List of Abbreviations and Symbols

$A[1..n]$	An array indexed from 1 to n of n elements.
$\mathbb{N}$	Set of all natural numbers, i.e., $\{1, 2, 3, \dots\}$.
$\mathbb{R}$	Set of all real numbers.
$\mathbb{Z}$	Set of all integers, i.e., $\{\dots, -2, -1, 0, 1, 2, \dots\}$.

Modifiers

To help you with what problems to try, problems marked with **[K]** are key questions that tests you on the core concepts, please do them first. Problems marked with **[H]** are harder problems that we recommend you to try after you complete all other questions (or perhaps you prefer a challenge). Good luck!!!

Contents

§1 Maximum flow

Exercise 1.1. [K] Several families are coming to a birthday celebration in a restaurant. You have arranged that v many tables will serve only vegetarian dishes, p many tables will not serve pork and r many remaining tables will serve food with pork. You know that V many families are all vegetarians, P_1 many families do not eat pork but do not mind eating vegetarian dishes, P_2 many families do not eat pork but hate vegetarian dishes. Also R_1 many families have no dietary restrictions and would also not mind eating vegetarian dishes or food without pork, R_2 many families have no dietary restrictions but hate vegetarian dishes but can eat food without pork. Finally, S many families are from Serbia and cannot imagine not eating pork. You are also given the number of family members in each family and the number of seats at each table.

In total, there are m families and n tables. You must place the guests at the tables so that their food preferences are respected and no two members from the same family sit at the same table. Your algorithm must run in time polynomial in m and n, and in case the problem has no solutions, your algorithm should output "no solution".

Exercise 1.2. [H] A band of m criminals has infiltrated a secure building, which is structured as an $n \times n$ square grid of rooms, each of which has a door on all of its sides. Thus,

- from an internal room, we can move to any of the four neighbouring rooms
- from a room on the side of the building (or edge room), we can move to three other rooms or leave the building
- from a corner room, we can move to two other rooms or leave the building

The criminals were able to shut down the building's security system before entering, but during their nefarious activities, the security system became operational again, so they decided to abort the mission and attempt to escape. The building has a sensor in each room, which becomes active when an intruder is detected, but only triggers the alarm if it is activated again. Thus, the criminals may be able to escape if they can all reach the outside of the building without any two of them passing through the same room.

Design an algorithm which runs in time polynomial in m and n and, given the m different rooms which the criminals occupy when the security system is reactivated, determines whether all m criminals can escape without triggering the alarm.

Exercise 1.3. [H] You have been told of the wonder and beauty of a very famous painting. It is painted in the hyper-modern style, and so it is simply an $n \times n$ grid of squares, with each square coloured either black or white.

You have never seen this picture for yourself but have been told some details of it by a friend. Your friend has told you the value of n and the number of white squares in each row and each column. Additionally, your friend has also been kind enough to tell you the specific colour of some squares: some squares are black, some are white, and the rest they simply could not remember.

The more details they tell you, the more amazing this painting becomes but you begin to wonder that perhaps it's simply too good to be true. Thus, you wish to design an algorithm which runs in time polynomial in n and determines whether or not such a painting can exist.

Exercise 1.4. [H] Alice is the manager of a café which supplies n different kinds of drink and m different kinds of dessert.

One day the materials are in short supply, so she can only make a_i cups of each drink type i and b_j servings of each dessert type j.

On this day, k customers come to the café and the ith of them has p_i favourite drinks $(c_{i,1}, c_{i,2}..., c_{i,p_i})$ and q_i favourite desserts $(d_{i,1}, d_{i,2}, \ldots, d_{i,q_i})$. Each customer wants to order one cup of any one of their favourite drinks and one serving of any one of their favourite desserts. If Alice refuses to serve them, or if all their favourite drinks or all their favourite desserts are unavailable, the customer will instead leave the café and provide a poor rating.

Alice wants to save the restaurant's rating. From her extensive experience with these k customers, she has listed out the favourite drinks and desserts of each customer, and she wants your help to decide which customers' orders should be fulfilled.

Design an algorithm which runs in time polynomial in n, m and k and determines the smallest possible number of poor ratings that Alice can receive, given that:

(a) all p_i and all q_i are 1 (i.e. each customer has only one favourite drink and one favourite dessert),

(b) there is no restriction on the p_i and q_i.

§2 Minimum cut

Exercise 2.1. [K] There are n cities (labelled $1, 2, \ldots, n$), connected by m bidirectional roads. Each road connects two different cities. A pair of cities may be connected by multiple roads. A well-known criminal is currently in city 1 and wishes to get to the city n via road. To catch them, the police have decided to block the minimum number of roads possible to make it impossible to get from city 1 to city n. However, some roads are major roads. In order to avoid disruption, the police cannot close any major roads.

Your goal is to find the minimum number of roads to block to prevent the criminal from going from city 1 to city n, or report that the police cannot stop the criminal. Design an algorithm which achieves this goal and runs in time polynomial in n and m.

Exercise 2.2. [K] In the country of Pipelistan there are several oil wells, several oil refineries and many distribution hubs all connected by oil pipelines. To visualise Pipelistan's oil infrastructure, just imagine a undirected graph with k source vertices (the oil wells), m sinks (refineries) and n vertices which are distribution hubs linking (unidirectional) pipelines incoming to this vertex with the outgoing pipelines from that vertex.

You are given the graph and the capacity $C(i, j)$ of each pipeline joining a vertex i with vertex j. You want to install the smallest possible number of flow meters on some of these pipelines so that the total throughput of oil from all the wells to all refineries can be computed exactly from the readings of all of these meters. Each meter shows the direction of the flow and the quantity of flow per minute. Design an algorithm which runs in time polynomial in k, m and n and decides on which pipelines to place the flow meters.

Exercise 2.3. [K] Assume that you are given a network flow graph with n vertices, including a source s, a sink t and two other distinct vertices u and v, and m edges. Design an algorithm which runs in time polynomial in n and m and returns the smallest capacity-cut among all cuts for which the vertex u is on the same side of the cut as the source s and vertex v is on the same side as the sink t.

Exercise 2.4. [K] Assume that you are given a network flow graph with n vertices, including a source s, a sink t and two other distinct vertices u and v, and m edges. Design an algorithm which returns a smallest capacity cut among all cuts for which vertices u and v are in the same side of the cut.

Exercise 2.5. [K] Given an undirected graph with vertices numbered $1, 2, \ldots, n$ and m edges, design an algorithm which runs in time polynomial in n and m and partitions the vertices into two disjoint subsets such that:

- vertex 1 and n are in different subsets, and
- the number of edges with both ends in the same subset is maximised.

Exercise 2.6. [K] You know that $n+2$ spies S, $s_1, s_2, \ldots, s_n$ and T are communicating through certain number of communication channels; in fact, for each i and each j you know if there is a channel through which spy s_i can send a secret message to spy s_j or if there is no such a channel (i.e., you know what the graph with spies as vertices and communication channels as edges looks like). Design an algorithm which runs in time polynomial in n and prevents spy S from sending a message to spy T by:

(a) compromising as few channels as possible;

(b) bribing as few of the other spies as possible.

§3 Maximum bipartite matching

Exercise 3.1. [K] You are manufacturing integrated circuits from a rectangular silicon board that is divided into an $m \times n$ grid of squares. Each integrated circuit requires two adjacent squares, either vertically or horizontally, that are cut out from this board. However, some squares in the silicon board are defective and cannot be used for integrated circuits. For each pair of coordinates (i, j), you are given a boolean $d_{i,j}$ representing whether the square in row i and column j is defective or not. Design an algorithm which runs in time polynomial in m and n and determines the maximum number of integrated circuits that can be cut out from this board.

Exercise 3.2. [H] You are the head of n spies, who are all wandering in a city. On one day you received a secret message that the bad guys in this city are going to arrest all your spies, so you'll have to arrange for your spies to run away and hide in strongholds. You have T minutes before the bad guys arrive. Your n spies are currently located at

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

and your m strongholds are located at

$$(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m).$$

The ith spy can move v_i units per minute, and each stronghold can hold only one spy.

Design an algorithm which runs in time polynomial in n and m determines which spies should be sent to which strongholds so that you have the maximum number of spies hiding from the bad guys.