The LLL Algorithm: Lattice Basis Reduction and applications to Approximate Shortest Vector Problem

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Recap: Euclidean Space and Inner Product

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We consider a real finite-dimensional vector space $\mathbb{R}^n$ equipped with the standard **Euclidean inner product**:

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This inner product induces the **Euclidean norm**:

$$\|\mathbf{u}\|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\sum_{i=1}^{n} \mathbf{u}_i^2}$$

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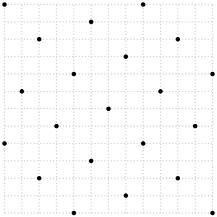

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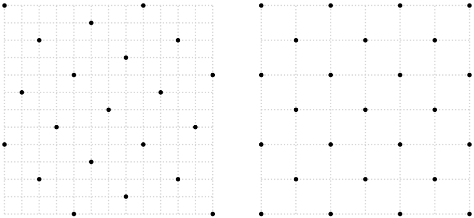

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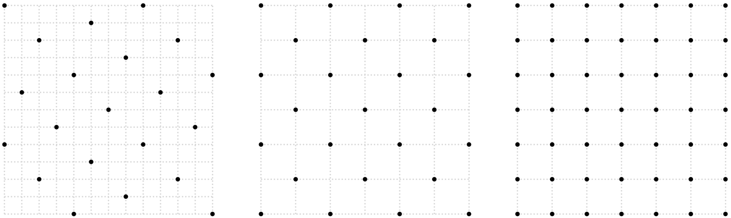

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Any lattice $\mathscr{L} \subseteq \mathbb{R}^n$ admits a maximal $\mathbb{Z}$-linearly independent family $(\mathbf{b}_i)_{1 \leq i \leq m}$, with $m \leq n$ such that:

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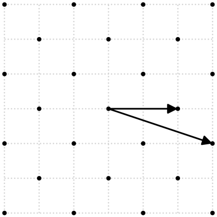

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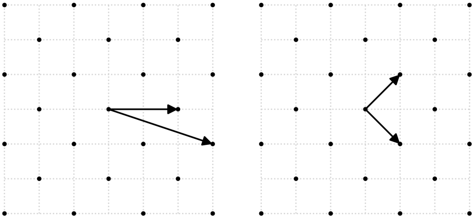

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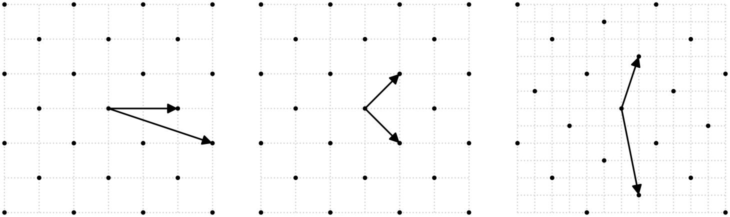

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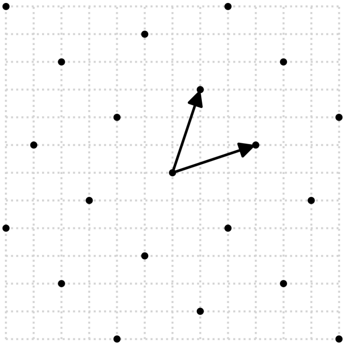

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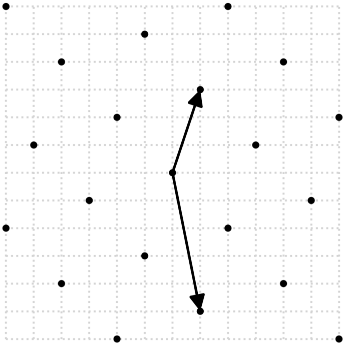
looks good



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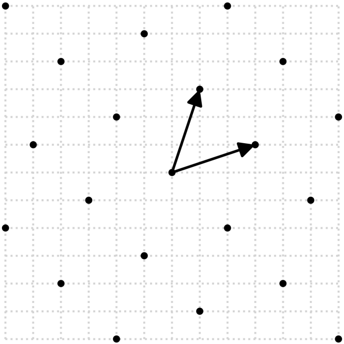

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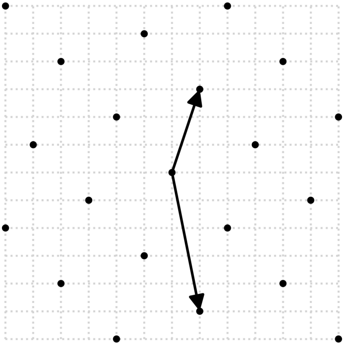

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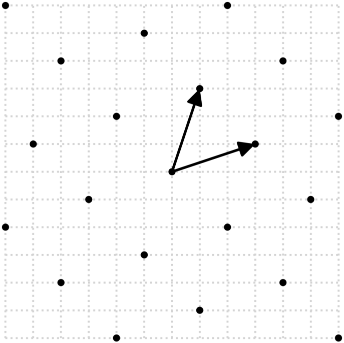

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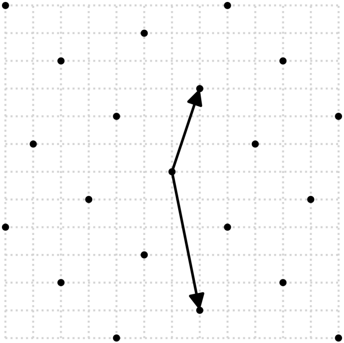

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→ notion of **quasi-orthogonal** (or **reduced**) bases.

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A basis $(\mathbf{b}_i)_{1 \le i \le n}$ of $\mathbb{R}^n$ is called **orthogonal** if

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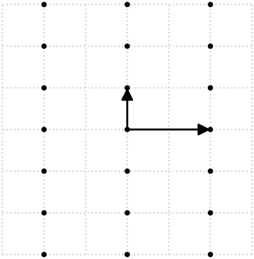

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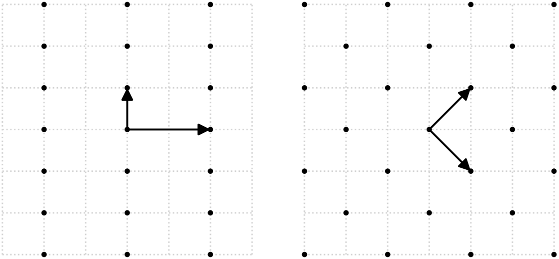

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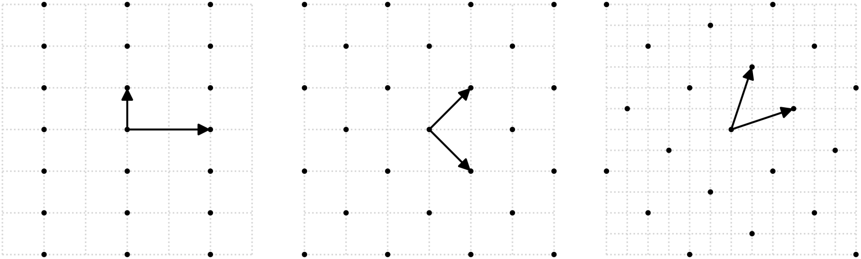

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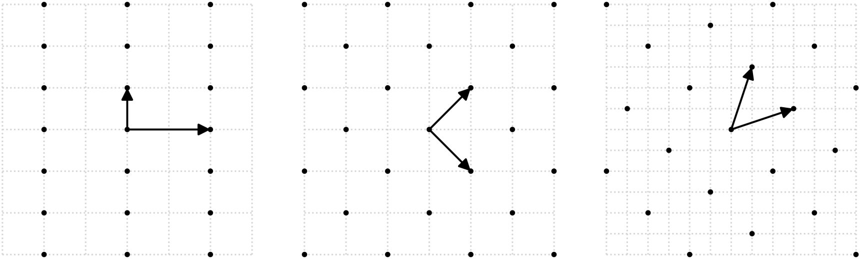

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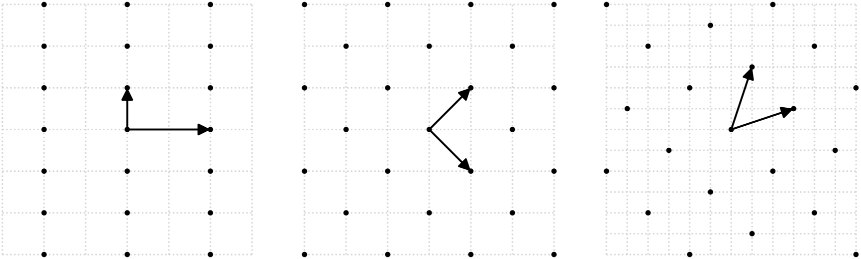

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How an we compute an orthogonal basis ?

→ **Gram-Schmidt orthogonalization process**

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Let $(\mathbf{b}_i)_{1\le i\le n}$ be a basis of $\mathbb{R}^n$. The associated orthogonal basis $(\mathbf{b}_i^*)_{1\le i\le n}$ is constructed via the **Gram–Schmidt orthogonalization process**:

$$\mathbf{b}_1^* := \mathbf{b}_1, \quad \mathbf{b}_i^* := \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{i,j}\mathbf{b}_j^*, \quad \mu_{i,j} := \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\|\mathbf{b}_j^*\|^2}.$$

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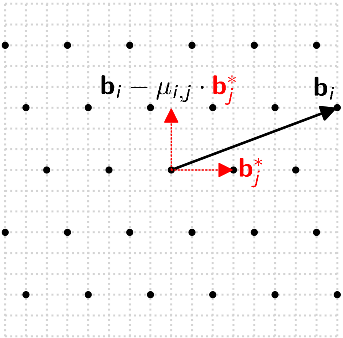

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The coefficients $\mu_{i,j}$ are called **Gram–Schmidt coefficients**.

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mu_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mu_{n,1} & \cdots & \mu_{n,n-1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \mathbf{b}_2^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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The resulting family $(\mathbf{b}_i^*)_{1 \leq i \leq n}$ is orthogonal.

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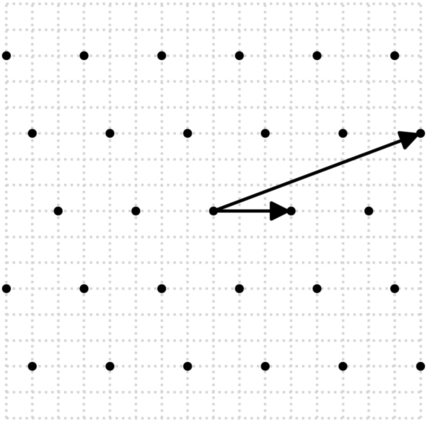

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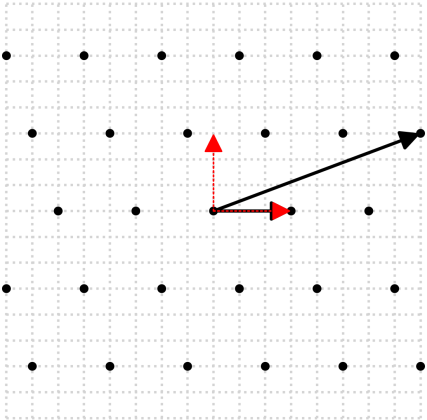

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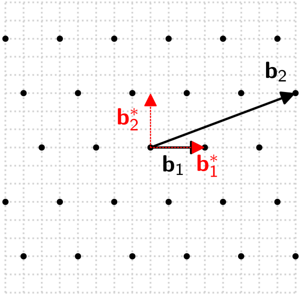

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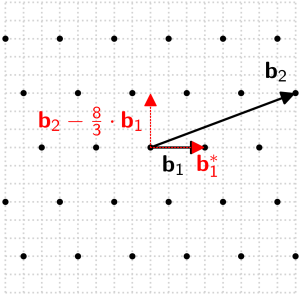

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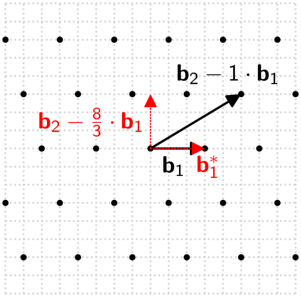
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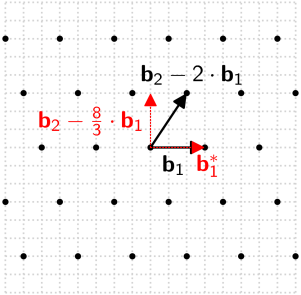
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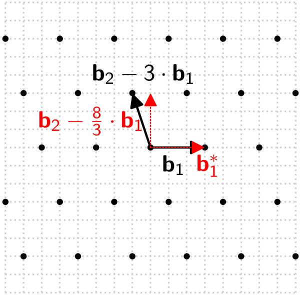

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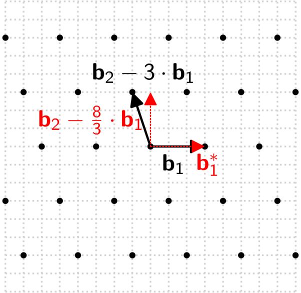
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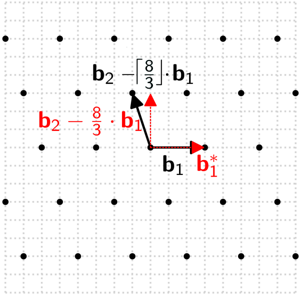

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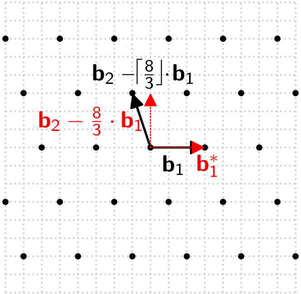

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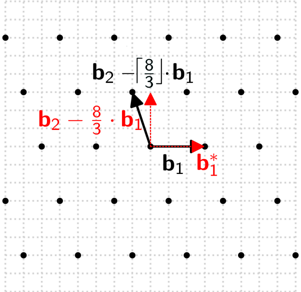
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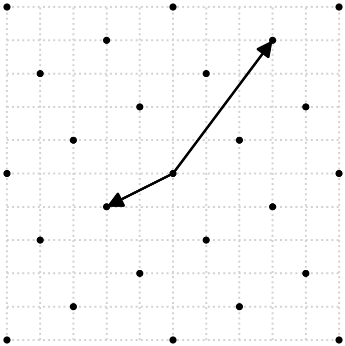
Definition: A basis is said to be **size-reduced** if:

$$\max_{1 \leq j < i \leq n} |\mu_{i,j}| \leq \frac{1}{2}$$

Why Size Reduction is Not Enough



A size-reduced basis.

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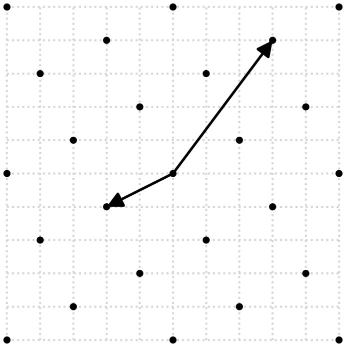

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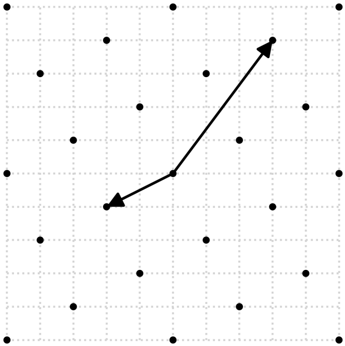

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Length reduction alone ***does not imply*** *almost-orthogonality!*

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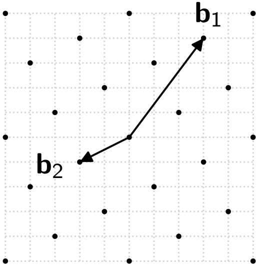

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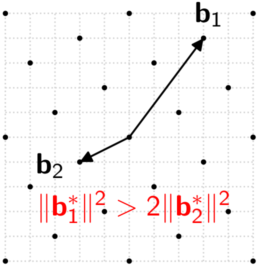

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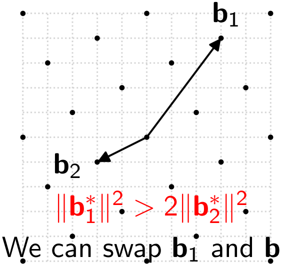

$$\|\mathbf{b}_1^*\|^2 > 2\|\mathbf{b}_2^*\|^2$$

We can swap $\mathbf{b}_1$ and $\mathbf{b}_2$.

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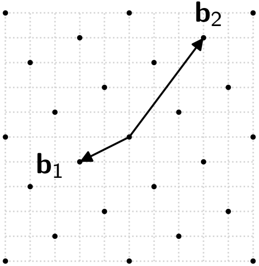

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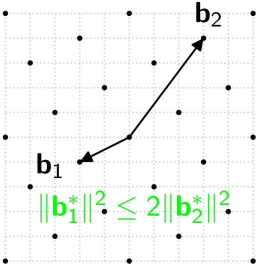

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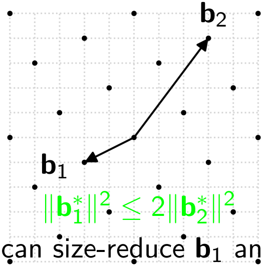

We can size-reduce $\mathbf{b}_1$ and $\mathbf{b}_2$!

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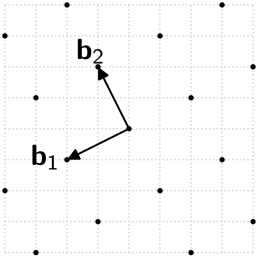

Definition: LLL– reduced Basis

A basis is called LLL–reduced if:

- It is size-reduced;
- It satisfies the Lovász condition.

Recap:The $\gamma - \mathrm{SVP}$ Problem

Definitions of λ_1, λ_2, ... are detailed in (Boudgoust 2023).

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$\gamma = 1$	exact SVP — NP-hard
$\gamma = \mathrm{poly}(n)$	relevant for **lattice-based cryptography**
$\gamma = 2^{\mathcal{O}(n)}$	solvable in **polynomial time** via LLL

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Let k be the largest index such that $\lambda_k \neq 0$. We can write

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$$\mathbf{b} = \sum_{i=1}^{k} \lambda_i \left(\mathbf{b}_i^* + \sum_{j=1}^{i} \mu_{ij} \mathbf{b}_j^* \right)$$

Lemma

Lemma. For any $\mathbf{b} \in \mathscr{L} \setminus \{0\}$ we have:

$$\|\mathbf{b}\| \geq \min_{1 \leq i \leq n} \|\mathbf{b}_i\|$$

Proof. Let $(\mathbf{b}_i)_{1 \leq i \leq n}$ of the lattice $\mathscr{L}$, and write:

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Theorem: Bound on First Vector in a Reduced Basis

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Theorem. Let $(\mathbf{b}_i)_{1\leq i\leq n}$ be a reduced basis of a lattice $\mathscr{L} \subseteq \mathbb{R}^n$, and let $\mathbf{b} \in \mathscr{L} \setminus \{0\}$. Then:

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$$\|\mathbf{b}_1\|^2 = \|\mathbf{b}_1^*\|^2 \leq 2\|\mathbf{b}_2^*\|^2 \leq 2^2\|\mathbf{b}_3^*\|^2 \leq \cdots \leq 2^{n-1}\|\mathbf{b}_n^*\|^2.$$

Thus,

$$\|\mathbf{b}\| \geq \min\{\|\mathbf{b}_1^*\|, \ldots, \|\mathbf{b}_n^*\|\} \geq 2^{-(n-1)/2}\|\mathbf{b}_1\|$$

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How can we compute a reduced basis in practice?

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→ Use the LLL (Lenstra 1982)(Lenstra, Lenstra, Lovasz) algorithm!

LLL Algorithm

LLL Algorithm

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$G \leftarrow copy(B)$
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while $i \leq n$ do
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    if $i > 1$ and $\boxed{\|bg^*_{i-1}\|^2 > 2\|bg^*_i\|^2}$ then    Lovász Condition
        Swap
        bg$_{i-1}$ and
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Let's compute a LLL reduced basis of $\mathscr{L}(B)$ with

$$B := \begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

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$$\overbrace{\begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}}^{B} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{9} & 1 & 0 \\ 0 & \frac{54}{101} & 1 \end{pmatrix}}^{U} \times \overbrace{\begin{pmatrix} -2 & 2 & 1 \\ \frac{19}{9} & \frac{8}{9} & \frac{22}{9} \\ \frac{88}{101} & \frac{154}{101} & -\frac{132}{101} \end{pmatrix}}^{B^*}$$

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$$\underbrace{\begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}}_{G} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{9} & 1 & 0 \\ 0 & \frac{54}{101} & 1 \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} -2 & 2 & 1 \\ \frac{19}{9} & \frac{8}{9} & \frac{22}{9} \\ \frac{88}{101} & \frac{154}{101} & -\frac{132}{101} \end{pmatrix}}_{G^*}$$

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We obtain the following LLL reduced basis:

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The vector $(2, 2, 0)$ is a shortest nonzero vector in the lattice, hence:

$$\lambda_1(\mathscr{L}) = 2\sqrt{2}.$$

LLL Complexity

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$G \leftarrow copy(B)$
$(G^*, U) \leftarrow$ Gram-Schmidt $G$   $\mathcal{O}(n^3)$
while $i \leq n$ do   How much?
    for $j = i-1, i-2, \ldots, 1$ do   $\mathcal{O}(n^2)$
        $\mathrm{bg}_i \leftarrow \mathrm{bg}_i - \lceil \mu_{i,j} \rfloor \mathrm{bg}_j$, update $(G^*, U)$   $\mathcal{O}(n)$
    if $i > 1$ and $\|$
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        Swap $\mathrm{bg}_{i-1}$ and $\mathrm{bg}_i$, update $(G^*, U)$   $\mathcal{O}(n)$
        $i \leftarrow i - 1$
```

Correctness

Key idea: Clearly, if the algorithm LLL **terminates**, the returned basis is by construction LLL-reduced.

Therefore, it remains ***to prove*** *that* LLL ***always terminates****.*

How can we prove the termination of the algorithm?

$$\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{i-1} \\ \mathbf{b}_i \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \mu_{2,1} & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \mu_{i\text{-}1,1} & \cdots & \mu_{i\text{-}1,i\text{-}2} & \ddots & \ddots & \vdots \\ \mu_{i,1} & \cdots & \mu_{i,i\text{-}2} & \mu_{i,i\text{-}1} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \mu_{n,1} & \cdots & \cdots & \cdots & \mu_{n,n\text{-}1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \vdots \\ \mathbf{b}_{i-1}^* \\ \mathbf{b}_i^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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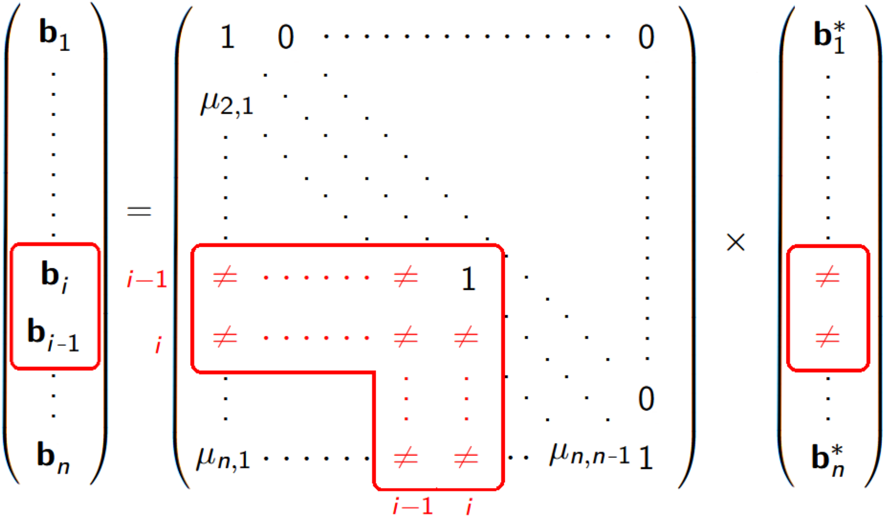

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$$\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \vdots \\ \mathbf{b}_i \\ \mathbf{b}_{i-1} \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{array}{c|ccccccc|} & 1 & 0 & \cdots & & & & 0 \\ & \mu_{2,1} & \ddots & \ddots & & & & \vdots \\ & \vdots & & \ddots & \ddots & & & \vdots \\ i-1 & \neq & \cdots & \neq & 1 & \ddots & & \vdots \\ i & \neq & \cdots & \neq & \neq & \ddots & \ddots & \vdots \\ & \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ & \mu_{n,1} & \cdots & \neq & \neq & \cdots & \mu_{n,n-1} & 1 \\ & & & i-1 & i & & & \end{array} \times \begin{pmatrix} \mathbf{b}_1^* \\ \vdots \\ \vdots \\ \neq \\ \neq \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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$$
\begin{pmatrix} \|\mathbf{b}_1\|^2 \\ \vdots \\ \vdots \\ \|\mathbf{b}_i\|^2 \\ \|\mathbf{b}_{i-1}\|^2 \\ \vdots \\ \|\mathbf{b}_n\|^2 \end{pmatrix}
=
\begin{array}{cc}
\begin{array}{r} \\ \\ \\ i-1 \\ i \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\mu_{2,1}^2 & \ddots & \ddots & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & & \vdots \\
\neq^2 & \cdots & \neq^2 & 1 & \ddots & & \vdots \\
\neq^2 & \cdots & \neq^2 & \neq^2 & \ddots & \ddots & \vdots \\
\vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\
\mu_{n,1}^2 & \cdots & \neq^2 & \neq^2 & \cdots & \mu_{n,n-1}^2 & 1
\end{pmatrix} \\
 & \begin{array}{ccccccc} \quad & \quad & i-1 & i & \quad & \quad & \quad \end{array}
\end{array}
\times
\begin{pmatrix} \|\mathbf{b}_1^*\|^2 \\ \vdots \\ \vdots \\ \|\neq\|^2 \\ \|\neq\|^2 \\ \vdots \\ \|\mathbf{b}_n^*\|^2 \end{pmatrix}
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\begin{pmatrix}
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\end{pmatrix}
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\\ \\ \\ i-1 \\ i \\ \\ \\ \\
\end{array}
\begin{array}{cccccc}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
\mu_{2,1}^2 & \ddots & & & & \vdots \\
\vdots & & \ddots & & & \vdots \\
\neq^2 & \cdots & \neq^2 & 1 & \ddots & \vdots \\
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\vdots & & \vdots & \vdots & \ddots & 0 \\
\mu_{n,1}^2 & \cdots & \neq^2 & \neq^2 & \cdots\ \mu_{n,n-1}^2 & 1 \\
 & & i-1 & i & &
\end{array}
\times
\begin{pmatrix}
\|\mathbf{b}_1^*\|^2 \\ \vdots \\ \vdots \\ \leq \frac{3}{4}\|\mathbf{b}_{i-1}^*\|^2 \\ \leq \|\mathbf{b}_{i-1}^*\|^2 \\ \vdots \\ \|\mathbf{b}_n^*\|^2
\end{pmatrix}
$$

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If we **swap**
bg$_i$ and
bg$_{i-1}$:
$\|\mathbf{d}_{i-1}^*\|$ decrease by a $\frac{3}{4}$ factor, so $\mathbf{d_{i-1}}$ decrease by a $\frac{3}{4}$ factor.

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Termination proof

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Algorithm 0: *LLL*

Input: A basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$
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$G \leftarrow copy(B)$
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  while $i \leq n$ do
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$\to$ LLL **compute** a reduced basis in **polynomial time**.

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Theorem.

$\rightarrow$ LLL **compute** a reduced basis in **polynomial time**.

$\rightarrow$ LLL **solve** $2^{\mathcal{O}(n)} - \mathrm{SVP}$ in **polynomial time**.

Thank you for your attention!

Questions?

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