Small/big-step correspondence: proper typeset proof

Here are the rules for our two semantics for arithmetic expressions.

First, the small step rules:

$$\frac{e_1 \mapsto e_1'}{(\texttt{Plus}\ e_1\ e_2) \mapsto (\texttt{Plus}\ e_1'\ e_2)}S_1$$

$$\frac{e_2 \mapsto e_2'}{(\texttt{Plus}\ (\mathrm{Num}\ n_1)\ e_2) \mapsto (\texttt{Plus}\ (\mathrm{Num}\ n_1)\ e_2')}S_2$$

$$\frac{}{(\texttt{Plus}\ (\mathrm{Num}\ n_1)\ (\mathrm{Num}\ n_2)) \mapsto (\mathrm{Num}\ (n_1 + n_2))}S_3$$

$$\frac{e_1 \mapsto e_1'}{(\texttt{Times}\ e_1\ e_2) \mapsto (\texttt{Times}\ e_1'\ e_2)}P_1$$

$$\frac{e_2 \mapsto e_2'}{(\texttt{Times}\ (\mathrm{Num}\ n_1)\ e_2) \mapsto (\texttt{Times}\ (\mathrm{Num}\ n_1)\ e_2')}P_2$$

$$\frac{}{(\texttt{Times}\ (\mathrm{Num}\ n_1)\ (\mathrm{Num}\ n_2)) \mapsto (\mathrm{Num}\ (n_1 \times n_2))}P_3$$

$$\frac{e_1 \mapsto e_1'}{(\texttt{Let}\ e_1\ (x.\ e_2)) \mapsto (\texttt{Let}\ e_1'\ (x.\ e_2))}L_1$$

$$\frac{}{(\texttt{Let}\ (\mathrm{Num}\ n_1)\ (x.\ e_2)) \mapsto e_2[x := (\mathrm{Num}\ n_1)]}L_3$$

And the big step rules:

$$\frac{}{(\mathrm{Num}\ n) \Downarrow n}N$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(\texttt{Plus}\ e_1\ e_2) \Downarrow (v_1 + v_2)}S$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(\texttt{Times}\ e_1\ e_2) \Downarrow (v_1 \times v_2)}P$$

$$\frac{e_1 \Downarrow v_1 \quad e_2[x := (\mathrm{Num}\ v_1)] \Downarrow v_2}{(\texttt{Let}\ e_1\ (x.\ e_2)) \Downarrow v_2}L$$

Now, to prove them equivalent, we have to prove two directions, to show that $s \overset{\star}{\mapsto} (\mathrm{Num}\ n)$ iff $s \Downarrow n$, where $\overset{\star}{\mapsto}$ (the *reflexive transitive closure*) indicates *zero or more steps*:

$$\frac{}{e \overset{\star}{\mapsto} e}\text{refl} \qquad \frac{e \mapsto e' \quad e' \overset{\star}{\mapsto} e''}{e \overset{\star}{\mapsto} e''}\text{trans}$$

If $s \Downarrow v$ then $s \overset{\star}{\mapsto} (\mathrm{Num}\ v)$

We will proceed by rule induction on the cases where $s \Downarrow v$.

Base Case ($s = (\mathtt{Num}\ n)$), from N

We must show that $(\mathtt{Num}\ n) \overset{\star}{\mapsto} (\mathtt{Num}\ n)$, obvious by rule refl.

Inductive Case ($s = (\mathtt{Plus}\ e_1\ e_2)$), from S

We know that $e_1 \Downarrow v_1$ and $e_2 \Downarrow v_2$, which gives us the inductive hypotheses:

- $\mathsf{IH}_1 - e_1 \overset{\star}{\mapsto} (\mathtt{Num}\ v_1)$
- $\mathsf{IH}_2 - e_2 \overset{\star}{\mapsto} (\mathtt{Num}\ v_2)$

Showing our overall goal:

$$\begin{array}{rcll} (\mathtt{Plus}\ e_1\ e_2) & \overset{\star}{\mapsto} & (\mathtt{Plus}\ (\mathtt{Num}\ v_1)\ e_2) & (\mathsf{IH}_1 \text{ with } S_1) \\ & \overset{\star}{\mapsto} & (\mathtt{Plus}\ (\mathtt{Num}\ v_1)\ (\mathtt{Num}\ v_2)) & (\mathsf{IH}_2 \text{ with } S_2) \\ & \overset{\star}{\mapsto} & (\mathtt{Num}\ (v_1 + v_2)) & (S_3) \end{array}$$

Inductive Case ($s = (\mathtt{Times}\ e_1\ e_2)$), from S

Extremely similar to $\mathtt{Plus}$, above.

Inductive Case ($s = (\mathtt{Let}\ e_1\ (x.\ e_2))$), from L

We know that $e_1 \Downarrow v_1$ and $e_2[x := (\mathtt{Num}\ v_1)] \Downarrow v_2$, which gives us the inductive hypotheses:

- $\mathsf{IH}_1 - e_1 \overset{\star}{\mapsto} (\mathtt{Num}\ v_1)$
- $\mathsf{IH}_2 - e_2[x := (\mathtt{Num}\ v_1)] \overset{\star}{\mapsto} (\mathtt{Num}\ v_2)$

Showing our overall goal:

$$\begin{aligned}
(\texttt{Let}\ e_1\ (x.e_2)) &\overset{\star}{\mapsto} (\texttt{Let}\ (\texttt{Num}\ v_1)\ (x.e_2)) & \text{(rule } L_1 \text{ with } \mathsf{IH}_1) \\
&\overset{\star}{\mapsto} e_2[x := (\texttt{Num}\ v_1)] & (L_2) \\
&\overset{\star}{\mapsto} (\texttt{Num}\ v_2) & (\mathsf{IH}_2)
\end{aligned}$$

Thus, by mathematical induction, we have shown one direction of the equivalence.

If $s \overset{\star}{\mapsto} (\texttt{Num}\ v)$ then $s \Downarrow v$

Doing rule induction on the assumption $s \overset{\star}{\mapsto} (\texttt{Num}\ v)$ leads to two cases.

Base case ($s = (\texttt{Num}\ v)$), from refl

We know that $(\texttt{Num}\ v) \Downarrow v$ from rule N.

Inductive case ($s \mapsto s'$ and $s' \overset{\star}{\mapsto} (\texttt{Num}\ v)$), from trans

We have the inductive hypothesis that $s' \Downarrow v$, so it suffices to prove the following lemma in order to discharge this case.

$$\frac{s \mapsto s' \qquad s' \Downarrow v}{s \Downarrow v}$$

Lemma: If $s \mapsto s'$ and $s' \Downarrow v$ then $s \Downarrow v$.

Written as a logical statement, this lemma is:

$$\forall v.\ s \mapsto s' \wedge s' \Downarrow v \Rightarrow s \Downarrow v$$

Equivalently, this can be stated as:

$$s \mapsto s' \Rightarrow \forall v.\ s' \Downarrow v \Rightarrow s \Downarrow v$$

This formulation lets us proceed by rule induction on the assumption $s \mapsto s'$, proving for each case for any arbitrary v:

$$\forall v.\ \frac{s' \Downarrow v}{s \Downarrow v}$$

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Base case from rule S_3

Here $s = (\texttt{Plus}\ (\texttt{Num}\ n)\ (\texttt{Num}\ m))$ and $s' = (\texttt{Num}\ n + m)$.

We have to show that $(\texttt{Plus}\ (\texttt{Num}\ n)\ (\texttt{Num}\ m)) \Downarrow v$ assuming that $(\texttt{Num}\ n + m) \Downarrow v$. The only way that assumption could hold, looking at the rules of $\Downarrow$, is if $v = n + m$ from rule N. Therefore we must show that $(\texttt{Plus}\ (\texttt{Num}\ n)\ (\texttt{Num}\ m)) \Downarrow n + m$, which is trivial from rules S and N.

Inductive case from rule S_1

Here $s = (\texttt{Plus}\ e_1\ e_2)$ and $s' = (\texttt{Plus}\ e_1'\ e_2)$. We know that $e_1 \mapsto e_1'$, giving the inductive hypothesis that:

$$\forall v.\ \frac{e_1' \Downarrow v}{e_1 \Downarrow v}\text{IH}$$

We must show that $(\texttt{Plus}\ e_1\ e_2) \Downarrow v$ assuming that $(\texttt{Plus}\ e_1'\ e_2) \Downarrow v$. Looking at the rules for $\Downarrow$, the only way that $(\texttt{Plus}\ e_1'\ e_2) \Downarrow v$ could hold is if $v = x + y$ and $e_1' \Downarrow x$ and $e_2 \Downarrow y$ (by rule P). By the inductive hypothesis, we have that $e_1 \Downarrow x$. Therefore, $(\texttt{Plus}\ e_1\ e_2) \Downarrow v$ as required.

Inductive case from rule S_2

Here $s = (\texttt{Plus}\ (\texttt{Num}\ n)\ e_2)$ and $s' = (\texttt{Plus}\ (\texttt{Num}\ n)\ e_2')$. We know that $e_2 \mapsto e_2'$, giving the inductive hypothesis that:

$$\forall v.\ \frac{e_2' \Downarrow v}{e_2 \Downarrow v}\text{IH}$$

We must show that $(\texttt{Plus}\ (\texttt{Num}\ n)\ e_2) \Downarrow v$ assuming that $(\texttt{Plus}\ (\texttt{Num}\ n)\ e_2') \Downarrow v$. Looking at the rules for $\Downarrow$, the only way that $(\texttt{Plus}\ (\texttt{Num}\ n)\ e_2') \Downarrow v$ could hold is if $v = n + y$ and $e_2' \Downarrow y$ (by rule P). By the inductive hypothesis, we have that $e_2 \Downarrow y$. Therefore, $(\texttt{Plus}\ (\texttt{Num}\ n)\ e_2) \Downarrow v$ as required.

Cases for `Times`

All analogous to the cases for `Plus`.

Inductive case from rule L_1

Here $s = (\texttt{Let}\ e_1\ (\texttt{x.}\ e_2))$ and $s' = (\texttt{Let}\ e_1'\ (\texttt{x.}\ e_2))$. We know that $e_1 \mapsto e_1'$, giving the inductive hypothesis that:

$$\forall v.\ \frac{e_1' \Downarrow v}{e_1 \Downarrow v}\mathrm{IH}$$

We must show that $(\texttt{Let}\ e_1\ (\texttt{x.}\ e_2)) \Downarrow v$ assuming that $(\texttt{Let}\ e_1'\ (\texttt{x.}\ e_2)) \Downarrow v$. Looking at the rules for $\Downarrow$, the only way for that assumption to hold is if there is some v_x such that $e_1' \Downarrow v_x$ and $e_2[x := (\texttt{Num}\ v_x)] \Downarrow v$. By the inductive hypothesis we have that $e_1 \Downarrow v_x$ and therefore we have that $(\texttt{Let}\ e_1\ (\texttt{x.}\ e_2)) \Downarrow v$ from rule L.

Base case from rule L_2

Here $s = (\texttt{Let}\ (\texttt{Num}\ n)\ (\texttt{x.}\ e_2))$ and $s' = e_2[x := (\texttt{Num}\ n)]$. We must show that $(\texttt{Let}\ (\texttt{Num}\ n)\ (\texttt{x.}\ e_2))$ assuming that $e_2[x := (\texttt{Num}\ n)] \Downarrow v$. This can be shown trivially by application of the rule L.