Mobile Robotics

1 EKF: Computing H matrix

The nonlinear mapping from state $\mathbf{u_t}$ to observation $\mathbf{z_t}$, with a given landmark position ($\mathbf{m_x}$,$\mathbf{m_y}$), is defined as follows-

$$\mathbf{\hat{z}_{t+1}} = \begin{bmatrix} \sqrt{(\hat{\mu}_{x,t+1} - m_x)^2 + (\hat{\mu}_{y,t+1} - m_y)^2} \\ \tan^{-1}(\frac{\hat{\mu}_{y,t+1} - m_y}{\hat{\mu}_{x,t+1} - m_x}) - \hat{\mu}_{\theta,t+1} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r_{t+1}}} \\ \hat{\psi}_{\mathbf{t+1}} \end{bmatrix} \tag{1}$$

Also,

$$\mathbf{\hat{z}_{t+1}} = h(\hat{\mu}_{\mathbf{t+1}}) \tag{2}$$

Applying Taylor's series to expand,

$$\mathbf{\hat{z}_{t+1}} = h(\hat{\mu}_{\mathbf{0}}) + \left(\frac{\partial h}{\partial \hat{\mu}_{\mathbf{t+1}}}\right)^T \Big|_{\hat{\mu}_0} (\hat{\mu} - \hat{\mu}_0) + (\hat{\mu} - \hat{\mu}_0)^T \left(\frac{\partial^2 h}{\partial \hat{\mu}^2_{\mathbf{t+1}}}\right) \Big|_{\hat{\mu}_0} (\hat{\mu} - \hat{\mu}_0) \tag{3}$$

Ignoring the higher order terms, $\mathbf{z_{t+1}}$ can be linearized about $\hat{\mu_{\mathbf{0}}}$ as

$$\mathbf{\hat{z}_{t+1}} = h(\hat{\mu}_{\mathbf{0}}) + H(\hat{\mu} - \hat{\mu}_0) \tag{4}$$

where,

$$H = \left(\frac{\partial h}{\partial \hat{\mu}_{\mathbf{t+1}}}\right) \Big|_{\hat{\mu}_0} = \begin{bmatrix} \frac{\partial \hat{r}_{t+1}}{\partial \mu_{x,t+1}} & \frac{\partial \hat{r}_{t+1}}{\partial \mu_{y,t+1}} & \frac{\partial \hat{r}_{t+1}}{\partial \mu_{\theta,t+1}} \\ \frac{\partial \hat{\psi}_{t+1}}{\partial \hat{\mu}_{x,t+1}} & \frac{\partial \hat{\psi}_{t+1}}{\partial \hat{\mu}_{y,t+1}} & \frac{\partial \hat{\psi}_{t+1}}{\partial \hat{\mu}_{\theta,t+1}} \end{bmatrix} \tag{5}$$

This results in the following H matrix-

$$H = \begin{bmatrix} \frac{\hat{\mu}_{x,t+1} - m_x}{\hat{r}_{t+1}} & \frac{\hat{\mu}_{y,t+1} - m_y}{\hat{r}_{t+1}} & 0 \\ \frac{-(\hat{\mu}_{y,t+1} - m_y)}{\hat{r}^2_{t+1}} & \frac{(\hat{\mu}_{x,t+1} - m_x)}{\hat{r}^2_{t+1}} & -1 \end{bmatrix} \tag{6}$$

Using

$$\mathbf{\hat{r}^2_{t+1}} = (\hat{\mu}_{x,t+1} - m_x)^2 + (\hat{\mu}_{y,t+1} - m_y)^2 \tag{7}$$

and

$$\hat{\psi}_{t+1} = \tan^{-1}(\frac{\hat{\mu}_{y,t+1} - m_y}{\hat{\mu}_{x,t+1} - m_x}) - \hat{\mu}_{\theta,t+1} \tag{8}$$

Let

$$\hat{\mu}_{x,t+1} - m_x = -\hat{r}_{t+1}\cos(\hat{\psi}_{t+1} + \hat{\mu}_{\theta,t+1}), \tag{9}$$

$$\hat{\mu}_{y,t+1} - m_y = -\hat{r}_{t+1}\sin(\hat{\psi}_{t+1} + \hat{\mu}_{\theta,t+1}), \tag{10}$$

From equations 9 and 10, H matrix in equation 5 can be rewritten as

$$H = \begin{bmatrix} -\cos(\hat{\psi}_{t+1} + \hat{\mu}_{\theta,t+1}) & -\sin(\hat{\psi}_{t+1} + \hat{\mu}_{\theta,t+1}) & 0 \\ \frac{\sin(\hat{\psi}_{t+1}+\hat{\mu}_{\theta,t+1})}{\hat{r}_{t+1}} & \frac{-\cos(\hat{\psi}_{t+1}+\hat{\mu}_{\theta,t+1})}{\hat{r}_{t+1}} & -1 \end{bmatrix} \tag{11}$$

2 Using motion model with control inputs as α, T, β

2.1 Advantage of using this model over the model with control inputs as ϕ, T

The motion model of a robot with control inputs as ϕ, **T** is given by-

$$\begin{bmatrix} \hat{\mu}_{x,t+1} \\ \hat{\mu}_{y,t+1} \\ \hat{\mu}_{\theta,t+1} \end{bmatrix} = \begin{bmatrix} \mu_{x,t} \\ \mu_{y,t} \\ \mu_{\theta,t} \end{bmatrix} + \begin{bmatrix} Tcos(\mu_{\theta,t} + \phi) \\ Tsin(\mu_{\theta,t} + \phi) \\ \phi \end{bmatrix} \tag{12}$$

The motion model of a robot with control inputs as α, **T**, β, is given by-

$$\begin{bmatrix} \hat{\mu}_{x,t+1} \\ \hat{\mu}_{y,t+1} \\ \hat{\mu}_{\theta,t+1} \end{bmatrix} = \begin{bmatrix} \mu_{x,t} \\ \mu_{y,t} \\ \mu_{\theta,t} \end{bmatrix} + \begin{bmatrix} Tcos(\mu_{\theta,t} + \alpha) \\ Tsin(\mu_{\theta,t} + \alpha) \\ \alpha + \beta \end{bmatrix} \tag{13}$$

The advantage of using the model f equation 13, over that of equation 12, is that it nullifies the error introduced by odometer's measurement (backlash error) or actuator's motion (which may be due to loose wheels or worn out actuators). For eg., if the odometer's reading is erroneous, say by $\pm\Delta\theta$, then by keeping the direction of α and β opposite with respect to each other, the net error gets nullified.

For instance, if the desired angle $\theta_d = \alpha - \beta$. However, the odometer reads the angles with errors, and measures $\alpha + \Delta\theta$ and $\beta + \Delta\theta$. The net rotation(assuming α in counter-clockwise(+) and β in clockwise(-) direction) θ is

$$\theta = (\alpha + \boldsymbol{\Delta}\theta) - (\beta + \boldsymbol{\Delta}\theta) = \alpha - \beta = \theta_{\mathbf{t}} \tag{14}$$

2.2 Motion model of the robot with control inputs as ϕ, T

The motion model for a differential drive robot with current state $\mathbf{x_t}$ and control $\mathbf{u_t}$ is as follows

$$\begin{bmatrix} \hat{\mu}_{x,t+1} \\ \hat{\mu}_{y,t+1} \\ \hat{\mu}_{\theta,t+1} \end{bmatrix} = \begin{bmatrix} \mu_{x,t} \\ \mu_{y,t} \\ \mu_{\theta,t} \end{bmatrix} + \begin{bmatrix} Tcos(\mu_{\theta,t} + \alpha) \\ Tsin(\mu_{\theta,t} + \alpha) \\ \alpha + \beta \end{bmatrix} \tag{15}$$

where,

$$\mathbf{u}_t = \begin{bmatrix} T \\ \alpha \\ \beta \end{bmatrix} \tag{16}$$

2.3 Jacobian of this motion model for the covariance update step of the EKF

The motion model can be generalized as

$$\hat{\mu}_{\mathbf{t+1}} = f(\mu_{\mathbf{t}}, \mathbf{u_t}) \tag{17}$$

The covariance of the estimated state is

$$\hat{\mathbf{\Sigma}}_{\mathbf{t+1}} = \mathbf{F}\mathbf{\Sigma_t}\mathbf{F^T} + \mathbf{G}\mathbf{\Sigma_u}\mathbf{G^T} \tag{18}$$

where

$$\mathbf{F} = \frac{\partial f}{\partial \mu_{\mathbf{t}}} = \begin{bmatrix} \frac{\partial \hat{\mu}_{x,t+1}}{\partial \mu_{x,t}} & \frac{\partial \hat{\mu}_{x,t+1}}{\partial \mu_{y,t}} & \frac{\partial \hat{\mu}_{x,t+1}}{\partial \mu_{\theta,t}} \\ \frac{\partial \hat{\mu}_{y,t+1}}{\partial \mu_{x,t}} & \frac{\partial \hat{\mu}_{y,t+1}}{\partial \mu_{y,t}} & \frac{\partial \hat{\mu}_{y,t+1}}{\partial \mu_{\theta,t}} \\ \frac{\partial \hat{\mu}_{\theta,t+1}}{\partial \mu_{x,t}} & \frac{\partial \hat{\mu}_{\theta,t+1}}{\partial \mu_{y,t}} & \frac{\partial \hat{\mu}_{\theta,t+1}}{\partial \mu_{\theta,t}} \end{bmatrix} \tag{19}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & -Tsin(\mu_{\theta,t} + \alpha) \\ 0 & 1 & Tcos(\mu_{\theta,t} + \alpha) \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$\mathbf{G} = \frac{\partial f}{\partial \mathbf{u_t}} = \begin{bmatrix} \frac{\partial \hat{\mu}_{x,t+1}}{\partial T} & \frac{\partial \hat{\mu}_{x,t+1}}{\partial \alpha} & \frac{\partial \hat{\mu}_{x,t+1}}{\partial \beta} \\ \frac{\partial \hat{\mu}_{y,t+1}}{\partial T} & \frac{\partial \hat{\mu}_{y,t+1}}{\partial \alpha} & \frac{\partial \hat{\mu}_{y,t+1}}{\partial \beta} \\ \frac{\partial \hat{\mu}_{\theta,t+1}}{\partial T} & \frac{\partial \hat{\mu}_{\theta,t+1}}{\partial \alpha} & \frac{\partial \hat{\mu}_{\theta,t+1}}{\partial \beta} \end{bmatrix} \tag{21}$$

$$\mathbf{G} = \begin{bmatrix} cos(\mu_{\theta,t} + \alpha) & -Tsin(\mu_{\theta,t} + \alpha) & 0 \\ sin(\mu_{\theta,t} + \alpha) & Tcos(\mu_{\theta,t} + \alpha) & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{22}$$

3 Uncertainity ellipse of a robot without updating its states in EKF

With a given motion model of the robot, the estimated mean $\hat{\mu}_{\mathbf{t+1}}$ and co-variance $\hat{\mathbf{\Sigma}}_{\mathbf{t+1}}$ of the states are determined as-

$$\hat{\mu}_{\mathbf{t+1}} = f(\mu_{\mathbf{t}}, \mathbf{u_t}) \tag{23}$$

$$\hat{\mathbf{\Sigma}}_{\mathbf{t+1}} = \mathbf{F}\mathbf{\Sigma_t}\mathbf{F^T} + \mathbf{G}\mathbf{\Sigma_u}\mathbf{G^T} \tag{24}$$

Here, **F** and **G** correspond to the Jacobians with respect to the state and the control vectors respectively.

$$\mathbf{F} = \frac{\partial f}{\partial \mu_{\mathbf{t}}}, \mathbf{G} = \frac{\partial f}{\partial \mathbf{u_t}} \tag{25}$$

and $\mathbf{\Sigma_t}$ is a positive semi-definite matrix. Rewriting equation 24 as

$$\hat{\mathbf{\Sigma}}_{\mathbf{t+1}} = \begin{bmatrix} \mathbf{F} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma_t} & 0 \\ 0 & \mathbf{\Sigma_u} \end{bmatrix} \begin{bmatrix} \mathbf{F}^T \\ \mathbf{G}^T \end{bmatrix} \tag{26}$$

Here, $\mathbf{\Sigma}_u$ includes the uncertainity associated with control input and the gaussian noise associated with state transition. Mathematically,

$$|\hat{\mathbf{\Sigma}}_{\mathbf{t+1}}| \geq |\mathbf{F}\mathbf{\Sigma_t}\mathbf{F^T}| + |\mathbf{G}\mathbf{\Sigma_u}\mathbf{G^T}| \tag{27}$$

The above inequality holds true, since $\mathbf{F\Sigma_t F^T}$ and $\mathbf{G\Sigma_u G^T}$ are symmetric and positive semi-definite matrices, as shown below-

$(\mathbf{F\Sigma_t F^T})^\mathbf{T} = \mathbf{F\Sigma_t^T F^T} = \mathbf{F\Sigma_t F^T}$
$(\mathbf{G\Sigma_u G^T})^\mathbf{T} = \mathbf{G\Sigma_u^T G^T} = \mathbf{G\Sigma_u G^T}$

For the motion model of equation 15, F matrix is as follows-

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & -Tsin(\mu_{\theta,t} + \phi) \\ 0 & 1 & Tcos(\mu_{\theta,t} + \phi) \\ 0 & 0 & 1 \end{bmatrix} \tag{28}$$

$\det(\mathbf{F}) = 1$, We may consider the following 2 cases to determine the behaviour of $\hat{\mathbf{\Sigma}}_{\mathbf{t+1}}$ with respect to time-

1. $\mathbf{\Sigma}_u = 0$,
   If either the control is assumed to be perfect or the gaussian noise associated with sate transition is assumed to be 0, then $\mathbf{\Sigma}_u = 0$. therefore,

   $$|\hat{\mathbf{\Sigma}}_{\mathbf{t+1}}| = |\mathbf{F}||\mathbf{\Sigma_t}||\mathbf{F^T}| = |\Sigma_t|, \tag{29}$$

   which implies that the variance of the estimated state will not grow with time and will remain same as the robot continues its motion.

2. $\mathbf{\Sigma}_u \neq 0$,
   $\hat{\mathbf{\Sigma}}_{\mathbf{t+1}} > \mathbf{\Sigma}_t$, since it is the sum of 2 positive semi-definite matrices, and therefore will keep on increasing with time.

   $$|\hat{\mathbf{\Sigma}}_{\mathbf{t+1}}| \geq |\mathbf{F}||\mathbf{\Sigma_t}||\mathbf{F^T}| + |\mathbf{G\Sigma_u G^T}| = |\mathbf{\Sigma_t}| + |\mathbf{G\Sigma_u G^T}| > |\mathbf{\Sigma_t}| \tag{30}$$

   Relationship between $|\hat{\Sigma}_{t+1}|$ and area of the ellipse-

For an n-Dimension Gaussian, the length of the axes of the Gaussian ellipsoid are determined by the eigen values of the covariance matrix Σ.
Consider a 2-D gaussian, where the length of the semi-major and semi-minor axes of the ellipse are determined as $\sqrt{\beta\lambda_1}$ and $\sqrt{\beta\lambda_2}$, where β is the chi-square value for a particular confidence level, α, and λ_1 and λ_2 are the eigen values of the matrix Σ. The area of the ellipse spanned by these axes is-
Area $= \Pi * \sqrt{\beta\lambda_1}\sqrt{\beta\lambda_2} = \Pi * \beta\sqrt{\lambda_1\lambda_2} = \Pi * \beta\sqrt{\det(\Sigma)}$,
Since, Σ is a positive semi-definite matrix,
$|\hat{\Sigma}_{t+1}| \geq |\Sigma_t| \Rightarrow \sqrt{|\hat{\Sigma}_{t+1}|} \geq \sqrt{|\Sigma_t|} \Rightarrow$ Area spanned by the covariance ellipse will vary similar to $|\hat{\Sigma}_{t+1}|$.

From the above 2 considerations and the relationship between area of gaussian ellipse and $|\hat{\Sigma}_{t+1}|$, it can be inferred that the uncertainity ellipse will be a non-decreasing function, with respect to time, if the robot keeps on moving without updating its states' knowledge.