

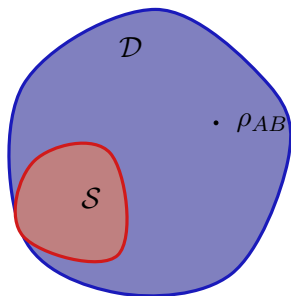
Bound Entanglement

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02/2014

Entanglement



... peculiar,
non-classical correlation
arising from the tensor
product structure in
combined quantum
systems

Bound Entanglement

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non-distillable entanglement

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Existence Explicite example of a bound entangled state

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Use Why should we be concerned about bound entanglement?

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Existence Explicite example of a bound entangled state

Use Why should we be concerned about bound entanglement?

Detection How do we decide for a given state whether it is (bound) entangled?

Roadmap

Showing the existence of BE

[Hor97] construct entangled state with negative partial transpose

[HHH98] NPT states are not distillable

The PPT Criterion

(Peres Criterion)

Definition

The partial transpose of a density matrix $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined as

$$\rho^\Gamma := (\mathcal{I}_A \otimes T_B)\rho \quad (1)$$

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Matrix elements in the tensor basis

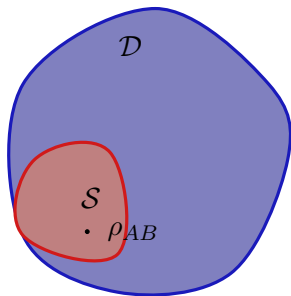
In the product basis

$$(\rho^\Gamma)_{ikjl} = \langle i \otimes k | \rho^\Gamma | j \otimes l \rangle \quad (2)$$

$$= \langle i \otimes l | \rho | j \otimes k \rangle = \rho_{iljk} \quad (3)$$

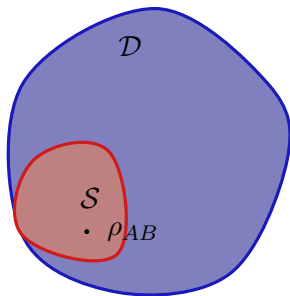
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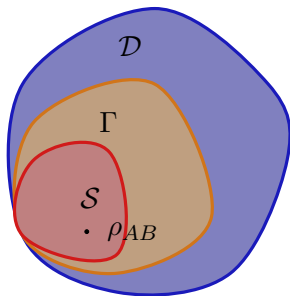


Entanglement criterion

$$\rho_{AB} \in \mathcal{S} \subset \mathcal{H}_A \otimes \mathcal{H}_B \Rightarrow \rho_{AB} \in \Gamma \quad (1)$$

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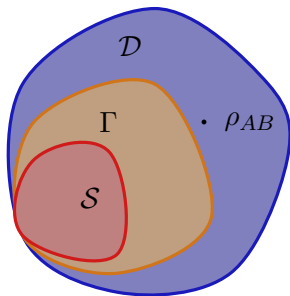


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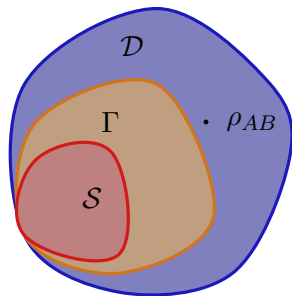


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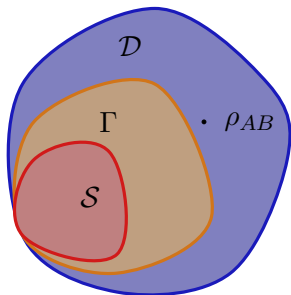
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Accuracy

The criterion is accurate in $2 \otimes 2$ and $2 \otimes 3$.
But not in higher dimensions.[HHH96]

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But not in higher dimensions. [HHH96]

Group of Entanglement Tests

Any Positive but not Completely Positive Map ('channel') serves as a similar entanglement test. [HHH96]

The Range Criterion [Hor97]

Decomposition of separable states

$\forall \rho \in \mathcal{S} :$

$$\rho = \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} p_{ij} |\phi_i\rangle\langle\phi_i| \otimes |\psi_j\rangle\langle\psi_j| \quad (2)$$

$$|\phi_i\rangle \in \mathcal{H}_A, |\psi_j\rangle \in \mathcal{H}_B$$

$$n_A < \dim \mathcal{H}_A, n_B < \dim \mathcal{H}_B$$

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→ features of convex sets, Caratheodories Thm [Car11]

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The range of an operator

$$\text{Ran } \rho := \{|\psi\rangle \in \mathcal{H}, \exists |\phi\rangle \in \mathcal{H} : |\psi\rangle = \rho|\phi\rangle\} \quad (3)$$

The Range Criterion [Hor97]

The range of separable density matrices

Any separable density matrix $\rho \in \mathcal{S} \subset \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as a convex combination of pure products states

$$\begin{aligned}\rho &= \sum_{i,j} p_{ij} |\psi_i\rangle\langle\psi_i| \otimes |\phi_j\rangle\langle\phi_j| = \sum_{i,j} p_{ij} |\psi_i \otimes \phi_j\rangle\langle\psi_i \otimes \phi_j| \\ \rho^\Gamma &= (\mathcal{I} \otimes T)\rho = \sum_{i,j} p_{ij} |\psi_i\rangle\langle\psi_i| \otimes \underbrace{|\phi_j\rangle\langle\phi_j|^T}_{|\phi_j^*\rangle\langle\phi_j^*|} \\ &= \sum_{ij} p_{ij} |\psi_i \otimes \phi_j^*\rangle\langle\psi_i \otimes \phi_j^*|\end{aligned}$$

and the set of product states $\{|\psi_i \otimes \phi_j\rangle\}_{ij}$ ($\{|\psi_i \otimes \phi_j^*\rangle\}_{ij}$) span the range of ρ (ρ^Γ)

The Range Criterion [Hor97]

Entanglement test

- find product vectors that span the range of ρ
- show that these vectors do **not** span the range of ρ^Γ

The first PPT entangled state[Hor97]

$$\rho_a = \frac{1}{8a+1} \begin{bmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix} \quad a \in (0, 1)$$

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Properties of ρ_a

- PPT
- ρ_a and ρ_a^Γ violate the range criterion

PPT and Distillability

Definition

A density matrix ρ is called distillable if

$$\begin{aligned} \exists \text{ LOCC } \Lambda : \mathcal{D}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}) &\rightarrow \mathcal{D}(\mathbb{C}^2) \quad \text{s.t. :} \\ \Lambda(\rho^{\otimes n}) &= |\Phi^+\rangle\langle\Phi^+| \quad \text{for some large } n \end{aligned}$$

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Cannot distill qubits from PPT density matrices ρ and $\rho^{\otimes N}$

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Non-distillable entanglement

Any PPT entangled state is not distillable.

Distillation of PPT states

Claim

PPT states are **not** distillable.

Distillation of PPT states

Distilling $\rho^{\otimes n}$

Assume ρ is distillable:

$$\Lambda(\rho^{\otimes n}) = \frac{1}{M} \sum_i (A_i \otimes B_i) \rho^{\otimes n} (A_i^\dagger \otimes B_i^\dagger) \quad \text{is entangled}$$

$$\Rightarrow \quad \exists \, i_0 \text{ s.t.: } \underbrace{(A_{i_0} \otimes B_{i_0}) \rho^{\otimes n} (A_{i_0}^\dagger \otimes B_{i_0}^\dagger)}_{=:\rho_{i_0}} \quad \text{is entangled}$$

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$$A_{i_0} = |0\rangle\langle\psi_A| + |1\rangle\langle\phi_A| \quad B_{i_0} = |0\rangle\langle\psi_B| + |1\rangle\langle\phi_B|$$

$$P_A(P_B) := \text{projector onto } \langle\psi_A, \phi_A\rangle(\langle\psi_B, \phi_B\rangle)$$

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$$\Rightarrow \exists \psi \text{ s.t.: } \langle \psi | (\rho')^{\Gamma_B} | \psi \rangle < 0$$

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$$\Rightarrow \rho \text{ is NPT}$$

Bound entanglement

a first summary

def entanglement that cannot be distilled

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existence only in higher dimensions (smallest BE systems in $3 \otimes 3$ and $4 \otimes 2$)

detection need a stronger criterion than PPT, e.g. range criterion

Quantum Key Distribution

with bound entangled resources

Secure key from BE

Bound entangled resources can be employed to distill secret key [HHHO05].¹

¹For further details see [HHHO09]

Quantum Key Distribution

with bound entangled resources

Shielded System with an adversary

$$A \otimes B \otimes \underbrace{A' \otimes B'}_{\text{shield}} \otimes E \quad (4)$$

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Assume: Eavesdropper Eve controls the whole environment (purification).

Quantum Key Distribution

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Shielded System with an adversary

$$|\psi\rangle\langle\psi| \in \mathcal{D}(A \otimes B \otimes \underbrace{A' \otimes B'}_{\text{shield}} \otimes E) \quad (4)$$

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Quantum Key Distribution

with bound entangled resources

Security and Key

After measuring in a product basis the systems A, B and tracing out A', B'

$$\rho_{ccq} = \sum_{ij} p_{ij} |e_i \otimes f_j\rangle \langle e_i \otimes f_j| \otimes \rho_{ij}^E \quad (4)$$

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Quantum Key Distribution

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Security and Key

$$\rho_{ccq} = \sum_{ij} p_{ij} |e_i \otimes f_j\rangle \langle e_i \otimes f_j| \otimes \rho^E \quad (4)$$

is **secure** and **has key** if $\{p_{ij}\} = \left\{ \frac{1}{d} \delta_{ij} \right\}$

Quantum Key Distribution

with bound entangled resources

Private States

$$\gamma = \frac{1}{d} \sum_{ij} |e_i f_i\rangle \langle e_j f_j|_{AB} \otimes U_i \sigma_{A'B'} U_j^\dagger \quad (4)$$

where U_i 's are arbitrary unitary transformations.

Quantum Key Distribution

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Key of private states

ρ is a private state $\Leftrightarrow \rho$ has key

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Private states and BE

Private states can be approximated with BE states.

Entanglement Tests

PPT all NPT states are entangled

→ does not detect BE

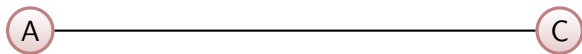
range compare the range of ρ and ρ^{Γ}

extendibility check extendibility of ρ [DPS05] (Semi definite program)

Further: ranges for randomly drawn states (concentration of measure) [ASY12a] [ASY12b]

- 1 Existence of Bound Entanglement
- 2 Applications of BE
- 3 Testing entanglement
- 4 Notions of Quantum Repeaters
 - Swap Operation on States
 - Composition of Channels
 - Hybrid Approach
- 5 Question

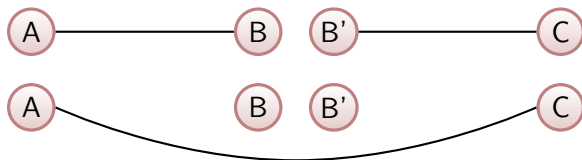
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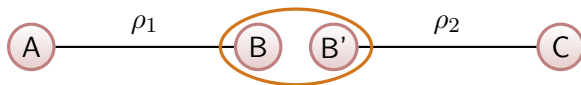
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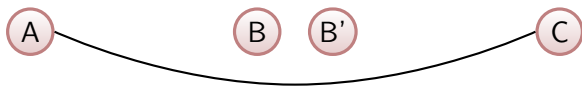
Swap Operator on States



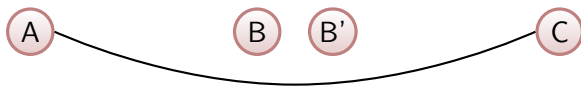
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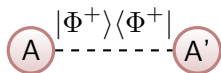


Swap operation

$$\text{swap}(\rho_A, \rho_B) = \frac{1}{N} \text{Tr}_{BB'} [(|\Phi^+\rangle\langle\Phi^+|_{BB'} \otimes \mathbb{1}_{AC}) (\rho_1 \otimes \rho_2)]$$

Composing Channels

corresponds to swapping CJ associated states



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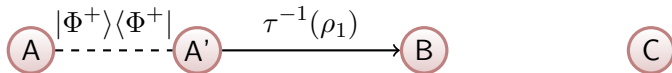
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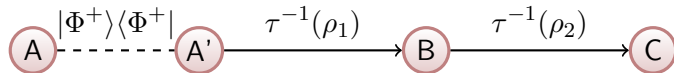
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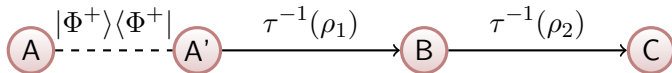
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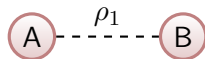


Swap in terms of channels

$$\tau^{-1}(\text{swap}(\rho_1, \rho_2)) = \tau^{-1}(\rho_2) \circ \tau^{-1}(\rho_1)$$

(flipping systems or equivalently transposing channels if necessary, up to normalization)

Hybrid approach



Hybrid approach



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Hybrid approach



Hybrid formulation

$$\text{swap}(\rho_1, \rho_2) = [\mathcal{I}_A \otimes \tau^{-1}(\rho_2)] (\rho_1) = [\mathcal{I}_C \otimes \tau^{-1}(\rho_1)^T] (\rho_2)$$

(flipping systems or equivalently transposing channels if necessary, up to normalization)

Swapped Bound Entanglement

Entanglement of swapped states from BE resources?

States: Image of the Swap operator

$$\text{swap} : \text{BE states} \times \text{BE states} \rightarrow \mathcal{S}$$

OR

$$\text{swap}(\text{BE states}, \text{BE states}) \cap \mathcal{D} \setminus \mathcal{S} \neq \emptyset$$

¹Defined by their image: $\Lambda_{\Gamma}(\mathcal{D}) \subset \Gamma$

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Composed Channels

If $\Lambda_1 : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ and $\Lambda_2 : \mathcal{D}(B) \rightarrow \mathcal{D}(C)$ are PPT channels¹,
is $\Lambda_2 \circ \Lambda_1 : \mathcal{D}(A) \rightarrow \mathcal{D}(C)$ separable?

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Bibliography I

Guillaume Aubrun, Stanislaw J. Szarek, and Deping Ye.
Entanglement thresholds for random induced states.
2012.

Guillaume Aubrun, Stanisław J. Szarek, and Deping Ye.
Phase transitions for random states and a semicircle law for the
partial transpose.
Phys. Rev. A, 85:030302, Mar 2012.

C. Carathéodory.
Über den variabilitätsbereich der fourier'schen konstanten von
positiven harmonischen funktionen.
Rendiconti del Circolo Matematico di Palermo, 32:193–217, 1911.

Bibliography II

Andrew C. Doherty, Pablo A. Parrilo, and Federico M. Spedalieri.
Detecting multipartite entanglement.
Phys. Rev. A, 71:032333, Mar 2005.

Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki.
Separability of mixed states: necessary and sufficient conditions.
Physics Letters A, 223(1–2):1 – 8, 1996.

Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki.
Mixed-state entanglement and distillation: Is there a “bound”
entanglement in nature?
Phys. Rev. Lett., 80:5239–5242, Jun 1998.

Bibliography III

Karol Horodecki, Michał Horodecki, Paweł Horodecki, and Jonathan Oppenheim.

Secure key from bound entanglement.

Phys. Rev. Lett., 94:160502, Apr 2005.

K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim.

General paradigm for distilling classical key from quantum states.

Information Theory, IEEE Transactions on, 55(4):1898 –1929, april 2009.

Paweł Horodecki.

Separability criterion and inseparable mixed states with positive partial transposition.

Physics Letters A, 232(5):333 – 339, 1997.