COMP3411: Artificial Intelligence

Automated Reasoning

C. Sammut & W. Wobcke

This Lecture

- Proof systems
  - Soundness, completeness, decidability
- Resolution and Refutation
- Horn clauses and SLD resolution
- Prolog

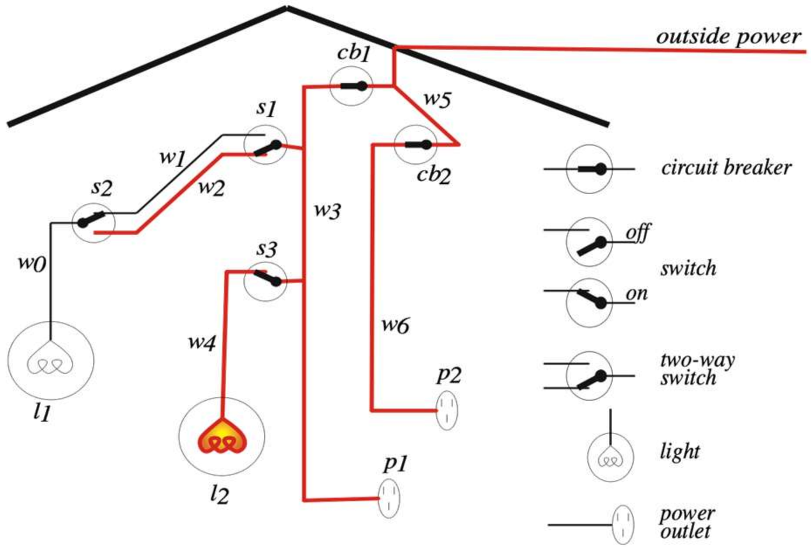
Summary So Far

- Propositional Logic
  - Syntax: Formal language built from $\wedge$, $\vee$, $\neg$, $\rightarrow$
  - Semantics: Definition of truth table for every formula
  - $S \models P$ if whenever all formulae in S are True, P is True
- Proof System
  - System of axioms and rules for deduction
  - Enables computation of proofs of P from S
- Basic Questions
  - Are the proofs that are computed always correct? (soundness)
  - If $S \models P$, is there always a proof of P from S (completeness)

Prove lamp l_2 is lit

$light\_l_1.$
$light\_l_2.$
$down\_s_1.$
$up\_s_2.$
$up\_s_3.$
$ok\_l_1.$
$ok\_l_2.$
$ok\_cb_1.$
$ok\_cb_2.$
$live\_outside.$

$lit\_l_1 \leftarrow live\_w_0 \land ok\_l_1$
$live\_w_0 \leftarrow live\_w_1 \land up\_s_2.$
$live\_w_0 \leftarrow live\_w_2 \land down\_s_2.$
$live\_w_1 \leftarrow live\_w_3 \land up\_s_1.$
$live\_w_2 \leftarrow live\_w_3 \land down\_s_1.$
$lit\_l_2 \leftarrow live\_w_4 \land ok\_l_2.$
$live\_w_4 \leftarrow live\_w_3 \land up\_s_3.$
$live\_p_1 \leftarrow live\_w_3.$
$live\_w_3 \leftarrow live\_w_5 \land ok\_cb_1.$
$live\_p_2 \leftarrow live\_w_6.$
$live\_w_6 \leftarrow live\_w_5 \land ok\_cb_2.$
$live\_w_5 \leftarrow live\_outside.$



$lit\_l_2 \leftarrow live\_w_4 \land ok\_l_2$
$lit\_l_2 \leftarrow live\_w_3 \land up\_s_3 \land ok\_l_2$
$lit\_l_2 \leftarrow live\_w_5 \land ok\_cb_1 \land up\_s_3 \land ok\_l_2$
$lit\_l_2 \leftarrow live\_outside \land ok\_cb_1 \land up\_s_3 \land ok\_l_2$
$lit\_l_2 \leftarrow ok\_cb_1 \land up\_s_3 \land ok\_l_2$
$lit\_l_2 \leftarrow up\_s_3 \land ok\_l_2$
$lit\_l_2 \leftarrow ok\_l_2$
lit_l_2

Mechanising Proof

- A proof of a formula P from a set of premises S is a sequence of lines in which any line in the proof is

1. An axiom of logic or premise from S, or
2. A formula deduced from previous lines of the proof using a rule of inference and the last line of the proof is the formula P

- Formally captures the notion of mathematical proof
- S proves P ($S \vdash P$) if there is a proof of P from S; alternatively, P follows from S
- Example: Resolution proof

Soundness, Completeness and Decidability

- A proof system is **decidable** if there is a mechanical procedure that, when asked whether $S \vdash P$, can always answer **True** or **False** correctly
  - i.e. the procedure is sound, complete and terminating

- **Soundness** means that the algorithm never returns **True** when it shouldn't.
  - i.e it is able to prove all consequences of any set of premises (including infinite sets)
  - Whenever $S \vdash P$, if every formula in S is **True**, P is also True
  - Whenever $S \vdash P$, $S \vDash P$

- **Completeness** means it always returns **True** when it should.
  - i.e it is able to prove all consequences of any set of premises (including infinite sets)
  - Whenever P is entailed by S, there is a proof of P from S
  - Whenever $S \vDash P$, $S \vdash P$

Resolution

- A common type of proof system based on refutation
- Well suited to computer implementation
- Decidable in the case of Propositional Logic
- Generalises to First-Order Logic (see next set of lectures)
- Needs all formulae to be converted to clausal form

Normal Forms

- To make it easier to mechanise a proof, first rewrite all sentence in normal form
- E.g.

Rewrite:

$$lit\_l_2 \leftarrow live\_w_3 \wedge up\_s_3 \wedge ok\_l_2$$

as a disjunction:

$$lit\_l_2 \vee \neg live\_w_3 \vee \neg up\_s_3 \vee \neg ok\_l_2$$

because $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

Normal Forms

- A literal ℓ is a propositional variable or the negation of a propositional variable (P or $\neg P$)
- A clause is a disjunction of literals $\ell_1 \vee \ell_2 \vee \cdots \vee \ell_n$
- Conjunctive Normal Form (CNF) — a conjunction of clauses, e.g.

  $(P \vee Q \vee \neg R) \wedge (\neg S \vee \neg R)$ – or just one clause, e.g. $P \vee Q$
- Disjunctive Normal Form (DNF) — a disjunction of conjunctions of literals, e.g.

  $(P \wedge Q \wedge \neg R) \vee (\neg S \wedge \neg R)$ – or just one conjunction, e.g. $P \wedge Q$
- Every Propositional Logic formula can be converted to CNF and DNF
- Every Propositional Logic formula is equivalent to its CNF and DNF

Conversion to Conjunctive Normal Form

- Eliminate $\leftrightarrow$ rewriting $P \leftrightarrow Q$ as $(P \rightarrow Q) \wedge (Q \rightarrow P)$
- Eliminate $\rightarrow$ rewriting $P \rightarrow Q$ as $\neg P \vee Q$
- Use De Morgan's laws to push $\neg$ inwards (repeatedly)
  - Rewrite $\neg(P \wedge Q)$ as $\neg P \vee \neg Q$
  - Rewrite $\neg(P \vee Q)$ as $\neg P \wedge \neg Q$
- Eliminate double negations: rewrite $\neg\neg P$ as P
- Use the distributive laws to get CNF [or DNF] – if necessary
  - Rewrite $(P \wedge Q) \vee R$ as $(P \vee R) \wedge (Q \vee R)$ [for CNF]
  - Rewrite $(P \vee Q) \wedge R$ as $(P \wedge R) \vee (Q \wedge R)$ [for DNF]

Example Clausal Form

Clausal Form = set of clauses in the CNF

1. $\neg(P \rightarrow (Q \wedge R))$
2. $\neg(\neg P \vee (Q \wedge R))$
3. $\neg\neg P \wedge \neg(Q \wedge R)$
4. $\neg\neg P \wedge (\neg Q \vee \neg R)$
5. $P \wedge (\neg Q \vee \neg R)$

- Clausal Form: $\{P, \neg Q \vee \neg R\}$

Resolution Rule of Inference

$$A_1 \vee \cdots \vee A_m \vee B \qquad\qquad \neg B \vee C_1 \vee \cdots \vee C_n$$

$$A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n$$

where B is a propositional variable and A_i and C_j are literals

- B and $\neg B$ are complementary literals
- $A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n$ is the resolvent of the two clauses
- Special case: If no A_i and C_j , resolvent is empty clause, denoted $\square$ or $\perp$

Resolution Rule

- Consider $A_1 \vee \cdots \vee A_m \vee B$ and $\neg B \vee C_1 \vee \cdots \vee C_n$
  - Suppose both are True
  - If B is True, $\neg B$ is False so $C_1 \vee \cdots \vee C_n$ must be True
  - If B is False, $A_1 \vee \cdots \vee A_m$ must be True
  - Hence $A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n$ is True

Hence the resolution rule is sound

- Starting with true premises, any conclusion made using resolution must be true

Applying Resolution: Naive Method

- Convert knowledge base into clausal form
- Repeatedly apply resolution rule to the resulting clauses
- P follows from the knowledge base if and only if each clause in the CNF of P can be derived using resolution from the clauses of the knowledge base (or subsumption)
- Example
  - $\{P \to Q, Q \to R\} \vdash P \to R$
  - Clauses $\neg P \vee Q$, $\neg Q \vee R$, show $\neg P \vee R$
  - Follows from one resolution step (Q and $\neg Q$ cancel, leaving $\neg P \vee R$)

Proof by contradiction

- Assume negative of what you are trying to prove and see if that leads to a contradiction
- So, assume lit_l_2 is false, i.e. $\neg lit\_l_2$ is true

$light\_l_1.$

$light\_l_2.$

$down\_s_1.$

$up\_s_2.$

$up\_s_3.$

$ok\_l_1.$

$ok\_l_2.$

$ok\_cb_1.$

$ok\_cb_2.$

$live\_outside.$

$lit\_l_1 \leftarrow live\_w_0 \wedge ok\_l_1$

$live\_w_0 \leftarrow live\_w_1 \wedge up\_s_2.$

$live\_w_0 \leftarrow live\_w_2 \wedge down\_s_2.$

$live\_w_1 \leftarrow live\_w_3 \wedge up\_s_1.$

$live\_w_2 \leftarrow live\_w_3 \wedge down\_s_1.$

$lit\_l_2 \leftarrow live\_w_4 \wedge ok\_l_2.$

$live\_w_4 \leftarrow live\_w_3 \wedge up\_s_3.$

$live\_p_1 \leftarrow live\_w_3.$

$live\_w_3 \leftarrow live\_w_5 \wedge ok\_cb_1.$

$live\_p_2 \leftarrow live\_w_6.$

$live\_w_6 \leftarrow live\_w_5 \wedge ok\_cb_2.$

$live\_w_5 \leftarrow live\_outside.$

$$lit\_l_2 \vee \neg live\_w_4 \vee \neg ok\_l_2$$
$$lit\_l_2 \vee \neg live\_w_3 \vee \neg up\_s_3 \vee \neg ok\_l_2$$
$$lit\_l_2 \vee \neg live\_w_5 \vee \neg ok\_cb_1 \vee \neg up\_s_3 \vee \neg ok\_l_2$$
$$lit\_l_2 \vee \neg live\_outside \vee \neg ok\_cb_1 \vee \neg up\_s_3 \vee \neg ok\_l_2$$
$$lit\_l_2 \vee \neg ok\_cb_1 \vee \neg up\_s_3 \vee \neg ok\_l_2$$
$$lit\_l_2 \vee \neg up\_s_3 \vee \neg ok\_l_2$$
$$lit\_l_2 \vee \neg ok\_l_2$$
$$lit\_l_2$$

but we assumed $\neg lit\_l_2$

$$lit\_l_2 \wedge \neg lit\_l_2$$

is a contradiction

Refutation Systems

- To show that P follows from S (i.e. $S \vdash P$) using refutation, start with S and ¬P in clausal form and derive a contradiction using resolution
- A contradiction is the "empty clause" (a clause with no literals)
- The empty clause $\square$ is unsatisfiable (always False)
- So if the empty clause $\square$ is derived using resolution, the original set of clauses is unsatisfiable (never all True together)
- That is, if we can derive $\square$ from the clausal forms of S and $\neg P$, these clauses can never be all True together
- Hence whenever the clauses of S are all True, at least one clause from $\neg P$ must be False, i.e. $\neg P$ must be False and P must be True
- By definition, $S \models P$ (so P can correctly be concluded from S)

Applying Resolution Refutation

- Negate query to be proven (resolution is a refutation system)
- Convert knowledge base and negated query into CNF
- Repeatedly apply resolution until either the empty clause (contradiction) is derived or no more clauses can be derived
- If the empty clause is derived, answer ‘true’ (query follows from knowledge base), otherwise answer ‘false’ (query does not follow from knowledge base)

Resolution: Example 1

$(G \lor H) \rightarrow (\neg J \land \neg K), G \vdash \neg J$

Clausal form of is $\{\neg G \lor \neg J, \neg H \lor \neg J, \neg G \lor \neg K, \neg H \lor \neg K, G\}$

1. $\neg G \lor \neg J$	[Premise]
2. $\neg H \lor \neg J$	[Premise]
3. $\neg G \lor \neg K$	[Premise]
4. $\neg H \lor \neg K$	[Premise]
5. G	[Premise]
6. J	[$\neg$ Query]
7. $\neg G$	[1, 6 Resolution]
8. $\square$	[5, 7 Resolution]

Resolution: Example 2

$P \rightarrow \neg Q, \neg Q \rightarrow R \vdash P \rightarrow R$

Recall $P \rightarrow R \Leftrightarrow \neg P \vee R$

Clausal form of $\neg(\neg P \vee R)$ is $\{P, \neg R\}$

1. $\neg P \vee \neg Q$	[Premise]
2. $Q \vee R$	[Premise]
3. P	[$\neg$ Query]
4. $\neg R$	[$\neg$ Query]
5. $\neg Q$	[1, 3 Resolution]
6. R	[2, 5 Resolution
7. $\square$	[4, 6 Resolution]

Resolution: Example 3

$\vdash ((P \vee Q) \wedge \neg P) \rightarrow Q$

Clausal form of $\vdash ((P \vee Q) \wedge \neg P) \rightarrow Q$ is $\{P \vee Q, \neg P, \neg Q\}$

$1. P \vee Q$	[$\neg$ Query]
$2. \neg P$	[$\neg$ Query]
$3. \neg Q$	[$\neg$ Query]
$4. Q$	[1, 2 Resolution
$5. \square$	[3, 4 Resolution]

Rewriting negated query in CNF:

$\neg[((P \vee Q) \wedge \neg P) \rightarrow Q]$

$\neg[\neg((P \vee Q) \wedge \neg P) \vee Q]$

$\neg\neg((P \vee Q) \wedge \neg P) \wedge \neg Q$

$(P \vee Q) \wedge \neg P \wedge \neg Q$

Now write in clausal form:

$\{P \vee Q, \neg P, \neg Q\}$

Soundness and Completeness Again

For Propositional Logic

- Resolution refutation is sound, i.e. it preserves truth (if a set of premises are all true, any conclusion drawn from those premises must also be true)
- Resolution refutation is complete, i.e. it is capable of proving all consequences of any knowledge base (not shown here!)
- Resolution refutation is decidable, i.e. there is an algorithm implementing resolution which when asked whether $S \vdash P$, can always answer 'true' or 'false' (correctly)

Heuristics in Applying Resolution

- Clause elimination — can disregard certain types of clauses
  - Pure clauses: contain literal L where $\neg L$ doesn't appear elsewhere
  - Tautologies: clauses containing both L and $\neg L$
  - Subsumed clauses: another clause is a subset of the literals
- Ordering strategies
  - Resolve unit clauses (only one literal) first
  - Start with query clauses
  - Aim to shorten clauses

Horn Clauses

Using a less expressive language makes proof procedure easier.

- Review
  - literal – proposition variable or negation of proposition variable
  - clause – disjunction of literals
- Definite Clause – exactly one positive literal
  - e.g. $B \vee \neg A_1 \vee \ldots \vee \neg A_n$, i.e. $B \leftarrow A_1 \wedge \ldots \wedge A_n$
- Negative Clause – no positive literals
  - e.g. $\neg Q_1 \vee \neg Q_2$ (negation of a query)
- Horn Clause – clause with at most one positive literal

Prolog

- Horn clauses in First-Order Logic
- SLD resolution
- Depth-first search strategy with backtracking
- User control
  - Ordering of clauses in Prolog database (facts and rules)
  - Ordering of subgoals in body of a rule
- Prolog is a programming language based on resolution refutation relying on the programmer to exploit search control rules

Prolog Clauses

P :– Q, R, S.

$P \leftarrow Q \land R \land S.$

$P \lor \neg(Q \land R \land S)$

$P \lor \neg Q \lor \neg R \lor \neg S$

Queries:

?- Q, R, S

$\perp \leftarrow Q \land R \land S$

$\neg(Q \land R \land S)$

$\neg Q \lor \neg R \lor \neg S$

Prolog DB = set of clauses

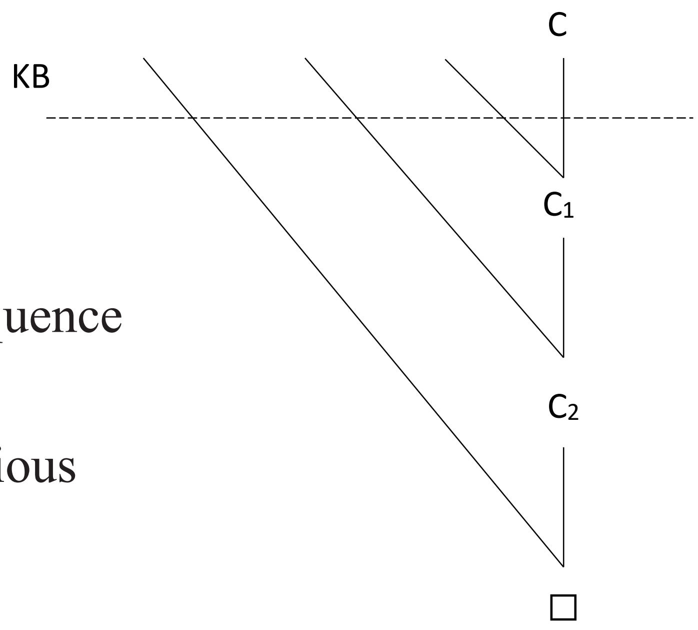
$P \rightarrow Q \equiv \neg P \lor Q$

$P \leftarrow Q \equiv P \lor \neg Q$

$\perp \equiv$ false (i.e. a contradiction)

SLD Resolution – $\vdash_{SLD}$

- **S**elected literals **L**inear form **D**efinite clauses resolution
- SLD refutation of a clause C from a set of clauses KB is a sequence
  1. First clause of sequence is C
  2. Each intermediate clause C_i is derived by resolving the previous clause C_{i-1} and a clause from KB
  3. The last clause in the sequence is $\square$



- For a definite KB and negative clause query Q: $KB \cup Q \vdash \square$ if and only if $KB \cup Q \vdash_{SLD} \square$

Prolog Example

```
r.                  % facts
u.
v.

q :- r, u.          % rules
s :- v.
p :- q, r, s.

?- p.               % query
true
```

Example Execution of Prolog interpreter

```
r.
u.
v.

q :- r, u.
s :- v.
p :- q, r, s.

?- p.
```

Initial goal set = {p}

1. {q, r, s}	because p :- q, r, s.
2. {r, u, r, s}	because q :- r, u.
3. {u, r, s}	because r.
4. {r, s}	because u.
5. {s}	because r.
6. {v}	because s :- v
7. {}	because v.
8. => true	because empty clause

- In each step, we remove the first element in the goal set and replace it with the body of the clause whose head matches that element. E.g. remove *p* and replace by *q*, *r*, *s*.
- **Note**: The simple Prolog interpreter isn't smart enough to remove the duplication of *r* in step 2.

Prolog Interpreter

Input: A query Q and a logic program KB

Output: 'true' if Q follows from KB, 'false' otherwise

Initialise current goal set to $\{Q\}$

while the current goal set is not empty do

Choose G from the current goal set; (first in goal set)

Make a copy G' :- $B_1, \ldots, B_n$ of a clause from KB

(try all in KB) (if no such rule, try alternative rules)

Replace G by $B_1, \ldots, B_n$ in current goal set

if current goal set is empty:

output 'true'

else output 'false'

Inefficient and not how a real Prolog interpreter works

- Depth-first, left-right with backtracking

Conclusion: Propositional Logic

- Propositions built from $\wedge$, $\vee$, $\neg$, $\rightarrow$
- Sound, complete and decidable proof systems (inference procedures)
  - Resolution refutation
  - Prolog for special case of definite clauses
  - Limited expressive power
  - Cannot express ontologies (no relations)
- First-Order Logic can express knowledge about objects, properties and relationships between objects