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Trajectories Theory and Cosmogravity

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I - Schwarzschild Metrics

I.1 External Metric

We wish to study the motion of a particle (massive or photon) in the external gravitational field of a centro-symmetric mass, without rotation, placed at the origin of spatial coordinates r, θ, φ and time t. The model is a static space-time with spherical symmetry of metric :

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) + e^{\nu(r)}c^2dt^2$$

In **general relativity**, for a mass M without rotation, spherically symmetric, placed at the origin of the coordinates, in an empty space-time, the solution of Einstein's equations is :

$$\lambda + \nu = 0 \qquad e^\nu = 1 - \frac{r_s}{r} \qquad \text{with} \qquad \boxed{r_s = 2\,\frac{GM}{c^2}}$$

hence the expression of Schwarzschild's metric :

$$\boxed{ds^2 = -\frac{dr^2}{1 - \dfrac{r_s}{r}} - r^2(d\theta^2 + sin^2\theta d\varphi^2) + \left(1 - \frac{r_s}{r}\right)c^2dt^2}$$

c the speed of light and G the gravitational constant .
r_s is called ≪ Schwarzschild rayon ≫ or ≪ black hole horizon ≫. It represents the limit of the region from which light and matter cannot escape.

This Schwarzschild solution is of remarkable importance since it is also the only solution to Einstein's equations outside of any spherically symmetric body of mass M, without rotation. This metric is therefore not limited to describe only black holes, it is also valid outside a star, planet, or any other spherically symmetric body without rotation. This result is known as the Birkhoff theorem (Gauss theorem in Newtonian gravitation).
The trajectories of a free particle are the geodesics of metric.
By reason of symmetry the trajectory is flat (we will take $\theta = \dfrac{\pi}{2}$).

I.1.1 Massive particle

The first integrals found using the Lagrangian $\mathcal{L} = \dfrac{ds}{d\tau}$ are written :

$$(S^e_{pm}1) \quad \frac{dt}{d\tau}(r) = \frac{E_e}{1 - \dfrac{r_s}{r}} \quad and \quad (S^e_{pm}2) \quad \frac{d\varphi}{d\tau}(r) = \frac{c\, L_e}{r^2}$$

With two integration constants : E_e dimensionless and L_e one length. These functions reported in the expression $ds = c\, d\tau$ involve :

$$(S^e_{pm}3) \quad (\frac{dr}{d\tau})^2 + V^e_{Spm} = c^2 E_e^2 \quad where \quad V^e_{Spm}(r) = c^2(1 - \frac{r_s}{r})(1 + \frac{L_e^2}{r^2})$$

These relations verify the equations of the geodesics.
By deriving with respect to τ the equation $(S^e_{pm}3)$ and for non-circular orbits, we deduce the following differential equation used to make the simulation :

$$(S^e_{pm}4) \quad \frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{dV^e_{Spm}}{dr} = \frac{c^2}{2r^4}(-r_s r^2 + 2rL_e^2 - 3r_s L_e^2) = f^e_{Spm}(r)$$

with the initial conditions : $r = r_0$, $U_r(r_0) = \dfrac{dr}{d\tau}(r_0)$ and $U_\varphi(r_0) = r_0 \dfrac{d\varphi}{d\tau}(r_0)$ which implies :

$$L_e = \frac{U_\varphi(r_0) r_0}{c} \quad and \quad c^2 E_e^2 = U_r(r_0)^2 + c^2(1 - \frac{r_s}{r_0})(1 + \frac{U_\varphi(r_0)^2}{c^2})$$

Note : the values of $U_r(r_0)$ and $U_\varphi(r_0)$ (respectively radial-coordinate velocity and tangential-coordinate velocity) are not limited by c (they check the metric equation)
see Éric Gourgoulhon - Relativité générale .
The study of the function V^e_{Spm} allows us to deduce that there can exist two circular orbits of radii :

$$\frac{L_e}{r_s}\left(L_e + \sqrt{L_e^2 - 3r_s^2}\right) \quad (stable) \quad and \quad \frac{L_e}{r_s}\left(L_e - \sqrt{L_e^2 - 3r_s^2}\right) \quad (unstable).$$

The equation $S^e_{pm}4$ is solved numerically by the Runge-Kutta method.
The observer who has stayed away from the black hole sees his colleague moving more and more slowly and eventually freezing when he reaches the r_s horizon. A traveler who falls into the black hole arrives at the center (r=0) in a finite time while his colleague has the impression that he remains frozen on the horizon. (and, in practice, disappears because of the spectral shift).

I.1.2 Photon

We keep the relations in t and φ with a different parameter λ of the proper time τ since for a photon we have always $d\tau = 0$.

$$(S^e_{ph}1) \quad \frac{dt}{d\lambda}(r) = \frac{E_e}{1 - \dfrac{r_s}{r}} \quad and \quad (S^e_{ph}2) \quad \frac{d\varphi}{d\lambda}(r) = \frac{c\, L_e}{r^2}$$

These functions reported in the expression $ds = 0$ involve :

$$(S^e_{ph}3) \quad (\frac{dr}{d\lambda})^2 + V^e_{Sph} = c^2 E_e^2 \quad where \quad V^e_{Sph}(r) = c^2(1 - \frac{r_s}{r})\frac{L_e^2}{r^2}$$

These relations verify the equations of the geodesics.
By deriving with respect to λ the equation $(S^e_{ph}3)$ and for non-circular orbits, we deduce the following differential equation used to make the simulation :

$$(S^e_{ph}4) \quad \frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{dV^e_{Sph}}{dr} = \frac{c^2}{2r^4}(2rL_e^2 - 3r_s L_e^2) = f^e_{Sph}(r)$$

with the initial conditions : $r = r_0$, $U_r(r_0) = \dfrac{dr}{d\lambda}(r_0)$ and $U_\varphi(r_0) = r_0 \dfrac{d\varphi}{d\lambda}(r_0)$ which implies :

$$L_e = \frac{U_\varphi(r_0) r_0}{c} \quad and \quad c^2 E_e^2 = U_r(r_0)^2 + (1 - \frac{r_s}{r_0}) U_\varphi(r_0)^2$$

Note : the values of $U_r(r_0)$ and $U_\varphi(r_0)$ are not limited by c (they check the metric equation).
The study of the function V^e_{Sph} shows that there is an unstable circular orbit of radius $\dfrac{3}{2} r_s$.
The equation $S^e_{ph}4$ is solved numerically by the Runge-Kutta method.

I.2 Internal Metric

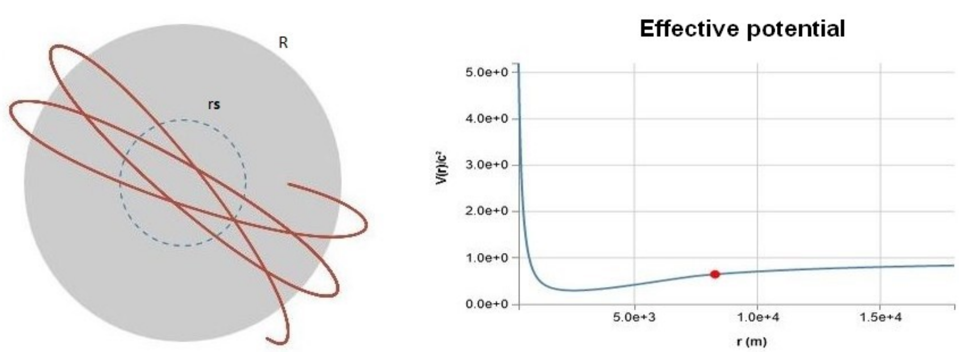
We wish to study the motion of a particle (subject only to gravitation) inside a **constant density**, centro-symmetric, non-rotating object. The mass is placed at the origin of spatial coordinates r, θ, φ and time t. ¡The model is a static space-time with spherical symmetry. The solution of Einstein's equations gives the metric :

$$\boxed{ds^2 = -\alpha(r) dr^2 \;-\; r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \;+\; \beta(r)^2 c^2 dt^2}$$

where $\boxed{\alpha(r) = 1 - \dfrac{r^2 r_s}{R^3}}$ $\boxed{\beta(r) = \frac{3}{2}\sqrt{1 - \dfrac{r_s}{R}} \;-\; \frac{1}{2}\sqrt{1 - \dfrac{r^2 r_s}{R^3}}}$ R object radius

The trajectories of a free particle are the geodesics of metric.
By reason of symmetry the trajectory is flat (we will take $\theta = \dfrac{\pi}{2}$).
For the establishment of this metric see Henri Andrillat - Introduction à l'étude des cosmologies .

I.2.1 Massive particle



M=2e30 kg r0=5e3 m R=7.5e3 m Uphi(r0)=4e7 m/s Ur(r0)=1.5e8 m/s

The first integrals found using the Lagrangian $\mathcal{L} = \dfrac{ds}{d\tau}$ are written :

$$(S^i_{pm}1) \quad \beta(r)^2 \frac{dt}{d\tau}(r) = E_i \quad and \quad (S^i_{pm}2) \quad \frac{d\varphi}{d\tau}(r) = \frac{c\, L_i}{r^2}$$

With two integration constants : E_i dimensionless and L_i one length. These functions reported in the expression $ds = c\, d\tau$ involve :

$$(S^i_{pm}3) \quad (\frac{dr}{d\tau})^2 + V^i_{Spm}(r) = c^2 E_i^2 \quad where \quad V^i_{Spm}(r) = c^2 E_i^2 - \; c^2 \alpha(r) \left[\frac{E_i^2}{\beta(r)^2} - \frac{L_i^2}{r^2} - 1 \right]$$

These relations verify the equations of the geodesics.

By deriving with respect to τ the equation $(S^i_{pm}3)$ and for non-circular orbits, we deduce the following differential equation used to make the simulation :

$$(S^i_{pm}4)\quad \frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{dV^i_{Spm}}{dr} = -\frac{c^2\, r\, r_s}{R^3}\left[\frac{E_i^2}{\beta(r)^2} - \frac{L_i^2}{r^2} - 1\right] + \frac{c^2\,\alpha(r)}{2}\left[\frac{-E_i^2\, r\, r_s}{\beta(r)^3\sqrt{\alpha(r)}R^3} + 2\frac{L_i^2}{r^3}\right] = f^i_{Spm}(r)$$

with the initial conditions : $r = r_0$, $U_r(r_0) = \dfrac{dr}{d\tau}(r_0)$ and $U_\varphi(r_0) = r_0\,\dfrac{d\varphi}{d\tau}(r_0)$ which implies :

$$L_i = \frac{r_0}{c}\,U_\varphi(r_0) \qquad c^2\,E_i^2 = \beta^2(r_0)\left[\frac{U_r^2(r_0)}{\alpha(r_0)} + U_\varphi^2(r_0) + c^2\right]$$

Note : the values of $U_r(r_0)$ and $U_\varphi(r_0)$ are not limited by c (they check the metric equation). The equation $S^i_{pm}4$ is solved numerically by the Runge-Kutta method.

I.2.2 Photon

We keep relations in t and φ :

$$(S^i_{ph}1)\quad \beta(r)^2\frac{dt}{d\lambda}(r) = E_i \quad \textit{and} \qquad (S^i_{ph}2)\quad \frac{d\varphi}{d\lambda}(r) = \frac{c\,L_i}{r^2}$$

These functions reported in the expression $ds = 0$ involve :

$$(\frac{dr}{d\lambda})^2 + V^i_{Sph}(r) = c^2E_i^2 \quad \textit{and} \quad V^i_{Sph}(r) = c^2E_i^2 - \; c^2\alpha(r)\left[\frac{E_i^2}{\beta(r)^2} - \frac{L_i^2}{r^2}\right]$$

By deriving with respect to λ the equation $(S^i_{ph}3)$ and for non-circular orbits, we deduce the following differential equation used to make the simulation :

$$(S^i_{ph}4)\quad \frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{dV^i_{Sph}}{dr} = -\frac{c^2\, r\, r_s}{R^3}\left[\frac{E_i^2}{\beta(r)^2} - \frac{L_i^2}{r^2}\right] + \frac{c^2\,\alpha(r)}{2}\left[\frac{-E_i^2\, r\, r_s}{\beta(r)^3\sqrt{\alpha(r)}R^3} + 2\frac{L_i^2}{r^3}\right] = f^i_{Sph}(r)$$

with the initial conditions : $r = r_0$, $U_r(r_0) = \dfrac{dr}{d\lambda}(r_0)$ and $U_\varphi(r_0) = r_0\,\dfrac{d\varphi}{d\lambda}(r_0)$ which implies :

$$L_i = \frac{r_0}{c}\,U_\varphi(r_0) \qquad c^2\,E_i^2 = \beta^2(r_0)\left[\frac{U_r^2(r_0)}{\alpha(r_0)} + U_\varphi^2(r_0)\right]$$

Note : the values of $U_r(r_0)$ and $U_\varphi(r_0)$ are not limited by c (they check the metric equation). The equation $S^i_{ph}4$ is solved numerically by the Runge-Kutta method.

II - Kerr Metric

II.1 General Theory

We wish to study the motion of a particle (massive or photon) in the gravitational field of a centro-symmetric, rotating mass, placed at the origin of the coordinates .
In **general relativity** , for a mass M in rotation, spherically symmetric, placed at the origin of the Boyer-Lindquist coordinates r, θ, φ, t, in an empty space-time, the solution of Einstein's equations is the Kerr metric :

$$\boxed{ds^2 = \frac{-\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_s\, r\, a^2}{\rho^2}\sin^2\theta)\sin^2\theta\, d\varphi^2 + \frac{2\, r_s\, r\, a}{\rho^2}\sin^2\theta\, c\, dt\, d\varphi + (1 - \frac{r_s\, r}{\rho^2})\, c^2 dt^2}$$

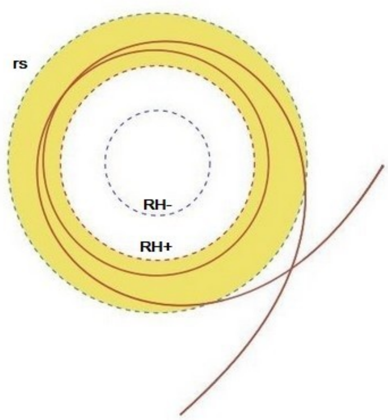
where $\boxed{\rho^2(r) = r^2 + a^2\cos^2\theta}$ $\boxed{\Delta(r) = r^2 - r_s\, r + a^2}$ $\boxed{a = \dfrac{J}{c\,M}}$ (J angular momentum)

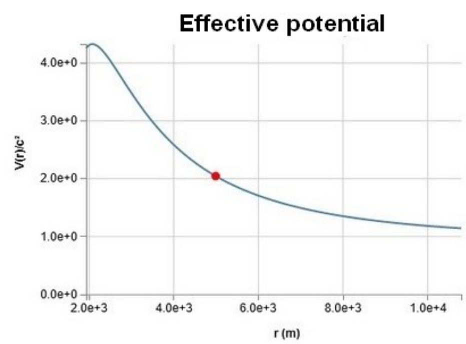
Unlike Schwarzschild metric, there is no equivalent of Birkhoff's theorem in Kerr metric. This geometry therefore only describes rotating black holes, not space-time external to other objects such as rotating stars or planets.
The trajectories of a free particle are the geodesics of the metric.
Only flat trajectories will be studied (with $\theta = \frac{\pi}{2}$).
The event horizon corresponds to the change of sign of g_{rr} , i.e. to the solutions of the equation $\Delta = 0$. If $a < \frac{r_s}{2}$ we get two values :

$\boxed{R_{H+} = \frac{r_s + \sqrt{r_s^2 - 4a^2}}{2}}$ and $\boxed{R_{H-} = \frac{r_s - \sqrt{r_s^2 - 4a^2}}{2}}$ so $\boxed{\Delta(r) = (r - R_{H+})(r - R_{H-})}$

The domain between r_s and R_{H+} is called ergoregion (in Schwarzschild metric there is no ergoregion $a = 0$ and $R_{H+} = r_s$) see Éric Gourgoulhon - Relativité générale .

II.2 Massive particle





M=2e30kg r0=4.5e3 m a=1.418e3 m Uphi(r0)=7e8 m/s Ur(r0)=-4.25e8 m/s

The first integrals found using the Lagrangian $\mathcal{L} = \frac{ds}{d\tau}$ are written :

$$(K_{pm}1)\quad \frac{dt}{d\tau}(r) = \frac{1}{\Delta(r)}\left[(r^2 + a^2 + \frac{r_s}{r}a^2)\,E - \frac{r_s\,a}{r}\,L\right] \quad \textit{and} \quad (K_{pm}2)\quad \frac{d\varphi}{d\tau}(r) = \frac{c}{\Delta(r)}\left[\frac{r_s\,a}{r}\,E + (1 - \frac{r_s}{r})\,L\right]$$

With two integration constants : E dimensionless and L one length.
These functions reported in the expression $ds = c\,d\tau$ involve :

$$(K_{pm}3)\quad (\frac{dr}{d\tau})^2 + V_{Kpm} = c^2E^2 \quad \textit{where} \quad V_{Kpm}(r) = c^2 - \frac{r_s}{r}\,c^2 - \frac{c^2}{r^2}\,(\,a^2\,(E^2-1) - L^2\,) - \frac{r_s\,c^2}{r^3}\,(L - a\,E)^2$$

These relations verify the equations of the geodesics.
Deriving from τ the equation $(K_{pm}3)$ and for non-circular orbits, we deduce the following differential equation used to make the simulation :

$$(K_{pm}4)\quad \frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{dV_{Kpm}}{dr} = -\frac{c^2}{2\,r^4}\left[r_s\,r^2 + 2\,r\,(a^2(E^2-1) - L^2) + 3\,r_s\,(L - aE)^2\right] = f_{Kpm}(r)$$

with the initial conditions $r = r_0$, $\Delta_0 = \Delta(r_0)$, $U_r(r_0) = \frac{dr}{d\tau}(r_0)$ and $U_\varphi(r_0) = r_0\,\frac{d\varphi}{d\tau}(r_0)$ which implies :

$$c^2\,E^2 = \frac{1}{r_0^2\,\Delta_0}\left[\,U_r^2(r_0)\,(r_0 - r_s)r_0^3 + c^2\,r_0(r_0 - r_s)\Delta_0 + \Delta_0^2\,U_\varphi^2(r_0)\,\right]$$

$$L = \frac{1}{c\,(r_0 - r_s)}\left[\,\Delta_0\,U_\varphi(r_0) - r_s\,a\,c\,E\,\right]$$

Note : the values of $U_r(r_0)$ and $U_\varphi(r_0)$ are not limited by c (they check the metric equation). The equation $K_{pm}4$ is solved numerically by the Runge-Kutta method.

II.3 Photon

We keep relations in t and φ :

$$(K_{ph}1)\quad \frac{dt}{d\lambda}=\frac{1}{\Delta(r)}\left[(r^2+a^2+\frac{r_s}{r}\,a^2)\,E-\frac{r_s\,a}{r}\,L\right]\qquad and \qquad (K_{ph}2)\quad \frac{d\varphi}{d\lambda}=\frac{c}{\Delta(r)}\left[\frac{r_s\,a}{r}\,E+(1-\frac{r_s}{r})\,L\right]$$

These functions reported in the expression $ds=0$ involve :

$$(K_{ph}3)\quad (\frac{dr}{d\lambda})^2+V_{Kph}(r)=c^2\,E^2\quad where\quad V_{Kph}(r)=-\frac{c^2}{r^2}\,(\,a^2\,E^2-L^2\,)-\frac{r_s\,c^2}{r^3}\,(L-a\,E)^2$$

These relations verify the equations of the geodesics.
By deriving with respect to λ the equation $(K_{ph}3)$ and for non-circular orbits, we deduce the following differential equation used to make the simulation :

$$(K_{ph}4)\quad \frac{d^2r}{d\lambda^2}=-\frac{1}{2}\frac{dV_{Kph}}{dr}=-\frac{c^2}{2\,r^4}\,\left[2\,r\,(a^2E^2-L^2)+3\,r_s\,(L-aE)^2\right]=f_{Kph}(r)$$

with the initial conditions $r=r_0,\ \Delta_0=\Delta(r_0),\ U_r(r_0)=\dfrac{dr}{d\lambda}(r_0)$ and $U_\varphi(r_0)=r_0\,\dfrac{d\varphi}{d\lambda}(r_0)$) which implies :

$$c^2\,E^2=\frac{1}{r_0^2\,\Delta_0}\,[\,U_r^2(r_0)\,(r_0-r_s)r_0^3+\Delta_0^2\,U_\varphi^2(r_0)\,]$$

$$L=\frac{1}{c\,(r_0-r_s)}\,[\,\Delta_0\,U_\varphi(r_0)-r_s\,a\,c\,E\,]$$

Note : the values of $U_r(r_0)$ and $U_\varphi(r_0)$ are not limited by c (they check the metric equation). There are two unstable circular orbits of radii (see James M. Bardeen) :

$$r_s\{1+\cos[\frac{2}{3}\arccos(\frac{2a}{r_s})]\}\quad and\quad r_s\{1+\cos[\frac{2}{3}\arccos(\frac{-2a}{r_s})]\}$$

The equation $K_{ph}4$ is solved numerically by the Runge-Kutta method.

III - Appendix

III.1 Euler-Lagrange equations

Let $\mathcal{L}\,(x^1(\lambda),x^2(\lambda),...,x^n(\lambda)\,,\,\dot{x}^1(\lambda),\dot{x}^2(\lambda),...,\dot{x}^n(\lambda))$ a function of 2n independent variables where $\dot{x}^i=\dfrac{dx^i}{d\lambda}$.

So, the value of the integral $\displaystyle\int_{\lambda_a}^{\lambda_b}\mathcal{L}\,(\lambda)\,d\lambda$ is extreme for curves $\{x^i(\lambda)\}_{i\in\{1,2...,n\}}$

that verify Euler-Lagrange's equations :

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^k}-\frac{\partial\mathcal{L}}{\partial x^k}\ =\ 0\quad \forall k\in\{1,2,...n\}$$

III.2 Metrics and geodesics

For an exhaustive study see Eric Gourgoulhon - Geometry and physics of black holes
and Eric Gourgoulhon - Relativité générale .

Metrics

In a space-time, we represent by ds^2 the infinitesimal interval between two events marked by the coordinates $(x^1, x^2, x^3, x^4 = ct)$ and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3, ct + cdt)$ where

$$ds^2 \;=\; \sum_{i=1}^{4}\sum_{j=1}^{4} g_{ij}(x^1, x^2, x^3, x^4)\, dx^i\, dx^j \;=\; g_{ij}\, dx^i\, dx^j \quad (\textit{with Einstein summation notation})$$

By abuse of language, we will speak of « distance » between these two events. The 4x4 symmetric matrix of the 16 functions g_{ij} (coefficients of the metric) allowing an inverse (whose coefficients are noted $g^{\alpha\beta}$) at any point where the metric is defined.

Basic properties of this « distance » :

- It's an invariant for any change in coordinates

$$ds^2 \;=\; \sum_{i=1}^{4}\sum_{j=1}^{4} g_{ij}(x^1, x^2, x^3, x^4)\, dx^i\, dx^j \;=\; \sum_{i=1}^{4}\sum_{j=1}^{4} \bar{g}_{pq}(y^1, y^2, y^3, y^4)\, dy^p\, dy^q \qquad (g_{ij}\, dx^i\, dx^j \;=\; \bar{g}_{pq}\, dy^p\, dy^q)$$

- It's spelled $ds = c\, d\tau$ where τ is the eigentime measured (by clock) between events (x^1, x^2, x^3, x^4) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3, x^4 + dx^4)$

- For a photon, you always have $ds = 0$

Geodesics

The curves that make the « distance » between two events in space-time extreme are called geodesics. They verify the differential equations :

$$\frac{d^2x^k}{d\lambda^2} \;+\; \Gamma^k_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \;=\; 0 \quad k \in \{1, 2, 3, 4\}$$

with the connection coefficients (Christoffel symbols of second kind)

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{2} g^{ip}\left(\frac{\partial g_{pk}}{\partial x^j} + \frac{\partial g_{pj}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^p}\right)$$

With the function : $\mathcal{L}\,(..x^p(\lambda).., ..\dot{x}^q(\lambda)..) \;=\; \sqrt{\epsilon\, g_{ij}\,\frac{dx^i}{d\lambda}\,\frac{dx^j}{d\lambda}} \;=\; \frac{ds}{d\lambda}$

where $\epsilon = 1$ for timelike curves $(ds^2 > 0)$ and $\epsilon = -1$ for spacelike curves ($ds^2 < 0$)
(for a signature $- \; - \; - +$) .

The curves $\{x^k(\lambda)\}_{k\in\{1,2,3,4\}}$ that make the integral $\displaystyle\int_a^b ds \;=\; \int_{\lambda_a}^{\lambda_b} \mathcal{L}\,(\lambda)\, d\lambda \;=\; s(b) - s(a)$ extreme are the geodesics.

To find first integrals of geodesic equations, we can look for solutions

to Euler-Lagrange's equations : $\displaystyle\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^k} - \frac{\partial \mathcal{L}}{\partial x^k} \;=\; 0 \quad k \in \{1, 2, 3, 4\}$

III.3 Application

For Schwarzschild and Kerr metrics the functions g_{ij} are independent of t and φ :

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \quad and \quad \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\Longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{t}} = constante1 \quad and \quad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = constante2$$

By imposing $\theta = \dfrac{\pi}{2}$ the combination of the two previous relationships gives the equations $S1, S2, K1, K2$.

III.4 Runge-Kutta method of order 4

Method for numerically solving the differential equation : $\dfrac{d^2 r}{d\lambda^2}(r) = f(r)$
with initial conditions $\lambda = \lambda_0$, $r = r_0$ and $\dfrac{dr}{d\lambda}(r_0) = U_r(r_0)$.
We calculate the values r_n , $y'_n = \dfrac{dr}{d\lambda}(r_n)$ and φ_n starting from $(r_0 \ , \ U_r(r_0) \ , \ \varphi_0)$ with a h step for the variable λ.

$$k_1 = f(r_n) \quad , \quad k_2 = f(r_n + \frac{h}{2}\, y'_n) \quad , \quad k_3 = f(r_n + \frac{h}{2}\, y'_n + \frac{h^2}{4}\, k_1) \quad , \quad k_4 = f(r_n + h\, y'_n + \frac{h^2}{2}\, k_2)$$

$$r_{n+1} = r_n + h\, y'_n + \frac{h^2}{6}\,(k_1 + k_2 + k_3)$$

$$y'_{n+1} = y'_n + \frac{h}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4)$$

$$\varphi_{n+1} = \varphi_n + d\varphi \quad see \quad S2 \quad or \quad K2$$

see here