Assignment

Homework HW4

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Musical Acoustics

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Design of a recorder flute and prediction of its sound

a) Bore dimensioning

The recorder's design evolved from a cylindrical shape before the baroque era to a distinctive conical shape afterward. This shift facilitated precise construction by tapering the bore, making manual tuning more manageable. The resulting geometry is the following:

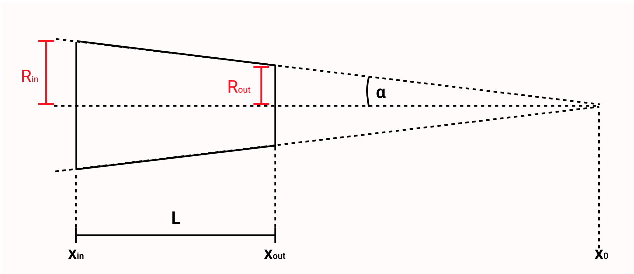

Figure 1: Truncated cone geometry (resonator)

In the case under analysis, the recorder's length is $L = 0.45\ m$ and its conical resonator is defined by the conical semiangle $\alpha = 0.75°$ as depicted in figure 1. The input radius of the cone R_{in} and the output one R_{out} must be correctly evaluated in order for the instrument to play the note E4 ($f_0 = 329.63\ Hz$) when all the finger holes are closed.
It is known from theory that the resonance condition for a recorder occurs when the impedance of the conical resonator Z_{cone} in series with the mouth impedance Z_{mouth} reaches a minimum. As concern the mouth impedance, it is given by the following expression:

$$Z_{mouth} = j\omega_0 M \tag{1}$$

where M is the air mass at the mouth window. This last parameter can be evaluated by exploiting the following expression for the end correction at the mouth:

$$\Delta L = \frac{MS_{in}}{\rho} \tag{2}$$

where $S_{in} = \pi R_{in}^2$ is the resonator cross section at the mouth and $\rho = 1.225\ kg/m^3$ is the air density. This last relationship is valid only for $k\Delta L << 2\pi$ and it can be demonstrated that in this low frequency range, the virtual elongation for a treble (alto) recorder is given by $\Delta L = 0.04\ m$. As a consequence, inverting the equation 2, the following result for M has been obtained:

$$M = \frac{\Delta L \rho}{S_{in}} \tag{3}$$

Regarding the input impedance of the incomplete cone Z_{cone}, it can be expressed, neglecting the radiation load impedance, as follows:

$$Z_{cone} = \frac{j\rho c}{S_{in}} \frac{sin(k_0 L')sin(k_0\theta_1)}{sin(k_0(L' + \theta_1))} \tag{4}$$

where $L' = L + 0.85 \cdot R_{out}$ is the acoustic length of the bore which takes into account also its virtual elongation; in particular, the baffled end correction has been applied since in a recorder the resonator foot is baffled. The angle θ_1 is related to x_{out} which represents the distance from the open end of the bore to the truncated apex of the cone whose position is x_0 and it can be computed as:

$$\theta_1 = \frac{tan^{-1}(k_0 x_{out})}{k_0} \tag{5}$$

where $k_0 = \frac{\omega_0}{c} = \frac{2\pi f_0}{c}$ is the wavenumber associated to the angular frequency $\omega_0 = 2\pi f_0$, and, after some geometrical considerations, it can be also proven that $x_{out} = \frac{r_{out}}{tan(\alpha)}$. Finally, the total impedance Z_{tot} is given by the following expression:

$$Z_{tot} = Z_{mouth} + Z_{cone} = j\omega \frac{\Delta L \rho}{\pi R_{in}^2} + \frac{j\rho c}{\pi R_{in}^2} \frac{sin(k_0 L') sin(tan^{-1}(k_0 \frac{r_{out}}{tan(\alpha)}))}{sin(k_0 L' + tan^{-1}(k_0 \frac{r_{out}}{tan(\alpha)}))} \tag{6}$$

It is now possible to express Z_{tot} as a function of R_{in} only, since the following relationship is geometrically valid: $R_{out} = R_{in} - L \cdot tan(\alpha)$.
Plotting the total impedance as function of R_{in}, it is possible to notice a minimum in the impedance Z_{tot} for $R_{in} \approx 4.3\ cm$:

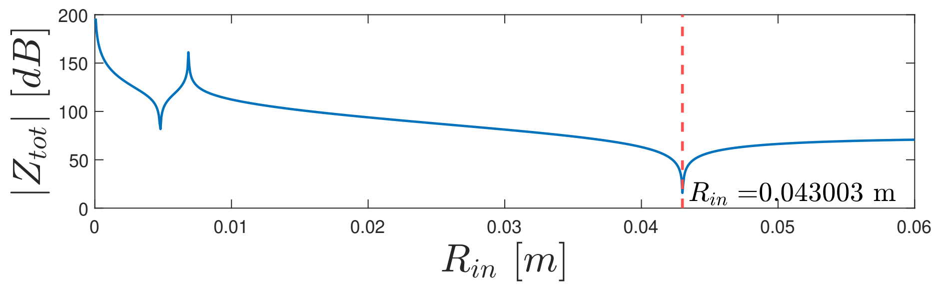

Figure 2: Total input impedance Z_{tot} as function of R_{in}

consequently, $R_{out} = R_{in} - L \cdot tan(\alpha) \approx 3.7\ cm$.
The two diameters will be therefore $D_{in} = 2R_{in} = 8.6\ cm$ and $D_{out} = 2R_{out} = 7.2\ cm$.

b) First hole positioning

In general, the note played by a woodwind instrument changes by opening one or more finger holes, thus changing the acoustic length of the air column. In this section, it is required to find the position of the last finger hole (the one closest to the resonator foot) in order to produce the note F4 ($f_1 = 349.23\ Hz$) when it is open. To simplify the problem, the finger hole diameter D_h is assumed to be equal to the bore diameter at the resonator foot D_{out} and the virtual elongation of the hole, denoted with l, is equal to the end correction Δ of the open end of the resonator. In addition, since the variation of the radius is negligible towards the resonator foot, it is possible to approximate this last part of the conical bore with a straight pipe.

The figure below illustrates the specific geometry to which all the calculations refer:

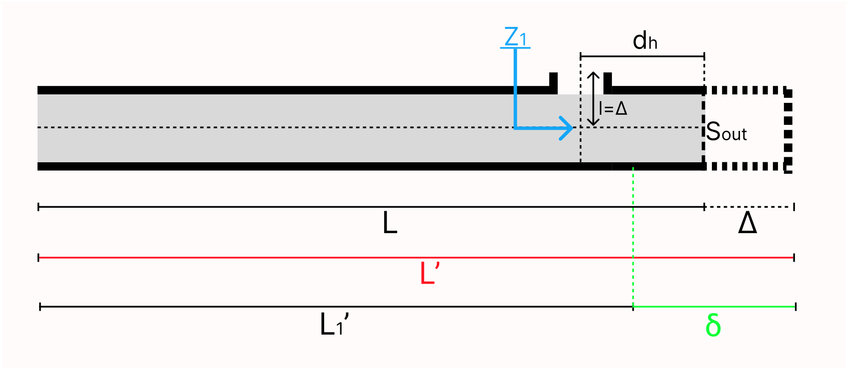

Figure 3: Geometry of the resonator with the last finger hole

If the hole is closed, then, neglecting the small perturbation caused by the air under the finger, the input impedance Z_1 (depicted in blue in figure 3) at the connection of the hole is given by:

$$Z_1^{(cl)} = j\frac{\rho c}{S_{out}} tan(k_1(d_h + \Delta)) \tag{7}$$

where d_h is the distance from the center of the hole to the end of the pipe; $k_1 = \frac{\omega_1}{c}$ is the wavenumber associated to the angular frequency $\omega_1 = 2\pi f_1$; Δ is the end correction of the foot resonator which is due to non-zero radiation impedance at the open end.
When the hole is open, the impedance Z_1 changes and it can be computed by evaluating the impedance of the hole Z_{h1} and the impedance of the final portion of pipe Z_{p1}. This is due to the fact that when the hole is open, the two acoustic elements (the hole and the final portion of the pipe) share the same pressure while the volume flow is split, as a consequence Z_1 is computed considering the parallel of the two impedances mentioned above: $Z_1^{(op)} = Z_{h1}//Z_{p1}$. At this point it is convenient to switch to the admittance notation ($Y = 1/Z$) in order to simplify the computations. The admittances are computed as:

$$Y_{h1} = -j\frac{S_{out}}{\rho c} cot(k_1 l) = -j\frac{S_{out}}{\rho c} cot(k_1 \Delta) \quad ; \quad Y_{p1} = -j\frac{S_{out}}{\rho c} cot(k_1(d_h + \Delta)) \tag{8}$$

The admittance $Y_1^{(op)} = 1/Z_1^{(op)}$ can be computed as the sum of the two admittances just evaluated:

$$Y_1^{(op)} = Y_{h1} + Y_{p1} = -j\frac{S_{out}}{\rho c}[cot(k_1\Delta) + cot(k_1(d_h + \Delta))] \tag{9}$$

Assuming that both d_h and Δ are small compared to the wavelength, the cotangent function can be approximated as the inverse of its argument $cot(x) \approx \frac{1}{x}$, obtaining a final expression for $Y_1^{(op)}$ from which the impedance $Z_1^{(op)}$ can be evaluated:

$$Y_1^{(op)} \approx -j\frac{S_{out}}{\rho c k_1}\left[\frac{d_h + 2\Delta}{\Delta(d_h + \Delta)}\right] \rightarrow Z_1^{(op)} = \frac{1}{Y_1^{(op)}} \approx j\frac{\rho c k_1}{S_{out}}\left[\frac{\Delta(d_h + \Delta)}{d_h + 2\Delta}\right] = j\frac{\rho c k_1}{S_{out}}\Delta' \tag{10}$$

By comparing the two acoustic lengths of equations 7 (closed hole) and 10 (open hole), it is possible to study how much the acoustic length of the resonator reduces after the opening of the finger hole. In particular, the acoustic reduction δ is given by:

$$\delta = d_h + \Delta - \Delta' = d_h + \frac{\Delta^2}{d_h + 2\Delta} \tag{11}$$

This result demonstrates how the opening of the finger hole leads to an acoustic short circuit: the acoustic length of the resonator is indeed diminished when the hole is open compared to the scenario where the finger hole is closed. As a consequence, the new acoustic length of the entire resonator when the hole is open is given by: $L'_1 = L' - \delta$.
It is now possible to compute the input impedance of the recorder characterized by the acoustic length L'_1 and mouth surface S_{in}, according to the following expression:

$$Z_{in}^{(op)} = \frac{j\rho c}{S_{in}} \frac{sin(k_1 L'_1) sin(k_1 \theta'_1)}{sin(k_1 (L'_1 + \theta'_1))} + j\omega_1 M \tag{12}$$

where the angle θ'_1 is related to $x' = \delta + x_{out} - \Delta$ which represents the distance of the end of the modified acoustic length from the apex of the cone and it can be computed as follows:

$$\theta_1 = \frac{tan^{-1}(k_1 x')}{k_1} \tag{13}$$

Since all the formulas mentioned above can be expressed as functions of d_h, it is possible to evaluate the input impedance $Z_{in}^{(op)}$ (impedance with the hole opened) in order to find the value of d_h for which the impedance presents a minimum, and therefore a resonance at the desired frequency. Plotting the total input impedance $Z_{in}^{(op)}$ as function of d_h, a minimum of the impedance it is observable for $d_h \approx 0.0167\ m$ as shown in the figure below:

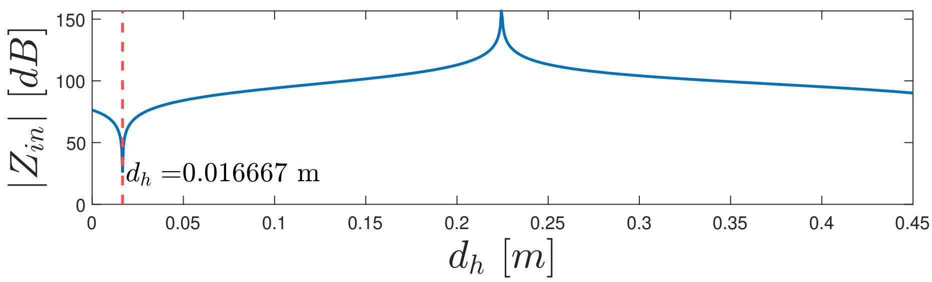

Figure 4: Total input impedance as function of d_h

Finally, the first hole's coordinate in relation to the length of the resonator is given by:

$$x_{h1} = L - d_h = 0.433\ m \tag{14}$$

c) Second hole positioning

In this section, it is required to find the position of the second last finger hole in order for the recorder to produce the note G4 ($f_2 = 392\ Hz$) when the two finger holes are open. The analytical procedure is similar to the one of the previous section but, in this case, the reference geometry is the following:

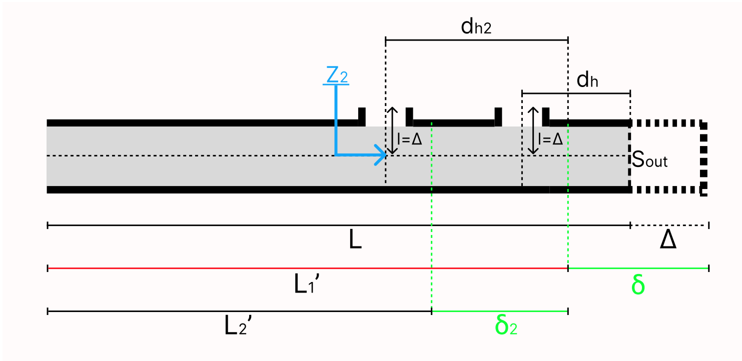

Figure 5: Geometry of the resonator with the last two finger holes

It is known from the previous calculations that when the last finger hole is open, the acoustic length of the resonator is reduced to L_1'; as a consequence, for the evaluation of the second hole position, it is possible to consider the length of the resonator to be equal to L_1'. The input impedance Z_2 when the second finger hole is closed is given by:

$$Z_2^{(cl)} = j\frac{\rho c}{S_{out}} tan(k_2 d_{h2}) \tag{15}$$

where d_{h2} is the distance from the center of the second hole to the end of the acoustic length L_1' and $k_2 = \frac{\omega_2}{c}$ is the wavenumber associated to the angular frequency $\omega_2 = 2\pi f_2$.
When the second hole gets opened, the input impedance Z_2 changes and it can be computed considering the parallel between the impedance of the second hole Z_{h2} and the impedance of the remaining final portion of the pipe Z_{p2} for the same reason explained in the section 2: $Z_2^{(op)} = Z_{h2}//Z_{p2}$. By switching to the admittance notation ($Y_2^{(op)} = 1/Z_2^{(op)}$), the parallel impedance configuration can be computed summing the associated admittances:

$$Y_2^{(op)} = Y_{h2} + Y_{p2} = \frac{1}{Z_{h2}} + \frac{1}{Z_{p2}} = -j\frac{S_{out}}{\rho c} cot(k_2 \Delta) - j\frac{S_{out}}{\rho c} cot(k_2 d_{h2}) \tag{16}$$

and assuming that both d_{h2} and Δ are smaller compared to the wavelength it is possible to approximate $Y_2^{(op)}$ as follows:

$$Y_2^{(op)} \approx -j\frac{S_{out}}{\rho c k_2}\left[\frac{d_{h2} + \Delta}{\Delta d_{h2}}\right] \tag{17}$$

At this point it is possible to compute the impedance by inverting the last expression:

$$Z_2^{(op)} = \frac{1}{Y_2^{(op)}} \approx j\frac{\rho c k_2}{S_{out}}\left[\frac{\Delta d_{h2}}{d_{h2} + \Delta}\right] = j\frac{\rho c k_2}{S_{out}}\Delta'' \tag{18}$$

By comparing the two acoustic lengths of equations 15 and 18, the reduction in acoustic length when the second hole gets opened is given by: $\delta_2 = d_{h2} - \Delta'' = \frac{d_{h2}^2}{\Delta + d_{h2}}$. Once the formula of the acoustic length reduction has been obtained, it is possible to retrieve the new acoustic length of the equivalent cone which compose the resonator as $L_2' = L_1' - \delta_2$. Finally, the input impedance of the recorder when both the finger holes are open can be computed according to the following expression:

$$Z_{in}^{(op)} = \frac{j\rho c}{S_{in}} \frac{sin(k_2 L_2')sin(k_2\theta_2')}{sin(k_2(L_2' + \theta_2'))} + j\omega_2 M \tag{19}$$

where $\theta_2' = tan^{-1}(k_2 x'')/k_2$ and $x'' = \delta_2 + x'$ is the distance from the foot of the new acoustic length L_2' to the apex of the cone.
At this point, it is possible to express $Z_{in}^{(op)}$ as a function of d_{h2} only, aiming to determine the d_{h2} value that yields the minimum input impedance:

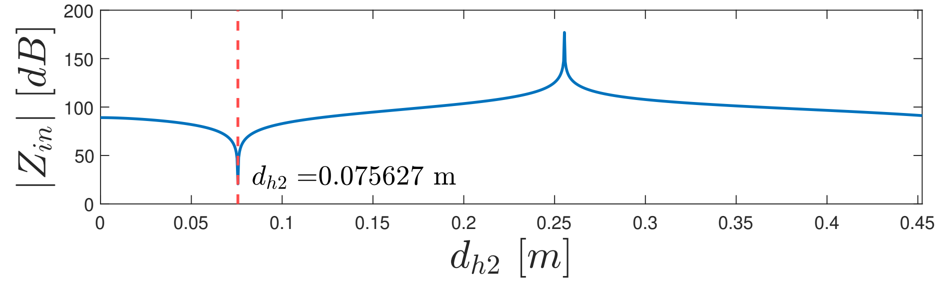

Figure 6: Total input impedance $Z_{in}^{(op)}$ as function of d_{h2}

As it can be seen from the figure above, the numerical solution given by the data extraction is: $d_{h2} = 0.0756\ m$. As a consequence, the coordinate of the second hole with respect to the resonator length can be evaluated as follows: $x_{h2} = L_1' - d_{h2} = 0.3768\ m$.

d) Prediction of the input impedance

In this last section, a method for the prediction of the input impedance of a recorder is presented. In particular, the theoretical approach has been adopted using the model of the tube as a transmission line. Approximate solutions can be found via spatial discretisation of the bore into a piecewise series of elemental sections as it can be seen in the figure below:

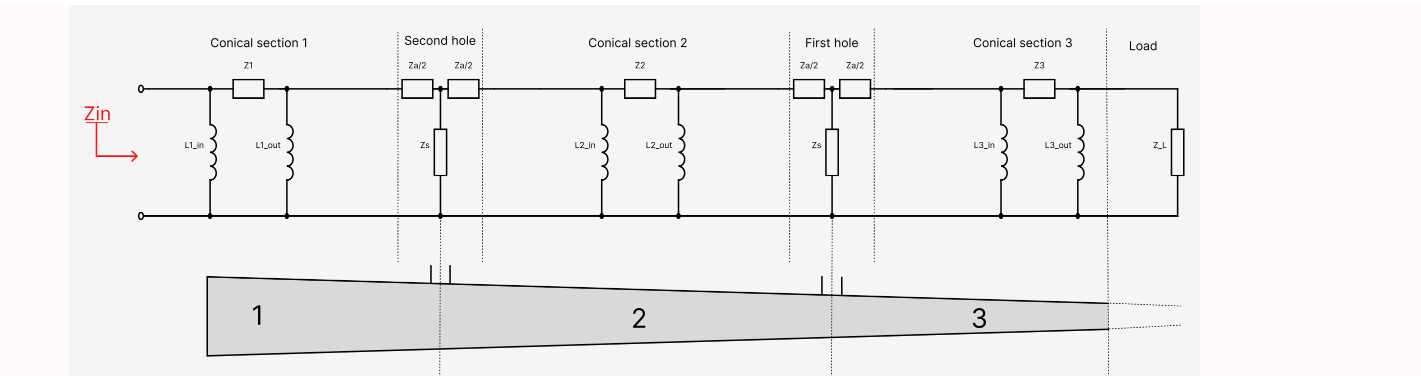

Figure 7: Equivalent circuit of the recorder under analysis. (The resonator geometry is only indicative and does not reflect the exact physical geometry of the problem)

The resonator has been modeled in such a way that the acoustic variables of pressure p and acoustic volume flow U are defined at various specific points along the air column axis.

According to Benade's [1] each conical section can be modeled through *an equivalent circuit consisting of a pair of inertances, a transformer, and a non-tapered duct that has the same length and small-end radius as the cone to be represented.* The transformers have been omitted from the network description, and this does not affect the response of the bore; the junction inertances at the entry $L_{n_{in}}$ and at the end $L_{n_{out}}$ of the n-th conical section are defined as:

$$L_{n_{in}} = \frac{\rho l_{n_{in}}}{S_{n_{in}}}; \quad L_{n_{out}} = \frac{\rho l_{n_{out}}}{S_{n_{out}}}; \tag{20}$$

where $\rho = 1.225\ kg/m^3$ is the air density, $l_{n_{in}}$ and $l_{n_{out}}$ are the distances of the input and output junctions of the n-th conical section from the cone apex, $S_{n_{in}}$ and $S_{n_{out}}$ are the cross sections at the left and the right end of the n-th conical section; the impedances Z_1, Z_2 and Z_3 account for the *non-tapered duct* of the Benade's model for a conical section.
The finger-holes have been modeled through the equivalent network given by the connection of the two impedances $Z_{a/2}$ with the impedance Z_s whose expressions are the following:

$$Z_{a/2} = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \cdot (-jkt_a) \ \rightarrow (closed\ or\ open\ hole) \tag{21}$$

$$Z_s = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \cdot \begin{cases} -j\ cot(kt) & \rightarrow (closed\ hole) \\ jkt_e & \rightarrow (open\ hole) \end{cases} \tag{22}$$

where the parameters t, t_a and t_e are related to the toneholes heights and their inner and outer acoustic end corrections. The analytical expressions of these parameters has been chosen experimentally in order for the results to be coherent with the recorder design process.
The radiation load impedance has been taken into account through the impedance Z_L. Finally, the expression of the input impedance Z_{in} (depicted in red in figure 7) has been evaluated analytically for the three different cases: both holes closed, only the last hole open, both holes open. The results are shown in the figure below:

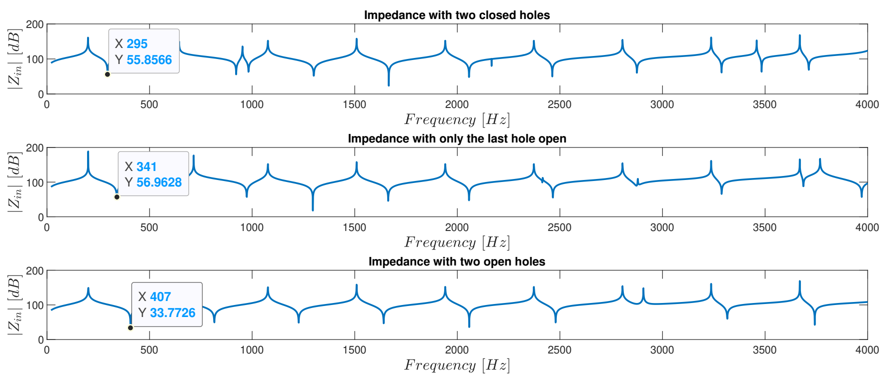

Figure 8: Prediction of the input impedance Z_{in} for the three different cases

Up to some approximation, results are coherent with the resonances for which the instrument has been designed in sections 1, 2, 3: the lowest resonance is obtained when both the finger-holes are closed and the highest one when both holes are open.

References

[1] A. H. Benade, "Equivalent circuits for conical waveguides", J. Acoust. Soc. Am. 83, pp.1764–1769 (1988).