Bound Entanglement

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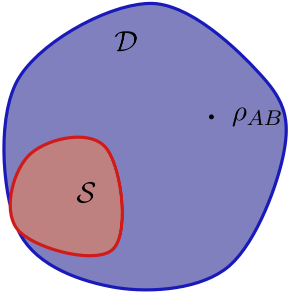
Entanglement



...peculiar, non-classical correlation arising from the tensor product structure in combined quantum systems

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non-distillable entanglement

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Existence Explicite example of a bound entangled state

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Existence: Explicite example of a bound entangled state

Use: Why should we be concerned about bound entanglement?

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Existence	Explicite example of a bound entangled state
Use	Why should we be concerned about bound entanglement?
Detection	How do we decide for a given state whether it is bound entangled?

The PPT Criterion

(Peres Criterion)

Definition

The partial transpose of a density matrix $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined as

$$\rho^{\Gamma} := (\mathcal{I}_A \otimes T_B)\rho \tag{1}$$

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Matrix elements in the tensor basis

In the product basis

$$\left(\rho^\Gamma\right)_{ikjl} = \langle i \otimes k|\ \rho^\Gamma\ |j \otimes l\rangle \tag{2}$$
$$= \langle i \otimes l|\ \rho\ |j \otimes k\rangle = \rho_{iljk} \tag{3}$$

The PPT Criterion

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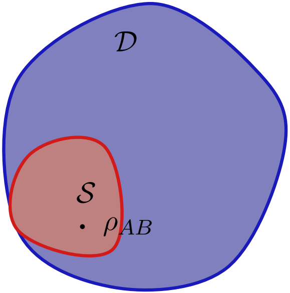

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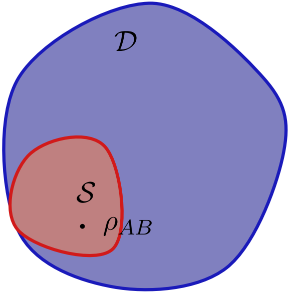

Entanglement criterion

$$\rho_{AB} \in \mathcal{S} \subset \mathcal{H}_A \otimes \mathcal{H}_B \;\Rightarrow\; \rho_{AB} \in \Gamma \tag{1}$$

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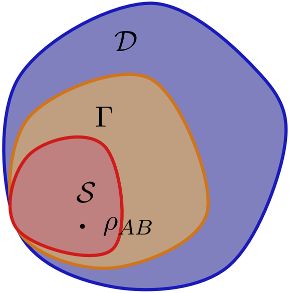

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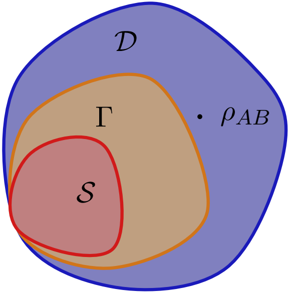

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$$\rho_{AB} \notin \Gamma \;\Rightarrow\; \rho_{AB} \text{ is entangled} \qquad (1)$$

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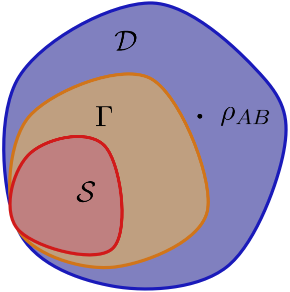


Entanglement criterion

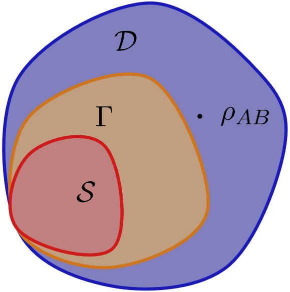
$$\rho_{AB} \notin \Gamma \;\Rightarrow\; \rho_{AB} \text{ is entangled} \tag{1}$$

Accuracy

The criterion is accurate in $2 \otimes 2$ and $2 \otimes 3$.
But not in higher dimensions.[HHH96]

The PPT Criterion

(Peres Criterion)



Group of Entanglement Tests

Any Positive but not Completely Positive Map ('channel') serves as a similar entanglement test.

The Range Criterion [Hor97]

Decomposition of separable states

$$
\begin{aligned}
\forall\ \rho \in \mathcal{S}\ \ :& \\
& \rho = \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} p_{ij}\ |\phi_i\rangle\langle\phi_i| \otimes |\psi_j\rangle\langle\psi_j| \qquad (1) \\
& |\phi_i\rangle \in \mathcal{H}_A,\ |\psi_j\rangle \in \mathcal{H}_B \\
& n_a < \dim \mathcal{H}_A,\ n_B < \dim \mathcal{H}_B
\end{aligned}
$$

The Range Criterion [Hor97]

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$\rightarrow$ features of convex sets, Caratheodories Thm [Car11]

The Range Criterion [Hor97]

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The range of an operator

$$\operatorname{Ran} \rho := \{|\psi\rangle \in \mathcal{H}, \exists\ |\phi\rangle \in \mathcal{H}\ :\ |\psi\rangle = \rho|\phi\rangle\} \tag{2}$$

The Range Criterion [Hor97]

The range of separable density matrices

Any separable density matrix $\rho \in \mathcal{S} \subset \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as a convex combination of pure products states

$$\rho = \sum_{i,j} p_{ij} |\psi_i\rangle\langle\psi_i| \otimes |\phi_j\rangle\langle\phi_j| = \sum_{i,j} p_{ij} |\psi_i \otimes \phi_j\rangle\langle\psi_i \otimes \phi_j|$$

$$\rho^\Gamma = (\mathcal{I} \otimes T)\rho = \sum_{i,j} |\psi_i\rangle\langle\psi_i| \otimes \underbrace{|\phi_j\rangle\langle\phi_j|^T}_{|\phi_j^*\rangle\langle\phi_j^*|}$$

$$= \sum_{ij} p_{ij} |\psi_i \otimes \phi_j^*\rangle\langle\psi_i \otimes \phi_j^*|$$

and the set of product states $\{|\psi_i \otimes \phi_j\rangle\}_{ij}$ ($\{|\psi_i \otimes \phi_j^*\rangle\}_{ij}$) span the range of ρ (ρ^Γ)

The Range Criterion [Hor97]

Entanglement test

- find product vectors that span the range of ρ
- show that these vectors do not span the range of ρ^{Γ}

The first bound entangled state

$$\rho_a = \frac{1}{8a+1}\begin{bmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix} \qquad a \in (0,1)$$

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Properties of ρ_a

- PPT
- ρ_a and ρ_a^Γ violate the range criterion

PPT and Distillability

Definition

A density matrix ρ is called distillable if

$$\begin{aligned} \exists\ \text{LOCC} \quad & \Lambda : \mathcal{D}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}) \to \mathcal{D}(\mathbb{C}^2) \quad \text{s.t. :} \\ & \Lambda(\rho^{\otimes n}) = |\Phi^+\rangle\langle\Phi^+| \quad \text{for some large } n \end{aligned}$$

Distillation of PPT states

Cannot distill qubits from PPT density matrices ρ and $\rho^{\otimes N}$

Non-distillable entanglement

Any PPT entangled state is not distillable.

Distillation of PPT states

Distilling $\rho^{\otimes n}$

Assume ρ is distillable:

$$\Lambda(\rho^{\otimes n}) = \frac{1}{M}\sum_i (A_i \otimes B_i)\ \rho^{\otimes n}\ (A_i^\dagger \otimes B_i^\dagger) \quad \text{is entangled}$$

$$\Rightarrow \quad \exists\ i_0 \text{ s.t.: } \underbrace{(A_{i_0} \otimes B_{i_0})\rho^{\otimes n}(A_{i_0}^\dagger \otimes B_{i_0}^\dagger)}_{=:\rho_{i_0}} \quad \text{is entangled}$$

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$$A_{i_0} = |0\rangle\langle\psi_A| + |1\rangle\langle\phi_A| \quad B_{i_0} = |0\rangle\langle\psi_B| + |1\rangle\langle\phi_B|$$

$$P_A(P_B) := \text{ projector onto } \langle\psi_A, \phi_A\rangle(\langle\psi_B, \phi_B\rangle)$$

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$$\Rightarrow \quad \exists\ \psi \text{ s.t.: } \quad \langle\psi|(\rho')^{\Gamma_B}|\psi\rangle < 0$$

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$$\Rightarrow \quad \rho \text{ is NPT}$$

Bound entanglement

a first summary

def entanglement that cannot be distilled $\rightarrow$ not a sufficient resource for many quantum informational protocols

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Bound entanglement

a first summary

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construction	PPT entangled states (any PPT is not distillable)
existence	only in higher dimensions (smallest BE systems in $3 \otimes 3$ and $4 \otimes 2$)
detection	need a stronger criterion than PPT, e.g. range criterion

Quantum Key Distribution

with bound entangled resources

Bound entangled resources can be employed to distill secret key [HHHO05].

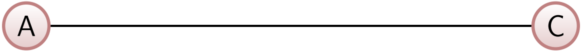
Deciding entanglement for PPT

PPT all NPT states are entangled

range criterion compare the range of ρ and ρ^Γ

extendibility check extendibility of ρ

Endeavour: Extending entanglement



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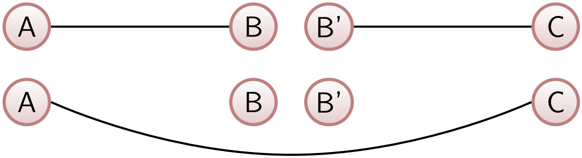

Swap Operator on States

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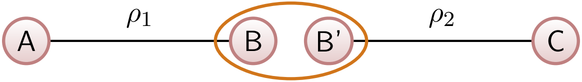

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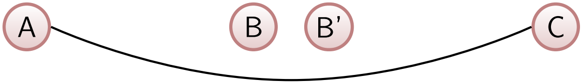

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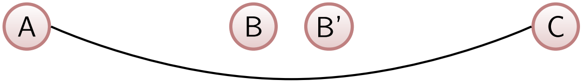
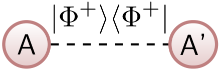


Swap operation

$$\mathrm{swap}(\rho_A, \rho_B) = \frac{1}{N}\,\mathrm{Tr}_{BB'}\left[\,(|\Phi^+\rangle\langle\Phi^+|_{BB'} \otimes \mathbb{1}_{AC})\ (\rho_1 \otimes \rho_2)\,\right]$$

Composing Channels

corresponds to swapping CJ associated states



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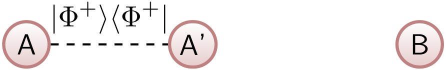

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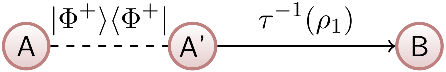

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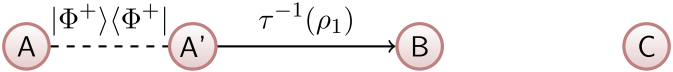

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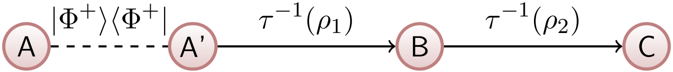

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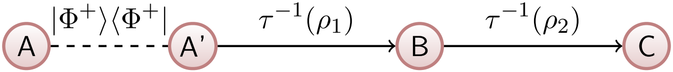
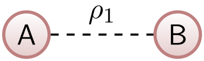


Swap in terms of channels

$$\tau^{-1}(\mathrm{swap}(\rho_1, \rho_2)) = \tau^{-1}(\rho_2) \circ \tau^{-1}(\rho_1)$$

(flipping systems or equivalently transposing channels if necessary, up to normalization)

Hybrid approach



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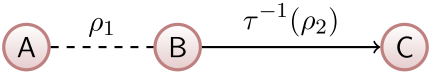

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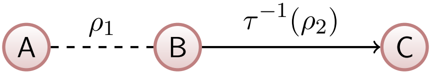

Hybrid formulation

$$\mathrm{swap}(\rho_1, \rho_2) = \left[\mathcal{I}_A \otimes \tau^{-1}(\rho_2)\right](\rho_1) = \left[\mathcal{I}_C \otimes \tau^{-1}(\rho_1)^T\right](\rho_2)$$

(flipping systems or equivalently transposing channels if necessary, up to normalization)

Swapped Bound Entanglement

Entanglement of swapped states from BE resources?

States: Image of the Swap operator

$$\mathrm{swap} : \text{BE states} \times \text{BE states} \to \mathcal{S}$$

OR

$$\mathrm{swap}(\text{BE states}, \text{BE states}) \cap \mathcal{D} \setminus \mathcal{S} \neq \emptyset$$

[1] Defined by their image: $\Lambda_\Gamma(\mathcal{D}) \subset \Gamma$

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Composed Channels

If $\Lambda_1 : \mathcal{D}(A) \to \mathcal{D}(B)$ and $\Lambda_2 : \mathcal{D}(B) \to \mathcal{D}(C)$ are PPT channels[1], is $\Lambda_2 \circ \Lambda_1 : \mathcal{D}(A) \to \mathcal{D}(C)$ separable?

[1] Defined by their image: $\Lambda_\Gamma(\mathcal{D}) \subset \Gamma$

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