^Td\hat{p}(t)+\int_0^T\hat{p}(t-)^Td(X(t)-\hat{X}(t))\\
&\qquad\quad+\int_0^T\operatorname{tr}([\sigma(t,X(t-),u(t),\alpha(t-))-\sigma(t,\hat{X}(t-),\hat{u}(t),\alpha(t-))]^T\hat{q}(t))dt\\
&\qquad\quad+\int_0^T\int_{\mathbb{R}_0}\sum_{m=1}^M\sum_{n=1}^L[\eta_{nm}(t,X(t-),u(t-),\alpha(t-),z)\\
&\qquad\qquad\qquad\qquad-\eta_{nm}(t,\hat{X}(t-),\hat{u}(t-),\alpha(t-),z)]\hat{r}_{nm}(t,z)\nu^m_{\alpha(t-)}(dz)dt\\
&\qquad\quad+\int_0^T\sum_{m=1}^D\sum_{n=1}^L[\gamma_{nm}(t,X(t-),u(t-),\alpha(t-))\\
&\qquad\qquad\qquad\qquad-\gamma_{nm}(t,\hat{X}(t-),\hat{u}(t-),\alpha(t-))]\hat{s}_{nm}(t)\lambda_m(t)dt\Bigg]\\
&\quad=E\Bigg[\int_0^T\Bigg\{(X(t-)-\hat{X}(t-))^T\Big(-\nabla_x\mathcal{H}(t,\hat{X}(t-),\hat{u}(t),\\
&\qquad\qquad\qquad\qquad\alpha(t-),\hat{p}(t-),\hat{q}(t),\hat{r}(t,\cdot),\hat{s}(t))\Big)\\
&\qquad\quad+\Big[b(t,X(t-),u(t),\alpha(t-))-b(t,\hat{X}(t-),\hat{u}(t),\alpha(t-))\Big]^T\hat{p}(t-)\\
&\qquad\quad+\operatorname{tr}\left(\Big[\sigma(t,X(t-),u(t),\alpha(t-))-\sigma(t,\hat{X}(t-),\hat{u}(t),\alpha(t-))\Big]^T\hat{q}(t)\right)\\
&\qquad\quad+\int_{\mathbb{R}_0}\sum_{m=1}^M\sum_{n=1}^L\big[\eta_{nm}(t,X(t-),u(t-),\alpha(t-),z)\\
&\qquad\qquad\qquad\qquad-\eta_{nm}(t,\hat{X}(t-),\hat{u}(t-),\alpha(t-),z)\big]\hat{r}_{nm}(t,z)\nu^m_{\alpha(t-)}(dz)\\
&\qquad\quad+\sum_{m=1}^D\sum_{n=1}^L\big[\gamma_{nm}(t,X(t-),u(t-),\alpha(t-))\\
&\qquad\qquad\qquad\qquad-\gamma_{nm}(t,\hat{X}(t-),\hat{u}(t-),\alpha(t-))\big]\hat{s}_{nm}(t)\lambda_m(t)\Bigg\}dt\Bigg].
\end{aligned}
\tag{3.2}
$$

By the definition (2.8) of $\mathcal{H}$, we have

$$
\begin{aligned}
&E\bigg[\int_0^T [f(t, X(t-), u(t), \alpha(t-)) - f(t, \hat{X}(t-), \hat{u}(t), \alpha(t-))]dt\bigg] \\
&\quad = E\Bigg[\int_0^T \Bigg\{\mathcal{H}(t, X(t-), u(t), \alpha(t-), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)) \\
&\qquad\qquad - \mathcal{H}(t, \hat{X}(t-), \hat{u}(t), \alpha(t-), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)) \\
&\qquad\qquad - \Big[b(t, X(t-), u(t), \alpha(t-)) - b(t, \hat{X}(t-), \hat{u}(t), \alpha(t-))\Big]^T \hat{p}(t-) \\
&\qquad\qquad - \mathrm{tr}\bigg(\Big[\sigma(t, X(t-), u(t), \alpha(t-)) - \sigma(t, \hat{X}(t-), \hat{u}(t), \alpha(t-))\Big]^T \hat{q}(t)\bigg) \\
&\qquad\qquad - \int_{\mathbb{R}_0} \sum_{m=1}^{M} \sum_{n=1}^{L} \big[\eta_{nm}(t, X(t-), u(t-), \alpha(t-), z) \\
&\qquad\qquad\qquad -\eta_{nm}(t, \hat{X}(t-), \hat{u}(t-), \alpha(t-), z)\big] \hat{r}_{nm}(t, z) \nu^m_{\alpha(t-)}(dz) \\
&\qquad\qquad - \sum_{m=1}^{D} \sum_{n=1}^{L} \big[\gamma_{nm}(t, X(t-), u(t-), \alpha(t-)) \\
&\qquad\qquad\qquad - \gamma_{nm}(t, \hat{X}(t-), \hat{u}(t-), \alpha(t-))\big] \hat{s}_{nm}(t)\lambda_m(t)\Bigg\}dt\Bigg].
\end{aligned}
\tag{3.3}
$$

Adding (3.2) and (3.3) gives

$$
\begin{aligned}
&J(x, e_i, u) - J(x, e_i, \hat{u}) \\
&\quad \geq E\Bigg[\int_0^T \Bigg\{\mathcal{H}(t, X(t-), u(t), \alpha(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)) \\
&\qquad\qquad - \mathcal{H}(t, \hat{X}(t-), \hat{u}(t), \alpha(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)) \\
&\qquad\qquad - (X(t-) - \hat{X}(t-))^T \nabla_x \mathcal{H}(t, \hat{X}(t-), \hat{u}(t), \alpha(t-), \hat{p}(t-), \\
&\qquad\qquad\quad \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t))\Bigg\}dt\Bigg].
\end{aligned}
$$

To show that the integrand on the right-hand side of the above equation is nonnegative $\mathcal{P}$-a.s., for almost every $t \in [0, T]$, we follow the argument in Framstad, Øksendal, and Sulem [9, pp. 83–84]. For the sake of completeness, we provide some details. To simplify the notation, write

$$
\begin{aligned}
H(t, x, u) &= \mathcal{H}(t, x, u, \alpha(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)) \\
\hat{H}(t, x) &= \inf_{u \in U} H(t, x, u).
\end{aligned}
$$

Then by condition 1,

$$H(t, \hat{X}(t-), \hat{u}(t)) = \hat{H}(t, \hat{X}(t-)),$$

and by condition 2,

$$H(t, x, u) \geq \hat{H}(t, x) \text{ for all } t, x.$$

Therefore, for any u,

$$H(t,x,u) - H(t,\hat{X}(t-),\hat{u}(t)) \geq \hat{H}(t,x) - \hat{H}(t,\hat{X}(t-)). \tag{3.4}$$

Fix $t \in [0,T]$. Since $x \to \hat{H}(t,x)$ is convex, it follows by a standard separating hyperplane argument (see, e.g., Rockafellar [17, Chapter 5, section 23]) that there exists a subgradient $a \in \mathbb{R}^L$ for $\hat{H}(t,x)$ at $x = \hat{X}(t)$, i.e.,

$$\hat{H}(t,x) - \hat{H}(t,\hat{X}(t-)) - \left\langle a, x - \hat{X}(t-)\right\rangle \geq 0 \text{ for all } x. \tag{3.5}$$

Define

$$g(x) := H(t,x,\hat{u}(t)) - H(t,\hat{X}(t-),\hat{u}(t)) - \left\langle a, x - \hat{X}(t-)\right\rangle.$$

Then by (3.4) and (3.5),

$$g(x) \geq 0 \text{ for all } x.$$

Furthermore,

$$g(\hat{X}(t-)) = 0.$$

Therefore

$$\nabla_x g(\hat{X}(t-)) = 0,$$

that is,

$$\nabla_x H(t,\hat{X}(t-),\hat{u}(t)) = a.$$

Substituting this into (3.5) and by (3.4) gives

$$\begin{aligned}
&H(t,X(t-),u(t)) - H(t,\hat{X}(t-),\hat{u}(t)) - \left\langle \nabla_x H(t,\hat{X}(t-),\hat{u}(t)), X(t-) - \hat{X}(t-)\right\rangle \\
&\quad \geq \hat{H}(t,X(t-)) - \hat{H}(t,\hat{X}(t-)) - \left\langle \nabla_x H(t,\hat{X}(t-),\hat{u}(t)), X(t-) - \hat{X}(t-)\right\rangle \\
&\quad \geq 0.
\end{aligned}$$

Therefore, we conclude that

$$J(x,e_i,u) - J(x,e_i,\hat{u}) \geq 0,$$

which proves that $\hat{u}$ is optimal. ∎

4. Relation to dynamic programming. As in the diffusion case, the adjoint processes $p(t), q(t), r(t,z), s(t)$ can also be expressed in terms of the derivatives of the value function $V(t,x,e_i)$.

To cast our optimal control problem (2.7) into a Markovian framework, we consider the Markovian (feedback) control, that is, the control $u(t)$ of the form $u(t,\alpha(t-), X(t-))$.

Write

$$J(t,x,e_i,u) := E_{t,x,e_i}\left(\int_t^T f(s,X(s),u(s),\alpha(s))ds + h(X(T),\alpha(T))\right) \quad \text{for all } u \in \mathcal{A},$$

where E_{t,x,e_i} is the conditional expectation given $X(t) = x, \alpha(t) = e_i$ under $\mathcal{P}$. Furthermore, we define

$$V(t,x,e_i) := \inf_{u \in \mathcal{A}} J(t,x,e_i,u)$$

for all $(t,x,e_i) \in [0,T] \times \mathbb{R}^L \times S$.

To show the relationship between the stochastic maximum principle and the dynamic programming principle, we need to use the following Itô formula for the Markov regime-switching jump-diffusion processes.

THEOREM 4.1. *Suppose we are given an L-dimensional process $X(t)$, $t \in [0,T]$, satisfying the SDE*

$$\begin{aligned} dX(t) &= b(t, X(t-), u(t), \alpha(t-))dt + \sigma(t, X(t-), u(t), \alpha(t-))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \eta(t, X(t-), u(t-), \alpha(t-), z)\widetilde{N}_\alpha(dt, dz) \\ &\quad + \gamma(t, X(t-), u(t-), \alpha(t-))d\widetilde{\Phi}(t) \end{aligned}$$

and the function $\phi(\cdot, \cdot, e_i) \in C^{1,2}([0,T] \times \mathbb{R}^L)$ for each $e_i \in S$. Then

$$\begin{aligned} &\phi(T, X(T), \alpha(T)) - \phi(0, X(0), \alpha(0)) \\ &\quad = \int_0^T \bigg(\frac{\partial \phi}{\partial t}(t, X(t-), \alpha(t-)) \\ &\qquad\qquad + \sum_{m=1}^L \frac{\partial \phi}{\partial x_m}(t, X(t-), \alpha(t-)) b_m(t, X(t-), u(t), \alpha(t-)) \bigg) dt \\ &\qquad + \frac{1}{2} \sum_{m=1}^L \sum_{n=1}^L \int_0^T \frac{\partial^2 \phi}{\partial x_m \partial x_n}(t, X(t-), \alpha(t-)) \sum_{l=1}^N (\sigma_{ml}\sigma_{nl})(t, X(t-), u(t), \alpha(t-))dt \\ &\qquad + \sum_{m=1}^M \int_0^T \int_{\mathbb{R}_0} \bigg(\phi(t, X(t-) + \eta^{(m)}(t, X(t-), u(t-), \alpha(t-), z), \alpha(t)) \\ &\qquad\qquad - \phi(t, X(t-), \alpha(t-)) - \sum_{n=1}^L \frac{\partial \phi}{\partial x_n}(t, X(t-), \alpha(t-)) \eta_{nm}(t, X(t-), \\ &\qquad\qquad u(t-), \alpha(t-), z) \bigg) \nu^m_{\alpha(t-)}(dz)dt \\ &\qquad + \sum_{m=1}^D \int_0^T \bigg(\phi(t, X(t-) + \gamma^{(m)}(t, X(t-), u(t-), \alpha(t-)), e_m) - \phi(t, X(t-), \alpha(t-)) \\ &\qquad\qquad - \sum_{n=1}^L \frac{\partial \phi}{\partial x_n}(t, X(t-), \alpha(t-)) \gamma_{nm}(t, X(t-), u(t-), \alpha(t-)) \bigg) \lambda_m(t)dt \\ &\qquad + \sum_{m=1}^L \frac{\partial \phi}{\partial x_m}(t, X(t-), \alpha(t-)) \sum_{n=1}^N \sigma_{mn}(t, X(t-), u(t), \alpha(t-))dW_n(t) \\ &\qquad + \sum_{m=1}^M \int_0^T \int_{\mathbb{R}_0} \bigg(\phi(t, X(t-) + \eta^{(m)}(t, X(t-), u(t-), \alpha(t-), z), \alpha(t)) \\ &\qquad\qquad - \phi(t, X(t-), \alpha(t-)) \bigg) \widetilde{N}^m_\alpha(dt, dz) \\ &\qquad + \sum_{m=1}^D \int_0^T \bigg(\phi(t, X(t-) + \gamma^{(m)}(t, X(t-), u(t-), \alpha(t-)), e_m) \\ &\qquad\qquad - \phi(t, X(t-), \alpha(t-)) \bigg) d\widetilde{\Phi}_m(t), \end{aligned}$$

where $\eta^{(m)}$ and $\delta^{(m)}$ are the mth columns of the martrices η and δ, respectively.

Proof. The result follows by the Itô formula for the semi-martingales; see Elliott [5, Theorem 12.21] or Protter [16, Chapter II]. ∎

From the standard dynamic programming principle (see, for example, Fleming and Soner [8]), the following Hamilton–Jacobi–Bellman equation and terminal boundary condition hold:

$$\frac{\partial V}{\partial t} + \inf_{u\in U}\{f(t,x,u,e_i) + \mathcal{L}^u V(t,x,e_i)\} = 0, \tag{4.1}$$

$$V(T,x,e_i) = h(x) \quad \text{for } e_i \in S, \tag{4.2}$$

where $\mathcal{L}^u$ is the infinitesimal generator of X defined by

$$\begin{aligned}
\mathcal{L}^u V(t,x,e_i) = {} & \sum_{m=1}^{L} \frac{\partial V}{\partial x_m}(t,x,e_i) b_m(t,x,u,e_i) \\
& + \frac{1}{2}\sum_{m=1}^{L}\sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_m \partial x_n}(t,x,e_i) \sum_{l=1}^{N} (\sigma_{ml}\sigma_{nl})(t,x,u,e_i) \\
& + \sum_{m=1}^{M} \int_{\mathbb{R}_0} \Bigg[V(t, x + \eta^{(m)}(t,x,u,e_i,z), e_i) - V(t,x,e_i) \\
& \qquad - \sum_{n=1}^{L} \frac{\partial V}{\partial x_n}(t,x,e_i)\eta_{nm}(t,x,u,e_i,z)\Bigg] \langle e_i, \nu^m(dz)\rangle \\
& + \sum_{m=1}^{D} \lambda_{im} \Bigg[V(t, x + \delta^{(m)}(t,x,u,e_i), e_m) - V(t,x,e_i) \\
& \qquad - \sum_{n=1}^{L} \frac{\partial V}{\partial x_n}(t,x,e_i)\delta_{nm}(t,x,u,e_i)\Bigg].
\end{aligned}$$

Now we present a theorem which establishes the relationship between the stochastic maximum principle and the dynamic programming principle.

THEOREM 4.2. *Assume that $V(\cdot,\cdot,e_i) \in C^{1,3}([0,T]\times\mathbb{R}^L)$ for each $e_i \in S$ and there exists $u^* := u^*(t,x,e_i)$ such that the infimum of (4.1) is attained at u^*. Let $X^*(t) := X^{(u^*)}$ be the corresponding state process for the control u^* and define the following processes:*

$$p_k(t) := \frac{\partial V}{\partial x_k}(t, X^*(t), \alpha(t)), \tag{4.3}$$

$$q_{km}(t) := \sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_k \partial x_n}(t, X^*(t), \alpha(t))\sigma_{nm}(t, X^*(t), u^*(t), \alpha(t)), \tag{4.4}$$

$$\begin{aligned}
r_{km}(t,z) := {} & \frac{\partial V}{\partial x_k}(t, X^*(t-) + \eta^{(m)}(t, X^*(t-), u^*(t-), \alpha(t-), z), \alpha(t)) \\
& - \frac{\partial V}{\partial x_k}(t, X^*(t-), \alpha(t-)),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
s_{km}(t) := {} & \frac{\partial V}{\partial x_k}(t, X^*(t-) + \gamma^{(m)}(t, X^*(t-), u^*(t-), \alpha(t-)), e_m) \\
& - \frac{\partial V}{\partial x_k}(t, X^*(t-), \alpha(t-)), \quad t \in [0,T].
\end{aligned} \tag{4.6}$$

Then $p(t), q(t), r(t,z), s(t),\ t \in \mathcal{T}$, *are the adjoint processes and satisfy the BSDE* (2.9).

Proof. Define

$$Y_k(t) := \frac{\partial V}{\partial x_k}(t, X^*(t), \alpha(t)) \text{ for } k = 1, \dots, L. \tag{4.7}$$

In what follows, to simplify the notation we denote the vector functions $(t, X^*(t), u^*(t, X^*(t), \alpha(t)), \alpha(t))$ and $(t, X^*(t-), u^*(t, X^*(t-), \alpha(t-)), \alpha(t-))$ by $(t, \alpha(t))$ and $(t, \alpha(t-))$, respectively. Using Itô's formula of Theorem 4.1, we can obtain the following dynamics for $Y_k(t)$:

$$\begin{aligned}
dY_k(t) = \Bigg\{ & \frac{\partial^2 V}{\partial t \partial x_k}(t, X^*(t-), \alpha(t-)) + \sum_{m=1}^{L} \frac{\partial^2 V}{\partial x_m \partial x_k}(t, X^*(t-), \alpha(t-)) b_m(t, \alpha(t-)) \\
& + \frac{1}{2} \sum_{m=1}^{L} \sum_{n=1}^{L} \frac{\partial^3 V}{\partial x_m \partial x_n \partial x_k}(t, X^*(t-), \alpha(t-)) \sum_{l=1}^{N} (\sigma_{ml}\sigma_{nl})(t, \alpha(t-)) \\
& + \sum_{m=1}^{M} \int_{\mathbb{R}_0} \Bigg[\frac{\partial V}{\partial x_k}(t, X^*(t-) + \eta^{(m)}(t, \alpha(t-), z), \alpha(t-)) \\
& \qquad - \frac{\partial V}{\partial x_k}(t, X^*(t-), \alpha(t-)) - \sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_n \partial x_k} V(t, X^*(t-), \\
& \qquad \alpha(t-)) \eta_{nm}(t, \alpha(t-), z) \Bigg] \nu^m(dz) \\
& + \sum_{m=1}^{D} \Bigg[\frac{\partial V}{\partial x_k}(t, X^*(t-) + \delta^{(m)}(t, \alpha(t-)), e_m) - \frac{\partial V}{\partial x_k}(t, X^*(t-), \alpha(t-)) \\
& \qquad - \sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_n \partial x_k} V(t, X^*(t-), \alpha(t-)) \delta_{nm}(t, \alpha(t-)) \Bigg] \lambda^m(t) \Bigg\} dt \\
& + \sum_{m=1}^{L} \frac{\partial^2 V}{\partial x_m \partial x_k}(t, X^*(t-), \alpha(t-)) \sum_{n=1}^{N} \sigma_{mn}(t, \alpha(t-)) dW_n(t) \\
& + \sum_{m=1}^{M} \int_{\mathbb{R}_0} \Bigg[\frac{\partial V}{\partial x_k}(t, X^*(t-) + \eta^{(m)}(t, \alpha(t-), z), \alpha(t-)) \\
& \qquad - \frac{\partial V}{\partial x_k}(t, X^*(t-), \alpha(t-)) \Bigg] \widetilde{N}_\alpha^m(dt, dz) \\
& + \sum_{m=1}^{D} \Bigg[\frac{\partial V}{\partial x_k}(t, X^*(t-) + \delta^{(m)}(t, \alpha(t-)), e_m) \\
& \qquad - \frac{\partial V}{\partial x_k}(t, X^*(t-), \alpha(t-)) \Bigg] d\widetilde{\Phi}_m(t).
\end{aligned} \tag{4.8}$$

To relate the above dynamics to the dynamic programming principle, define

$$F(t, x, e_i, u) := \frac{\partial V}{\partial t}(t, x, e_i) + f(t, x, u, e_i) + \mathcal{L}^u V(t, x, e_i).$$

Then differentiate $F(t, x, e_i, u^*(t, x, e_i))$ with respect to x_k and evaluate the derivative at $x = X^*(t-)$ and $e_i = \alpha(t-)$ to obtain

$$
\begin{aligned}
0 = {} & \frac{\partial f}{\partial x_k}(t, \alpha(t-)) + \frac{\partial^2 V}{\partial x_k \partial t}(t, X^*(t-), \alpha(t-)) \\
& + \sum_{m=1}^{L} \frac{\partial^2 V}{\partial x_m \partial x_k}(t, X^*(t-), \alpha(t-)) b_m(t, \alpha(t-)) \\
& + \sum_{m=1}^{L} \frac{\partial V}{\partial x_m}(t, X^*(t-), \alpha(t-)) \frac{\partial b_m}{\partial x_k}(t, \alpha(t-)) \\
& + \frac{1}{2} \sum_{m=1}^{L} \sum_{n=1}^{L} \frac{\partial^3 V}{\partial x_m \partial x_n \partial x_k}(t, X^*(t-), \alpha(t-)) \sum_{l=1}^{N} (\sigma_{nl}\sigma_{ml})(t, \alpha(t-)) \\
& + \frac{1}{2} \sum_{m=1}^{L} \sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_m \partial x_n}(t, X^*(t-), \alpha(t-)) \frac{\partial}{\partial x_k} \sum_{l=1}^{N} (\sigma_{nl}\sigma_{ml})(t, \alpha(t-)) \\
& + \sum_{m=1}^{M} \int_{\mathbb{R}_0} \Bigg[\sum_{n=1}^{L} \frac{\partial V}{\partial x_n}(t, X^*(t-) + \eta^{(m)}(t, \alpha(t-), z), \\
& \qquad \alpha(t-)) \left(\delta_{nk} + \frac{\partial \eta_{nm}}{\partial x_k}(t, \alpha(t-), z) \right) - \frac{\partial V}{\partial x_k}(t, X^*(t-), , \alpha(t-)) \\
& \qquad - \sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_n \partial x_k}(t, X^*(t-), , \alpha(t-)) \eta_{nm}(t, \alpha(t-), z) \\
& \qquad - \sum_{n=1}^{L} \frac{\partial V}{\partial x_n}(t, X^*(t-), \alpha(t-)) \frac{\partial \eta_{nm}}{\partial x_k}(t, \alpha(t-), z) \Bigg] \nu^m_{\alpha(t-)}(dz) \\
& + \sum_{m=1}^{D} \Bigg[\sum_{n=1}^{L} \frac{\partial V}{\partial x_n}(t, X^*(t-) + \gamma^{(m)}(t, \alpha(t-)), \alpha(t-)) \left(\delta_{nk} + \frac{\partial \gamma_{nm}}{\partial x_k}(t, \alpha(t-)) \right) \\
& \quad - \frac{\partial V}{\partial x_k}(t, X^*(t-), , \alpha(t-)) - \sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_n \partial x_k}(t, X^*(t-), , \alpha(t-)) \gamma_{nm}(t, \alpha(t-)) \\
& \quad - \sum_{n=1}^{L} \frac{\partial V}{\partial x_n}(t, X^*(t-), \alpha(t-)) \frac{\partial \gamma_{nm}}{\partial x_k}(t, \alpha(t-)) \Bigg] \lambda_m(t).
\end{aligned} \tag{4.9}
$$

Solving for $\frac{\partial^2 V}{\partial t \partial x_k}$ from (4.9) and substituting into (4.8), we have

$$
\begin{aligned}
dY_k(t) = -\Bigg\{ & \frac{\partial f}{\partial x_k}(t, \alpha(t-)) + \sum_{m=1}^{L} \frac{\partial V}{\partial x_m}(t, X^*(t-), \alpha(t-)) \frac{\partial b_m}{\partial x_k}(t, \alpha(t-)) \\
& + \frac{1}{2} \sum_{m=1}^{L} \sum_{n=1}^{L} \frac{\partial^2 V}{\partial x_m \partial x_n}(t, X^*(t-), \alpha(t-)) \frac{\partial}{\partial x_k} \sum_{l=1}^{N} (\sigma_{nl}\sigma_{ml})(t, \alpha(t-)) \\
& + \sum_{m=1}^{M} \int_{\mathbb{R}_0} \sum_{n=1}^{L} \Bigg[\frac{\partial V}{\partial x_n}(t, X^*(t-) + \eta^{(m)}(t, \alpha(t-), z), \alpha(t-)) \\
& \qquad - \frac{\partial V}{\partial x_n}(t, X^*(t-), \alpha(t-)) \Bigg] \frac{\partial \eta_{nm}}{\partial x_k}(t, \alpha(t-), z)) \nu^m_{\alpha(t-)}(dz)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{m=1}^{D}\sum_{n=1}^{L}\left[\frac{\partial V}{\partial x_n}(t, X^*(t-)+\gamma^{(m)}(t,\alpha(t-)),\alpha(t-))\right.\\
&\qquad\left.-\frac{\partial V}{\partial x_n}(t, X^*(t-),\alpha(t-))\right]\frac{\partial \gamma_{nm}}{\partial x_k}(t,\alpha(t-)))\lambda_m(t)\Bigg\}dt\\
&+\sum_{m=1}^{L}\frac{\partial^2 V}{\partial x_m \partial x_k}(t, X^*(t-),\alpha(t-))\sum_{n=1}^{N}\sigma_{mn}(t,\alpha(t-))dW_n(t)\\
&+\sum_{m=1}^{M}\int_{\mathbb{R}_0}\left[\frac{\partial V}{\partial x_k}(t, X^*(t-)+\eta^{(m)}(t,\alpha(t-),z),\alpha(t-))\right.\\
&\qquad\left.-\frac{\partial V}{\partial x_k}(t, X^*(t-),\alpha(t-))\right]\widetilde{N}_\alpha^m(dt,dz)\\
&+\sum_{m=1}^{D}\left[\frac{\partial V}{\partial x_k}(t, X^*(t-)+\delta^{(m)}(t,\alpha(t-)),e_m)\right.\\
&\qquad\left.-\frac{\partial V}{\partial x_k}(t, X^*(t-),\alpha(t-))\right]d\widetilde{\Phi}_m(t). \qquad (4.10)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{1}{2}\sum_{m=1}^{L}\sum_{n=1}^{L}\frac{\partial^2 V}{\partial x_m \partial x_n}(t, X^*(t-),\alpha(t-))\frac{\partial}{\partial x_k}\sum_{l=1}^{N}(\sigma_{nl}\sigma_{ml})(t,\alpha(t-))\\
&\quad=\frac{1}{2}\sum_{m=1}^{L}\sum_{n=1}^{L}\frac{\partial^2 V}{\partial x_m \partial x_n}(t, X^*(t-),\alpha(t-))\sum_{l=1}^{N}\left(\frac{\partial \sigma_{nl}}{\partial x_k}\sigma_{ml}+\sigma_{nl}\frac{\partial \sigma_{ml}}{\partial x_k}\right)(t,\alpha(t-))\\
&\quad=\sum_{l=1}^{N}\sum_{m=1}^{L}\left[\sum_{n=1}^{L}\frac{\partial^2 V}{\partial x_m \partial x_n}(t, X^*(t-),\alpha(t-))\sigma_{nl}(t,\alpha(t-))\right]\frac{\partial \sigma_{ml}}{\partial x_k}(t,\alpha(t-)).
\end{aligned}
$$

On the other hand,

$$
\operatorname{tr}\left(\frac{\partial \sigma^T}{\partial x_k}(t,\alpha(t-))q(t)\right)=\sum_{l=1}^{N}\left(\frac{\partial \sigma^T}{\partial x_k}(t,\alpha(t-))q(t)\right)_{ll}=\sum_{l=1}^{N}\sum_{m=1}^{L}q_{ml}(t)\frac{\partial \sigma_{ml}}{\partial x_k}(t,\alpha(t-)).
$$

Thus, from the definition of $\mathcal{H}(t,x,u,e_i,p,q,r,s), p(t), q(t), r(t,z),\ s(t)$, and $Y_k(t)$ in (2.8), (4.3)–(4.6), (4.7), respectively, we can rewrite (4.10) as

$$
\begin{aligned}
dp_k(t)&=\frac{\partial \mathcal{H}}{\partial x_k}(t, X(t-),u(t),\alpha(t-),p(t-),q(t),r(t,\cdot),s(t))dt\\
&\quad+\sum_{m=1}^{N}q_{km}(t)dW_m(t)+\sum_{m=1}^{M}\int_{\mathbb{R}_0}r_{km}(t,z)\widetilde{N}_\alpha^m(dt,dz)\\
&\quad+\sum_{m=1}^{D}s_{km}(t)d\widetilde{\Phi}_m(t).
\end{aligned}
$$

Furthermore, the boundary condition (4.2) of $V(t,x,e_i)$ and the definition of $p_k(t)$ in (4.3) lead to $p_k(T)=\nabla_{x_k}h(x)$.

Therefore, we have shown that $p(t), q(t), r(t,z)$, and $s(t)$ given by (4.3)–(4.6) solve the adjoint equation (2.9). $\square$

5. Application to mean-variance portfolio selection. In this section, we apply the stochastic maximum principle to solve the mean-variance portfolio selection problem in a Markov regime-switching jump-diffusion financial market.

5.1. Mean-variance portfolio selection problem. Consider a Markov regime-switching jump-diffusion financial market consisting of one risk-free asset and L risky assets. The risk-free asset's price process $S_0 = \{S_0(t) | t \in [0,T]\}$ is given by

$$dS_0(t) = r(t, \alpha(t-))S_0(t)dt \text{ for all } t \in [0,T], \quad S_0(0) = 1,$$

where $r(t, e_i) \geq 0,\ i = 1, \dots, D$, are bounded, deterministic functions on $[0,T]$ and can be seen as the interest rates in different market modes.

The other L risky assets are stocks whose price processes $S_k(t), k = 1, 2, \dots, L$, are described by the following SDEs

$$\begin{aligned} dS_k(t) = S_k(t-)\Bigg\{ & b_k(t, \alpha(t-))dt + \sum_{j=1}^{N} \sigma_{kj}(t, \alpha(t-))dW_j(t) \\ & + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \eta_{kj}(t, z, \alpha(t-))\widetilde{N}_\alpha^j(dt, dz) \Bigg\}. \end{aligned}$$

Here $W_j(t), j = 1, \dots, N$, are independent standard Brownian motion and

$$\tilde{N}_\alpha^j(dt, dz) := N_j(dt, dz) - \nu_{\alpha(t-)}^j(dz)dt, \ j = 1, 2, \dots, M,$$

are independent compensated Markov regime-switching Poisson random measures. Moreover, we shall assume that $b_k(t, \mathbf{e}_i) \geq r(t, \mathbf{e}_i)$ and that the compensated Markov-switching Poisson random measures and the Brownian motion are independent.

To simplify our description, we introduce the following notation:

$$\begin{aligned} \sigma(t, e_i) &:= (\sigma_{kj}(t, e_i))_{L \times N}, \quad \gamma(t, z, e_i) := (\gamma_{kj}(t, z, e_i))_{L \times M}, \\ D(\nu_{e_i}(dz)) &:= \mathrm{Diag}(\nu_{e_i}^1(dz), \nu_{e_i}^2(dz), \dots, \nu_{e_i}^M(dz)). \end{aligned}$$

We assume throughout this section that the following nondegeneracy condition is satisfied, that is,

$$\Theta(t, e_i) := \sigma(t, e_i)\sigma(t, e_i)^T + \int_{\mathbb{R}_0} \gamma(t, z, e_i)D(\nu_{e_i}(dz))\gamma(t, z, e_i)^T \geq \delta \mathbf{I} \tag{5.1}$$

for all $t \in [0,T]$ and $i = 1, \dots, D$. Here δ is some positive constant and I is the identity matrix. We also assume that all the functions $r(t, e_i), b_k(t, e_i), \sigma_{kj}(t, e_i), \gamma_{kj}(t, y, e_i)$ are measurable and uniformly bounded in t.

In what follows, we denote by $u_k(t), k = 0, 1, \dots, L$, the amount of the agent's wealth invested in the kth asset at time t. We call $u(\cdot) := (u_1(\cdot), \dots, u_L(\cdot))^T$ a portfolio strategy of the agent. Note that once $u(\cdot)$ is determined, the amount of the agent's wealth invested in the bank account is completely specified and equal to the current amount of the agent's wealth minus the sum of all $u_k(\cdot), k = 1, 2, \dots, L$. Denote by $X^u(t)$ the total wealth of the agent at time t corresponding to the portfolio strategy u. Suppose (1) the risky shares can be traded continuously over time, (2) there are no transaction costs and taxes in trading, and (3) the trading strategies are self-financing (i.e., there are no injections or withdrawals of funds). Then one has

$$
\begin{aligned}
dX^u(t) &= \frac{X^u(t) - \sum_{k=1}^L u_k(t)}{S_0(t)} dS_0(t) + \sum_{k=1}^L \frac{u_k(t)}{S_k(t-)} dS_k(t) \\
&= r(t, \alpha(t-))X^u(t)dt + \sum_{k=1}^L u_k(t) \Big[b_k(t, \alpha(t-)) - r(t, \alpha(t-)) \Big] dt \\
&\quad + \sum_{k=1}^L u_k(t) \sum_{j=1}^N \sigma_{kj}(t, \alpha(t-)) dW_j(t) \\
&\quad + \sum_{k=1}^L u_k(t) \sum_{j=1}^M \int_{\mathbb{R}_0} \eta_{kj}(t, z, \alpha(t-)) \widetilde{N}^j_\alpha(dt, dz). \qquad (5.2)
\end{aligned}
$$

Setting

$$B(t, e_i) := (b_1(t, e_i) - r(t, e_i), \ldots, b_L(t, e_i) - r(t, e_i)), i = 1, \ldots, D,$$

we can rewrite the wealth equation (5.2) as

$$
\begin{aligned}
dX^u(t) = &\Big[r(t, \alpha(t-))X^u(t) + u(t)^T B(t, \alpha(t-)) \Big] dt + u(t)^T \sigma(t, \alpha(t-)) dW(t) \\
&+ u(t)^T \int_{\mathbb{R}_0} \eta(t, z, \alpha(t-)) \widetilde{N}_\alpha(dt, dz).
\end{aligned}
$$

DEFINITION 5.1. *A portfolio $u(\cdot)$ is said to be admissible if $u(\cdot)$ is $\mathbb{F}$-progressively measurable and satisfies the following three conditions:*

1.

$$E\left[\int_0^T \sum_{k=1}^L u_k(t)^2 dt \right] < \infty;$$

2.

$$
\begin{aligned}
E\Big[&\int_0^T |r(t, \alpha(t-))X^u(t) + u(t)^T B(t, \alpha(t-))| dt \\
&+ \int_0^T u(t)\sigma(t, \alpha(t-))\sigma(t, \alpha(t-))^T u^T(t) dt \\
&+ \int_0^T u(t)\eta(t, z, \alpha(t-))\eta(t, z, \alpha(t-))^T u^T(t) dt \Big] < \infty;
\end{aligned}
$$

3. *the SDE for X^u has a unique strong solution.*

The set of all admissible strategies is denoted by $\mathcal{U}$.

The agent's objective is to find an admissible portfolio $u(\cdot)$ such that the expected terminal wealth satisfies $EX^u(T) = d$ for some $d \in \Re$ while the risk measured by the variance of the terminal wealth

$$\mathrm{Var} X^u(T) := E[X^u(T) - EX^u(T)]^2 = E[X^u(T) - d]^2$$

is minimized. Finding such a portfolio $u(\cdot)$ is referred to as the mean-variance portfolio selection problem. In particular, we formulate the mean-variance portfolio selection problem as follows.

DEFINITION 5.2. *The mean-variance portfolio selection is the following constrained stochastic optimization problem, parameterized by $d \in R$:*

$$\begin{cases} \textit{minimize} & J_{MV}(x_0, e_i, u(\cdot)) := E_{x_0,e_i}[X^u(T) - d]^2 \\ \textit{subject to} & \begin{cases} EX^u(T) = d, \\ u(\cdot) \in \mathcal{U}, \\ (X^u(t), u(\cdot)) \ \ \textit{satisfy} \ \ (5.2), \end{cases} \end{cases} \tag{5.3}$$

where E_{x_0,e_i} is the expectation with respect to the probability measure

$$\mathcal{P}_{x_0,e_i}(\cdot) := \mathcal{P}(\cdot | X^u(0) = x_0, \alpha(0) = e_i).$$

Note that the mean-variance problem (5.3) is a dynamic optimization problem with a constraint $E[X^u(T)] = d$. Here we apply the Lagrange multiplier technique to handle this constraint. Define

$$J_{MVL}(x_0, e_i, u(\cdot), \zeta) := E_{x_0,e_i}[X^u(T) - d]^2 + 2\zeta E_{x_0,e_i}[X^u(T) - d].$$

In this way the mean-variance problem (5.3) can be solved via the following stochastic optimal control problem (for every fixed ζ):

$$\begin{cases} \text{minimize} & J_{MVL}(x_0, e_i, u(\cdot), \zeta) = E_{x_0,e_i}[X^u(T) - (d-\zeta)]^2 - \zeta^2 \\ \text{subject to} & u(\cdot) \in \mathcal{U} \text{ and } (X^u(t), u(\cdot)) \ \ \text{satisfy} \ \ (5.2). \end{cases} \tag{5.4}$$

Clearly this problem has the same optimal strategy as the following optimization problem:

$$\begin{cases} \text{minimize} & J(x_0, e_i, u(\cdot), \vartheta) = E_{x_0,e_i}[X^u(T) - \vartheta]^2 \\ \text{subject to} & u(\cdot) \in \mathcal{U} \text{ and } (X^u(t), u(\cdot)) \ \ \text{satisfy } (5.2), \end{cases} \tag{5.5}$$

where we let $\vartheta = d - \zeta$. Thus the above optimal control problem turns out to be a quadratic loss minimization problem and we shall solve it using the stochastic maximum principle.

5.2. Solving quadratic loss minimization problem by the stochastic maximum principle. In this case, the Hamiltonian (2.8) has the following form:

$$\begin{aligned} \mathcal{H}(t, x, u, e_i, p, q, r, s) = {} & \left(r(t, e_i)x + \sum_{k=1}^{L} B_k(t, e_i)u_k \right) p + \sum_{j=1}^{N} \sum_{k=1}^{L} u_k \sigma_{kj}(t, e_i) q_j \\ & + \sum_{j=1}^{M} \sum_{k=1}^{L} \int_{\mathbb{R}_0} u_k \eta_{kj}(t, z, e_i) r_j(t, z) \nu_{e_i}^j(dz). \end{aligned}$$

Therefore the adjoint equation (2.9) is

$$\begin{cases} dp(t) = -r(t, \alpha(t-))p(t-)dt + \sum_{j=1}^{N} q_j(t) dW_j(t) \\ \qquad\qquad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} r_j(t, z) \widetilde{N}_\alpha^j(dt, dz) + \sum_{j=1}^{D} s_j(t) d\widetilde{\Phi}_j(t), \\ p(T) = 2X(T) - 2\vartheta \quad \text{a.s.} \end{cases} \tag{5.6}$$

To find a solution $(p(t), q(t), r(t, \cdot), s(t))$ to (5.6), we try a process $\{p(t) | t \in [0, T]\}$ of the following form:

$$p(t) = \phi(t, \alpha(t))X(t) + \psi(t, \alpha(t)), \tag{5.7}$$

where $\phi(\cdot, e_i), \psi(\cdot, e_i), i = 1, \dots, D$, are deterministic, differentiable functions which are to be determined. From (5.6), ϕ, ψ must satisfy the following terminal boundary condition:

$$\phi(T, e_i) = 2 \quad \text{and} \quad \psi(T, e_i) = -2\vartheta, \quad i = 1, \dots, D.$$

Applying Itô's formula (see Theorem 4.1) to the right-hand side of (5.7) leads to

$$\begin{aligned}
dp(t) = \Bigg\{ & \phi'(t, \alpha(t-))X(t-) + \psi'(t, \alpha(t-)) \\
& + \phi(t, \alpha(t-)) \left[r(t, \alpha(t-))X(t-) + \sum_{k=1}^{L} B_k(t, \alpha(t-))u_k(t) \right] \\
& + \sum_{j=1}^{D} \Big[(\phi(t, e_j) - \phi(t, \alpha(t-)))X(t-) + \psi(t, e_j) - \psi(t, \alpha(t-)) \Big] \lambda_j(t) \Bigg\} dt \\
& + \sum_{j=1}^{N} \phi(t, \alpha(t-)) \sum_{k=1}^{L} u_k(t)\sigma_{kj}(t, \alpha(t-))dW_j(t) \\
& + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \left[\phi(t, \alpha(t-)) \sum_{k=1}^{L} u_k(t)\eta_{kj}(t, z, \alpha(t-)) \right] \widetilde{N}_{\alpha}^{j}(dt, dz) \\
& + \sum_{j=1}^{D} \Big[(\phi(t, e_j) - \phi(t, \alpha(t-)))X(t-) + \psi(t, e_j) - \psi(t, \alpha(t-)) \Big] d\widetilde{\Phi}_j(t).
\end{aligned}$$

Comparing the coefficients with (5.6), we get

$$\begin{aligned}
- r(t, \alpha(t-))p(t-) = {} & \phi'(t, \alpha(t-))X(t-) + \psi'(t, \alpha(t-)) \\
& + \phi(t, \alpha(t-)) \left[r(t, \alpha(t-))X(t-) + \sum_{k=1}^{L} B_k(t, \alpha(t-))u_k(t) \right] \\
& + \sum_{j=1}^{D} \Big[(\phi(t, e_j) - \phi(t, \alpha(t-)))X(t-) \\
& \qquad + \psi(t, e_j) - \psi(t, \alpha(t-)) \Big] \lambda_j(t),
\end{aligned} \tag{5.8}$$

$$q_j(t) = \phi(t, \alpha(t-)) \sum_{k=1}^{L} u_k(t)\sigma_{kj}(t, \alpha(t-)), \tag{5.9}$$

$$r_j(t, z) = \phi(t, \alpha(t-)) \sum_{k=1}^{L} u_k(t)\eta_{kj}(t, z, \alpha(t-)), \tag{5.10}$$

$$s_j(t) = (\phi(t, e_j) - \phi(t, \alpha(t-)))X(t-) + \psi(t, e_j) - \psi(t, \alpha(t-)). \tag{5.11}$$

Let $\hat{u} \in U$ be a candidate for an optimal control and let $\hat{X}(t)$ be the corresponding wealth process. Suppose $(\hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t))$ is the corresponding solution of the adjoint equation. Then

$$
\begin{aligned}
&\mathcal{H}(t, \hat{X}(t-), u, \alpha(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t)) \\
&\quad = \left(r(t, \alpha(t-))\hat{X}(t-) + \sum_{k=1}^{L} B_k(t, \alpha(t-)) u_k \right) \hat{p}(t-) + \sum_{j=1}^{N} \sum_{k=1}^{L} u_k \sigma_{kj}(t, \alpha(t-)) \hat{q}_j(t) \\
&\qquad + \sum_{j=1}^{M} \sum_{k=1}^{L} \int_{\mathbb{R}_0} u_k \eta_{kj}(t, z, \alpha(t-)) \hat{r}_j(t, z) \nu^j_{\alpha(t-)}(dz).
\end{aligned}
$$

Since $\mathcal{H}(t, \hat{X}(t-), u, \alpha(t-), \hat{p}(t-), \hat{q}(t), \hat{r}(t, \cdot), \hat{s}(t))$ is a linear expression in $u_k, k = 1, \ldots, L$, the coefficients of u_k should vanish at optimality, i.e., for $k = 1, 2, \ldots, L$,

$$
\begin{aligned}
&B_k(t, \alpha(t-))\hat{p}(t-) + \sum_{j=1}^{N} \sigma_{kj}(t, \alpha(t-)) \hat{q}_j(t) \\
&\qquad\qquad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \eta_{kj}(t, z, \alpha(t-)) \hat{r}_j(t, z) \nu^j_{\alpha(t-)}(dz) = 0.
\end{aligned}
$$

We can write the above L equations in the following vector form:

$$
B(t, \alpha(t-))\hat{p}(t-) + \sigma(t, \alpha(t-))\hat{q}(t) + \int_{\mathbb{R}_0} \eta(t, z, \alpha(t-)) \mathrm{Diag}(\nu_\alpha(dz)) \hat{r}(t, z) = 0, \tag{5.12}
$$

where $\hat{q}(t) := (\hat{q}_1(t), \ldots, \hat{q}_N(t))^T$ and $\hat{r}(t, z) := (\hat{r}_1(t, z), \ldots, \hat{r}_M(t, z))^T$. Substituting from (5.9) and (5.10)

$$
\hat{q}(t) = \phi(t, \alpha(t-))\sigma(t, \alpha(t-))^T \hat{u}(t) \quad \text{and} \quad \hat{r}(t, z) = \phi(t, \alpha(t-))\eta(t, z, \alpha(t-))^T \hat{u}(t)
$$

into (5.12), we obtain

$$
\begin{aligned}
\hat{u}(t) &= -\Theta(t, \alpha(t-))^{-1} \phi(t, \alpha(t-))^{-1} B(t, \alpha(t-)) \hat{p}(t-) \\
&= -\Theta(t, \alpha(t-))^{-1} B(t, \alpha(t-)) \left[\hat{X}(t-) + \frac{\psi(t, \alpha(t-))}{\phi(t, \alpha(t-))} \right],
\end{aligned} \tag{5.13}
$$

where $\Theta(t, e_i)$ is defined by (5.1). To obtain the expression of the functions ϕ and ψ, we set $X(t) := \hat{X}(t), p(t) := \hat{p}(t)$ and $u(t) := \hat{u}(t)$ in (5.8) and then substitute for $\hat{p}(t)$ from (5.7) and for $\hat{u}(t)$ from (5.13). This leads to a linear equation in $\hat{X}(t-)$. Setting the coefficient of $\hat{X}(t-)$ equal to zero, we get the two equations

$$
\begin{aligned}
&\phi'(t, e_i) + \left[2r(t, e_i) - B^T(t, e_i)\Theta(t, e_i)^{-1} B(t, e_i) \right] \phi(t, e_i) \\
&\qquad + \sum_{j=1}^{D} \lambda_{ij} [\phi(t, e_j) - \phi(t, e_i)] = 0,
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
&\psi'(t, e_i) + \left[r(t, e_i) - B^T(t, e_i)\Theta(t, e_i)^{-1} B(t, e_i) \right] \psi(t, e_i) \\
&\qquad + \sum_{j=1}^{D} \lambda_{ij} [\psi(t, e_j) - \psi(t, e_i)] = 0
\end{aligned} \tag{5.15}
$$

with the terminal boundary conditions

$$
\phi(T, e_i) = 2 \quad \text{and} \quad \psi(T, e_i) = -2\vartheta, \quad i = 1, \ldots, D. \tag{5.16}
$$

By applying the standard procedure to obtain the Feynman–Kac representation of the solution to a system of differential equations (see, for example, Kloeden and Platen [10]), we obtain

$$\phi(t,e_i) = 2E\bigg[\exp\bigg\{\int_t^T \Big(\rho(s,\alpha(s)) - 2r(s,\alpha(s))\Big)ds\bigg\}|\alpha(t)=e_i\bigg], \tag{5.17}$$

$$\psi(t,e_i) = -2\vartheta E\bigg[\exp\bigg\{\int_t^T \Big(\rho(s,\alpha(s)) - r(s,\alpha(s))\Big)ds\bigg\}|\alpha(t)=e_i\bigg], \tag{5.18}$$

where we set

$$\rho(t,e_i) := B^T(t,e_i)\Theta(t,e_i)^{-1}B(t,e_i).$$

Now to solve the original mean-variance problem, we need to determine the value function $V(t,x,e_i)$ of the quadratic loss minimization problem, which is defined by

$$V(t,x,e_i) := \inf_{u\in\mathcal{U}} E[(X^u(T)-\vartheta)^2|X^u(t)=x,\alpha(t)=e_i]. \tag{5.19}$$

From the relationship between $\hat{p}(t)$ and the value function $V(t,\hat{X}(t),\alpha(t))$ (see Theorem 4.2) and the expression of $\hat{p}(t)$ in (5.7), we get

$$V(t,\hat{X}(t),\alpha(t)) = \frac{1}{2}\phi(t,\alpha(t))\hat{X}^2(t) + \psi(t,\alpha(t))\hat{X}(t) + \kappa(t,\alpha(t)),$$

where $\kappa(t,\alpha(t))$ is some appropriate function which must be determined. Note that

$$V(T,x,e_i) = (x-\vartheta)^2, \quad i=1,\dots,D.$$

Consequently, from the boundary conditions (5.16) for ϕ and ψ, it is easy to see

$$\kappa(T,e_i) = \vartheta^2, \quad i=1,\dots,D. \tag{5.20}$$

Under the optimal strategy $\hat{u}(t)$ in (5.13), the corresponding wealth process $\hat{X}(t)$ is modeled by

$$\begin{aligned} d\hat{X}(t) &= \big[r(t,\alpha(t-)) + \hat{u}(t)^T B(t,\alpha(t-))\big]dt + \hat{u}(t)^T\sigma(t,\alpha(t-))dW(t) \\ &\quad + \hat{u}(t)^T\int_{\mathbb{R}_0}\gamma(t,z,\alpha(t-))\widetilde{N}_\alpha(dt,dz). \end{aligned} \tag{5.21}$$

Applying the Itô formula to $V(t,\hat{X}(t),\alpha(t))$ (see Theorem 4.1) leads to

$$\begin{aligned} &dV(t,\hat{X}(t),\alpha(t)) \\ &\quad = \bigg\{\frac{1}{2}\phi'(t,\alpha(t-))\hat{X}^2(t-) + \psi'(t,\alpha(t-))\hat{X}(t-) + \kappa'(t,\alpha(t-)) \\ &\qquad + \big[\phi(t,\alpha(t-))X(t-) + \psi(t,\alpha(t-))\big]\big[r(t,\alpha(t-))\hat{X}(t-) + \hat{u}(t)^T B(t,\alpha(t-))\big] \\ &\qquad + \frac{1}{2}\phi(t,\alpha(t-))\hat{u}(t)^T(\sigma\sigma^T)(t,\alpha(t-))\hat{u}(t) \\ &\qquad + \frac{1}{2}\sum_{j=1}^M\int_{\mathbb{R}_0}\phi(t,\alpha(t-))\bigg[\sum_{k=1}^L \hat{u}_k(t)\gamma_{kj}(t,z,\alpha(t-))\bigg]^2\nu^j_{\alpha(t-)}(dz) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{D}\bigg(\frac{1}{2}\phi(t,e_j)\hat{X}^2(t-)+\psi(t,e_j)\hat{X}(t-)+\kappa(t,e_j)\\
&\qquad -\frac{1}{2}\phi(t,\alpha(t-))\hat{X}^2(t-)-\psi(t,\alpha(t-))\hat{X}(t-)-\kappa(t,\alpha(t-))\bigg)\lambda_j(t)\bigg\}dt\\
&+\frac{\partial V}{\partial x}(t,\hat{X}(t-),\alpha(t-))\hat{u}(t)^T\sigma(t,\alpha(t-))dW(t)\\
&+\sum_{j=1}^{M}\int_{\mathbb{R}_0}\Big(V(t,\hat{X}(t),\alpha(t-))-V(t,\hat{X}(t-),\alpha(t-))\Big)\widetilde{N}_\alpha^j(dt,dz)\\
&+\sum_{j=1}^{D}\Big(V(t,\hat{X}(t-),e_j)-V(t,\hat{X}(t-),\alpha(t-))\Big)d\widetilde{\Phi}_j(t).
\end{aligned}
$$

Substituting the $\hat{u}(t)$ of (5.13) into the above equation, noting that ϕ,ψ are solutions to the differential equations (5.14), (5.15), and using the identity

$$
\begin{aligned}
&\sum_{j=1}^{M}\int_{\mathbb{R}_0}\left[\sum_{k=1}^{L}\hat{u}_k(t)\gamma_{kj}(t,z,\alpha(t-))\right]^2\nu^j_{\alpha(t-)}(dz)\\
&\quad=\sum_{j=1}^{M}\int_{\mathbb{R}_0}\left[\hat{u}(t)^T\gamma(t,z,\alpha(t-))e_j\right]^2\nu^j_{\alpha(t-)}(dz)\\
&\quad=\sum_{j=1}^{M}\int_{\mathbb{R}_0}\hat{u}(t)^T\gamma(t,z,\alpha(t-))e_j\cdot e_j^T\gamma(t,z,\alpha(t-))^T\hat{u}(t)\nu^j_{\alpha(t-)}(dz)\\
&\quad=\int_{\mathbb{R}_0}\hat{u}(t)^T\gamma(t,z,\alpha(t-))\mathrm{Diag}(\nu_\alpha(dz))\gamma(t,z,\alpha(t-))^T\hat{u}(t),
\end{aligned}
$$

we can rewrite the Itô expansion of $V(t,\hat{X}(t),\alpha(t))$ as

$$
\begin{aligned}
&dV(t,\hat{X}(t),\alpha(t))\\
&\quad=\left\{\kappa'(t,\alpha(t-))-\frac{1}{2}\frac{\psi^2(t,\alpha(t-))}{\phi(t,\alpha(t-))}\rho(t,\alpha(t-))+\sum_{j=1}^{D}\big(\kappa(t,e_j)-\kappa(t,\alpha(t-))\big)\lambda_j(t)\right\}dt\\
&\qquad+\frac{\partial V}{\partial x}(t,\hat{X}(t-),\alpha(t-))\hat{u}(t)^T\sigma(t,\alpha(t-))dW(t)\\
&\qquad+\sum_{j=1}^{M}\int_{\mathbb{R}_0}\Big(V(t,\hat{X}(t),\alpha(t-))-V(t,\hat{X}(t-),\alpha(t-))\Big)\widetilde{N}_\alpha^j(dt,dz)\\
&\qquad+\sum_{j=1}^{D}\Big(V(t,\hat{X}(t-),e_j)-V(t,\hat{X}(t-),\alpha(t-))\Big)d\widetilde{\Phi}_j(t).
\end{aligned}
$$

Since $\hat{u}(t)$ is the optimal strategy, $V(t,\hat{X}(t),\alpha(t))$ should be a martingale. To ensure the martingale property of $V(t,\hat{X}(t),\alpha(t))$, the dt part of $V(t,\hat{X}(t),\alpha(t))$ must be equal to 0, that is,

$$
\kappa'(t,\alpha(t-))-\frac{1}{2}\frac{\psi^2(t,\alpha(t-))}{\phi(t,\alpha(t-))}\rho(t,\alpha(t-))+\sum_{j=1}^{D}\big[\kappa(t,e_j)-\kappa(t,\alpha(t-))\big]\lambda_j(t)=0.
$$

This implies

$$\kappa'(t,e_i) - \frac{1}{2}\frac{\psi^2(t,e_i)}{\phi(t,e_i)}\rho(t,e_i) + \sum_{j=1}^{D}\lambda_{ij}\left[\kappa(t,e_j)-\kappa(t,e_i)\right] = 0. \tag{5.22}$$

Combining the terminal boundary condition (5.20) and the standard procedure to the Feyman–Kac representation of a system of differential equations, we have the following expression for $\kappa(t,e_i)$:

$$\kappa(t,e_i) = \vartheta^2\left(1 - E\left[\int_t^T \rho(s,\alpha(s))\frac{\psi(s,\alpha(s))^2}{\phi(s,\alpha(s))}ds|\alpha(t)=e_i\right]\right). \tag{5.23}$$

The above analysis yields the following theorem for the quadratic loss minimization problem (5.5).

Theorem 5.3. *The optimal strategy for the quadratic loss minimization problem* (5.5) *is given by*

$$\hat{u}(t,\hat{X}(t-),\alpha(t-)) = -\Theta(t,\alpha(t-))^{-1}B(t,\alpha(t-))\left[\hat{X}(t-) + \frac{\psi(t,\alpha(t-))}{\phi(t,\alpha(t-))}\right]$$

and the corresponding optimal value function is given by

$$V(t,x,e_i) = \frac{1}{2}\phi(t,e_i)x^2 + \psi(t,e_i)x + \kappa(t,e_i),$$

where $\phi(t,e_i), \psi(t,e_i)$*, and* $\kappa(t,e_i)$ *are given by* (5.17)*,* (5.18)*, and* (5.23)*, respectively.*

5.3. The solution of the mean-variance problem. Denote by $V_{MV}(0,x_0,e_i)$ and $V_{MVL}(0,x_0,e_i)$ the optimal value functions for problem (5.3) and problem (5.4), respectively. Observing the relationship between the control problem (5.4) and the control problem (5.5) and the solution of the control problem (5.5) established in the previous subsection, we have the following result:

$$\begin{aligned} V_{MVL}(0,x_0,e_i) &= V(0,x_0,e_i) - \zeta^2 \\ &= \frac{1}{2}\phi(0,e_i)x_0^2 + \psi(0,e_i)x_0 + \kappa(0,e_i) - \zeta^2. \end{aligned}$$

Write

$$\begin{aligned} \widetilde{\phi}(t,e_i) &:= \frac{1}{2}\phi(t,e_i), \\ \widetilde{\psi}(t,e_i) &:= -\frac{\psi(t,e_i)}{2\vartheta} = -\frac{\psi(t,e_i)}{2(d-\zeta)}, \\ \widetilde{\kappa}(t,e_i) &:= \frac{\kappa(t,e_i)}{\vartheta^2} = \frac{\kappa(t,e_i)}{(d-\zeta)^2}. \end{aligned}$$

Then, we can rewrite $V_{MVL}(0,x_0,e_i)$ as

$$V_{MVL}(0,x_0,e_i) = \widetilde{\phi}(t,e_i)x_0^2 - 2(d-\zeta)\widetilde{\psi}(t,e_i)x_0 + (d-\zeta)^2\widetilde{\kappa}(t,e_i) - \zeta^2.$$

Note that $J_{MV}(x_0,e_i,u(\cdot))$ is strictly convex in $u(\cdot)$ and the constraint function $EX^u(T) - d$ is affine in $u(\cdot)$. Therefore, we can apply the well-known Lagrange duality theorem (see Luenberger [11, Theorem 1, p. 224]) to obtain that

$$V_{MV}(0,x_0,e_i) = \sup_{\zeta\in R} V_{MVL}(0,x_0,e_i).$$

Observing that $V_{MVL}(0, x_0, e_i)$ is a quadratic function in ζ and the quadratic coefficient is equal to

$$\widetilde{\kappa}(0, e_i) - 1 = -E\left[\int_t^T \rho(s, \alpha(s))\frac{\psi(s, \alpha(s))^2}{\phi(s, \alpha(s))} ds|\alpha(t) = e_i\right] < 0,$$

so $V_{MVL}(0, x_0, e_i)$ attains its maximum at the point

$$\zeta^* = d + \frac{d - \widetilde{\psi}(0, e_i)x_0}{\widetilde{\kappa}(0, e_i) - 1}.$$

Substituting ζ^* into $V_{MVL}(0, x_0, e_i)$, we obtain the maximum value as follows:

$$\begin{aligned}\sup_{\zeta \in R} V_{MVL}(0, x_0, e_i) = {} & \frac{\widetilde{\kappa}(0, e_i)}{1 - \widetilde{\kappa}(0, e_i)}\left[d - \frac{\widetilde{\psi}(0, e_i)}{\widetilde{\kappa}(0, e_i)}x_0\right]^2 \\ & + \frac{\widetilde{\phi}(0, e_i)\widetilde{\kappa}(0, e_i) - \widetilde{\psi}(0, e_i)^2}{\widetilde{\kappa}(0, e_i)}x_0^2.\end{aligned}$$

That is,

$$V_{MV}(0, x_0, e_i) = \frac{\widetilde{\kappa}(0, e_i)}{1 - \widetilde{\kappa}(0, e_i)}\left[d - \frac{\widetilde{\psi}(0, e_i)}{\widetilde{\kappa}(0, e_i)}x_0\right]^2 + \frac{\widetilde{\phi}(0, e_i)\widetilde{\kappa}(0, e_i) - \widetilde{\psi}(0, e_i)^2}{\widetilde{\kappa}(0, e_i)}x_0^2.$$

The above analysis yields the following theorem.

THEOREM 5.4. *The efficient portfolio of the mean-variance problem* (5.3) *corresponding to the expected terminal value d, as a function of time t, the wealth level x, and the Markov chain state e_i, is*

$$u^*(t, x, e_i) = -\left[x - (d - \zeta^*)\frac{\widetilde{\psi}(t, e_i)}{\widetilde{\phi}(t, e_i)}\right]\Theta(t, e_i)^{-1}B(t, e_i), \tag{5.24}$$

where

$$\zeta^* = d + \frac{d - \widetilde{\psi}(0, e_i)x_0}{\widetilde{\kappa}(0, e_i) - 1}.$$

Furthermore, the efficient frontier (or optimal value function) for the mean-variance problem (5.3) *is*

$$\begin{aligned}VarX^*(T) &= V_{MV}(0, x_0, e_i) \\ &= \frac{\widetilde{\kappa}(0, e_i)}{1 - \widetilde{\kappa}(0, e_i)}\left[d - \frac{\widetilde{\psi}(0, e_i)}{\widetilde{\kappa}(0, e_i)}x_0\right]^2 + \frac{\widetilde{\phi}(0, e_i)\widetilde{\kappa}(0, e_i) - \widetilde{\psi}(0, e_i)^2}{\widetilde{\kappa}(0, e_i)}x_0^2,\end{aligned} \tag{5.25}$$

where $\widetilde{\phi}(t, e_i), \widetilde{\psi}(t, e_i)$, and $\widetilde{\kappa}(t, e_i)$ are given by

$$\begin{aligned}\widetilde{\phi}(t, e_i) &= E\left[\exp\left\{\int_t^T \big(\rho(s, \alpha(s)) - 2r(s, \alpha(s))\big)ds\right\}|\alpha(t) = e_i\right], \\ \widetilde{\psi}(t, e_i) &= E\left[\exp\left\{\int_t^T \big(\rho(s, \alpha(s)) - r(s, \alpha(s))\big)ds\right\}|\alpha(t) = e_i\right], \\ \widetilde{\kappa}(t, e_i) &= 1 - E\left[\int_t^T \rho(s, \alpha(s))\frac{\psi(s, \alpha(s))^2}{\phi(s, \alpha(s))}ds|\alpha(t) = e_i\right].\end{aligned}$$

5.4. The solution of the mean-variance problem without the Markov regime-switching Poisson random jumps. The mean-variance problem in the financial market without Markov regime-switching Poisson random jumps has been considered by Zhou and Yin [20]. In this subsection, we shall show the connection between our results and those of Zhou and Yin [20].

Let $(p_{ij}(t))_{D\times D}$ be the transition probability matrix of the Markov chain $\{\alpha(t)\}$. Then

$$(p_{ij}(t))_{D\times D} = \exp\{\Lambda t\}.$$

The following lemma gives the relationships among $\widetilde{\phi}(t,e_i), \widetilde{\psi}(t,e_i)$, and $\widetilde{\kappa}(t,e_i)$.

LEMMA 5.5. *For each $i = 1, 2, \dots, D$, let*

$$\theta(t,e_i) := \sum_{k=1}^{D}\sum_{j=1}^{D}\int_t^T p_{ik}(s-t)\lambda_{kj}\widetilde{\phi}(t,e_j)\left[\frac{\widetilde{\psi}(s,e_j)}{\widetilde{\phi}(t,e_j)} - \frac{\widetilde{\psi}(s,e_k)}{\widetilde{\phi}(t,e_k)}\right]^2 ds. \tag{5.26}$$

Then

$$\widetilde{\kappa}(t,e_i) = \frac{\widetilde{\psi}(t,e_i)^2}{\widetilde{\phi}(t,e_i)} + \theta(t,e_i),\ \ t\in[0,T].$$

Proof. Define

$$\begin{aligned} H(t,e_i) &:= \frac{\widetilde{\psi}(t,e_i)}{\widetilde{\phi}(t,e_i)}, \quad i = 1,\dots,D, \\ G(t,e_i) &:= \widetilde{\kappa}(t,e_i) - \frac{\widetilde{\psi}(t,e_i)^2}{\widetilde{\phi}(t,e_i)} = \widetilde{\kappa}(t,e_i) - H(t,e_i)^2\widetilde{\phi}(t,e_i), i = 1,\dots,D. \end{aligned} \tag{5.27}$$

Thus we can show that $H(t,e_i)$ and $G(t,e_i)$ satisfy the following differential equation systems:

$$\begin{cases} H'(t,e_i) = r(t,e_i)H(t,e_i) - \frac{1}{\widetilde{\phi}(t,e_i)}\sum_{j=1}^D \lambda_{ij}\widetilde{\phi}(t,e_j)[H(t,e_j) - H(t,e_i)], \\ H(T,e_i) = 1,\ i = 1,2,\dots,D, \end{cases} \tag{5.28}$$

$$\begin{cases} G'(t,e_i) = -\sum_{j=1}^D \lambda_{ij}\widetilde{\phi}(t,e_j)[H(t,e_j) - H(t,e_i)]^2 - \sum_{j=1}^D \lambda_{ij}G(t,e_j), \\ G(T,e_i) = 0,\ i = 1,2,\dots,D. \end{cases} \tag{5.29}$$

Let

$$f_i(t) := \sum_{j=1}^D \lambda_{ij}\widetilde{\phi}(t,e_j)[H(t,e_j) - H(t,e_i)]^2, i = 1,\dots,D,$$

and setting

$$\boldsymbol{f}(t) := (f_1(t),\dots,f_D(t))'.$$

Then, from Bronson [2, Chapter 8], we can give an explicit solution of (5.29) as follows:

$$G(t,e_i) = e_i'\int_t^T \exp\{\Lambda(s-t)\}\boldsymbol{f}(s)ds, i = 1,\dots,D.$$

Therefore we can rewrite the solution of (5.29) as

$$G(t,e_i)=\sum_{k=1}^{D}\sum_{j=1}^{D}\int_t^T p_{ik}(s-t)\lambda_{kj}\widetilde{\phi}(t,e_j)\left[\frac{\widetilde{\psi}(s,e_j)}{\widetilde{\phi}(t,e_j)}-\frac{\widetilde{\psi}(s,e_k)}{\widetilde{\phi}(t,e_k)}\right]^2 ds=\theta(t,e_i).$$

From the definitions of $G(t,e_i)$ and $\theta(t,e_i)$, we obtain the result

$$\begin{aligned}\widetilde{\kappa}(t,e_i)&=\frac{\widetilde{\psi}(t,e_i)^2}{\widetilde{\phi}(t,e_i)}+\theta(t,e_i)\\&=\widetilde{\phi}(t,e_i)H^2(t,e_i)+\theta(t,e_i),\ t\in[0,T],\ i=1,\dots,D.\qquad\square\end{aligned}$$

Note that if we set $\gamma(t,z,e_i)=0$, the Markov regime-switching jump-diffusion market reduces to a financial market without any Markov regime-switching Poisson random jumps. Thus, if we set $\gamma(t,x,e_i)=0$ in Theorem 5.4, we obtain the corresponding solution of the mean-variance problem.

THEOREM 5.6. *The efficient portfolio of the mean-variance problem* (5.3) *corresponding to the case of* $\gamma(t,z,e_i)=0$ *is given by*

$$u^*(t,x,e_i)=-\big[x-(d-\zeta^*)H(t,e_i)\big]\big[\sigma(t,e_i)\sigma^T(t,e_i)\big]^{-1}B(t,e_i),$$

where $H(t,e_i)$ *is defined by* (5.27) *and*

$$\zeta^*=d+\frac{d-\widetilde{\phi}(0,e_i)H(0,e_i)x_0}{\widetilde{\phi}(0,e_i)H^2(0,e_i)+\theta(0,e_i)-1}.$$

Furthermore, the efficient frontier (or optimal value function) for the mean-variance problem (5.3) *is*

$$\begin{aligned}VarX^*(T)&=V_{MV}(0,x_0,e_i)\\&=\frac{\widetilde{\phi}(0,e_i)H^2(0,e_i)+\theta(0,e_i)}{1-\widetilde{\phi}(0,e_i)H^2(0,e_i)-\theta(0,e_i)}\left[d-\frac{\widetilde{\phi}(0,e_i)H(0,e_i)}{\widetilde{\phi}(0,e_i)H^2(0,e_i)+\theta(0,e_i)}x_0\right]^2\\&\quad+\frac{\widetilde{\phi}(0,e_i)\theta(0,e_i)}{\widetilde{\phi}(0,e_i)H^2(0,e_i)+\theta(0,e_i)}x_0^2,\end{aligned}$$

where $\theta(t,e_i)$ *is defined by* (5.26) *and* $\widetilde{\phi}(t,e_i)$ *is given by*

$$\widetilde{\phi}(t,e_i)=E\left[\exp\left\{\int_t^T\big(\rho(s,\alpha(s))-2r(s,\alpha(s))\big)ds\right\}|\alpha(t)=e_i\right].$$

Considering the above result, we see it agrees with Theorem 5.1 of Zhou and Yin [20].

6. Conclusion. We have proved a sufficient stochastic maximum principle in a general Markov regime-switching jump-diffusion setting and explored its relationship with the dynamic programming principle in a Markovian situation. Applications to mean-variance portfolio selection were presented and explicit solutions to the problem obtained in some particular cases.

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