1. a. $SSR = \Sigma(\hat{y}_i - \bar{y})^2$ $\quad SSE = \Sigma e_i^2$ $\quad SST = \Sigma(y_i - \bar{y})^2$

$$F = \frac{R^2}{1-R^2}(n-2) = \frac{\frac{SSR}{SST}}{1-\frac{SSR}{SST}}(n-2) = \frac{\frac{SSR}{SST}}{\frac{SSE}{SST}}(n-2) = \frac{SSR}{SSE}(n-2)$$

$$= \frac{SSR}{SSE/(n-2)} = \frac{SSR/1}{SSE/(n-2)} \sim F_{1,n-2}$$

b. $$E(F) = E\left(\frac{\hat{\beta}_1^2 \Sigma(x_i-\bar{x})^2}{S_e^2}\right)$$

$$= E\left(\frac{\frac{\hat{\beta}_1^2 \Sigma(x_i-\bar{x})^2}{\sigma^2}}{\frac{(n-2)S_e^2}{\sigma^2(n-2)}}\right)$$

$$= E\left(\frac{\hat{\beta}_1^2 \Sigma(x_i-\bar{x})^2}{\sigma^2}\right) \cdot E\left[\frac{(n-2)S_e^2}{\sigma^2}\right]^{-1} \cdot (n-2)$$

$$= E\left[\Gamma\left(\tfrac{1}{2},2\right)\right] \cdot E\,\Gamma\left(\tfrac{n-2}{2},2\right)^{-1} \cdot (n-2)$$

$$= \frac{\Gamma\left(\frac{1}{2}+1\right)\cdot 2^1}{\Gamma\left(\frac{1}{2}\right)} \cdot \frac{\Gamma\left(\frac{n}{2}-1-1\right)\cdot 2^{-1}}{\Gamma\left(\frac{n}{2}-1\right)} \cdot (n-2)$$

$$= \frac{2\cdot\frac{1}{2}\sqrt{\pi}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}-2\right)\cdot 2^{-1}}{\Gamma\left(\frac{n}{2}-2+1\right)} \cdot (n-2)$$

$$= \frac{n-2}{2} \cdot \frac{\Gamma\left(\frac{n}{2}-2\right)}{\left(\frac{n}{2}-2\right)\Gamma\left(\frac{n}{2}-2\right)} = \frac{1}{2}\cdot\frac{n-2}{\left(\frac{n}{2}-2\right)}$$

$$= \frac{n-2}{n-4}$$

2. a. $F = \frac{W}{1-W}(n-2)$

$W = \frac{F}{n-2+F}$

$$P(W \le w) = P\left(\frac{1}{\frac{n-2}{F}+1} \le w\right) = P\left(\frac{1}{w} \le \frac{n-2}{F}+1\right) = P\left(F \le \frac{n-2}{\frac{1}{w}-1}\right) = P\left(F \le \frac{w(n-2)}{1-w}\right)$$

$$F_W(w) = F_F\left(\frac{w(n-2)}{1-w}\right)$$

$$f_W(w) = F_W'(w) = f_F\left(\frac{w(n-2)}{1-w}\right)\left[\frac{(n-2)(1-w)+w(n-2)}{(1-w)^2}\right] = \frac{n-2}{(1-w)^2} \cdot f_F\left(\frac{w(n-2)}{1-w}\right)$$

$$f(x) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}\left(1+\frac{n_1}{n_2}x\right)^{-\frac{1}{2}(n_1+n_2)}$$

let $n_1 = 1$, $n_2 = n-2$, $x = \frac{w(n-2)}{1-w}$

$$\frac{n-2}{(1-w)^2} f_F\left(\frac{w(n-2)}{1-w}\right) = \frac{\Gamma\left(\frac{1+n-2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}\left(\frac{1}{n-2}\right)^{\frac{1}{2}}\left(\frac{w(n-2)}{1-w}\right)^{\frac{1}{2}-1}\left(1+\frac{1}{n-2}\frac{w(n-2)}{1-w}\right)^{-\frac{1+n-2}{2}} \cdot \frac{n-2}{(1-w)^2}$$

$$= \frac{\Gamma\left(\frac{1+n-2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}(1-w)^{-\frac{3}{2}} w^{-\frac{1}{2}}\left(1+\frac{w}{1-w}\right)^{-\frac{n-1}{2}}$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)} \cdot (1-w)^{\frac{n-4}{2}} \cdot w^{-\frac{1}{2}} \quad \sim B\left(\frac{1}{2}, \frac{n-2}{2}\right)$$

$$\therefore R^2 \sim \mathrm{Beta}\left(\frac{1}{2}, \frac{n-2}{2}\right)$$

b. $$E(R^2) = \frac{\alpha}{\alpha+\beta} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{n-2}{2}} = \frac{1}{n-1}$$

$$\mathrm{Var}(R^2) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\frac{1}{2}\cdot\frac{n-2}{2}}{\left(\frac{1+n-2}{2}\right)^2\left(\frac{1+n-2}{2}+1\right)} = \frac{2n-4}{(n-1)^2(n+1)}$$

3.a. Using R. we find p-value < 2e-16.

b.

$$\frac{\dfrac{\hat{\beta}_1 - 0}{\dfrac{\sigma}{\sqrt{\Sigma(x_i-\bar{x})^2}}}}{\sqrt{\dfrac{\frac{(n-2)S_e^2}{\sigma^2}}{n-2}}} = \frac{\hat{\beta}_1\sqrt{\Sigma(x_i-\bar{y})^2}}{\sqrt{S_e^2}} = \frac{\hat{\beta}_1\sqrt{\Sigma(x_i-\bar{x})^2}}{S_e}$$

$$E\left(\frac{\hat{\beta}_1\sqrt{\Sigma(x_i-\bar{x})^2}}{S_e}\right) = \beta_1\sqrt{\Sigma(x_i-\bar{x})^2} \cdot E\left(S_e^2\right)^{-\frac{1}{2}}$$

$$= \beta_1\sqrt{\Sigma(x_i-\bar{y})^2} \cdot \frac{\Gamma\left(\frac{n-2}{2} - \frac{1}{2}\right)\cdot\left(\frac{2\sigma^2}{n-2}\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{n-2}{2}\right)}$$

c.

Using R. the power of test is 0.9209