a. $y = D\delta + \varepsilon$

$D = (X \ \ Z) \qquad \delta = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$

$$\begin{aligned}
Var(\hat{\delta}_G) &= \sigma^2 (D'D)^{-1} \\
&= \sigma^2 \left[\begin{pmatrix} X' \\ Z' \end{pmatrix} (X \ Z)\right]^{-1} \\
&= \sigma^2 \begin{pmatrix} X'X & X'Z \\ Z'X & Z'Z \end{pmatrix}^{-1}
\end{aligned}$$

$$Var(\hat{\delta}_G) = \begin{pmatrix} Var(\hat{\beta}_G) & cov(\hat{\beta}_G, \hat{\gamma}_G) \\ cov(\hat{\beta}_G, \hat{\gamma}_G) & Var(\hat{\gamma}_G) \end{pmatrix}$$

Partial regression

1. Y on X $\rightarrow e_1^*$
2. Z on X $\rightarrow e_2^*$
3. e_1^* on e_2^* $\rightarrow \hat{\gamma}_G$

$e_1^* = [(I-H)Y] = Y^*$

$e_2^* = (I-H)Z = Z^*$

$$\begin{aligned}
\hat{\gamma}_G &= (Z^{*\prime}Z^*)^{-1} Z^{*\prime} y^* \\
&= (Z'(I-H)'(I-H)Z)^{-1} Z'(I-H)'(I-H)Y \\
&= (Z'(I-H)Z)^{-1} Z'(I-H)Y
\end{aligned}$$

$$\begin{aligned}
Var(\hat{\gamma}_G) &= Var((Z'(I-H)Z)^{-1} Z'(I-H)Y) \\
&= (Z'(I-H)Z)^{-1} Z'(I-H) Var(Y) (I-H)'Z (Z'(I-H)Z)^{-1} \\
&= \sigma^2 (Z'(I-H)Z)^{-1}
\end{aligned}$$

$\hat{\beta}_G$ $\quad Y = X\beta + \varepsilon \rightarrow X'X\hat{\beta} = X'Y$ $\qquad \begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} X'Y \\ Z'Y \end{bmatrix}$

$\quad Y = D\delta + \varepsilon \rightarrow D'D\hat{\beta}_G = D'Y$

$$\hat{\beta}_G = (X'X)^{-1}X'Y - (X'X)^{-1}X'Z\hat{\gamma}_G = (X'X)^{-1}X'Y - (X'X)^{-1}X'Z(Z'(I-H)Z)^{-1}Z'(I-H)Y$$

$L = (X'X)^{-1}X'Z \quad R = (I-H) \quad M = (Z'RZ)^{-1} \qquad = \hat{\beta} - LMZ'RY$

$$\begin{aligned}
Var(\hat{\beta}_G) &= \sigma^2 (X'X)^{-1} + LMZ'R\,Var(Y)\,R'ZM'L' \quad \boxed{(M'=M)} \quad -2\,cov(\hat{\beta}, LMZ'RY) \\
&= \sigma^2 (X'X)^{-1} + \sigma^2 LML' - 2\,cov(\hat{\beta}, LMZ'RY) \\
&= \sigma^2 (X'X)^{-1} + \sigma^2 LML' - 2(X'X)^{-1}X'\,cov(Y,Y)RZML' \\
&= \sigma^2 [(X'X)^{-1} + LML' - 2(X'X)^{-1}\underbrace{X'R}_{0}ZML'] \\
&= \sigma^2 [(X'X)^{-1} + LML']
\end{aligned}$$

PS($cov(AX, BY) = A\,cov(X,Y)B'$)

PS($RX = 0$)

$$cov(\hat{\gamma}_G, \hat{\beta}_G) = cov(\hat{\gamma}_G, \hat{\beta} - L\hat{\gamma}_G) = cov(\hat{\gamma}_G, \hat{\beta}) - cov(\hat{\gamma}_G, L\hat{\gamma}_G) = cov(MZ'RY, (X'X)^{-1}X'Y) - Var(\hat{\gamma}_G)L'$$

$$= MZ'R\,Var(Y)\,X(X'X)^{-1} - \sigma^2 ML' \quad = -\sigma^2 ML'$$

$$\operatorname{cov}(\hat{\beta}_G, \hat{r}_G) = \operatorname{cov}(\hat{\beta} - L\hat{r}_G, \hat{r}_G) = \operatorname{cov}(\hat{\beta}, \hat{r}_G) - \operatorname{cov}(L\hat{r}_G, \hat{r}_G)$$

$$= 0 - L\operatorname{var}(\hat{r}_G)$$

$$= -\sigma^2 L M$$

$$\therefore \operatorname{Var}(\hat{\delta}_G) = \sigma^2 \begin{pmatrix} (X'X)^{-1} + LML' & -LM \\ -ML' & M \end{pmatrix}$$

c. $(y-xc)'(y-xc)$

$= (y-xc \pm x\hat{B})'(y-xc \pm x\hat{B})$

$= [(y-x\hat{B}) - x(c-\hat{B})]'(y-x\hat{B} - x(c-\hat{B}))$

$= (e - x(c-\hat{B}))'(e - x(c-\hat{B}))$

$= e'e - (c-\hat{B})'x'e - e'x(c-\hat{B}) + (c-\hat{B})'x'x(c-\hat{B})$

$\therefore (y-xc)'(y-xc) - (y-x\hat{B})'(y-x\hat{B})$

$= e'e - 0 - 0 + (c-\hat{B})x'x(c-\hat{B}) - e'e$

$= (c-\hat{B})x'x(c-\hat{B})$

Since $c \neq \hat{B}$ and x is full rank so $Null(x)$ is 0

$c-\hat{B} \neq 0$ and since $(c-\hat{B})x'x(c-\hat{B}) = \|x(c-\hat{B})\|^2 \geq 0$

$\therefore (c-\hat{B})x'x(c-\hat{B}) > 0$