Introduction to graphs

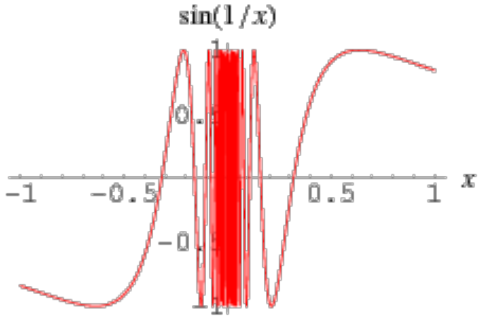
Programmazione Avanzata 2025-26

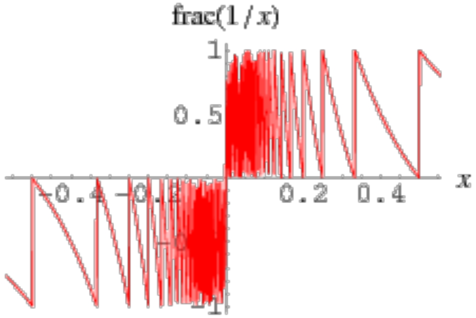
Graph: definition

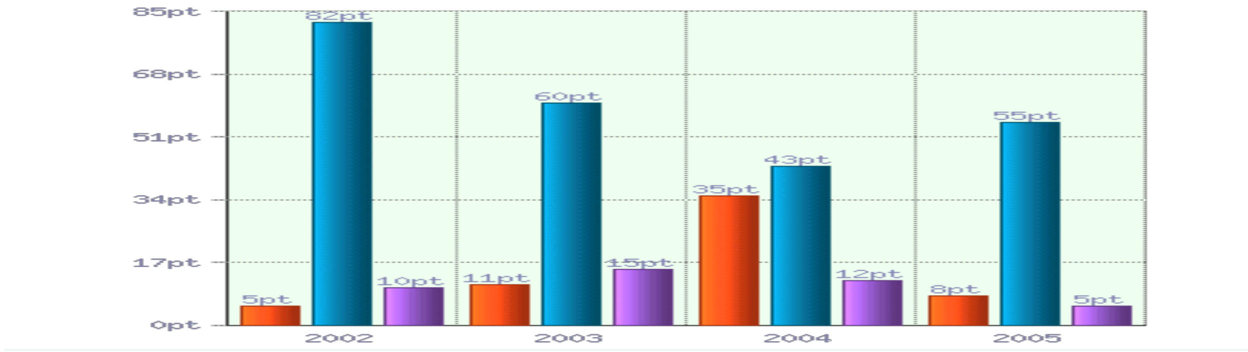
- A graph is a collection of points and lines connecting some (eventually empty) subset of them
- The points of a graph are tipically known as graph vertices, but may also be called “nodes” or simply “points”
- The lines connecting the vertices of a graph are called graph edges, but may also be called “arcs” or “lines”

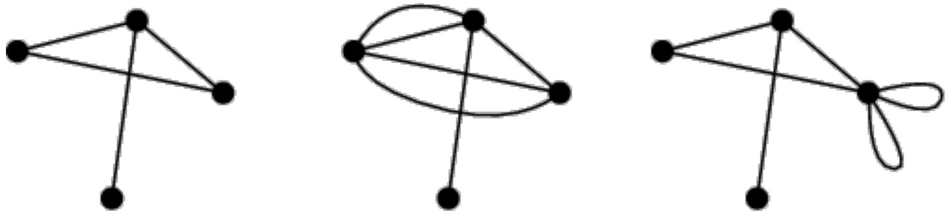
Warning: Graph ≠ Graph ≠ Graph

- Graph (plot)
- Graph (chart)
- Graph (maths)

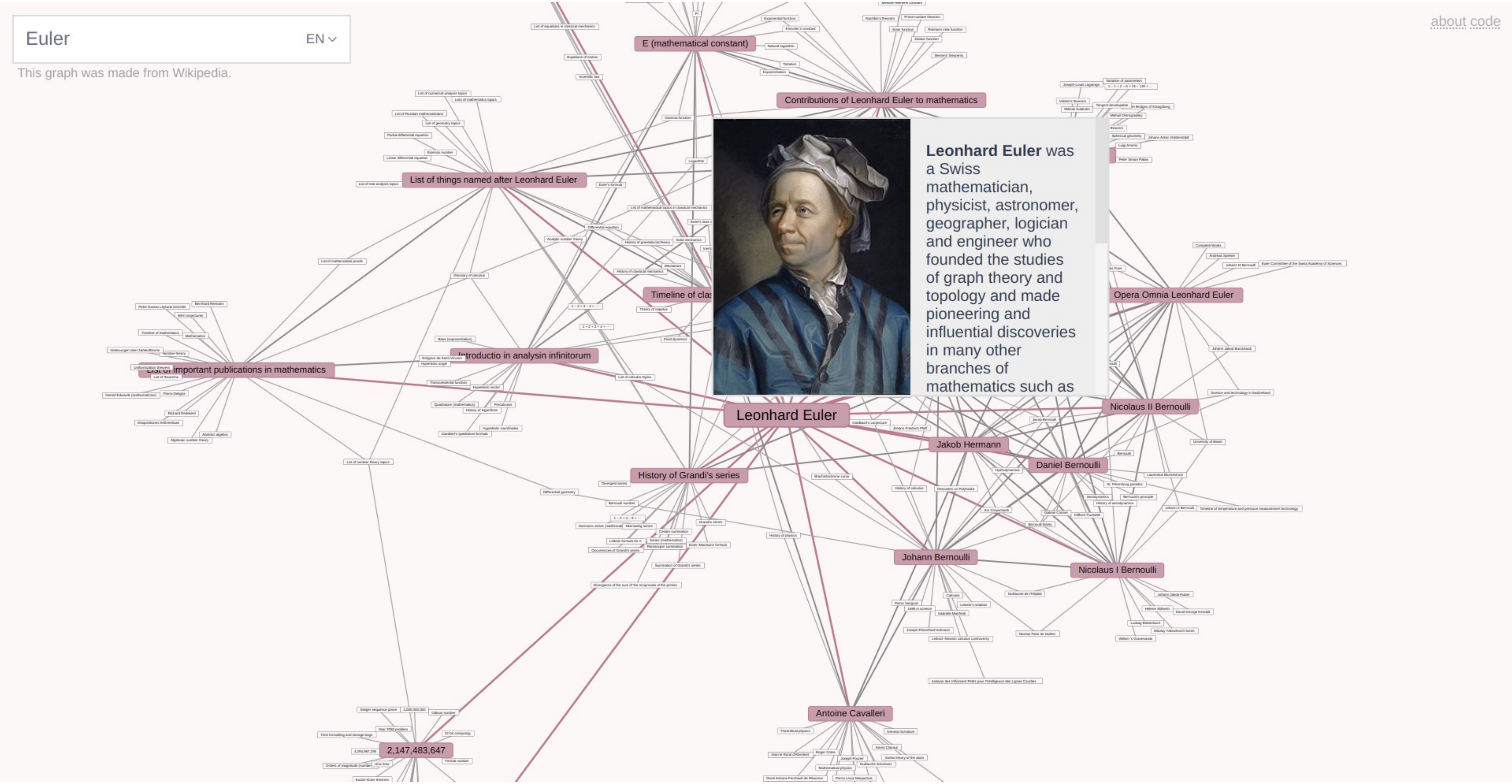








Example



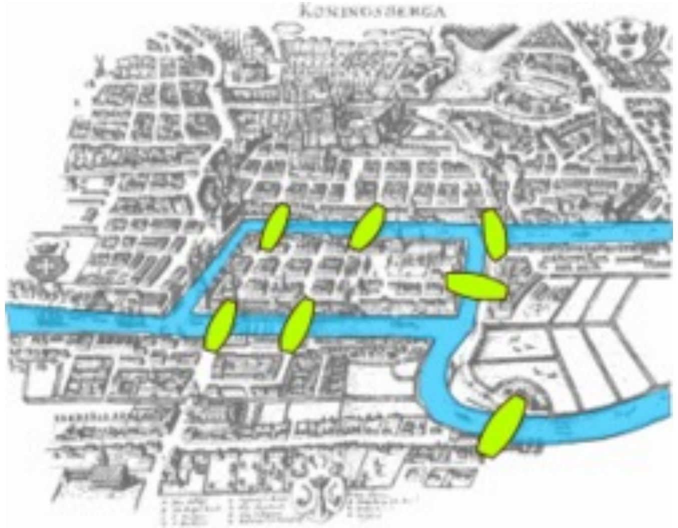
Euler
EN
This graph was made from Wikipedia.
about code
E (mathematical constant)
Contributions of Leonhard Euler to mathematics
List of things named after Leonhard Euler
Leonhard Euler was a Swiss mathematician, physicist, astronomer, geographer, logician and engineer who founded the studies of graph theory and topology and made pioneering and influential discoveries in many other branches of mathematics such as
Opera Omnia Leonhard Euler
Introductio in analysin infinitorum
List of important publications in mathematics
Leonhard Euler
Nicolaus II Bernoulli
Jakob Hermann
Daniel Bernoulli
History of Grandi's series
Johann Bernoulli
Nicolaus I Bernoulli
Antoine Cavalleri
2,147,483,647

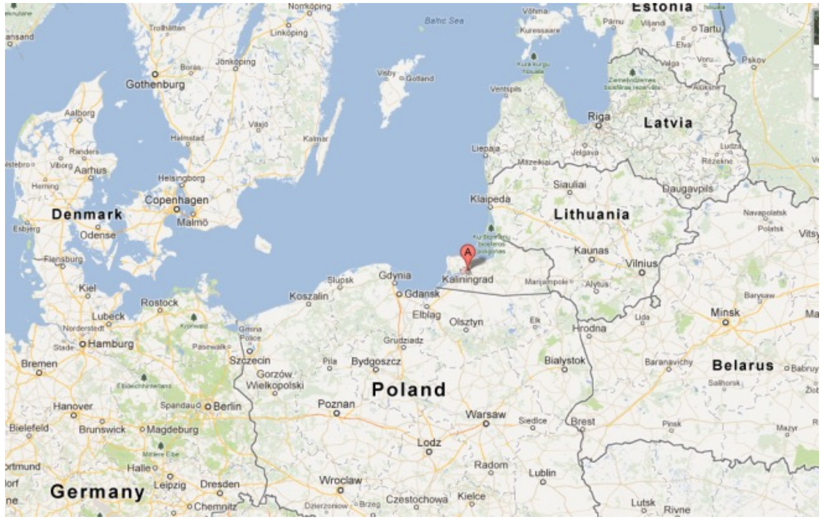
History

- The study of graphs is known as graph theory, and was first systematically investigated by D. König in the 1930s
- Euler’s proof about the *walk across all seven bridges of Königsberg* (1736), now known as the *Königsberg bridge problem*, is a famous precursor to graph theory
- Indeed, the study of various sorts of paths in graphs has many applications in real-world problems

Königsberg Bridge Problem

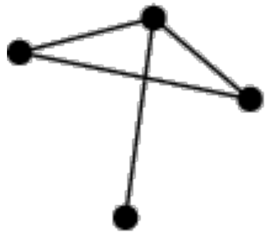
- Can the 7 bridges the of the city of Königsberg over the river Preger all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began?



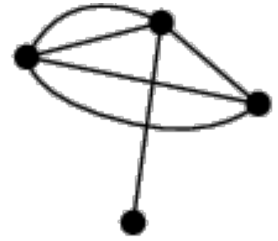


Types of graphs: edge cardinality

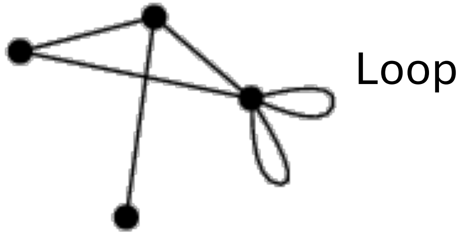
- Simple graph
  - At most one edge (i.e., either one edge or no edges) may connect any two vertices
- Multigraph
  - Multiple edges are allowed between vertices
- Loops
  - Edge between a vertex and itself
- Pseudograph
  - Multigraph with loops



Simple graph



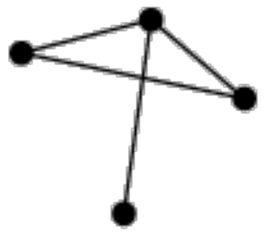
Multigraph



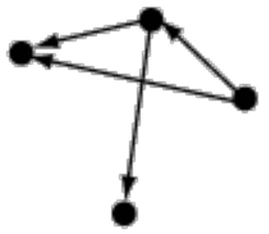
Pseudograph

Types of graphs: edge direction

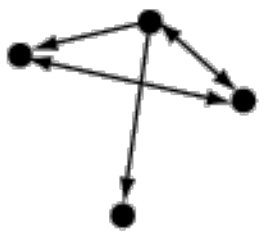
- Undirected
- Oriented
  - Edges have one direction (indicated by arrow)
- Directed
  - Edges may have one or two directions
- Network
  - Oriented graph with weighted edges



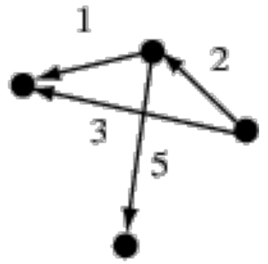
Undirected graph



Oriented graph



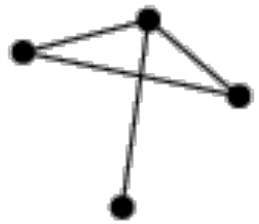
Directed graph



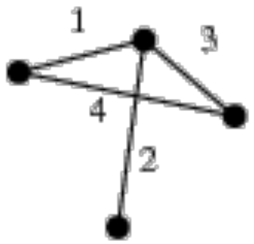
Network

Types of graphs: labeling

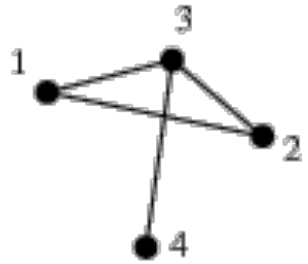
- Labels
  - None
  - On Vertices
  - On Edges
- Groups (=colors)
  - Of Vertices
    - No edge connects two identically colored vertices
  - Of Edges
  - Of both
    - Adjacent edges



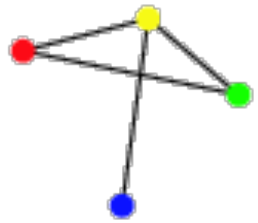
Unlabeled graph



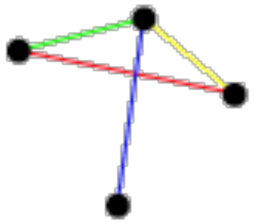
Edge-labeled graph



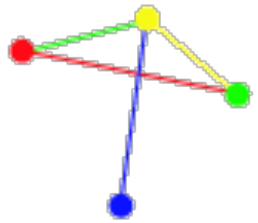
Vertex-labeled graph



Vertex-colored graph



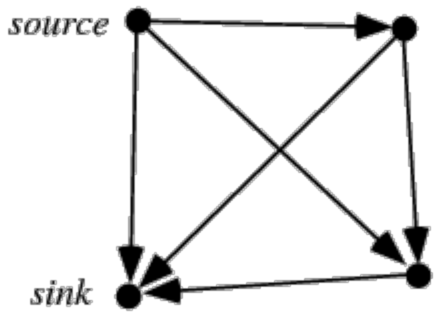
Edge-colored graph



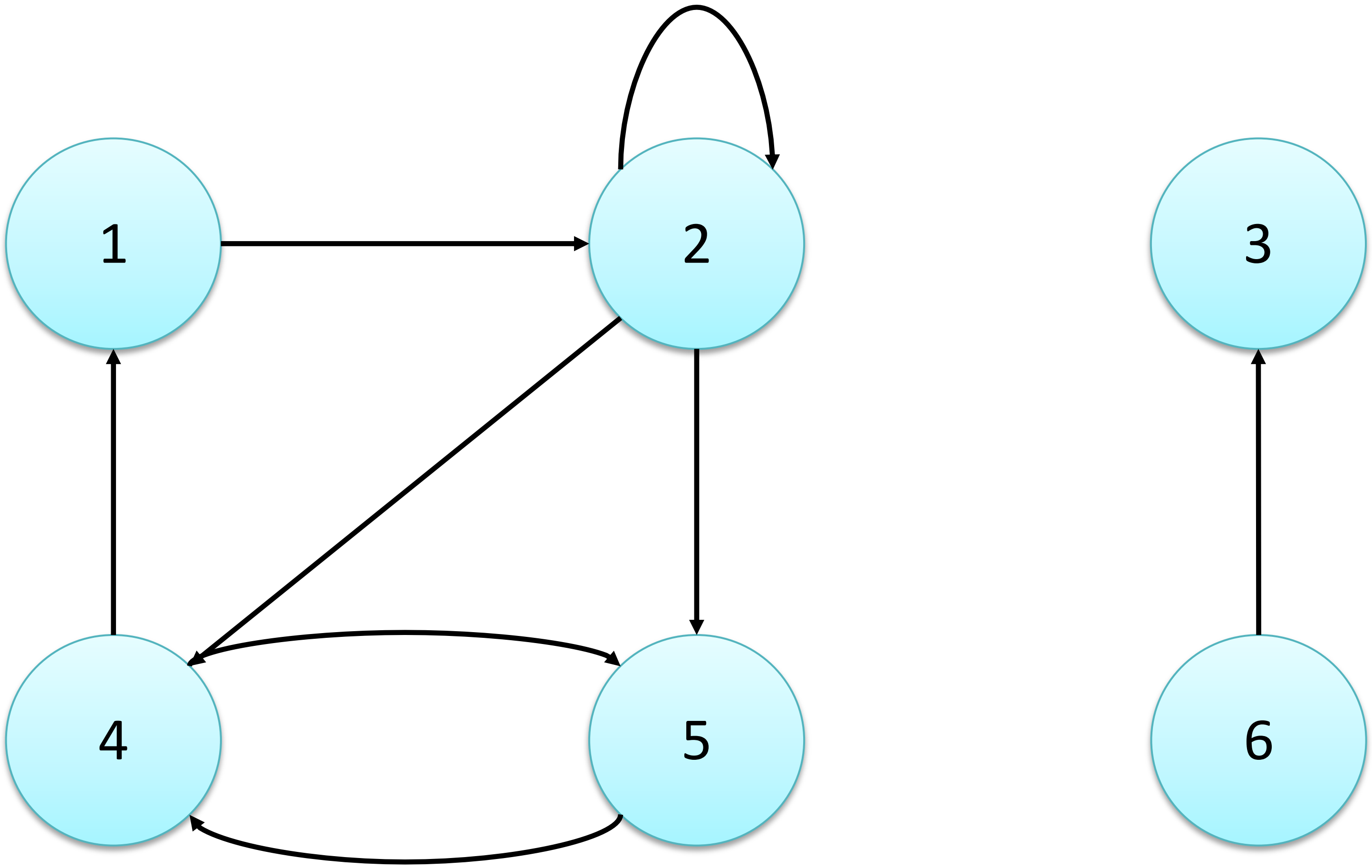
Vertex- and edge-colored graph

Directed and Oriented graphs

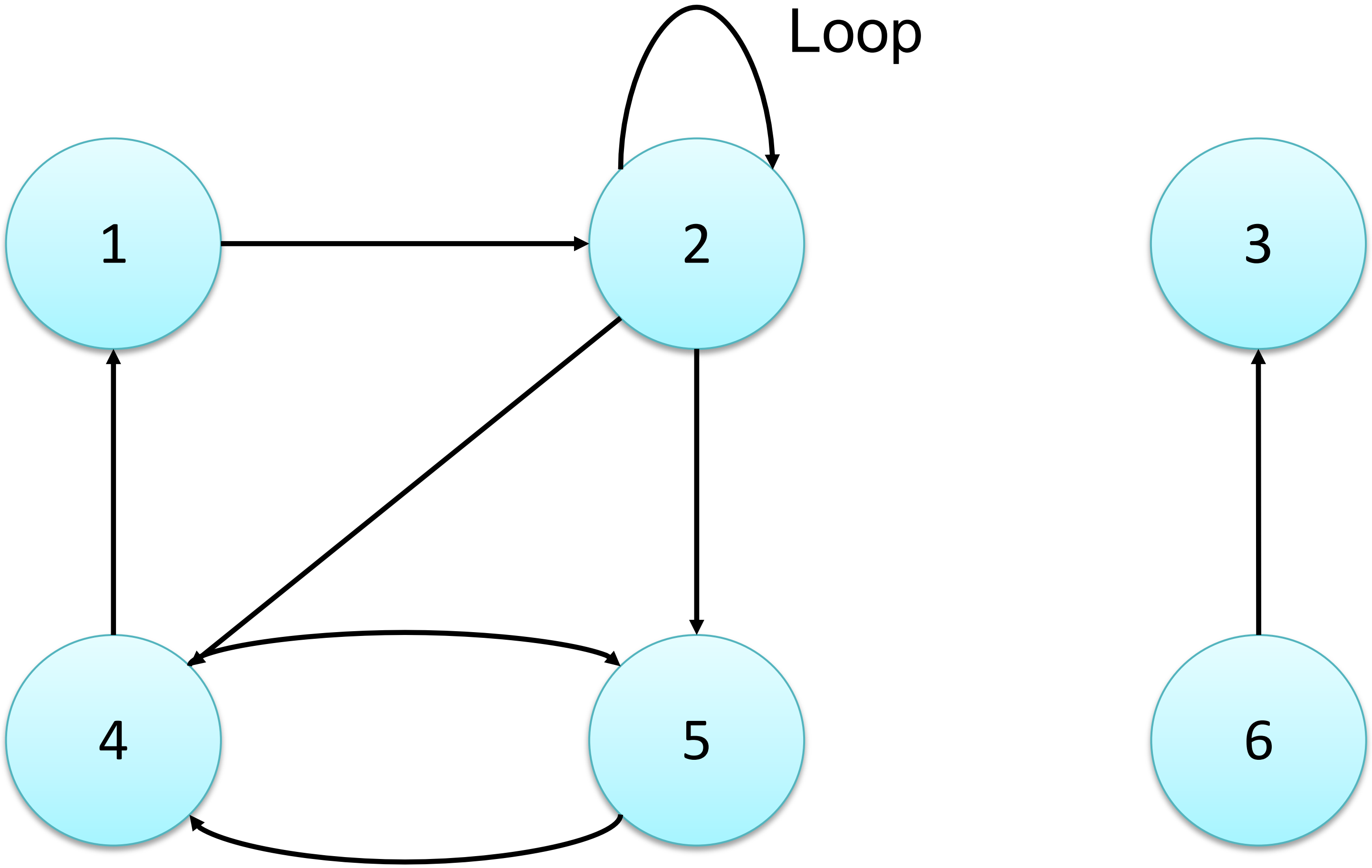
- A Directed Graph (*di-graph*) G is a pair (V,E), where
  - V is a (finite) set of vertices
  - E is a (finite) set of edges, that identify a binary relationship over V
    - $E \subseteq V \times V$



Example

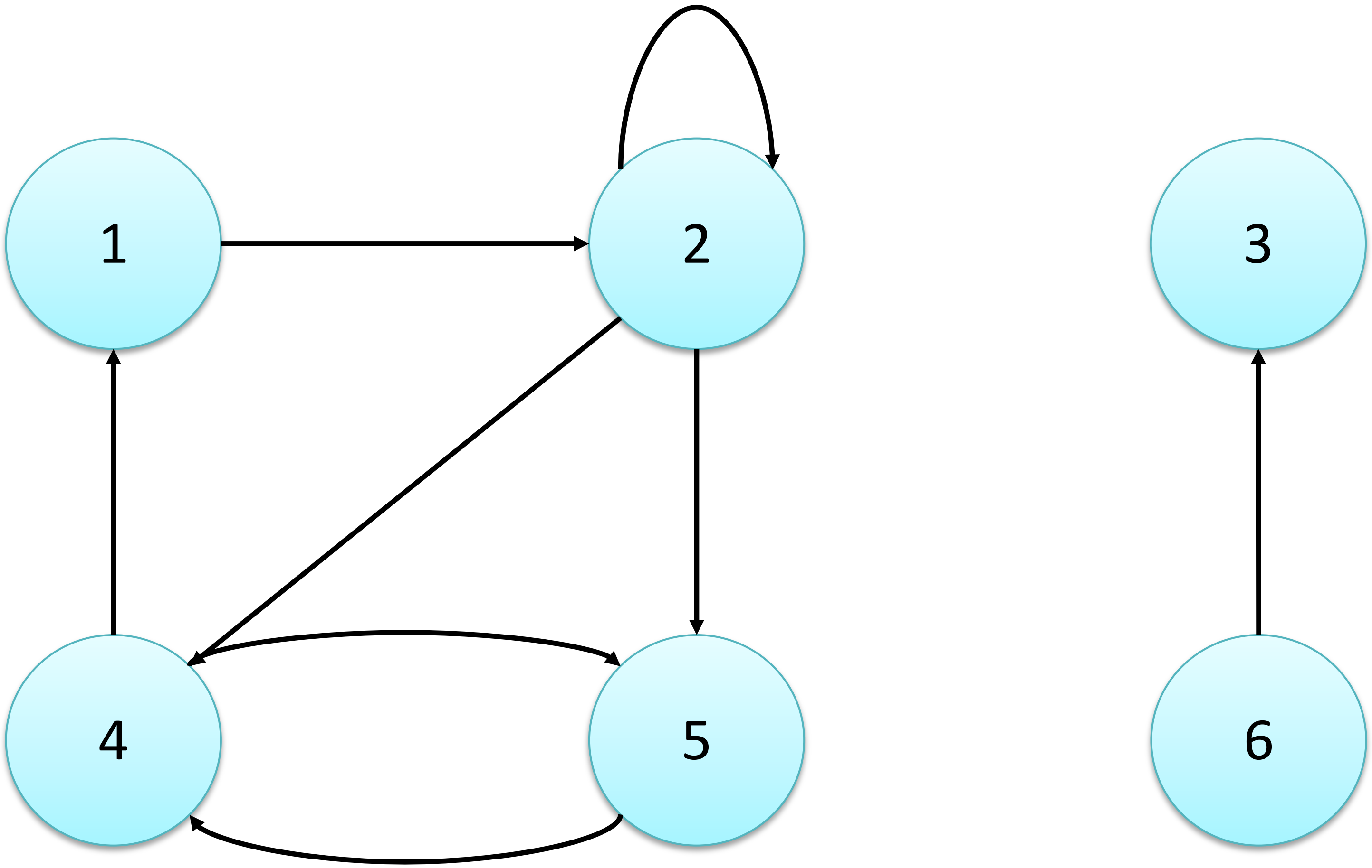


Example



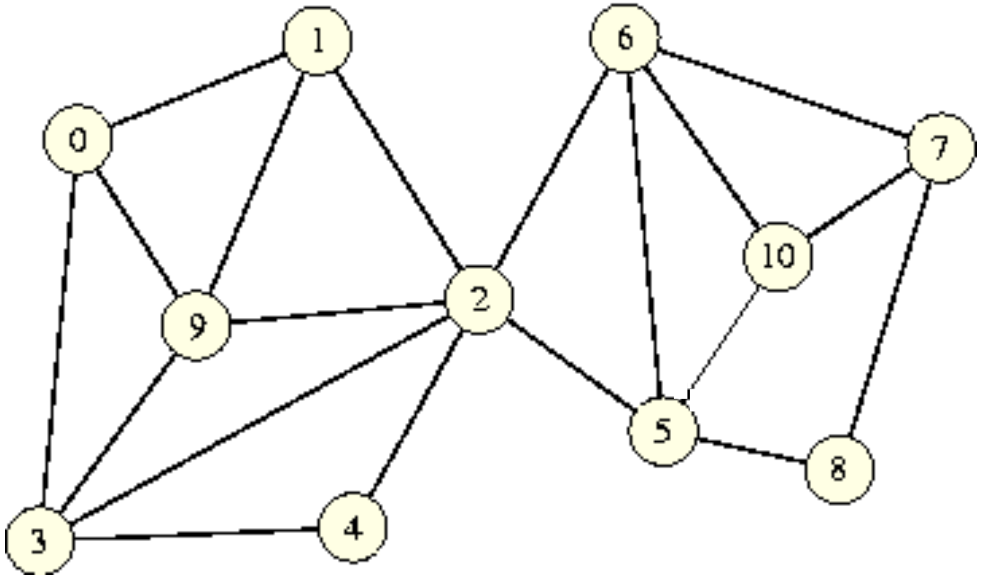
Example

V = {1,2,3,4,5,6}
E = { (1,2), (2,2), (2,5), (5,4), (4,5), (4,1), (2,4), (6,3) }



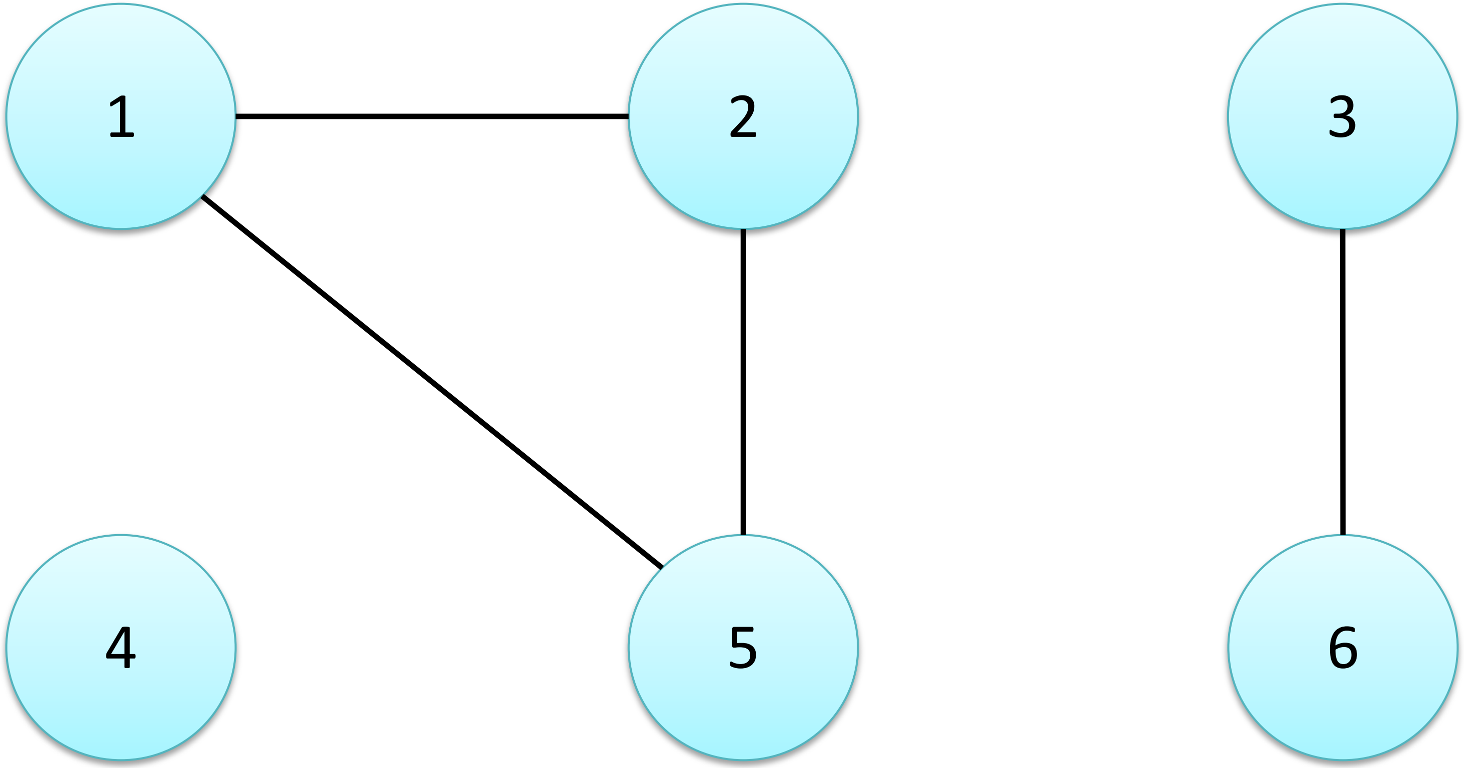
Undirected graph

- An Undirected Graph is still represented as a tuple G=(V,E), but the set E is made of non-ordered pairs of vertices



Example

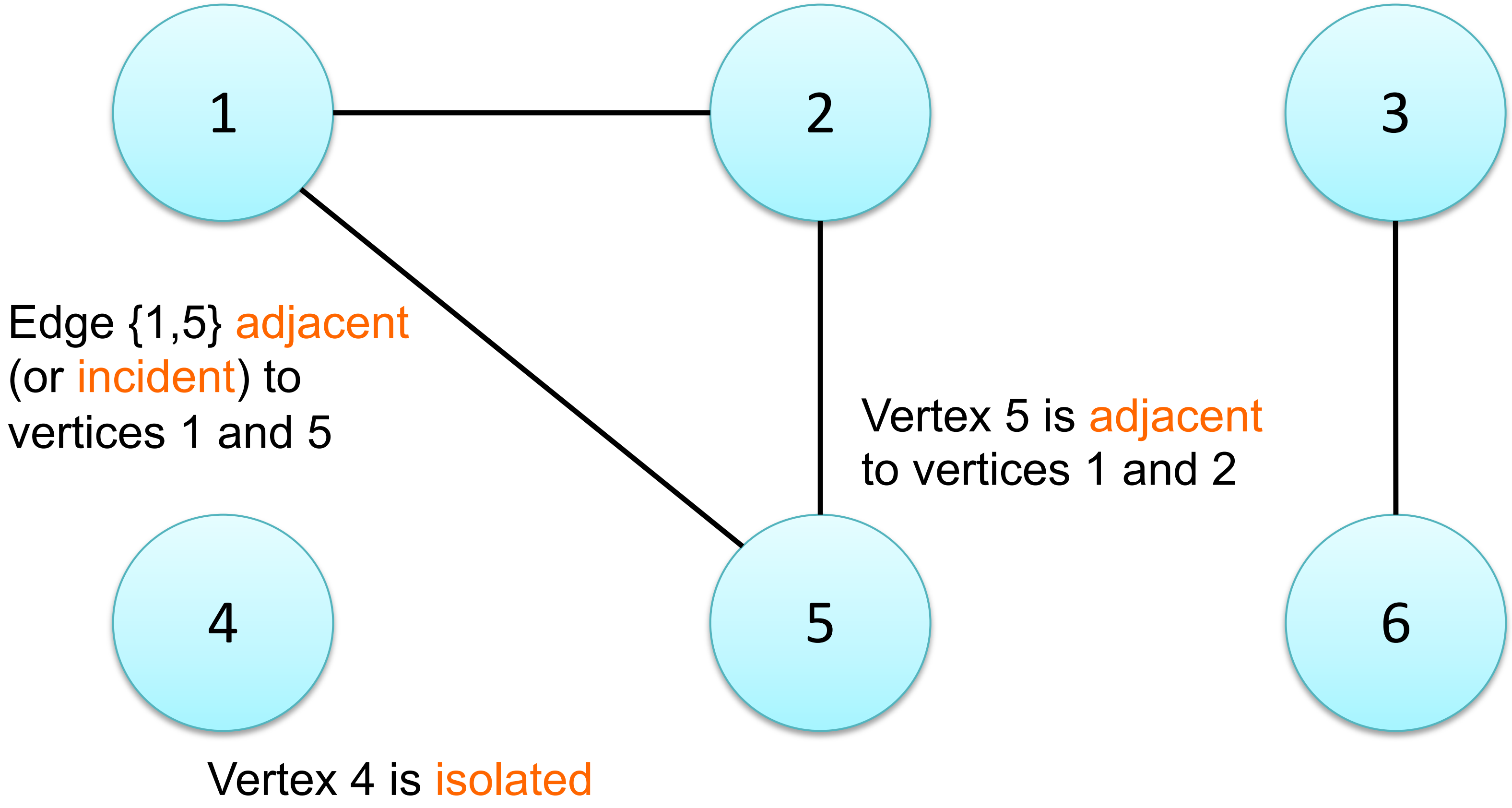
V = { 1, 2, 3, 4, 5, 6 }
E = { {1,2}, {2,5}, {5,1}, {6,3} }



Example

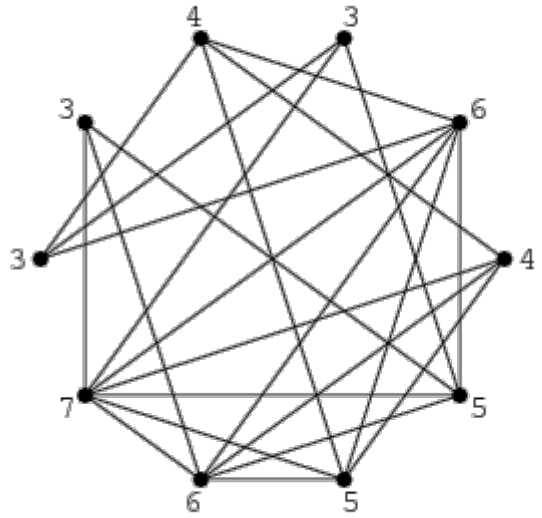
$V = \{ 1, 2, 3, 4, 5, 6 \}$

$E = \{ \{1,2\}, \{2,5\}, \{5,1\}, \{6,3\} \}$

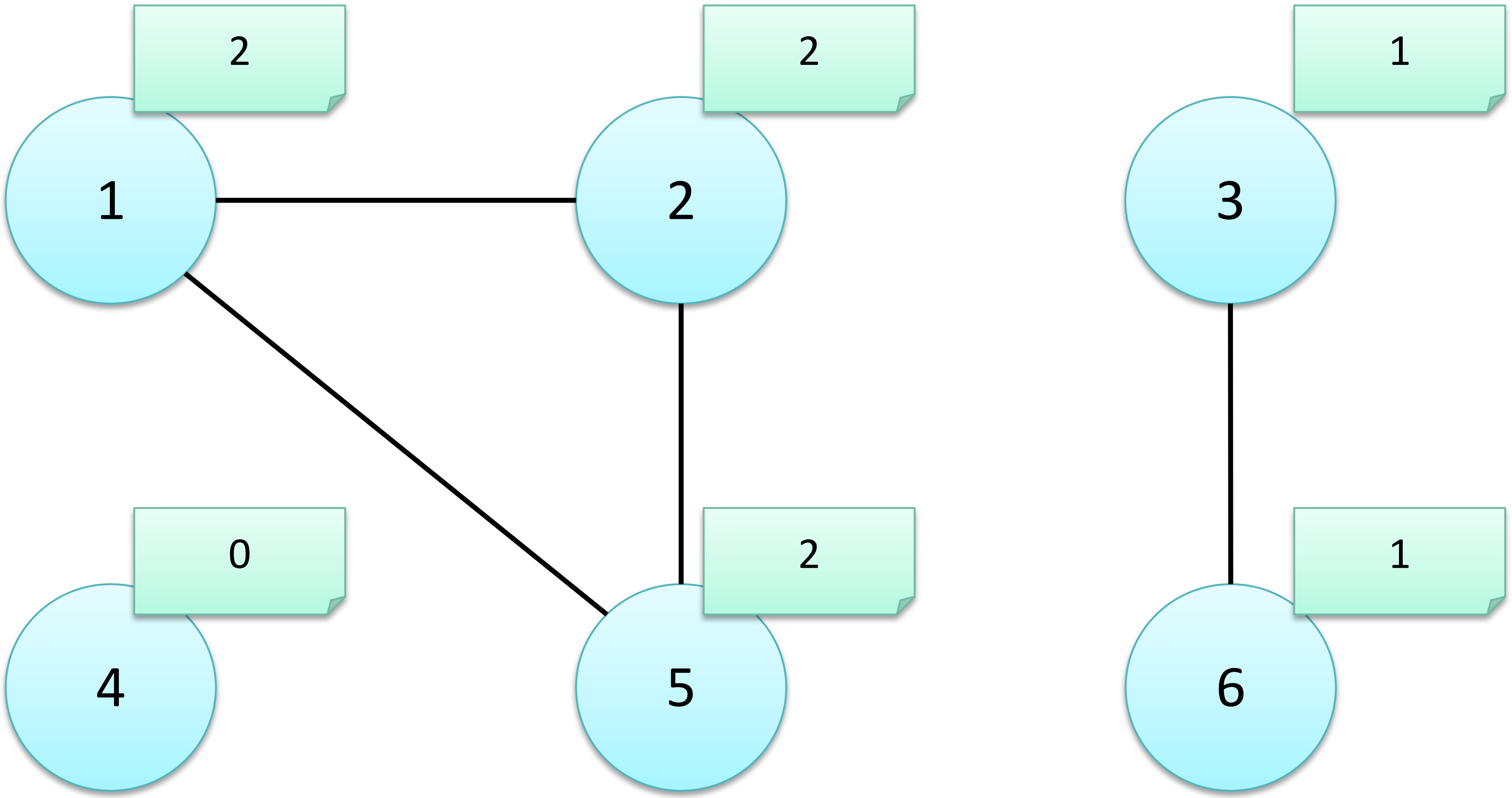


Degree

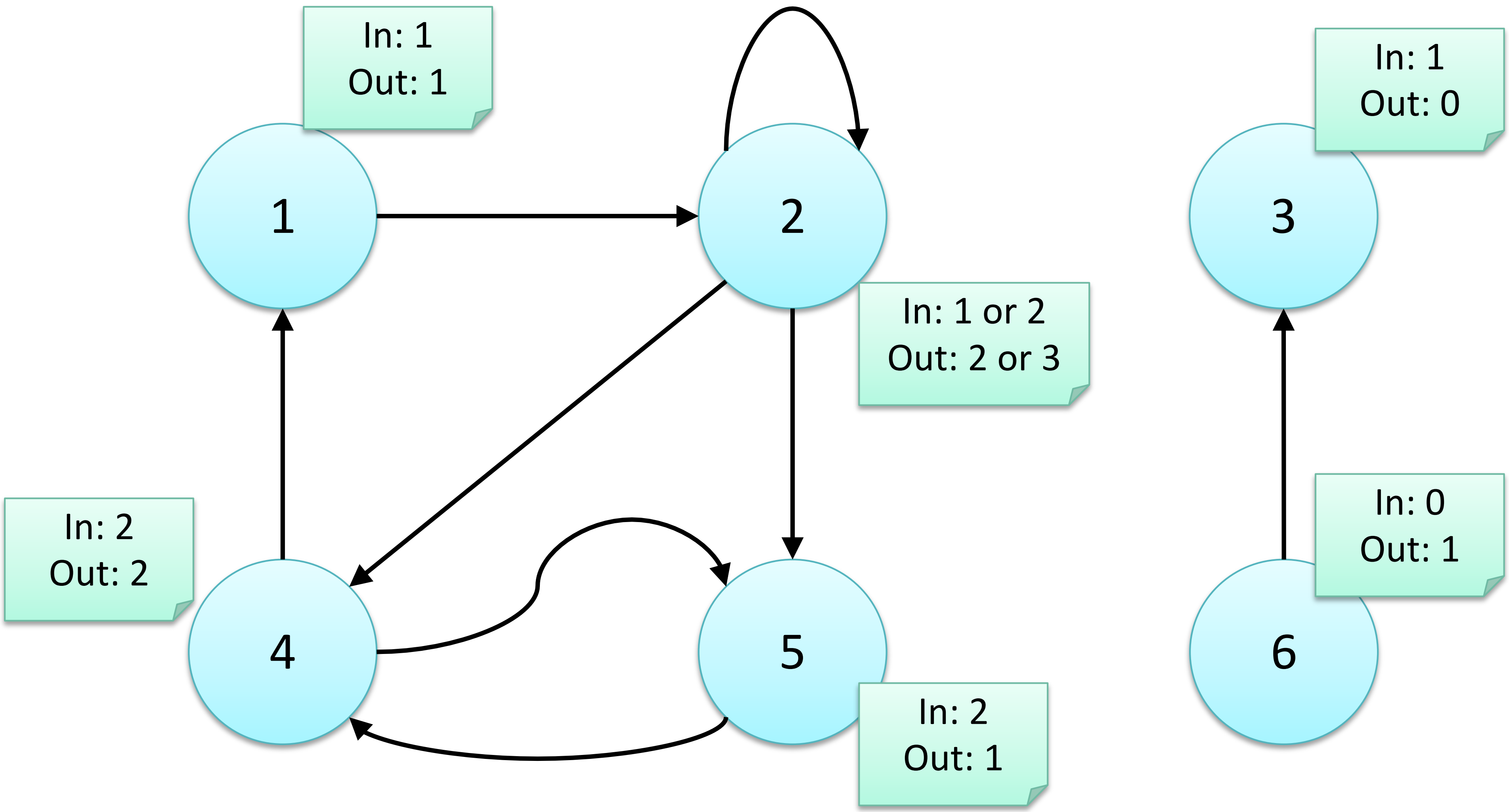
- In an undirected graph,
  - The degree of a vertex is the number of incident edges
- In a directed graph
  - The in-degree is the number of incoming edges
  - The out-degree is the number of departing edges
  - The degree is the sum of in-degree and out-degree
- A vertex with degree 0 is isolated



Degree



Degree

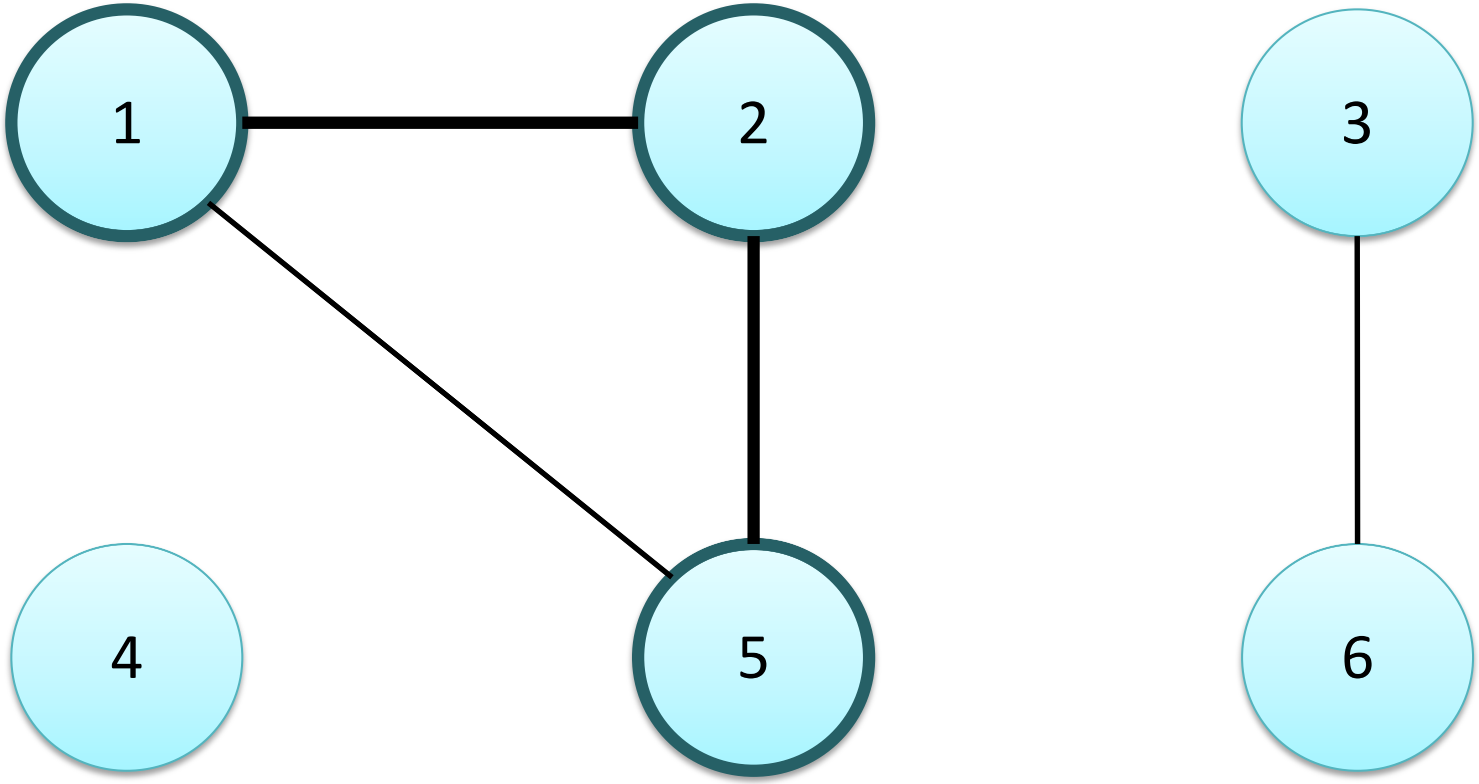

In: 1
Out: 1
1
2
In: 1 or 2
Out: 2 or 3
3
In: 1
Out: 0
In: 2
Out: 2
4
5
In: 2
Out: 1
6
In: 0
Out: 1

Paths

- A path on a graph $G=(V,E)$, also called a trail, is a sequence $\{v_1, v_2, \ldots, v_n\}$ such that:
  - $v_1, \ldots, v_n$ are vertices: $v_i \in V$
  - $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1},v_n)$ are graph edges: $(v_{i-1},v_i) \in E$
  - v_i are distinct (for "simple" paths).
- The length of a path is the number of edges $(n–1)$
- If there exist a path between v_A and v_B it is said that v_B is reachable from v_A

Paths

Path = (1, 2, 5)
Length = 2

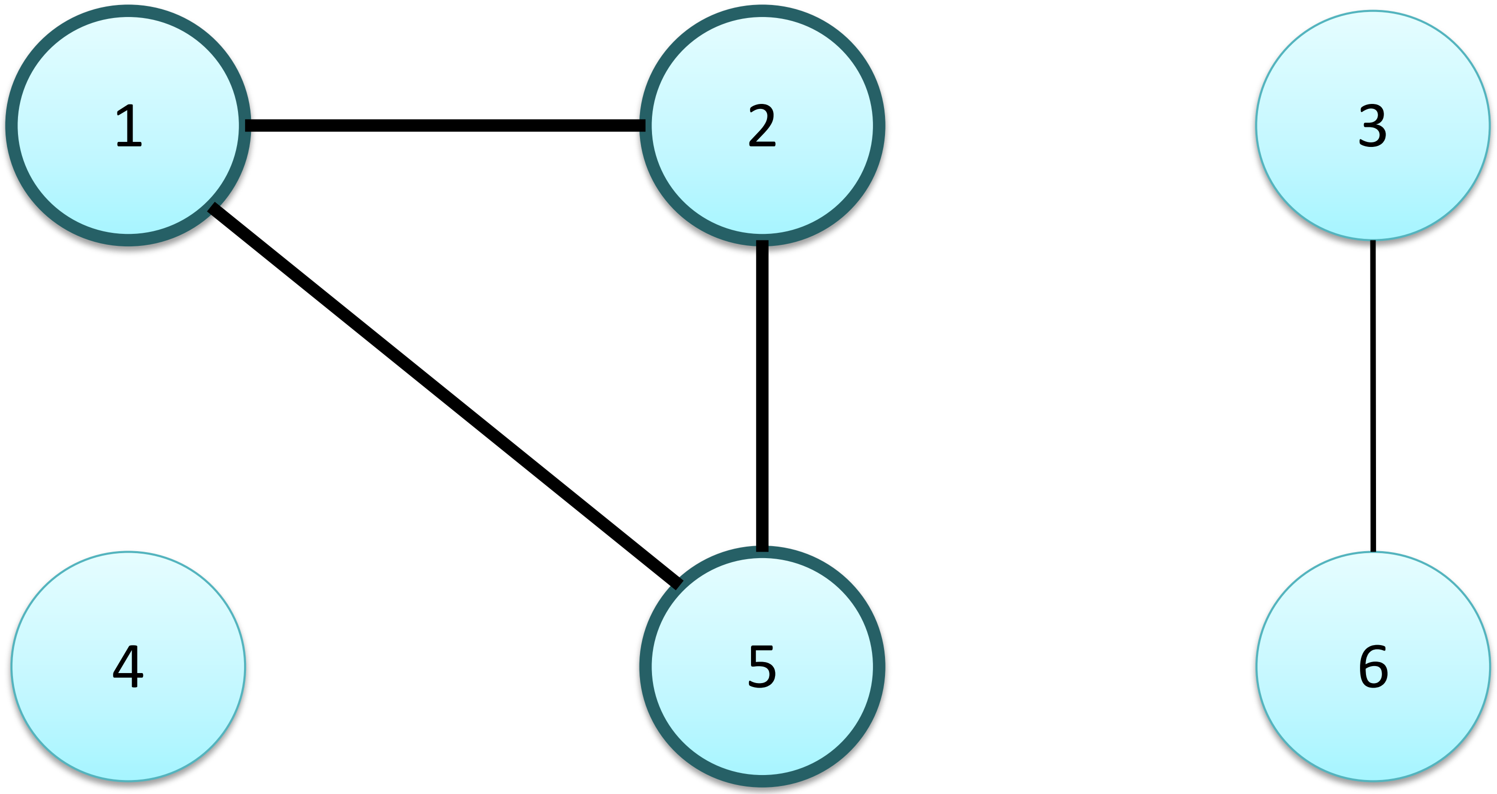


Cycles

- A cycle is a path where $v_1 = v_n$
- A graph with no cycles is said acyclic

Cycles

Path = (1, 2, 5, 1)
Length = 3

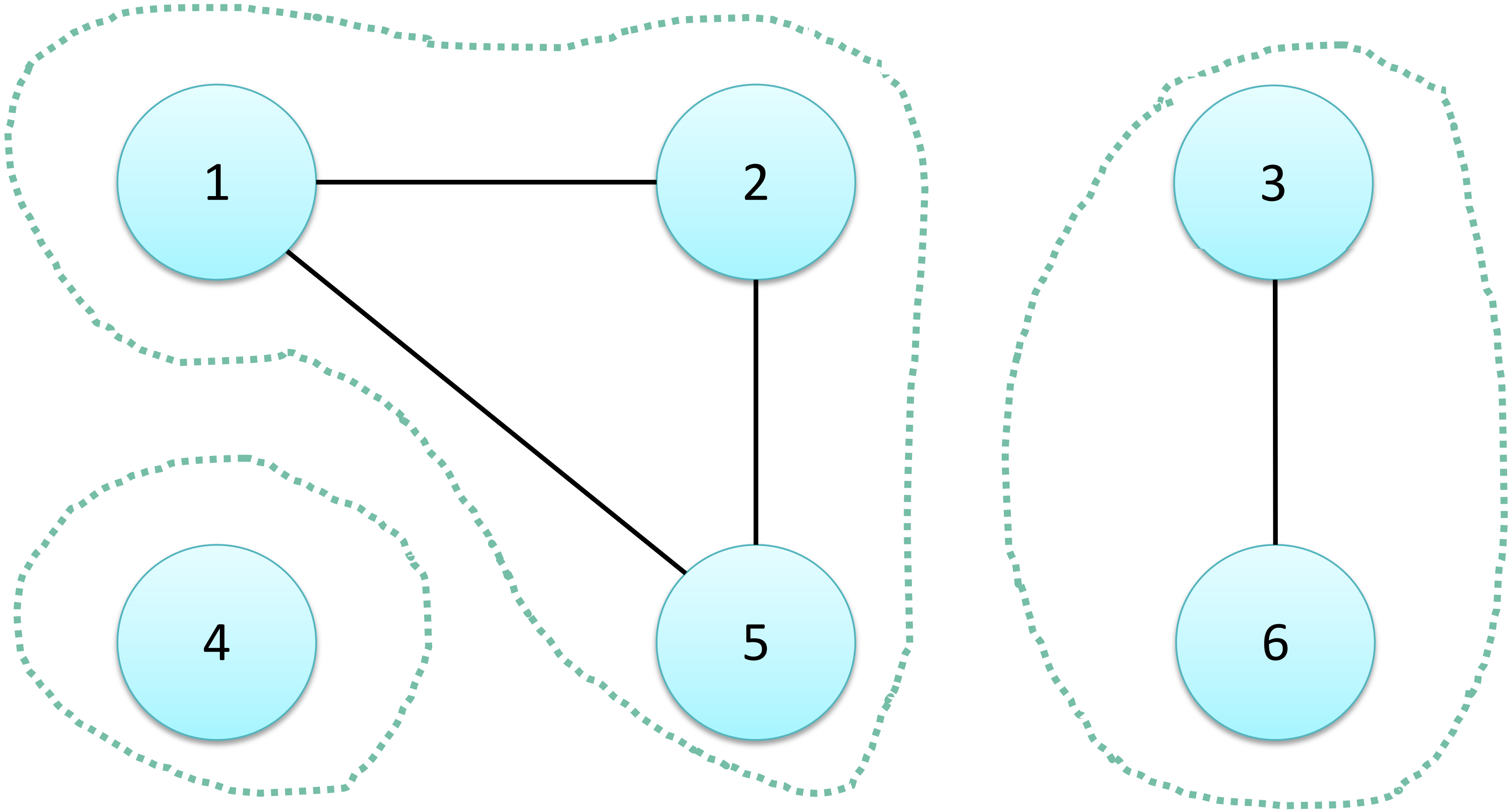


Reachability (undirected)

- An undirected graph is connected if, for every couple of vertices, there is path connecting them
- The connected sub-graphs of maximum size are called connected components
- A connected graph has exactly one connected component

Connected components

The graph is not connected;
connected components = 3
{ 4 } , { 1, 2, 5 }, { 3, 6 }

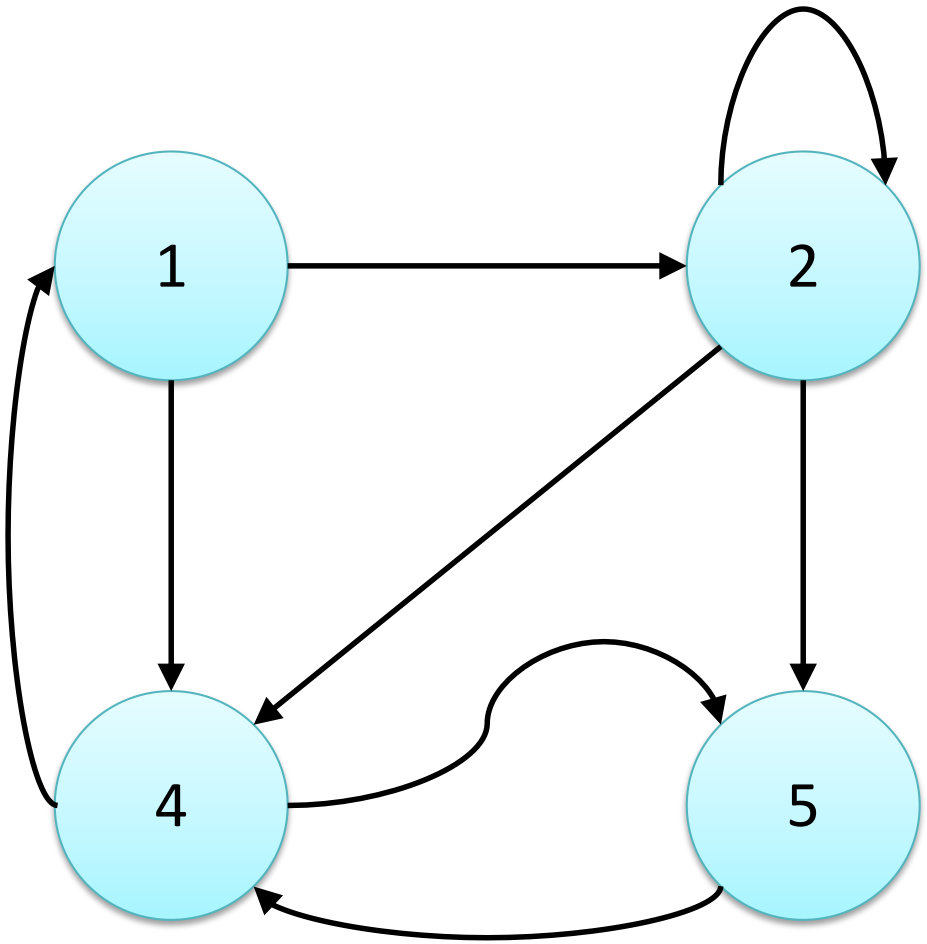


Reachability (directed)

- A directed graph is strongly connected if, for every ordered pair of vertices (v, v’), there exists at least one path connecting v to v’

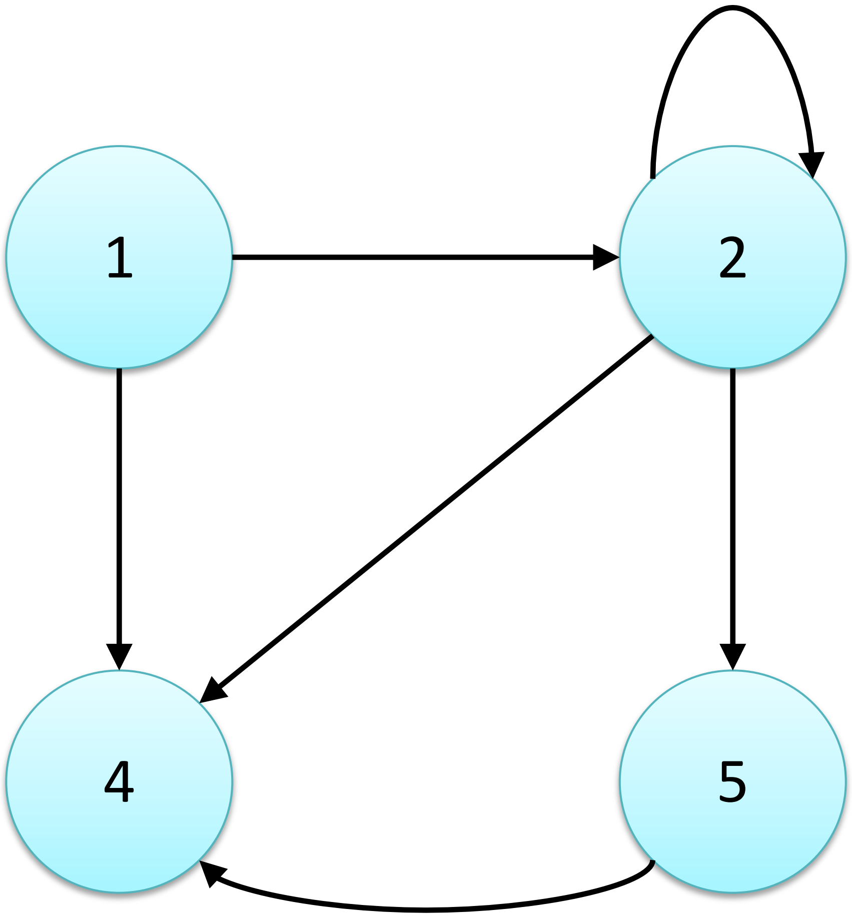
Example

The graph is
strongly connected



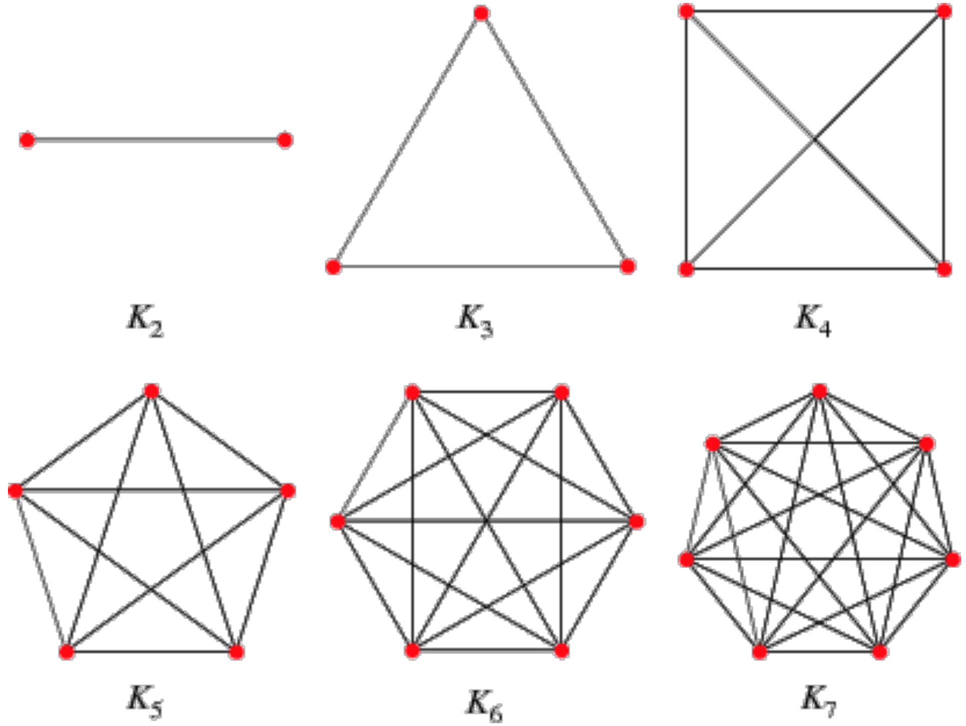
Example

The graph is not strongly connected



Complete graph

- A graph is complete if, for every pair of vertices, there is an edge connecting them (they are adjacent)
- Symbol: K_n



Complete graph: edges

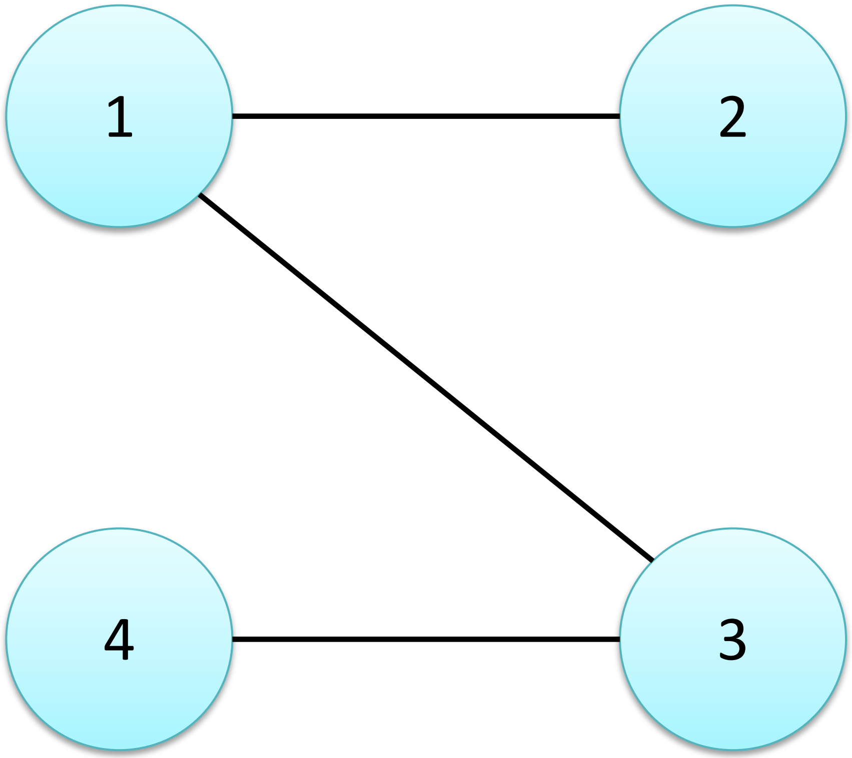
- Number of edges in a complete graph with *n* vertices
- Directed
  - No self loops: $n(n-1)$
  - With self loops: n^2
- Undirected
  - No self loops: $n(n-1)/2$
  - With self loops: $n(n+1)/2$

Density

- The density *d* of a graph $G=(V,E)$ is the ratio of the number of edges to the total number of possible edges
- It can be written that $d = |E(G)| \,/\, |E(K_{|V(G)|})|$

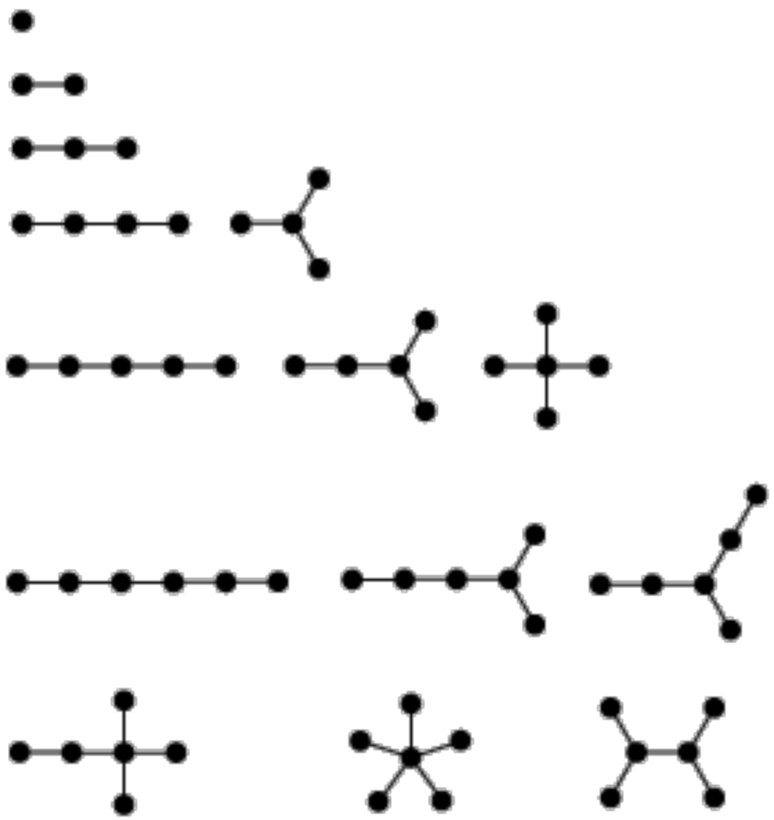
Example

Density = 0.5
Existing: 3 edges
Total: 6 possible edges



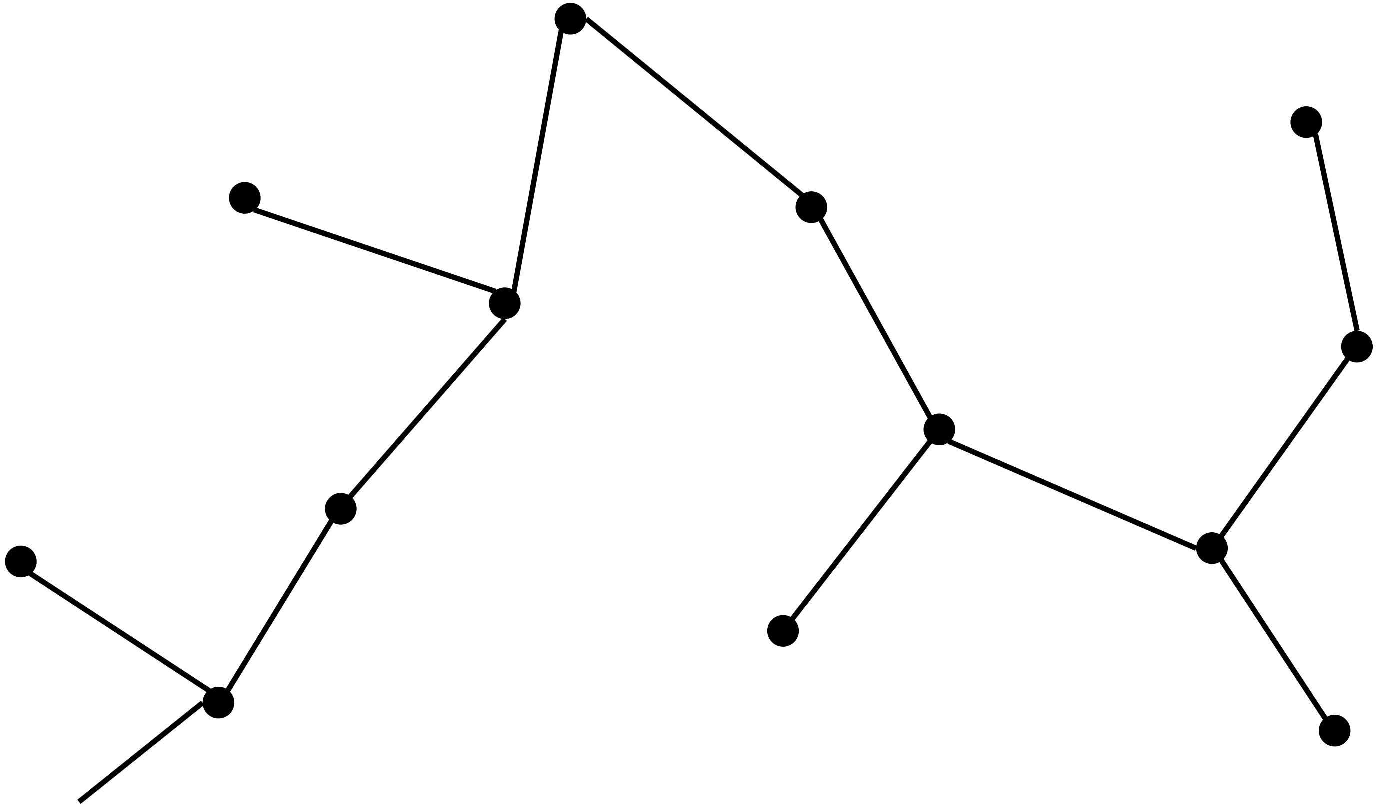
Trees and Forests

- An undirected acyclic graph is called forest
- An undirected acyclic connected graph is called tree



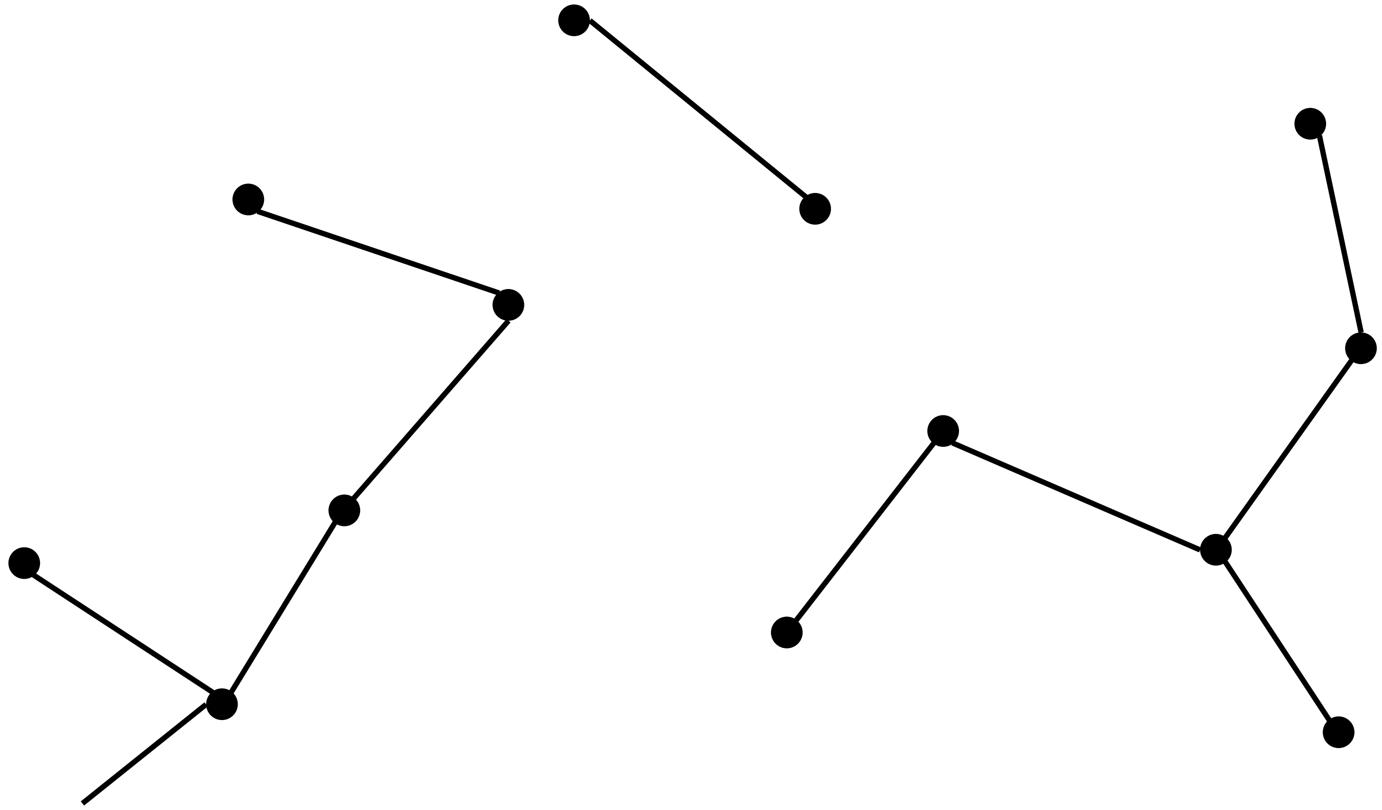
Example

Tree



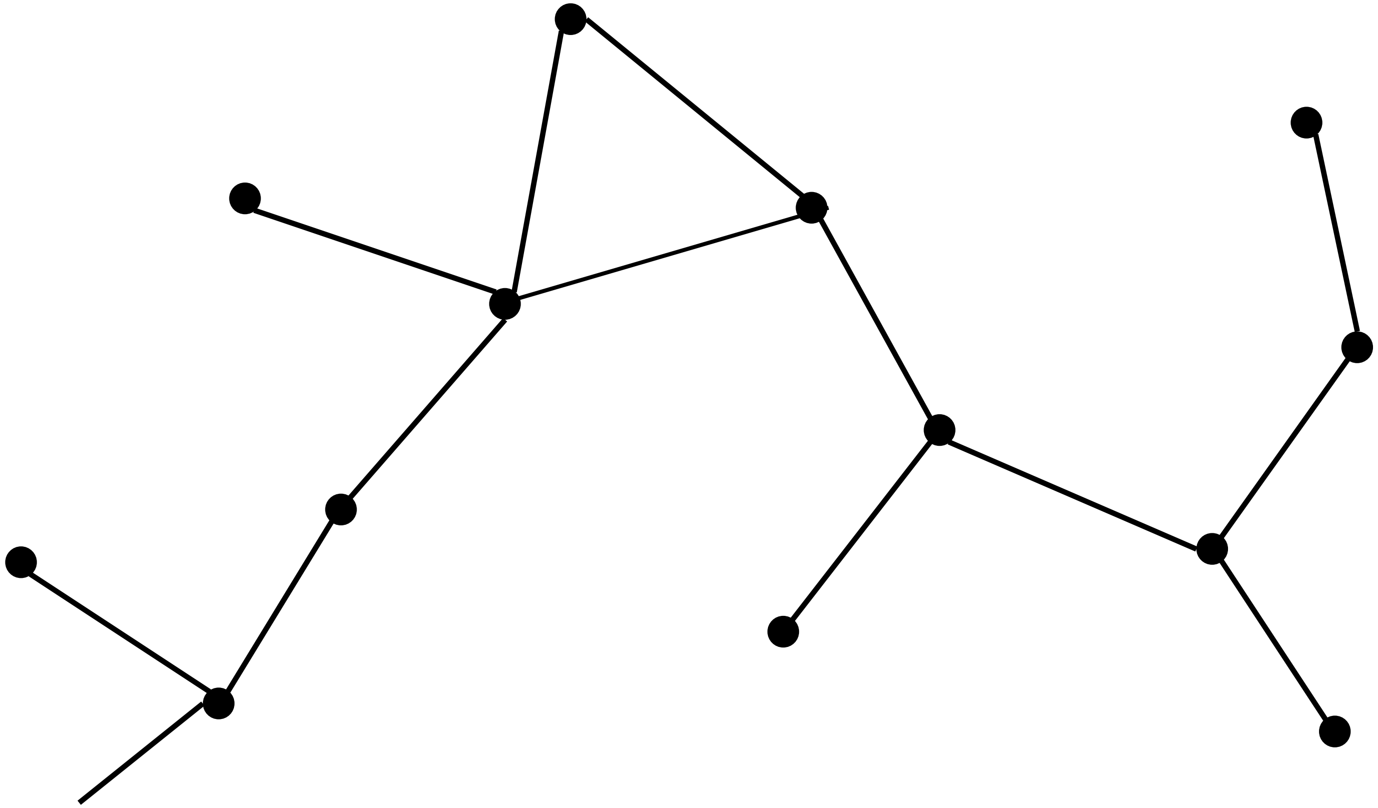
Example

Forest



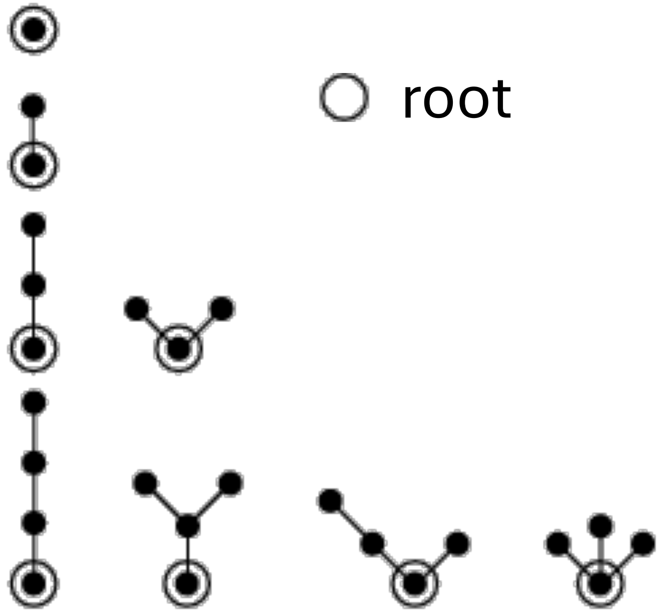
Example

This is neither a tree nor a forest (it contains a cycle)



Rooted trees

- In a tree, a special node may be singled out
- This node is called the "root" of the tree
- Any node of a tree can be the root

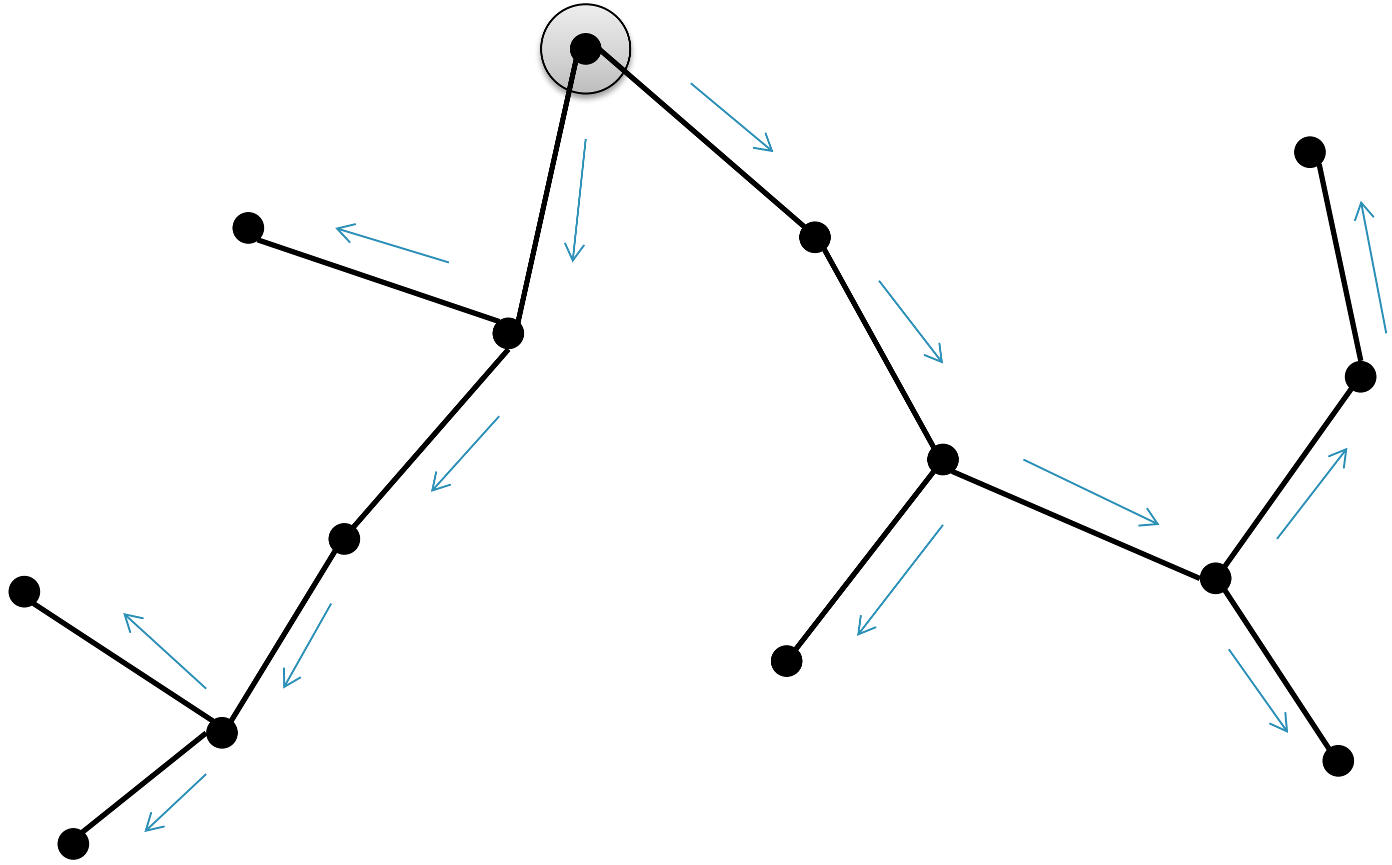


Tree (implicit) ordering

- The root node of a tree induces an ordering of the nodes
- The root is the “ancestor” of all other nodes/vertices
  - “children” are “away from the root”
  - “parents” are “towards the root”
- The root is the only node without parents
- All other nodes have exactly one parent
- The furthermost (children-of-children-of-children …) nodes are “leaves”

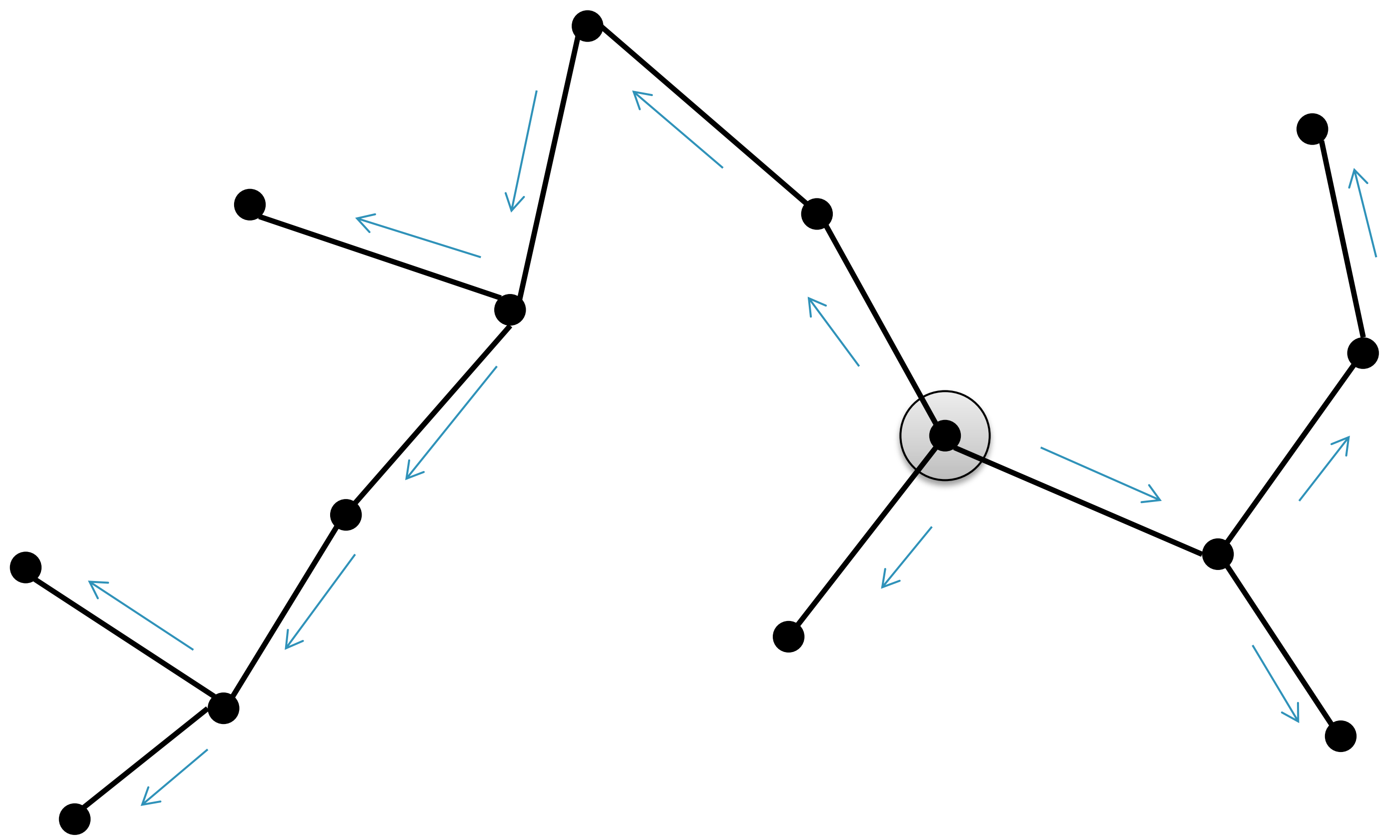
Example

Rooted tree



Example

Rooted tree



Weighted graphs

- A weighted graph is a graph in which each branch (edge) is given a numerical weight
- A weighted graph is therefore a special type of labeled graph in which the labels are numbers (which are usually taken to be positive)