Problem Set 5

5/5/2021

1.) Dry days happened 82% of the days

0.01 precipitation, or light rain happened 2% of days.

wet days $= 1 - 0.82 - 0.02 = 0.16$

state	
1	$P(X_n = 1) = 82\%$
2	$P(X_n = 2) = 2\%$
3	$P(X_n = 3) = 16\%$

$$\begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \mathbb{P} = & \begin{pmatrix} 0.91 & 0.01 & 0.08 \\ 0.57 & 0.16 & 0.27 \\ 0.4 & 0.07 & 0.53 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Transition matrix

a.) $\boxed{P(j=1 \mid i=2) = 0.01 \quad P(j=3 \mid i=3) = 0.07}$

b.) $i = 1 \quad j = 2 \quad n = 10$ use chapman Kolmogorov

$P(X_n = 2) = \tilde{\pi}_2(n)$

$\tilde{\pi}_0 \mathbb{P}^n \to \tilde{\pi}_0 (\mathbb{P}^{10})$

$$\mathbb{P}^n \cdot \tilde{\pi}_1 = (0.91, 0.01, 0.08) \cdot \begin{pmatrix} 0.91 & 0.01 & 0.08 \\ 0.57 & 0.16 & 0.27 \\ 0.4 & 0.07 & 0.53 \end{pmatrix}$$

$(\quad , \quad , \quad)$

$\tilde{\pi}_1(10) = (0.0112, 0.0011, 0.0068)$ $\boxed{P(X_{10}=2 \mid X_0=1) = 0.0011 = 0.11\%}$

$\tilde{\pi}_1(10) = (0.0221, 0.0021, 0.0133)$ $\boxed{P(X_{10}=2 \mid X_0=1) = 0.0133 = 1.33\%}$

c.) The probability of b is much smaller than a, and by law of total probability its more likely a light rain happens on day 10

2.) Random walks on a finite set $1, 2, \ldots, N$ movement between locations

$X_{n+1} = X_n \pm 1 \quad P(X_n = 1 + X_{n-1}) = P(X_n = X_{n-1} - 1)$, when $2 \leq X_n \leq n-1$ $\quad N = 5$

Edges $X_{n+1} = X_n + 1$ when $X_n = 1$, and $X_{n+1} = X_n - 1$

$[1] \; [2, \ldots, N-1] \; [N]$

a.) P

1 2 3 4 5

b.) $X_0 = 1$, π_i $\quad i = 1 \quad P(X_n = i) = \pi_i(n)$

$X = $ a vector $\quad P_n f(X_n) = \pi_1(3) = \pi P^n$

1, +

$0. * [0, 1, 0, 0, 0] = [0, 0, 0,$

$1 * [\frac{1}{2}, 0, \frac{1}{2}, 0, 0] = [\frac{1}{2}, 0, \frac{1}{2}$

no memory between locations

$X_{n+1} = X_n \pm 1 \quad P(X_n = 1 + X_{n-1}) = P(X_n = X_{n-1} - 1)$, when $2 \leq X_n \leq n-1$ $\qquad N = 5$

Else $X_{n+1} = X_n + 1$ when $X_n = 1$, and $X_{n+1} = X_n - 1$

[1] [2, ..., N-1] [N]

a.) P

$$\begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$i = 1$

b.) $X_0 = 1, \; i$. $\quad P(X_n = i) = \pi_i(n)$

$X_3 =$ a vector. $\quad Pmf(X_3) = \pi_1(3) = \tilde{\pi}_3 P^n$

$= \tilde{\pi}_0 P^3$

$X_3 = X_0$

$\tilde{\pi}_0 : X_1 = (0, 1, 0 0 0)$

1st

$0 \cdot [0, 1, 0, 0, 0] = [0, 0, 0, 0, 0]$

$1 \cdot [\frac{1}{2}, 0, \frac{1}{2}, 0, 0] = [\frac{1}{2}, 0, \frac{1}{2}, 0, 0]$

$0 \cdot [0, \frac{1}{2}, 0, \frac{1}{2}, 0] = [0, 0, 0, 0, 0]$

$0 \cdot [0, 0, \frac{1}{2}, 0, \frac{1}{2}] = [0, 0, 0, 0, 0]$

$0 \cdot [0, 0, 0, 1, 0] = [0, 0, 0, 0, 0]$ +

$\tilde{\pi}_1 = \tilde{\pi}_0 \cdot P = [\frac{1}{2}, 0, \frac{1}{2}, 0, 0]$

2nd

$\frac{1}{2} \cdot *$

$0 \cdot *$

$\frac{1}{2} \cdot *$

$0 \cdot *$

$0 \cdot *$

$$\begin{bmatrix} 0, \frac{1}{2}, 0, 0, 0 \\ 0, 0, 0, 0, 0 \\ 0, \frac{1}{4}, 0, \frac{1}{4}, 0 \\ 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0 \end{bmatrix} +$$

$\tilde{\pi}_2 = \tilde{\pi}_1 \cdot P = (0, \frac{3}{4}, 0, \frac{1}{4}, 0)$

3rd

$0 \cdot *$

$\frac{3}{4} \cdot *$

$0 \cdot *$

$\frac{1}{4} \cdot *$

$0 \cdot *$

$$\begin{bmatrix} \_\_\_\_ \\ \frac{3}{8}, 0, \frac{3}{8}, 0, 0 \\ \_\_\_\_ \\ 0, 0, \frac{1}{8}, 0, \frac{1}{8} \\ \_\_\_\_ \end{bmatrix}$$

$$\boxed{Pmf(X_3) = \tilde{\pi}_3(i) = (\frac{3}{8}, 0, \frac{1}{2}, 0, \frac{1}{8})}$$

3.) For all states, classify them as transient or recurrent, and classify markov chains as irreducible or reducible.

I hate puzzles

$$P = \begin{pmatrix} 0 & 1-p & p & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{pmatrix}$$

- No state is recurrent, as none are self reflexive in one step.
- Each state can communicate with all the states ∴ irreducible.
- Irreducible,

It's easy to prove it can grow up to any number... So prove it can famine

$x_i = 1 \quad x_j = 0$

$\binom{1}{0}(1-d)(d)^1 \cdot \frac{e^{-\lambda}\lambda^0}{0!} > 0$ ∴ irreducible across all S.,

$1 > \quad 1 > d > 0$ but only when $d < 1$.

$p(x) = \binom{n}{k} p^k (1-p)^{n-k}$

(c) $m_n = E[X_n] \quad m_{n+1} = b + (1-d) m_n \to b + (1-d) E[X_n]$

$= E\left[B + \binom{i}{j}(1-d)^{i-j}(d)^{j} \frac{e^{-\lambda}\lambda^{j}}{j!}\right] = \sum \text{Poisson}(B) + \sum_{i=0}^{n} x \, \overbrace{P(B \cdot (x_i)(1-d)) / X_{i-1})}^{(1-d)} \cdot P(X_i)$

$B = \bar{\lambda}$

$X_i \cdot P(X_i | X_{i-1}) = (1-d)$

because = probability of survive

$$\boxed{(1-d) \sum_{i=0}^{n} x_{i-1} P(x_{i-1}) = E(X_i) \cdot (1-d)}$$

d) $\lim_{n \to \infty} m_n = m^* = b + (1-d) m^*$ prove

$\lim_{n \to \infty} m_n = E(m_\infty) = b + (1-d) m^*$, as proven above the expectation is recursive.

e) Wright Fisher Model X_n Population size N. Show that $E[X_{n+1} | \{X_n = x\}] = x$ = Prove $X_{n+1} =$

$X_{n+1} = j$

Genetic drift

$\sum_{n=0}^{N} X_n P(X_n)$

$X_n = A$

$X_{n-1} = x$

i copies of allele, N individuals, k = # of copies of allele = $[0, \ldots, 2N]$

$P(X_n) = \binom{2N}{k}\left(\frac{i}{2N}\right)^k \left(1 - \frac{i}{2N}\right)^{2N-k}$

$\sum_{n=0}^{N} X_{n+1} \binom{2N}{X_{n+1}} \left(\frac{x}{2N}\right)^k \left(1 - \frac{x}{2N}\right)^{2N-k} = x = E(X$

4.) Open population. X_n = # organisms at time step n. At each time step, existing population is lost due to an independent probability, $d > 0$. P(death) = d   # dead = poisson variable, mean $b > 0$

$\to d$ = individual probability the organism will die

a.) $S = [0, 1, 2 \ldots] = \mathbb{R} \geq 0$

I can start at zero, and something will be added.

a poisson # of individuals are added to population, average is greater than zero. This component, $= b \approx \lambda$

b.) Prove irreducible = 1 class of equivalence

Binomial of how many didn't die, from previous generation.

technically... geometric component,

Pmf(X) = Poisson + Binomial (what X =, probability like) $1-d$

reducible if # cases > 1.

Because: Poisson is independent of Binomial / survivors,

Prove state space if fully communicatable.

Average of Poisson = b,

Binomial = $P(X_n = j \mid X_{n-1} = i)\, P(X_{n-1} = i)$

$$\binom{n}{k}(1-d)^{k}(d)^{n-k} \cdot e^{-\lambda}\frac{\lambda^{-k}}{k!} \to \binom{x_i}{x_j}(1-d)^{x_j}(d)^{x_i - x_j} \cdot e^{-\lambda}\frac{\lambda^{x_i - x_j}}{!x_j} > 0$$

I had answer, but I want to prove it in my own understanding

Its easy to prove it can grow up to any number... So prove it can famine.

$x_i = 1 \quad x_j = 0$

x = Binomial

$P(x) = \binom{n}{k} p^{k}(1-p)^{n-k}$

★ $$\binom{1}{0}(1-d)^{1}(d)^{1} \cdot e^{-\lambda}\frac{\lambda^{0}}{0!} > 0 \quad \therefore \text{ irreducible across all } S,$$

$1 > \quad 1 > d > 0$

but Only when $d < 1$.

c.) $m_n = E[X_n] \quad m_{n+1} = b + (1-d) m_n \to b + (1-d) E[X_n]$

$1 > \quad 1 > d > 0$ but only when $d < 1$.

(c) $m_n = E[X_n]$ $\quad m_{n+1} = b + (1-d) m_n \rightarrow b + (1-d) E[X_n]$

$m_n = E\left[B + \binom{i}{j}(1-d)^j (d)^{i-j} \frac{e^{-\lambda}\lambda^j}{j!}\right] = \sum \text{Poisson}(\lambda) + \sum_{i=0}^{i=n} x \cdot \overbrace{P(X_i \mid X_{i-1})}^{(1-d)} \cdot P(X_i)$ $\quad B = \lambda$

$P(X_i \mid X_{i-1}) = (1-d)$

because = probability of surviving

$\boxed{(1-d) \sum_{i=0}^{i=n} x_{i-1} P(x_{i-1}) = E(x_i) \cdot (1-d)}$

d) $\lim_{n\to\infty} m_n = m^* = b + (1-d) m^*$

prove

$m^* = b + m^* - d m^*$ $\qquad m^* = \cancel{dm^*} + m^* - \cancel{dm^*}$

$b = \cancel{m^*} - \cancel{m^*} + dm^*$ $\qquad m^* = m^*$

$b = dm^*$ $\qquad m^* = \frac{b}{d}$

5.) Wright Fisher Model X_n Population size N. Show that $E[X_{n+1} \mid \{X_n = x\}] = x$ = Prove $X_{n+1} = X_n$

Genetic drift $\qquad$ Trials $= N$ $\qquad \sum_{n=0}^{n=N} X_n P(X_n)$ $\qquad X_n = A$, $X_{n-1} = x$

i copies of allele, N individuals, k = # of copies of allele $= [0, \ldots, 2N]$

$P(X_n) = \binom{2N}{k}\left(\frac{i}{2N}\right)^k \left(1 - \frac{i}{2N}\right)^{2N-k}$

$\boxed{\sum_{n=0}^{n=N} X_{n+1} \cdot P(\text{success}) = 2N \cdot \frac{x}{2N} = x}$

$E[\text{Binomial}] = $ trials $\cdot$ probability of success