COMP3411 Tutorial - Week 10
Probability and Grammars

1. Probability

1.1. Conditional Probability

Suppose you are give the following information:

- Mumps causes fever 75% of the time
- The chance of a patient having mumps is $\frac{1}{15000}$
- The chance of a patient having fever is $\frac{1}{1000}$

Determine the conditional probability of a patient suffering from mumps given that they have don't have a fever, i.e. $P(Mumps \,|\, \neg Fever)$.

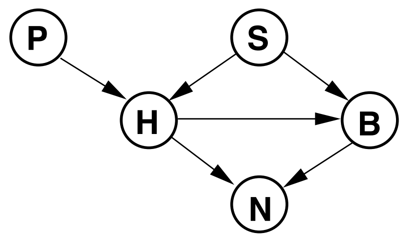
Solution:

$$
\begin{aligned}
P(Mumps \,|\, \neg Fever) &= \frac{P(\neg Fever \,|\, Mumps)\; P(Mumps)}{P(\neg Fever)} \\
&= \frac{(1 - P(Fever \,|\, Mumps))\; P(Mumps)}{1 - P(Fever)} \\
&= \frac{(1 - \frac{3}{4}) \times \frac{1}{15000}}{1 - \frac{1}{1000}} \\
&= 0.0000167
\end{aligned}
$$

1.2. Consider the following statements:

Headaches and blurred vision may be the result of sitting too close to a monitor. Headaches may also be caused by bad posture. Headaches and blurred vision may cause nausea. Headaches may also lead to blurred vision.

(i) Represent the causal links in a Bayesian network. Let H stand for "headache", B for "blurred vision", S for "sitting too close to a monitor", P for "bad posture" and N for "nausea". In terms of conditional probabilities, write a formula for the event that all five variables are true, i.e. $P(H \wedge B \wedge S \wedge P \wedge N)$.



$$P(H \wedge B \wedge S \wedge P \wedge N) = P(H \mid P \wedge S) \times P(B \mid S \wedge H) \times P(S) \times P(P) \times P(N \mid H \wedge B)$$

(ii) Suppose the following probabilities are given:

$$P(H \mid S, P) = 0.8 \qquad P(H \mid \neg S, P) = 0.4$$

$$P(H \mid S, \neg P) = 0.6 \qquad P(H \mid \neg S, \neg P) = 0.02$$

$$P(B \mid S, H) = 0.4 \qquad P(B \mid \neg S, H) = 0.3$$

$$P(B \mid S, \neg H) = 0.2 \qquad P(B \mid \neg S, \neg H) = 0.01$$

$$P(S) = 0.1$$

$$P(P) = 0.2$$

$$P(N \mid H, B) = 0.9 \qquad P(N \mid \neg H, B) = 0.3$$

$$P(N \mid H, \neg B) = 0.5 \qquad P(N \mid \neg H, \neg B) = 0.7$$

Furthermore, assume that some patient is suffering from headaches but not from nausea. Calculate joint probabilities for the 8 remaining possibilities (that is, according to whether S, B, P are true or false).

$P(H \wedge B \wedge S \wedge P \wedge \neg N) = P(H \mid S \wedge P).P(B \mid H \wedge S).P(S).P(P).P(\neg N \mid H \wedge B)$
$= 0.8 \times 0.4 \times 0.1 \times 0.2 \times 0.1$
$= 0.00064$

$P(H \wedge \neg B \wedge S \wedge P \wedge \neg N) = P(H \mid S \wedge P).P(\neg B \mid H \wedge S).P(S).P(P).P(\neg N \mid H \wedge \neg B)$
$= 0.8 \times 0.6 \times 0.1 \times 0.2 \times 0.5$
$= 0.00480$

$P(H \wedge B \wedge \neg S \wedge P \wedge \neg N) = P(H \mid \neg S \wedge P).P(B \mid H \wedge \neg S).P(\neg S).P(P).P(\neg N \mid H \wedge B)$
$= 0.4 \times 0.3 \times 0.9 \times 0.2 \times 0.1$
$= 0.00216$

$P (H \wedge \neg B \wedge \neg S \wedge P \wedge \neg N) = P (H \mid \neg S \wedge P).P (\neg B \mid H \wedge \neg S).P (\neg S).P (P).P (\neg N \mid H \wedge \neg B)$
$= 0.4 \times 0.7 \times 0.9 \times 0.2 \times 0.5$
$= 0.02520$

$P(H \wedge B \wedge S \wedge \neg P \wedge \neg N) = P(H|S \wedge \neg P).P(B|H \wedge S).P(S).P(\neg P).P(\neg N|H \wedge B)$
$= 0.6 \times 0.4 \times 0.1 \times 0.8 \times 0.1$
$= 0.00192$

$P(H \wedge \neg B \wedge S \wedge \neg P \wedge \neg N) = P(H|S \wedge \neg P).P(\neg B \mid H \wedge S).P(S).P(\neg P).P(\neg N \mid H \wedge \neg B)$
$= 0.6 \times 0.6 \times 0.1 \times 0.8 \times 0.5$
$= 0.0144$

$P (H \wedge B \wedge \neg S \wedge \neg P \wedge \neg N) = P (H \mid \neg S \wedge \neg P).P (B \mid H \wedge \neg S).P (\neg S).P (\neg P).P (\neg N \mid H \wedge B)$
$= 0.02 \times 0.3 \times 0.9 \times 0.8 \times 0.1$
$= 0.000432$

P (H ∧ ¬B ∧ ¬S ∧ ¬P ∧ ¬N) = P (H | ¬S∧¬P).P (¬B | H∧¬S).P (¬S).P (¬P).P (¬N | H∧¬B)
= 0.02×0.7×0.9×0.8×0.5
= 0.00504

(iii) What is the probability that the patient suffers from bad posture given that they are suffering from headaches but not from nausea?

$$P(P \mid H \wedge \neg N) = \frac{P(P \wedge H \wedge \neg N)}{P(H \wedge \neg N)} = \frac{0.0328}{0.054592} = 0.60082$$

Note:

$$P(P \wedge H \wedge \neg N) = \sum_{b,s} P(H \wedge b \wedge s \wedge P \wedge \neg N)$$
$$= 0.00064 + 0.00480 + 0.00216 + 0.02520$$
$$= 0.0328$$

$$P(H \wedge \neg N) = \sum_{b,s,p} P(H \wedge b \wedge s \wedge p \wedge \neg N)$$
$$= 0.00064 + 0.00480 + 0.00216 + 0.02520 + 0.00192 + 0.0144 + 0.000432$$
$$+ 0.00504$$
$$= 0.05452$$

1.3 Consider the “burglar alarm” Bayesian network from the lectures.

Derive, using Bayes’ Rule, an expression for P(Burglary|Alarm) in terms of the conditional probabilities represented in the network. Then calculate the value of this probability.

Is this number what you expected? Explain what is going on.

Solution:

Let *A* stand for “Alarm”, *B* for “Burglary” and *E* for “Earthquake”.

Then by Bayes’ Rule:

P (B|A) = P (A|B).P(B)/P (A)
= (P(A|B∧E).P (E).P (B)+P(A|B∧¬E).P(¬E).P (B))/P (A),

and as in lectures

$P(A) = P(A|B \wedge E).P(E).P(B)+P(A|B \wedge \neg E).P(\neg E).P(B)+P(A|\neg B \wedge E).P(E).P(\neg B)+$
$P(A|\neg B \wedge \neg E).P(\neg E).P(\neg B)$

So $P(B|A) = (0.95 \times 0.002 \times 0.001 + 0.94 \times 0.998 \times 0.001)/P(A)$ and
$P(A) = 0.95 \times 0.002 \times 0.001 + 0.94 \times 0.998 \times 0.001 + 0.29 \times 0.002 \times 0.999 +$

$0.001 \times 0.998 \times 0.999$

Thus $P(B|A) = 0.00094002/0.002516442 = 0.3735512$

Intuitively, the "true positives" (when there really is a burglary) account for roughly only 10/26 of the cases when the alarm is ringing (around 0.001 of the time), while the "false positives" account for 16/26 cases (6/26 when the alarm is ringing because of an earthquake, due to a false positive rate around 0.3 and prior of 0.002, so around 0.0006 of the time, and 10/26 when there is neither a burglary nor an earthquake, due to a false positive rate of 0.001 and a prior close to 1, so around 0.001 of the time). The rough calculation is $10/26 = 0.001/(0.001 + 0.0006 + 0.001)$. That is, the false positives significantly outweigh the true positives in this scenario.

4. Definite Clause Grammars

The following definite clause grammar parses a very small subset of English:

```
sentence -->
     noun_phrase,
     verb_phrase.

noun_phrase -->
     determiner,
     noun.

verb_phrase -->
     verb,
     noun_phrase.

determiner --> [a].
determiner --> [the].

noun --> [cat].
noun --> [cats].
noun --> [mouse].
noun --> [mice].

verb --> [scares].
verb --> [hates].
verb --> [hate].
```

4.1. Extend the grammar to handle number agreement. That is a query like:

```
?- sentence([the, cat, hate, the mouse], []).
```

should fail.

Solution:

```
sentence -->
      noun_phrase(Number),
      verb_phrase(Number).

noun_phrase(Number) -->
      determiner(Number),
      noun(Number).

verb_phrase(Number) -->
      verb(Number),
      noun_phrase(_).

determiner(singular) --> [a].
determiner(_) --> [the].

noun(singular) --> [cat].
noun(plural) --> [cats].
noun(singular) --> [mouse].
noun(plural) --> [mice].

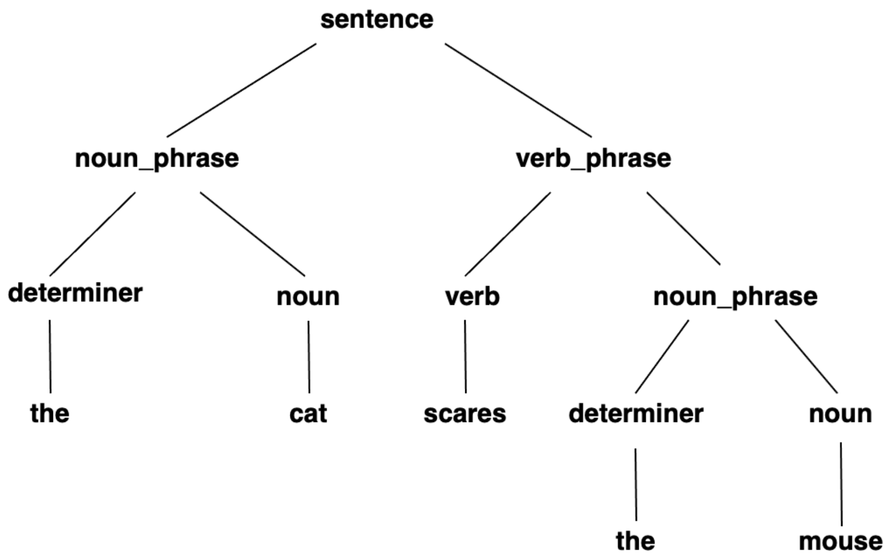
verb(singular) --> [scares].
verb(singular) --> [hates].
verb(plural) --> [hate].
```

4.2.Further extend the grammar so that it constructs a parse tree, like:

```
?- sentence(X, [the, cat, scares, the, mouse], Y).

X = sentence(noun_phrase(determiner(the), noun(cat)),
verb_phrase(verb(scares), noun_phrase(determiner(the),
noun(mouse))))
```

The output represents the parse tree:



Solution:

```
sentence(sentence(NP, VP)) -->
     noun_phrase(Number, NP),
     verb_phrase(Number, VP).

noun_phrase(Number, noun_phrase(Det, Noun)) -->
     determiner(Number, Det),
     noun(Number, Noun).

verb_phrase(Number, verb_phrase(V, NP)) -->
     verb(Number, V),
     noun_phrase(_, NP).

determiner(singular, determiner(a)) --> [a].
determiner(_, determiner(the)) --> [the].

noun(singular, noun(cat)) --> [cat].
noun(plural, noun(cats)) --> [cats].
noun(singular, noun(mouse)) --> [mouse].
noun(plural, noun(mice)) --> [mice].

verb(singular, verb(scares)) --> [scares].
verb(singular, verb(hates)) --> [hates].
verb(plural, verb(hate)) --> [hate].
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