2021 年期末试题解析

彭康学导团

1 选择题

1.1 D.

令 $\varphi(x) = \sqrt{x^2+1}$, $f(x) = \sqrt{x^2+2}$, $g(x) = \sqrt{x^2+3}$, 则 $\varphi(x) < f(x) < g(x)$, 且

$$\lim_{x\to\infty}(g(x) - \varphi(x)) = 0$$

但 $\lim\limits_{x\to\infty} f(x)$ 不存在.

1.2 C.

令 $f(x) = \int_1^x \dfrac{\sin t}{t}\,\mathrm{d}t - \ln x$, 则 $f'(x) = \dfrac{\sin x - 1}{x} \le 0$. 即 f 在 $(0,+\infty)$ 上单调递减. 又

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\int_1^x \frac{\sin t}{t}\,\mathrm{d}t - \ln x\right] = +\infty, f(1) = 0$$

故在 $(0,1)$ 上 $f(x) > 0$.

1.3 A.

设 $h(x) = \dfrac{f(x)}{g(x)}$, 则

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} < 0$$

故 $h(x)$ 在 (a,b) 上单调递减, 有 $h(b) < h(x) < h(a)$, 即

$$\frac{f(b)}{g(b)} < \frac{f(x)}{g(x)} < \frac{f(a)}{g(a)}$$

于是 $f(x)g(a) < f(a)g(x)$.

1.4 B.

因为

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{\int_0^x f(t)\,\mathrm{d}t}{x} = \lim_{x\to 0} f(x) = f(0)$$

又 $g(x)$ 在 $x = 0$ 处无定义, 故 $x = 0$ 是 $g(x)$ 的可去间断点.

1.5 C.

因为

$$\int_0^x x f'(x)\,\mathrm{d}x = \int_0^x x\,\mathrm{d}f(x) = {\color{red}x f(x)\,|_0^x} - {\color{blue}\int_0^x f(x)\,\mathrm{d}x}$$

其中红色部分代表矩形 $OBAC$ 的面积, 蓝色部分代表曲边梯形 $OBAD$ 的面积. 故原式代表曲边三角形 ACD 的面积.

2 填空题

2.1 e^{x+1}.

因为

$$f(x+1) = \lim_{n \to \infty} \left(1 + \frac{x+2}{n-2}\right)^n$$

$$= \lim_{n \to \infty} \left(1 + \frac{x+2}{n-2}\right)^{\frac{n-2}{x+2} \cdot (x+2) \cdot \frac{n}{n-2}}$$

$$= e^{x+2}$$

于是 $f(x) = e^{x+1}$.

2.2 $\dfrac{5}{2}$.

因为

$$f(x) = \begin{cases} x^2, & x > 2 \\ a + \dfrac{3}{2}, & x = 2 \\ ax - 1, & x < 2 \end{cases}$$

故 $f(x)$ 在 $x = 2$ 处连续 $\Leftrightarrow \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2) \Leftrightarrow a = \dfrac{5}{2}$.

2.3 3.

因为

$$\int_0^\pi f''(x) \sin x \, dx = \int_0^\pi \sin x \, df'(x)$$

$$= f'(x) \sin x \, |_0^\pi - \int_0^\pi f'(x) \, d\sin x$$

$$= -\int_0^\pi f'(x) \cos x \, dx = -f(x) \cos x \, |_0^\pi + \int_0^\pi f(x) \, d\cos x$$

$$= f(\pi) + f(0) - \int_0^\pi f(x) \sin x \, dx$$

于是

$$\int_0^\pi [f(x) + f''(x)] \sin x \, dx = \int_0^\pi f(x) \sin x \, dx + \int_0^\pi f''(x) \sin x \, dx$$

$$= \int_0^\pi f(x) \sin x \, dx + f(\pi) + f(0) - \int_0^\pi f(x) \sin x \, dx$$

$$= f(\pi) + f(0)$$

故 $f(\pi) + f(0) = 5$, $f(0) = 3$.

2.4 $4x\left(e^{-x^4}+6\right)$.

$$\lim_{\alpha\to 0}\frac{f(x+\alpha)-f(x-\alpha)}{\alpha}=\lim_{\alpha\to 0}f'(x+\alpha)+f'(x-\alpha)=2f'(x)=4x\left(e^{-x^4}+6\right)$$

2.5 $\dfrac{1}{p+1}$.

$$\lim_{n\to+\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}}=\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p$$
$$=\int_0^1 x^p\mathrm{d}x=\left.\frac{1}{p+1}x^{p+1}\right|_0^1=\frac{1}{p+1}$$

3 计算题

3.1 对原式泰勒展开, 得

$$\lim_{x\to 0}\frac{e^x\sin x-x-x^2}{(e^x-1)\sin^2 x}=\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2}\right)\left(x-\frac{x^3}{6}\right)-x-x^2+o\left(x^3\right)}{x^3}$$
$$=\frac{x+x^2+\frac{x^3}{2}-\frac{x^3}{6}-x-x^2+o\left(x^3\right)}{x^3}$$
$$=\frac{\frac{x^3}{3}+o\left(x^3\right)}{x^3}=\frac{1}{3}$$

3.2 容易发现 $f(x)$ 的定义域为 $\mathbb{R}$, 且在 $x\neq 0$ 时 $f(x)$ 可导, 故只需考虑分段点处的情况.

若使 f 在 $x=0$ 处可导, 则 f 在 $x=0$ 处连续, 有 $\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^+}f(x)=f(0)$. 又

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\sin x+2ae^x=2a \tag{1}$$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}9\arctan x+2b(x-1)^3=-2b \tag{2}$$

由 $(1),(2)$ 可得 $a=-b$.

因为 f 在 $x=0$ 处可导, 有 $f'_-(0)=f'_+(0)$, 又

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^-}\frac{\sin x+2ae^x-2a}{x}=2a+1 \tag{3}$$

$$f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{9 \arctan x + 2b(x-1)^3 + 2b}{x}$$

$$= \lim_{x \to 0^+} \frac{9 \arctan x + 2b \cdot x \cdot \left[(x-1)^2 - (x-1) + 1\right]}{x} \quad (4)$$

$$= \lim_{x \to 0^+} 9 + 2b \cdot (x^2 - 3x + 3) = 9 + 6b$$

即 $2a + 1 = 9 + 6b$, 解得 $a = 1, b = -1$.

3.3 因为 $f'(x) = 1 - \dfrac{2}{1 + x^2}$, 解得驻点为 $x = \pm 1$. 又 $f''(x) = \dfrac{4x}{\left(1 + x^2\right)^2}$, 拐点为 $x = 0$.
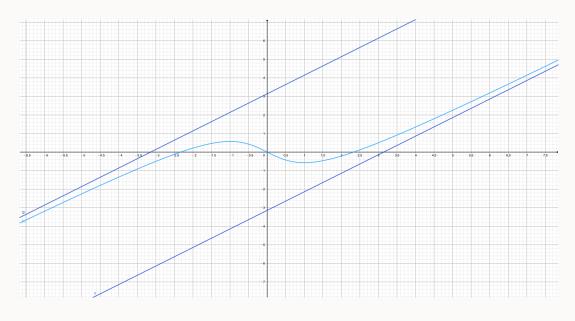
故

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
f'	$+$	0	$-$	$-$	$-$	0	$+$
f''	$-$	$-$	$-$	0	$+$	$+$	$+$
f	↑下凹	极大	↓下凹	拐点	↓上凹	极小	↑上凹

f 的单增区间为 $(-\infty, -1) \cup (1, +\infty)$, 单减区间为 $(-1, 1)$.

$$f_{\max} = f(-1) = \frac{\pi}{2} - 1, \quad f_{\min} = f(1) = 1 - \frac{\pi}{2}$$

f 的图像在 $(-\infty, 0)$ 下凹, 在 $(0, +\infty)$ 上凹, $(0, 0)$ 是拐点.

渐近线: $x \to +\infty$ 方向为 $y = x - \pi$, $x \to -\infty$ 方向为 $y = x + \pi$.

3.4-1 令 $\sqrt{\dfrac{x}{1-x}}$，则 $\dfrac{1}{1+x}=1-t^2$，$\mathrm{d}x=\mathrm{d}\left(\dfrac{1}{1-t^2}\right)$. 于是

$$\int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,\mathrm{d}x = \int_0^{\frac{\sqrt{3}}{2}} \arcsin t\,\mathrm{d}\left(\frac{1}{1-t^2}\right)$$

$$= \frac{1}{1-t^2}\Big|_0^{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1-t^2}\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \frac{4\pi}{3} - I_1$$

其中 $I_1 = \displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1-t^2}\frac{\mathrm{d}t}{\sqrt{1-t^2}}$，令 $t=\sin u$，有

$$I_1 = \int_0^{\frac{\pi}{3}} \frac{\mathrm{d}u}{\cos^2 u} = \tan u\,|_0^{\frac{\pi}{3}} = \sqrt{3}$$

故答案为 $\dfrac{4\pi}{3} - \sqrt{3}$.

3.4-2

$$\int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,\mathrm{d}x = \int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,\mathrm{d}(1+x)$$

$$= (1+x)\arcsin\sqrt{\frac{x}{x+1}}\,\Big|_0^3 - \int_0^3 (1+x)\cdot\frac{1}{2\sqrt{x}(1+x)}\,\mathrm{d}x$$

$$= \frac{4\pi}{3} - \int_0^3 \frac{\mathrm{d}x}{2\sqrt{x}} = \frac{4\pi}{3} - \sqrt{3}$$

3.5-1 令 $x=\tan t$，有

$$\int \frac{x^3\,\mathrm{d}x}{\sqrt{1+x^2}} = \int \frac{\tan^3 t\sec^2 t}{\sec t}\,\mathrm{d}t$$

$$= \int \tan^2 t\cdot\tan t\sec t$$

$$= \int (\sec^2 t - 1)\,\mathrm{d}\sec t$$

$$= \frac{1}{3}\sec^3 t - \sec t + C = \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \left(1+x^2\right)^{\frac{1}{2}} + C$$

3.5-2

$$\frac{x^3\,\mathrm{d}x}{\sqrt{1+x^2}} = \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}\,\mathrm{d}\left(1+x^2\right)$$

$$= \int x^2\,\mathrm{d}\sqrt{1+x^2} = \int \left(\sqrt{1+x^2}\right)^2\,\mathrm{d}\sqrt{1+x^2} - \int \mathrm{d}\sqrt{1+x^2}$$

$$= \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \left(1+x^2\right)^{\frac{1}{2}} + C$$

3.5-3

$$\int \frac{x^3 \, dx}{\sqrt{1+x^2}} = \frac{1}{2} \int \frac{x^2}{\sqrt{1+x^2}} \, d\left(1+x^2\right)$$

$$= \frac{1}{2} \int \frac{1+x^2-1}{\sqrt{1+x^2}} \, d\left(1+x^2\right)$$

$$= \frac{1}{2} \int \left(\left(1+x^2\right)^{\frac{1}{2}} - \left(1+x^2\right)^{-\frac{1}{2}}\right) \, d\left(1+x^2\right)$$

$$= \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \left(1+x^2\right)^{\frac{1}{2}} + C$$

3.6 设被积函数为 $g(x)$, 则 $g(x)$ 有奇点 $x=0, x=2$. 设

$$I = \int_{-1}^{3} g(x) \, dx = \int_{-1}^{0} g(x) \, dx + \int_{0}^{2} g(x) \, dx + \int_{2}^{3} g(x) \, dx \stackrel{\text{def}}{=} I_1 + I_2 + I_3$$

又 $f(0-0) = -\infty, f(0+0) = +\infty, f(2-0) = -\infty, f(2+0) = +\infty$, 故

$$I_1 = \arctan f(x) \, |_{-1}^{0} = \arctan(f(0-0)) - \arctan(f(-1)) = -\frac{\pi}{2} - 0 = -\frac{\pi}{2}$$

$$I_2 = \arctan f(x) \, |_{0}^{2} = \arctan(f(2-0)) - \arctan(f(0+0)) = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi$$

$$I_3 = \arctan f(x) \, |_{2}^{3} = \arctan(f(3)) - \arctan(f(2+0)) = \arctan \frac{32}{27} - \frac{\pi}{2}$$

于是 $I = \arctan \dfrac{32}{27} - 2\pi.$

3.7 由题意有

$$dW = \pi \left(y - \frac{y}{4}\right) g(H-y) \, dy = \frac{3}{4}\pi g(Hy - y^2) \, dy$$

故

$$W = \frac{3}{4}\pi g \int_{0}^{H} (Hy - y^2) \, dy = \frac{1}{8}\pi g H^3$$

4

令 $t = \ln(2x-1)$, 则

$$\frac{dt}{dx} = \frac{2}{2x-1}.$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{2}{2x-1}\frac{dy}{dt} \qquad \frac{d^2y}{dx^2} = -\frac{4}{(2x-1)^2}\frac{dy}{dt} + \frac{4}{(2x-1)^2}\frac{d^2y}{dt^2}$$

代入原式, 化简得

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} - 2y = \frac{\mathrm{e}^t}{2} - \frac{1}{4}$$

因此

$$\lambda^2 + \lambda - 2 = 0 \qquad \Longrightarrow \qquad \lambda_1 = -2, \lambda_2 = 1.$$

齐次通解:

$$\tilde{y} = c_1 \mathrm{e}^{-2t} + c_2$$

$$\mathrm{e}^t \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} - 2y = \frac{\mathrm{e}^t}{2},$$

设特解为 $y_1^* = At\mathrm{e}^t$, 解得

$$y_1^* = \frac{t}{6}\mathrm{e}^t \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} - 2y = -\frac{1}{4},$$

易见特解为

$$y_2^* = \frac{1}{8}$$

通解为

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^t + \frac{t}{6}\mathrm{e}^t + \frac{1}{8} = \frac{c_1}{(2x-1)^2} + c_2(2x-1) + \frac{2x-1}{6}\ln(2x-1) + \frac{1}{8}$$

5

解法一: $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (\lambda-2)^2(3-\lambda) \quad \therefore \lambda_1 = \lambda_2 = 2, \lambda_3 = 3$

对 $\lambda_1 = \lambda_2 = 2, (A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, 解 $(A-2I)^2 x = 0$, 得:

$$\vec{r}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{r}_0^{(2)} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{r}_1^{(1)} = (A-2I)\vec{r}_0^{(1)} = 0, \quad \vec{r}_1^{(2)} = (A-2I)\vec{r}_0^{(2)} = (-1,0,0)^T$$

$$\vec{x}_1(t) = \mathrm{e}^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x}_2(t) = \mathrm{e}^{2t}\left(\vec{r}_0^{(2)} + t\vec{r}_1^{(2)}\right) = \mathrm{e}^{2t}\begin{pmatrix} -t \\ -1 \\ 1 \end{pmatrix}$$

对 $\lambda_3 = 3$, 解得特征向量为: $\vec{r}_3 = (0, 0, 1)^T$. $\quad \vec{x}_3(t) = \mathrm{e}^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

齐次方程的基解矩阵为 $X_1(t) = (\vec{x}_1(t), -\vec{x}_2(t), \vec{x}_3(t)) = \begin{pmatrix} \mathrm{e}^{2t} & t\mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & -\mathrm{e}^{2t} & \mathrm{e}^{3t} \end{pmatrix}$

$$
\begin{aligned}
X(t) = X_1(t)X_1^{-1}(0) &= \begin{pmatrix} \mathrm{e}^{2t} & \mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & -\mathrm{e}^{2t} & \mathrm{e}^{3t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \mathrm{e}^{2t} & t\mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & -\mathrm{e}^{2t} & \mathrm{e}^{3t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{2t} & t\mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{3t} - \mathrm{e}^{2t} & \mathrm{e}^{3t} \end{pmatrix}
\end{aligned}
$$

$$
X(t-\tau)\vec{f}(\tau) = X(t-\tau) \begin{pmatrix} \tau \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{2(t-\tau)} & (t-\tau)\mathrm{e}^{2(t-\tau)} & 0 \\ 0 & \mathrm{e}^{2(t-\tau)} & 0 \\ 0 & \mathrm{e}^{3(t-\tau)} - \mathrm{e}^{2(t-\tau)} & \mathrm{e}^{3(t-\tau)} \end{pmatrix} \begin{pmatrix} \tau \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t\mathrm{e}^{2(t-\tau)} \\ \mathrm{e}^{2(t-\tau)} \\ \mathrm{e}^{3(t-\tau)} - \mathrm{e}^{2(t-\tau)} \end{pmatrix}
$$

$$
\int_0^t X(t-\tau)\vec{f}(\tau)\mathrm{d}\tau = \begin{pmatrix} \frac{t}{2}\mathrm{e}^{2t} - \frac{t}{2} \\ \frac{1}{2}\mathrm{e}^{2t} - \frac{1}{2} \\ -\frac{1}{2}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{3t} + \frac{1}{6} \end{pmatrix}
$$

<p style="text-align:center">方程组通解为</p>

$$
\begin{aligned}
\vec{x} = X(t)\vec{C} + \int_0^t X(t-\tau)\vec{f}(\tau)\mathrm{d}\tau &= \begin{pmatrix} \mathrm{e}^{2t} & t\mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{3t} - \mathrm{e}^{2t} & \mathrm{e}^{3t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} + \begin{pmatrix} \frac{t}{2}\mathrm{e}^{2t} - \frac{t}{2} \\ \frac{1}{2}\mathrm{e}^{2t} - \frac{1}{2} \\ -\frac{1}{2}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{3t} + \frac{1}{6} \end{pmatrix} \\
&= \begin{pmatrix} C_1\mathrm{e}^{2t} + \left(C_2 + \frac{1}{2}\right)t\mathrm{e}^{2t} - \frac{t}{2} \\ \left(C_2 + \frac{1}{2}\right)\mathrm{e}^{2t} - \frac{1}{2} \\ \left(C_2 + C_3 + \frac{1}{3}\right)\mathrm{e}^{3t} - \left(C_2 + \frac{1}{2}\right)\mathrm{e}^{2t} + \frac{1}{6} \end{pmatrix} \quad \text{其中} \vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \text{任意.}
\end{aligned}
$$

解法二：

$$
\begin{cases} \dot{x}_1 = 2x_1 + x_2 \quad +t & \cdots (1) \\ \dot{x}_2 = \qquad\quad 2x_2 \qquad +1 & \cdots (2) \\ \dot{x}_3 = \qquad\quad x_2 + 3x_3 & \cdots (3) \end{cases}
$$

其中 (2) 式的解为

$$x_2 = \mathrm{e}^{\int 2\mathrm{d}t}\left[\int \mathrm{e}^{-\int 2\mathrm{d}t}\mathrm{d}t + C_2\right]$$

代入 (1), 得:

$$\dot{x}_1 - 2x_1 = C_2\mathrm{e}^{2t} - \frac{1}{2} + t,\, x_1 = \mathrm{e}^{2t}\left[\int\left(C_2\mathrm{e}^{2t} - \frac{1}{2} + t\right)\mathrm{e}^{-2t}\,\mathrm{d}t + C_1\right] = C_1\mathrm{e}^{2t} + C_2 t\mathrm{e}^{2t} - \frac{t}{2}$$

代入 (3), 得:

$$\dot{x}_3 - 3x_3 = C_2\mathrm{e}^{2t} - \frac{1}{2},\, x_3 = \mathrm{e}^{3t}\left[\int\left(C_2\mathrm{e}^{2t} - \frac{1}{2}\right)\mathrm{e}^{-3t}\,\mathrm{d}t + C_3\right] = -C_2\mathrm{e}^{2t} + C_3\mathrm{e}^{3t} + \frac{1}{6}$$

故方程组通解为

$$\vec{x} = C_1\mathrm{e}^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\mathrm{e}^{2t}\begin{pmatrix} t \\ 1 \\ -1 \end{pmatrix} + C_3\mathrm{e}^{3t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -t/2 \\ -1/2 \\ 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{e}^{2t} & t\mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & -\mathrm{e}^{2t} & \mathrm{e}^{3t} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} + \begin{pmatrix} -t/2 \\ -1/2 \\ 1/6 \end{pmatrix}$$

6

(1) 构造函数 $F(x) = e^{\sin(x)}\mathscr{F}(x)$

$$F'(x) = \cos(x)e^{\sin(x)}\mathscr{F}(x) + e^{\sin(x)}\mathscr{F}'(x) = (\cos(x)\mathscr{F}(x) + \mathscr{F}'(x))e^{\sin(x)}$$

$\mathscr{F}(x)$ 在 $(0, 2\pi)$ 可导, $e^{\sin(x)}$ 在 $(0, 2\pi)$ 可导, 故 $F(x)$ 在 $(0, 2\pi)$ 可导, $\mathscr{F}(x)$ 在 $[0, 2\pi]$ 连续, $F(x)$ 在 $[0, 2\pi]$ 连续

(2) $F(0) = 1$, $F(\pi) = 3$, $F(2\pi) = 2$, 又由于 $F(x)$ 连续, 因此必有 a, b 满足 $0 < a < \pi < b < 2\pi$ 使得

$$F(a) = F(b)$$

且 $F(x)$ 在 $[a, b]$ 上连续, 在 (a, b) 上可导, 故由罗尔定理知, $\exists \xi \in (a, b)$ 使得

$$F'(\xi) = 0$$

即

$$e^{\sin(\xi)}(\mathscr{F}'(\xi) + \mathscr{F}(\xi)\cos(\xi)) = 0$$

而 $e^{\sin(\xi)} \neq 0$ 故在 $(0, 2\pi)$ 上至少有一点 ξ, $\mathscr{F}'(\xi) + \mathscr{F}(\xi)\cos(\xi) = 0$

7

(1) 由题干条件 $f(-x) + f(x) = A$, 考虑代换 $t = -x$, 有

$$\int_{-a}^{a} f(x)g(x)dx \stackrel{t=-x}{=} -\int_{a}^{-a} f(-t)g(-t)dt$$

$$= \int_{-a}^{a} f(-t)g(t)dt,$$

$$\text{则} \int_{-a}^{a} f(x)g(x)dx = \frac{1}{2}\left[\int_{-a}^{a} f(x)g(x)dx + \int_{-a}^{a} f(-x)g(x)dx\right]$$

$$= \frac{1}{2}\int_{-a}^{a} [f(x) + f(-x)]g(x)dx$$

$$= \frac{A}{2}\int_{-a}^{a} g(x)dx \stackrel{(\text{偶})}{=} A\int_{0}^{a} g(x)dx.$$

(2) 考虑到反正切函数特殊性，记 $h(x) = \arctan(e^x)$ 猜想 $h(x) + h(-x) = C$. 下面进行证明:

$$\frac{\mathrm{d}}{\mathrm{d}x}[h(x) + h(-x)] = \frac{e^x}{1 + e^{2x}} + \frac{-e^{-x}}{1 + e^{-2x}} = 0$$

即

$$h(x) + h(-x) = C = 2h(0) = \frac{\pi}{2}$$

由 (1) 中结论有

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \cdot \arctan(e^x)dx = \frac{\pi}{2}\int_{0}^{\frac{\pi}{2}} \cos^4 xdx = \frac{\pi}{2} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{3\pi^2}{32}$$

注: 华里士公式 (Wallis) 公式（书 $P_{2.11}$ 例3.21）

$$I_n = \int_{0}^{\frac{\pi}{2}} \sin^n xdx = \int_{0}^{\frac{\pi}{2}} \cos^n xdx = \begin{cases} \dfrac{(n-1)!!}{n!!}, & n\text{为奇数} \\ \dfrac{(n-1)!!}{n!!} \cdot \dfrac{\pi}{2}, & n\text{为偶数} \end{cases}$$