1. Consider the random variable X characterized by the following distribution function:

$$F(x)=\begin{cases} 0 & , \quad x<0 \\ \dfrac{x}{2} & , \quad 0 \le x < 1 \\ 1-\dfrac{1}{2x^2} & , \quad x \ge 1 \end{cases}$$

a) Show that *F* is actually a distribution function and represents it graphically.
b) Is the random variable continuous, discrete or mixed? Justify.
c) If it is continuous compute the mass probability function of *X*.
d) Compute $P(X=-1)$, $P(X=0)$, $P(X=1)$, $P(1/2 \le X < 1)$, $P(1/2 \le X \le 1)$, $P(X>1)$, $P(X \ge 1)$ and $P(X \ge 2)$.

2. Consider the random variable X, discrete, with the following probability function:

| x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 0.1 | 0.3 | 0.4 | 0.2 |

a) Show that *f* is actually a probability function and represents it graphically.
b) Compute $P(X=1)$, $P(0 \le X < 1)$, $P(1 \le X \le 2)$, $P(1 \le X < 3)$, $P(X>1)$, $P(X \ge 1)$ and $P(X \ge 2)$.
c) Compute $E(X)$, $E(2X+3)$, $V(X)$ e $V(2X+3)$.

3. Suppose the duration in hours of a certain type of lamps has the following probability density function:

$$f(x)=\begin{cases} \dfrac{200}{x^3} & , \quad x>10 \\ 0 & , \quad \text{other values} \end{cases}$$

a) What is the probability of such a lamp not to malfunction during the first 150 hours of use?
b) What is the probability of damaging in these first 150 hours?
c) What is the average duration of such a lamp?

4. Consider the following bivariate distribution $p(x, y)$ of two discrete random variables $X$ and $Y$.

| $Y$ | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|
| | $y_1$ | 0.01 | 0.02 | 0.03 | 0.1 | 0.1 |
| | $y_2$ | 0.05 | 0.1 | 0.05 | 0.07 | 0.2 |
| | $y_3$ | 0.1 | 0.05 | 0.03 | 0.05 | 0.04 |

$X$

Compute:

a. The marginal distributions $p(x)$ and $p(y)$.
b. The conditional distributions $p(x|Y = y_1)$ and $p(y|X = x_3)$.

5. The continuous random variable $X$ has the following probability density function

$$f(x) = \begin{cases} a + bx & 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

where $a$ and $b$ are constants.

(a) Show that $10a + 25b = 2$

Given that $E(X) = \frac{35}{12}$

(b) find a second equation in $a$ and $b$,

(c) hence find the value of $a$ and the value of $b$.

(d) Find, to 3 significant figures, the median of $X$.

(e) Comment on the skewness. Give a reason for your answer.

6. The length of time, in minutes, that a customer queues in a Post Office is a random variable, $T$, with probability density function

$$f(t) = \begin{cases} c(81 - t^2) & 0 \leqslant t \leqslant 9 \\ 0 & \text{otherwise} \end{cases}$$

where $c$ is a constant.

(a) Show that the value of $c$ is $\frac{1}{486}$

(b) Show that the cumulative distribution function $F(t)$ is given by

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{6} - \frac{t^3}{1458} & 0 \leqslant t \leqslant 9 \\ 1 & t > 9 \end{cases}$$

(c) Find the probability that a customer will queue for longer than 3 minutes.

A customer has been queueing for 3 minutes.

(d) Find the probability that this customer will be queueing for at least 7 minutes.

Three customers are selected at random.

(e) Find the probability that exactly 2 of them had to queue for longer than 3 minutes.

7. Consider the following density probability function:

$$f(x) = \begin{cases} a \cdot x & , \quad 0 \le x \le 1 \\ a & , \quad 1 \le x \le 2 \\ -a \cdot x + 3 \cdot a & , \quad 2 \le x \le 3 \\ 0 & , \quad \text{other values} \end{cases}$$

a) Find the value of $a$.

b) Write the distribution function $F$.

8. Consider (X, Y) a continuous random variable and the function

$$f(x,y)=\begin{cases} k\cdot x\cdot e^{-y} & , \quad 0<x<1\,,\ y>0 \\ 0 & , \ \text{other values} \end{cases}$$

a) Find the value of *k*, so that f is a density probability function.

b) Find the marginal density functions of *x* and of *y*.

c) Find the conditional density function of *x* given *y* and of *y* given *x*.

1. Compute the derivative $f'(x)$ for

$$f(x) = \log(x^4)\sin(x^3)\,.$$

2. Compute the derivative $f'(x)$ of the logistic sigmoid

$$f(x) = \frac{1}{1+\exp(-x)}\,.$$

3. Compute the derivative $f'(x)$ of the function

$$f(x) = \exp(-\tfrac{1}{2\sigma^2}(x-\mu)^2)\,,$$

where $\mu,\ \sigma \in \mathbb{R}$ are constants.

4. Compute the Taylor polynomials $T_n$, $n = 0, \ldots, 5$ of $f(x) = \sin(x) + \cos(x)$ at $x_0 = 0$.

5. Graph the surface determined by $z = f(x,y) = 10 - 3x^2 - 7y^2$ .

6. Obtain a contour map of the function $z = f(x,y) = 10 - 3x^2 - 7y^2$ .

7. Graph the surface $z(x,y)$ defined implicitly by the equation $xy + z\cosh(z-1) = 1$ .

8. Obtain a contour map for the function $z(x,y)$ defined implicitly by the equation $xy + z\cosh(z-1) = 1$ .

9. Obtain plane sections $x = c$ for the surface defined by $z = f(x,y) = 10 - 3x^2 - 7y^2$ .

10. For each $f$ and $(a,b)$ , obtain $f_x$ and $f_y$ both at $(x,y)$ and at $(a,b)$ .

$f = x\sin(y) + y\sin(x)$ ; $(a,b) = (\pi/3, \pi/6)$

$f = xy^2 - 3y - 2$ ; $(a,b) = (3,2)$

$f = \sin(xy)\cos(x/y)$ ; $(a,b) = (1,-1)$

$f = e^{x^2/y}\ln(y^2/x)$ ; $(a,b) = (2,-2)$

11. For each $f$ and $(a,b)$ , obtain all second partial derivatives, both at $(x,y)$ and at $(a,b)$

$f = \dfrac{xy}{x^2+y^2}$ ; $(a,b) = (2,3)$

$f = \dfrac{x-y}{x+y}$ ; $(a,b) = (-3,2)$

$f = \sin(xy)$ ; $(a,b) = (\pi/6, \pi/3)$

$f = \ln(x/y)$ ; $(a, b) = (2, -3)$

12. The composition of $f(x, y) = 3 - x^2 - y^2$ with $x(t) = t, y(t) = t^2$ forms the function $F(t) = f(x(t), y(t))$ .
Obtain $F'(t)$ by an appropriate form of the chain rule, and again by writing the rule for $F$ explicitly.

13. The composition of $f(x, y) = \sin(2x - 3y)$ , with $x(t) = t + 1/t$ , $y(t) = t - 1/t$ forms the function $F(t) = f(x(t), y(t))$ . Obtain $F'(t)$ by an appropriate form of the chain rule, and again by writing the rule for $F$ explicitly. Show that the results agree.

14. The composition of $f(x, y) = \ln(3x^2 + 4y^2)$ with $x(r, s) = 3r + 2s, y(r, s) = 5r - 7s$ forms the function $F(r, s) = f(x(r, s), y(r, s))$ . Obtain the partial derivatives $F_r$ and $F_s$ by appropriate forms of the chain rule, and again by writing the rule for $F$ explicitly. Show that the results agree.