Seminar Computational Mathematics

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1 AMB+ Model

The AMB+ Model is given by:

$$
\begin{aligned}
\partial_t \phi &= -\nabla \cdot (J + \sqrt{2DM}\Lambda) \\
&= -\nabla \cdot (M(-\nabla \mu_\lambda + \zeta(\nabla^2 \phi)\nabla\phi) + \sqrt{2DM}\Lambda) \\
&= -\nabla \cdot (M(-\nabla(\frac{\partial F}{\partial \phi} + \lambda|\nabla\phi|^2) + \zeta(\nabla^2\phi)\nabla\phi) + \sqrt{2DM}\Lambda)
\end{aligned}
$$

Where we have F and its functional derivative $\frac{\partial F}{\partial \phi}$ given by

$$
\begin{aligned}
F &= \int \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 + \frac{\epsilon}{2}|\nabla\phi|^2 dr \\
\frac{\partial F}{\partial \phi} &= a\phi + b\phi^3 - \epsilon\nabla^2\phi
\end{aligned}
$$

The parameters $a, b, \lambda, \epsilon, \zeta, M, D \in \mathbb{R}$ and Λ is standard normal gaussian noise.
Overall we have:

$$
\begin{aligned}
\partial_t \phi &= -\nabla \cdot (M(-\nabla(a\phi + b\phi^3 - \epsilon\nabla^2\phi + \lambda|\nabla\phi|^2) + \zeta(\nabla^2\phi)\nabla\phi) + \sqrt{2DM}\Lambda) \\
&= aM\nabla^2\phi + bM\nabla^2(\phi^3) - \epsilon M\nabla^4\phi + \lambda M\nabla^2|\nabla\phi|^2 - \zeta M\nabla \cdot (\nabla^2\phi)\nabla\phi - \nabla \cdot \sqrt{2DM}\Lambda) \\
&=: L(\phi) + N(\phi) - \nabla \cdot \sqrt{2DM}\Lambda
\end{aligned}
$$

Defining

$$
\begin{aligned}
L(\phi) &= aM\nabla^2\phi - \epsilon M\nabla^4\phi \\
N(\phi) &= bM\nabla^2(\phi^3) + \lambda M\nabla^2|\nabla\phi|^2 - \zeta M\nabla \cdot (\nabla^2\phi)\nabla\phi
\end{aligned}
$$

2 Pseudospectral Fourier Method applied on AMB+

We can now use the pseudospectral Fourier method as follows:

$$
\begin{aligned}
\partial_t \widehat{\phi} &= \widehat{L} \cdot \widehat{\phi} + \widehat{N(\phi)} - \sqrt{2DM} i (\vec{k} \cdot \widehat{\Lambda}) \\
&= (-aM\vec{k}^2 - \epsilon M \vec{k}^4)\widehat{\phi} + \widehat{N(\phi)} - \sqrt{2DM} i (\vec{k} \cdot \widehat{\Lambda})
\end{aligned}
$$

We discretize using a semi-implicit Euler scheme (with scaling for the stochastic part):

$$
\begin{aligned}
\widehat{\phi}^{(n+1)} &= \widehat{\phi}^{(n)} + \tau((-aM\vec{k}^2 - \epsilon M\vec{k}^4)\widehat{\phi}^{(n+1)} + \widehat{N(\phi^{(n)})}) - \sqrt{2DM\tau} i (\vec{k} \cdot \widehat{\Lambda}^{(n)}) \\
\implies \widehat{\phi}^{(n+1)} &= \frac{\widehat{\phi}^{(n)} + \tau \widehat{N(\phi^{(n)})} - \sqrt{2DM\tau} i (\vec{k} \cdot \widehat{\Lambda}^{(n)})}{1 - \tau(-\epsilon M \vec{k}^4 - aM\vec{k}^2)}
\end{aligned}
$$

Here we can compute the nonlinear term, given $\phi^{(n)}, \nabla\phi^{(n)}, \nabla^2\phi^{(n)}$, as

$$
\widehat{N(\phi^{(n)})} = -bM\vec{k}^2 \widehat{\phi^{(n)}}^3 - \lambda M \vec{k}^2 \widehat{|\nabla\phi^{(n)}|^2} - \zeta M i \vec{k} \cdot \widehat{(\nabla^2\phi^{(n)})\nabla\phi^{(n)}}
$$

Substitution into the time-stepping scheme above leads to the overall formula

$$
\widehat{\phi}^{(n+1)} = \frac{\widehat{\phi}^{(n)} + \tau(-M\vec{k}^2 \widehat{(b{\phi^{(n)}}^3 + \lambda|\nabla\phi^{(n)}|^2)} - \zeta M i \vec{k} \cdot \widehat{(\nabla^2\phi^{(n)})\nabla\phi^{(n)}}) - \sqrt{2DM\tau} i (\vec{k} \cdot \widehat{\Lambda}^{(n)})}{1 + \tau(\epsilon M \vec{k}^4 + aM\vec{k}^2)}
$$

where we compute $\phi^{(n)}, \nabla\phi^{(n)}, \nabla^2\phi^{(n)}$ from $\widehat{\phi}^{(n)}$ by

$$
\begin{aligned}
\phi^{(n)} &= \mathcal{F}^{-1}(\widehat{\phi}^{(n)}) \\
\nabla\phi^{(n)} &= \mathcal{F}^{-1}(i\vec{k}\widehat{\phi}^{(n)}) \\
\nabla^2\phi^{(n)} &= \mathcal{F}^{-1}(-\vec{k}^2\widehat{\phi}^{(n)})
\end{aligned}
$$

which immediately implies an algorithmic implementation.

3 Algorithm/Implementation

Algorithm 1: Solving AMB+ with Fourier Pseudospectral Method and semi-implicit Euler

Data: $\phi^{(0)} \in \mathbb{R}^d, a, b, \lambda, \epsilon, \zeta, M, D, \tau \in \mathbb{R}, N \in \mathbb{N},$

Result: $\phi^{(N+1)}$

$\widehat{\phi}^{(0)} \leftarrow FFT(\phi^{(0)})$

Precompute $\vec{k}^2, \vec{k}^4$

`// Time-stepping loop`

for $n = 0, \ldots, N$ **do**

  Compute $\nabla\phi^{(n)}, \nabla^2\phi^{(n)}$ from $\widehat{\phi}^{(n)}$ using iFFT

  Sample random white noise $\Lambda^{(n)}$

  Compute $\widehat{\phi}^{(n+1)}$ as above

  Compute $\phi^{(n+1)}$ from $\widehat{\phi}^{(n+1)}$ using iFFT

end