Prof. Salma Elmalaki
Fall 2022

Homework 5

Topic: NP and NP-Completeness

1. *(10 Points)* **Interval Scheduling.** Consider the decision version of the problem:

> Given a collection of intervals in the timeline, and a bound k, does the collection contain a subset of non-overlapping intervals of size at least k?

For each statement below, state whether it is "True", "False", or "Unknown", and briefly justify your answer.

   (a) *(2 Points)* Interval Scheduling is in P.

   (b) *(2 Points)* Interval Scheduling is in NP.

   (c) *(2 Points)* Interval Scheduling is NP-complete.

   (d) *(2 Points)* Interval Scheduling $\leq_p$ Vertex Cover.

   (e) *(2 Points)* Independent Set $\leq_p$ Interval Scheduling.

2. *(35 Points)* **Vertex Cover on Trees.**

For a graph $G = (V, E)$ with $|V| = n$ nodes and $|E| = m$ edges, a *vertex cover* is defined as a subset of nodes $S \subset V$ s.t. that all edges e are covered (*i.e.,* each edge is touched by at least one node in S). Let us restrict our attention specifically to graphs that are trees.

   (a) *(15 Points)* Design a greedy algorithm that finds the minimum size vertex cover on a tree; or argue that such an algorithm does not exist.

   (b) *(15 Points)* Design a dynamic programming algorithm that finds a minimum size vertex cover on a tree in $O(m+n)$ time. *(Note: justify why the running time of your algorithm is indeed $O(m+n)$.)*

   (c) *(5 Points)* Is the polynomial time algorithm in (b) in contradiction with the fact that the vertex cover is a well-known NP-complete problem? Please explain.

3. *(30 Points)* **The Path Selection Problem.**

Consider the path selection problem stated below:

> Given a directed graph $G = (V, E)$, a set of paths $P_1, P_2, ...P_c$, and an integer $k > 0$, is it possible to select at least k out of the c paths so that no two of the selected paths share any nodes?

   (a) *(20 points)* Prove that the path selection problem is NP-complete.

   (b) *(10 points)* Is this a decision or an optimization problem? In which of the six broad categories of NP-complete problems does this problem belong?

4. *(25 Points)* **The Traveling Salesman Problem.**

   Let's consider the optimization version of the traveling salesman problem (TSP) : We are given a graph $G(V, E)$ with $n = |V|$ nodes. Each edge of the graph has an associated distance d_{ij}. The goal is to find a *tour*, *i.e.,* a cycle that passes through every node exactly once, with minimum total distance.

   (a) *(6 Points)* State the decision version of the TSP problem. Show that the decision version of TSP is in NP.

   (b) *(6 Points)* How would you solve the TSP problem using a brute force approach? What would be the running time of that approach?

   (c) *(13 Points)* Develop a dynamic programming approach that solves this problem and report the running time of your algorithm.