COMP3161/COMP9164
Preliminaries Exercises

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1. **Strange Loops:** The following system, based on a system called Miu, is perhaps famously mentioned in Douglas Hofstadter's book, *Gödel, Escher, Bach.*

$$\frac{}{\texttt{MI}\ \textsc{Miu}}1 \quad \frac{x\texttt{I}\ \textsc{Miu}}{x\texttt{IU}\ \textsc{Miu}}2 \quad \frac{\texttt{M}x\ \textsc{Miu}}{\texttt{M}xx\ \textsc{Miu}}3 \quad \frac{x\texttt{III}y\ \textsc{Miu}}{x\texttt{U}y\ \textsc{Miu}}4 \quad \frac{x\texttt{UU}y\ \textsc{Miu}}{xy\ \textsc{Miu}}5$$

(a) [⋆] Is MUII Miu derivable? If so, show the derivation tree. If not, explain why not.

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{MI}\ \textsc{Miu}}1}{\texttt{MII}\ \textsc{Miu}}3}{\texttt{MIIII}\ \textsc{Miu}}3}{\texttt{MIIIIIIII}\ \textsc{Miu}}3}{\texttt{MIIIIIIIIU}\ \textsc{Miu}}2}{\texttt{MIIIIIUU}\ \textsc{Miu}}4}{\texttt{MIIIII}\ \textsc{Miu}}5}{\texttt{MUII}\ \textsc{Miu}}4$$

(b) [⋆⋆] Is $\dfrac{x\texttt{IU}\ \textsc{Miu}}{x\texttt{I}\ \textsc{Miu}}$ admissible? Is it derivable? Justify your answer.

Solution: It is not derivable, but it is admissible. It is not derivable because there is no way to construct a tree that looks like this:

$$\dfrac{\dfrac{x\texttt{IU}\ \textsc{Miu}}{\vdots}}{x\texttt{I}\ \textsc{Miu}}$$

I believe it is, however, *admissible* because it does not change the language Miu. I think there is no string x that could be judged x Miu with this rule that could not be so judged without it, although I welcome any counterexample or proof.

(c) [⋆⋆⋆⋆] Perhaps famously, MU Miu is not admissible. Prove this using rule induction. *Hint*: Try proving something related to the number of Is in the string.

Solution: We will prove that the number of Is in any string in Miu is not divisible by three. Seeing as MU has zero Is (a multiple of three), if we prove the above, we prove that MU is not admissible.

Base Case (From rule 1). We see that the string MI has only one I, which is not a multiple of three, hence we have shown our goal.

Inductive case (From rule 2). Given that the number of Is in xI is not divisible by three (our inductive hypothesis), we can easily see that the number of Is in xIU is identical and therefore is similarly not divisible by three.

Inductive case (From rule 3). Let n be the number of Is in Mx. Our inductive hypothesis is that $3 \nmid n$. The number of Is in Mxx, clearly $2n$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid 2n$.

Inductive case (From rule 4). Let n be the number of $\texttt{I}$s in $x\texttt{III}y$. Our inductive hypothesis is that $3 \nmid n$. The number of $\texttt{I}$s in $x\texttt{U}y$, clearly $n-3$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid (n-3)$
Inductive case (From rule 5). Given that the number of $\texttt{I}$s in $x\texttt{UU}y$ is the same as the number of $\texttt{I}$s in xy, our inductive hypothesis trivially proves our goal.

Thus, by induction, no string in $\textsc{Miu}$ has a number of $\texttt{I}$s divisible by three. Therefore, $\texttt{MU}$ $\textsc{Miu}$ is not admissible. □

(d) Here is another language, which we'll call $\textsc{Mi}$:

$$\frac{}{\texttt{MI}\ \textsc{Mi}}A \qquad \frac{\texttt{M}x\ \textsc{Mi}}{\texttt{M}xx\ \textsc{Mi}}B \qquad \frac{x\texttt{IIIIII}y\ \textsc{Mi}}{xy\ \textsc{Mi}}C$$

i. [★★★] Prove using rule induction that all strings in $\textsc{Mi}$ could be expressed as follows, for some k and some i, where $2^k - 6i > 0$ (where $\texttt{C}^n$ is the character $\texttt{C}$ repeated n times):

$$\texttt{M}\,\texttt{I}^{2^k-6i}$$

Solution:

Base case (From rule A). $\texttt{MI} = \texttt{M}\,\texttt{I}^{2^k-6i}$ when $2^k - 6i = 1$, i.e when $k = 0$ and $i = 0$.

Inductive case (From rule B) Given that $\texttt{M}x = \texttt{MI}^{2^a-6b}$ (our inductive hypothesis), we must show that $\texttt{M}xx = \texttt{M}\,\texttt{I}^{2^k-6i}$ for some k and some i. As $x = \texttt{I}^{2^a-6b}$ (from I.H), it is easy to see that $xx = \texttt{I}^{2(2^a-6b)} = \texttt{I}^{2^{a+1}-6(2b)} = \texttt{I}^{2^k-6i}$ for $k = a+1$ and $i = 2b$.

Inductive case (from rule C) Given that $x\texttt{IIIIII}y = \texttt{M}\,\texttt{I}^{2^a-6b}$ (our inductive hypothesis). We must show that $xy = \texttt{M}\,\texttt{I}^{2^k-6i}$ for some k and i. It should be clear to see that this rule simply subtracts six $\texttt{I}$ characters, and therefore $xy = \texttt{MI}^{2^a-6(b+1)}$, hence $k = a$ and $i = b+1$.

Thus, all strings in $\textsc{Mi}$ can be expressed as $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$ □

ii. We will now prove the opposite claim that, for all k and i, assuming $2^k - 6i > 0$:

$$\texttt{M}\,\texttt{I}^{2^k-6i}\ \textsc{Mi}$$

To prove this we will need a few lemmas which we will prove separately.

α) [★★] Prove, using induction on the natural number k (i.e when $k = 0$ and when $k = k' + 1$), that $\texttt{M}\,\texttt{I}^{2^k}$ $\textsc{Mi}$

Solution:

Base case (when $k = 0$). We have to show $\texttt{MI}$ $\textsc{Mi}$, which is true by rule A.
Inductive case (when $k = k'+1$) We have to show $\texttt{MI}^{2^{k'+1}}$ $\textsc{Mi}$, with the inductive hypothesis that $\texttt{MI}^{2^{k'}}$ $\textsc{Mi}$. Equivalently, we have to show $\texttt{MI}^{2^{k'}}\texttt{I}^{2^{k'}}$ $\textsc{Mi}$, as follows:

$$\cfrac{\cfrac{}{\texttt{MI}^{2^{k'}}\ \textsc{Mi}}I.H}{\texttt{MI}^{2^{k'}}\texttt{I}^{2^{k'}}\ \textsc{Mi}}B$$

Therefore, by induction on the natural number k, we have shown $\forall k.\texttt{M}\,\texttt{I}^{2^k}$ $\textsc{Mi}$. □

β) [★★] Prove, using induction on the natural number i, that $\texttt{M}\,\texttt{I}^k$ $\textsc{Mi}$ implies $\texttt{M}\,\texttt{I}^{k-6i}$ $\textsc{Mi}$, assuming $k - 6i > 0$.

Solution:

Base case (when $i = 0$). We must show that $\texttt{M}\,\texttt{I}^k$ $\textsc{Mi}$ implies $\texttt{M}\,\texttt{I}^{k-0}$ $\textsc{Mi}$, which is obviously a tautology.

Inductive case (when $i = i' + 1$) We must show that $\mathtt{M}\,\mathtt{I}^{k}$ Mi implies $\mathtt{M}\,\mathtt{I}^{k-6(i'+1)}$ Mi, given the inductive hypothesis that $\mathtt{M}\,\mathtt{I}^{k-6i'}$ Mi. Note that our I.H can be restated as $\mathtt{MIIIIII}\,\mathtt{I}^{k-6(i'+1)}$ Mi due to our assumption that $k - 6(i' + 1) > 0$. With this, we can prove our goal as shown:

$$\dfrac{\dfrac{}{\mathtt{MIIIIII}\,\mathtt{I}^{k-6(i'+1)}\ \textsc{Mi}}\,I.H}{\mathtt{M}\,\mathtt{I}^{k-6(i'+1)}\ \textsc{Mi}}\,C$$

□

Therefore, our goal is shown by induction.

Hence, as we know $\mathtt{M}\,\mathtt{I}^{2^k}$ Mi for all k from lemma α, we can conclude from lemma β that $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ Mi for all k and all i where $2^k - 6i > 0$ by modus ponens.

These two parts prove that the language Mi is exactly characterised by the formulation $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ where $2^k - 6i > 0$. A very useful result!

iii. [$\star$] Hence prove or disprove that the following rule is admissible in Mi:

$$\dfrac{\mathtt{M}x x\ \textsc{Mi}}{\mathtt{M}x\ \textsc{Mi}}\,\textsc{Lem}_1$$

Solution: We know from part i that $\mathtt{M}xx$ Mi $\implies x^2 = \mathtt{I}^{2^k-6i}$ for some k and some i where $2^k - 6i > 0$.
This rule is *not admissible* as it adds strings to the language. As $2^4 - 6 = 10$, we know $\mathtt{MI}^{10}$ is in the language. This rule would make $\mathtt{MI}^5$ a string in the language which it is not as there is no k and i such that $2^k - 6i = 5$.

iv. [$\star$] Why is the following rule **not** admissible in Mi?

$$\dfrac{xy\ \textsc{Mi}}{x\mathtt{IIIIII}y\ \textsc{Mi}}\,\textsc{Lem}_2$$

Solution: The rule is not admissible as it adds strings to the language. This allows us to *add* six I characters to any string in Mi and judge it in Mi, which results in additional strings. For example, applying the rule to MI (which is in Mi), gives us $\mathtt{M}\,\mathtt{I}^7$, when our existing formulation of Mi ($\mathtt{M}\,\mathtt{I}^{2^k-6i}$) clearly only allows for even amounts of Is.

v. [$\star\star\star$] Prove that, for all s, s Mi $\implies$ s Miu. Note that using straightforward rule induction appears to necessitate $\textsc{Lem}_2$ above, which we know is not admissible. Try proving using the characterisation we have already developed.

Solution: We shall show that all strings in Mi, characterised by $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ where $2^k - 6i > 0$, are also in Miu. That is, we shall show that $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ Miu.
To start, we shall prove inductively on k that $\mathtt{M}\,\mathtt{I}^{2^k}$ Miu for all k.

Base case (Where $k = 0$). We must show MI Miu, which we know trivially from rule 1.

Inductive case (where $k = k' + 1$). We must show $\mathtt{M}\,\mathtt{I}^{2^{k'+1}}$ Miu, given the inductive hypothesis that $\mathtt{M}\,\mathtt{I}^{2^{k'}}$ Miu. Note we can restate our proof goal as $\mathtt{M}\,\mathtt{I}^{2^{k'}}\,\mathtt{I}^{2^{k'}}$ Miu

$$\dfrac{\dfrac{}{\mathtt{M}\,\mathtt{I}^{2^{k'}}\ \textsc{Miu}}\,I.H}{\mathtt{M}\,\mathtt{I}^{2^{k'}}\,\mathtt{I}^{2^{k'}}\ \textsc{Miu}}\,B$$

Thus we have shown by induction that $\mathtt{M}\,\mathtt{I}^{2^k}$ Miu for all k.

Next we must prove that $\mathtt{M}\,\mathtt{I}^{k}$ Miu implies $\mathtt{M}\,\mathtt{I}^{k-6i}$ Miu for all i, assuming $k - 6i > 0$.

Base case (where $i = 0$), we must show that $\mathtt{M}\,\mathtt{I}^k$ Miu implies $\mathtt{M}\,\mathtt{I}^{k-0}$ Miu, which is trivially a tautology.
Inductive case (where $i = i' + 1$) we must show that $\mathtt{M}\,\mathtt{I}^{k-6(i'+1)}$ Miu given the inductive hypothesis $\mathtt{M}\,\mathtt{I}^{k-6i'}$ Miu. As we know $k - 6(i' + 1) > 0$, we can restate our inductive hypothesis as $\mathtt{MIIIIII}\,\mathtt{I}^{k-6(i'+1)}$ Miu, and easily prove our goal:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathtt{MIIIIII}\,\mathtt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,I.H}{\mathtt{MUIII}\,\mathtt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,4}{\mathtt{MUU}\,\mathtt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,4}{\mathtt{M}\,\mathtt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,5$$

Thus, by modus ponens we can see that $\mathtt{M}\,\mathtt{I}^{2^k-6i}$ Miu for all k and i where $2^k - 6i > 0$. As this is the exact characterisation of Mi, we have proven that s Mi $\implies$ s Miu for all s. □

2. **Counting Sticks:** The following language (also presented in a similar form by Douglas Hofstadter, but the original invention is not his) is called the $\Phi\Psi$ system. Unlike the Miu language discussed above, this language is not comprised of a single judgement, but of a ternary *relation*, written $x\ \Phi\ y\ \Psi\ z$, where x, y and z are strings of hyphens (i.e '`-`'), which may be empty (ϵ). The system is defined as follows:

$$\dfrac{}{\epsilon\ \Phi\ x\ \Psi\ x}\,B \qquad \dfrac{x\ \Phi\ y\ \Psi\ z}{\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z}\,I$$

(a) [⋆] Prove that `--` Φ `---` Ψ `-----`.

Solution:

$$\dfrac{\dfrac{\dfrac{}{\epsilon\ \Phi\ \texttt{---}\ \Psi\ \texttt{---}}\,B}{\texttt{-}\ \Phi\ \texttt{---}\ \Psi\ \texttt{----}}\,I}{\texttt{--}\ \Phi\ \texttt{---}\ \Psi\ \texttt{-----}}\,I$$

(b) [⋆] Is the following rule admissible? Is it derivable? Explain your answer

$$\dfrac{\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z}{x\ \Phi\ y\ \Psi\ z}\,I'$$

Solution: It is not derivable (as it cannot be shown with a proof tree), but it is admissible. We know this because the language definition for $\Phi\Psi$ is unambiguous, so the only way for $\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z$ to hold is if this was established by rule I. Therefore, we can deduce that $x\ \Phi\ y\ \Psi\ z$, as this is the premise of rule I. We can often "flip" or invert rules in this way, but *only if the language definition is unambiguous.*

(c) [⋆⋆] Show that $x\ \Phi\ \epsilon\ \Psi\ x$, for all hyphen strings x, by doing induction on the length of the hyphen string (where $x = \epsilon$ and $x = \texttt{-}x'$).

Solution:

Base case (where $x = \epsilon$). We must show that $\epsilon\ \Phi\ \epsilon\ \Psi\ \epsilon$, which is trivially true by rule B.

Inductive case (where $x = \texttt{-}x'$) We have the inductive hypothesis $x'\ \Phi\ \epsilon\ \Psi\ x'$, and must show that $\texttt{-}x'\ \Phi\ \epsilon\ \Psi\ \texttt{-}x'$. Our goal trivially reduces to our induction hypothesis by rule I.

Therefore we have shown $x\ \Phi\ \epsilon\ \Psi\ x$ for all x by induction on x. □

(d) [⋆⋆⋆] Show that if $\texttt{-}x\ \Phi\ y\ \Psi\ z$ then $x\ \Phi\ \texttt{-}y\ \Psi\ z$, for all hyphen strings x, y and z, by doing induction on the size of x.

Solution: We shall do induction on x where we keep y and z as arbitrary.

Base case. (where $x = \epsilon$). We must show that if $\texttt{-}\ \Phi\ y\ \Psi\ z$ then $\epsilon\ \Phi\ \texttt{-}y\ \Psi\ z$. Observe that the only way for $\texttt{-}\ \Phi\ y\ \Psi\ z$ to hold is if $z = \texttt{-}y$. Therefore we must show that $\epsilon\ \Phi\ \texttt{-}y\ \Psi\ \texttt{-}y$ which is true by rule B.

Inductive case. (From rule I, where $x = -x'$). We have the inductive hypothesis that $-x'\ \Phi\ y'\ \Psi\ z'$ implies $x'\ \Phi\ -y'\ \Psi\ z'$ for any y', z'.

We must show that $--x'\ \Phi\ y\ \Psi\ z$ implies $-x'\ \Phi\ -y\ \Psi\ z$. Observe that the only way that $--x'\ \Phi\ y\ \Psi\ z$ could hold is if $z = -k$ for some k, then by the rule I' which we have already shown to be admissible, we know that $-x'\ \Phi\ y\ \Psi\ k$. Using our induction hypothesis where $y' = y$ and $z' = k$, we can establish that $x'\ \Phi\ -y\ \Psi\ k$, and therefore by rule I we can finally conclude that $-x'\ \Phi\ -y\ \Psi\ -k$ as required. □

(e) [⋆⋆] Show that $x\ \Phi\ y\ \Psi\ z$ implies $y\ \Phi\ x\ \Psi\ z$.

Solution: We show this by rule induction on the premise with the rules of $\Phi\Psi$.

Base case. (From rule B, where $\epsilon\ \Phi\ y\ \Psi\ y$). We must show that $y\ \Phi\ \epsilon\ \Psi\ y$. We proved this, most fortunately, above in part (c).

Inductive case. (From rule I, where $-x'\ \Phi\ y\ \Psi\ -z'$). We have the inductive hypothesis that

$$y\ \Phi\ x'\ \Psi\ z'$$

We must show that $y\ \Phi\ -x'\ \Psi\ -z'$.

$$\dfrac{\dfrac{\dfrac{}{y\ \Phi\ x'\ \Psi\ z'}\,I.H}{-y\ \Phi\ x'\ \Psi\ -z'}\,I}{y\ \Phi\ -x'\ \Psi\ -z'}\,(d)$$

Thus we have shown by induction that $x\ \Phi\ y\ \Psi\ z$ implies $y\ \Phi\ x\ \Psi\ z$. □

(f) [⋆⋆] Have you figured out what the $\Phi\Psi$ system actually is? Prove that if $-^x\ \Phi\ -^y\ \Psi\ -^z$, then $z = -^{x+y}$ (where $-^x$ is a hyphen string of length x).

Solution: We proceed by rule induction on the premise.

Base case. (From rule B, where $-^0\ \Phi\ -^y\ \Psi\ -^y$), we must show that $0+y = y$, which holds trivially.
Inductive case (From rule I, where $-^{x'+1}\ \Phi\ -^y\ \Psi\ -^{z'+1}$), we have the inductive hypothesis that $z' = x' + y$. We must show that $z' + 1 = (x' + 1) + y$, or, equivalently, that $z' = x' + y$, which is just our I.H.
Thus we have shown by rule induction that the $\Phi\Psi$ system is in fact unary addition. □

3. **Ambiguity and Simultaneity**: Here is a simple grammar for a functional programming language [1]:

$$\dfrac{x \in \mathbb{N}}{x\ \mathit{Expr}}\textsc{Var.} \qquad \dfrac{e_1\ \mathit{Expr} \quad e_2\ \mathit{Expr}}{e_1 e_2\ \mathit{Expr}}\textsc{Appl.} \qquad \dfrac{e\ \mathit{Expr}}{\lambda e\ \mathit{Expr}}\textsc{Abst.} \qquad \dfrac{e\ \mathit{Expr}}{(e)\ \mathit{Expr}}\textsc{Paren.}$$

(a) [⋆] Is this grammar ambiguous? If not, explain why not. If so, give an example of an expression that has multiple parse trees.

Solution: Yes, the expression 1 2 3 could be parsed two different ways, i.e:

$$\dfrac{\dfrac{\dfrac{}{1\ \mathit{Expr}}\textsc{Var.} \quad \dfrac{}{2\ \mathit{Expr}}\textsc{Var.}}{1\ 2\ \mathit{Expr}}\textsc{Appl.} \quad \dfrac{}{3\ \mathit{Expr}}\textsc{Var.}}{1\ 2\ 3\ \mathit{Expr}}\textsc{Appl.}$$

Or:

$$\dfrac{\dfrac{}{1\ \mathit{Expr}}\textsc{Var.} \quad \dfrac{\dfrac{}{2\ \mathit{Expr}}\textsc{Var.} \quad \dfrac{}{3\ \mathit{Expr}}\textsc{Var.}}{2\ 3\ \mathit{Expr}}\textsc{Appl.}}{1\ 2\ 3\ \mathit{Expr}}\textsc{Appl.}$$

[1] if you're interested, it's called *lambda calculus*, with *de Bruijn indices* syntax, not that it's relevant to the question!

(b) [⋆⋆] Develop a new (unambiguous) grammar that encodes the left associativity of application, that is 1 2 3 4 should be parsed as ((1 2) 3) 4 (modulo parentheses). Furthermore, lambda expressions should extend as far as possible, i.e $\lambda 1\ 2$ is equivalent to $\lambda(1\ 2)$ not $(\lambda 1)2$.

Solution:

$$\frac{x \in \mathbb{N}}{x\ \mathit{AExpr}}\textsc{AVar.} \qquad \frac{e_1\ \mathit{PExpr} \quad e_2\ \mathit{AExpr}}{e_1 e_2\ \mathit{PExpr}}\textsc{AAppl.} \qquad \frac{e\ \mathit{LExpr}}{\lambda e\ \mathit{LExpr}}\textsc{AAbs.}$$

$$\frac{e\ \mathit{LExpr}}{(e)\ \mathit{AExpr}}\textsc{AParen.}y \qquad \frac{e\ \mathit{PExpr}}{e\ \mathit{LExpr}}\textsc{Shunt}_1 \qquad \frac{e\ \mathit{AExpr}}{e\ \mathit{PExpr}}\textsc{Shunt}_2$$

(c) [⋆⋆⋆] Prove that all expressions in your grammar are representable in *Expr*, that is, that your grammar describes only strings that are in *Expr*.

Solution: We shall prove the following simultaneously:

- $x\ \mathit{LExpr} \Rightarrow x\ \mathit{Expr}$
- $x\ \mathit{PExpr} \Rightarrow x\ \mathit{Expr}$
- $x\ \mathit{AExpr} \Rightarrow x\ \mathit{Expr}$

Proof. Base case (From rule AVar., where $x\ \mathit{AExpr}$ for some $x \in \mathbb{N}$). We must show $x\ \mathit{Expr}$, trivial by rule Var.

Inductive case. (From rule AAppl., where $e_1e_2\ \mathit{PExpr}$). We know $e_1\ \mathit{PExpr}$, and $e_2\ \mathit{AExpr}$ which give rise to inductive hypotheses $e_1\ \mathit{Expr}$ (I.H$_1$) and $e_2\ \mathit{Expr}$ (I.H$_2$). We must show that $e_1e_2\ \mathit{Expr}$.

$$\frac{\dfrac{}{e_1\ \mathit{Expr}}IH_1 \quad \dfrac{}{e_2\ \mathit{Expr}}IH_2}{e_1e_2\ \mathit{Expr}}$$

Inductive case. (From rule AAbs., where $\lambda x\ \mathit{LExpr}$). We know that $x\ \mathit{LExpr}$, giving inductive hypothesis $x\ \mathit{Expr}$. The rule Abs. then proves our goal: $\lambda x\ \mathit{Expr}$.

Inductive case. (From rule AParen., where $(x)\ \mathit{AExpr}$). We know that $x\ \mathit{LExpr}$, giving inductive hypothesis $x\ \mathit{Expr}$. Then by rule Paren. we show our goal $(x)\ \mathit{Expr}$.

The inductive case for the rules Shunt$_1$ and Shunt$_2$ are trivial as they do not alter the expression.

Thus, by induction, $s\ \mathit{LExpr} \lor s\ \mathit{PExpr} \lor s\ \mathit{AExpr} \implies s\ \mathit{Expr}$. We can state this more succinctly thanks to the Shunt rules as $s\ \mathit{LExpr} \implies s\ \mathit{Expr}$.

□

4. **Regular Expressions:** Consider this language used to describe regular expressions consisting of:

- single characters, written c
- Sequential composition, written $R; R$
- Nondeterministic choice, written $R \mid R$.
- Kleene star, written $R\star$.
- Grouping parentheses.

$$\frac{\mathsf{c}\ \mathbf{Char}}{\mathsf{c}\ \mathbf{R}} \qquad \frac{a\ \mathbf{R} \quad b\ \mathbf{R}}{a; b\ \mathbf{R}} \qquad \frac{a\ \mathbf{R} \quad b\ \mathbf{R}}{a \mid b\ \mathbf{R}} \qquad \frac{a\ \mathbf{R}}{a \star \mathbf{R}} \qquad \frac{a\ \mathbf{R}}{(a)\ \mathbf{R}}$$

(a) [⋆] In what way is this grammar *ambiguous*? Identify an expression with multiple parse trees.

Solution: Precedence between choice and sequential composition is not enforced, and the associativity of the two binary operators is not clarified either.

$\texttt{a}; \texttt{b} \mid \texttt{c} \mid \texttt{d}$ encounters both the precedence and the associativity issue.

(b) [$\star$] Devise an alternative grammar that is unambiguous, order of operations should be such that

$$\texttt{a}; \texttt{b}; \texttt{c} \star \ \mid \texttt{a}; \texttt{d} \mid \texttt{e}$$

is parsed with the grouping indicated by the parentheses in:

$$(\texttt{a}; (\texttt{b}; (\texttt{c}\star))) \mid ((\texttt{a}; \texttt{d}) \mid \texttt{e})$$

Solution:

$$\frac{\texttt{c}\ \textbf{Char}}{\texttt{c}\ \textbf{RAtom}} \qquad \frac{a\ \textbf{RAtom} \qquad b\ \textbf{RSeq}}{a; b\ \textbf{RSeq}} \qquad \frac{a\ \textbf{RSeq} \qquad b\ \textbf{RChoice}}{a \mid b\ \textbf{RChoice}} \qquad \frac{a\ \textbf{RAtom}}{a \star \textbf{RAtom}} \qquad \frac{a\ \textbf{RChoice}}{(a)\ \textbf{RAtom}}$$

$$\frac{a\ \textbf{RAtom}}{a\ \textbf{RSeq}} \qquad \frac{a\ \textbf{RSeq}}{a\ \textbf{RChoice}}$$

5. **Key Combinations:** Consider the language used to document key combinations:

$$\frac{x \in \{\text{a}, \text{b}, \ldots, \text{Shift}\}}{\boxed{x}\ \ \textbf{K}}\,\textit{Key} \qquad \frac{c_1\ \textbf{K} \quad c_2\ \textbf{K}}{c_1 \boldsymbol{+} c_2\ \textbf{K}}\,\textit{Hold} \qquad \frac{c_1\ \textbf{K} \quad c_2\ \textbf{K}}{c_1 c_2\ \textbf{K}}\,\textit{Then} \qquad \frac{c\ \textbf{K}}{(c)\ \textbf{K}}\,\textit{Paren}$$

For example $\boxed{\text{Ctrl}} \boldsymbol{+} \boxed{\text{C}}$ is a string in this language.

(a) [$\star$] Find an example of ambiguity in this language.

Solution: The very next question offers an example:

$$\boxed{q}\ \ \boxed{w} \boldsymbol{+} \boxed{e}\ \ \boxed{r}\ \ \boxed{t}$$

This can be parsed using *Hold* first or using *Then* first. It also shows an associativity issue with the *Then* rule.

(b) [$\star$] Eliminate ambiguity such that

$$\boxed{q}\ \ \boxed{w} \boldsymbol{+} \boxed{e}\ \ \boxed{r}\ \ \boxed{t}$$

is parsed with this grouping:

$$(\ \boxed{q}\ ((\ \boxed{w} \boldsymbol{+} \boxed{e}\ )(\ \boxed{r}\ \ \boxed{t}\ )))$$

and such that

$$\boxed{\text{Ctrl}} \boldsymbol{+} \boxed{\text{Shift} \Uparrow} \boldsymbol{+} \boxed{\text{Q}}$$

is parsed with the following grouping:

$$(\ \boxed{\text{Ctrl}} \boldsymbol{+} \boxed{\text{Shift} \Uparrow}\ ) \boldsymbol{+} \boxed{\text{Q}}$$

Solution:

$$\frac{x \in \{\text{a}, \text{b}, \ldots, \text{Shift}\}}{\boxed{x}\ \ \textbf{KA}}\,\textit{Key} \qquad \frac{c_1\ \textbf{KH} \quad c_2\ \textbf{KA}}{c_1 \boldsymbol{+} c_2\ \textbf{KH}}\,\textit{Hold} \qquad \frac{c_1\ \textbf{KH} \quad c_2\ \textbf{KT}}{c_1 c_2\ \textbf{KT}}\,\textit{Then} \qquad \frac{c\ \textbf{KT}}{(c)\ \textbf{KA}}\,\textit{Paren}$$

$$\frac{a\ \textbf{KA}}{a\ \textbf{KH}} \qquad \frac{a\ \textbf{KH}}{a\ \textbf{KT}}$$