Assignment

Homework HW3

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Musical Acoustics

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Synthesis of the guitar sound

a) Derivation of the bridge impedance

The characteristics of the sound generated by an acoustic guitar depend on the vibratory interaction of the whole instrument to a given string excitation. Each string is coupled to the instrument body at the bridge, consequently the bridge of a guitar is the location where the energy of the strings is transferred to the top plate. This interaction can be described in terms of the bridge's mechanical impedance which relates the force exerted by the vibrating string on the bridge to its resulting velocity due to this force.
In the case under analysis, the bridge impedance of the Martin D28 has been computed considering the simplified mechanical model of a guitar known as *two mass model*, figure 1(a). The vibrating strings apply a force F(t) to the top plate, whose mass and stiffness are represented by m_p and K_p. A second piston of mass m_h represents the mass of air constituting the virtual neck in proximity of the soundhole, and the volume V of enclosed air acts as the second spring. In order to simplify the computation of the bridge impedance, it is possible to describe this mechano-acoustic system via an analogous electric one. In particular, the equivalent electrical circuit in figure 1(b) is an acoustical impedance representation and it has been obtained adopting the *impedance analogy* according to which the equivalent voltage is the force applied to the top plate divided by the top plate area, and the equivalent currents are volume velocities. In this case, where both mechanical and acoustical components are present, it is useful to recall the fundamental relationship between mechanical impedance and acoustic impedance: $Z_m = Z_a S^2$. Given this expression, it is possible to obtain the equivalent electrical circuit employing the following relationships between the two domains:

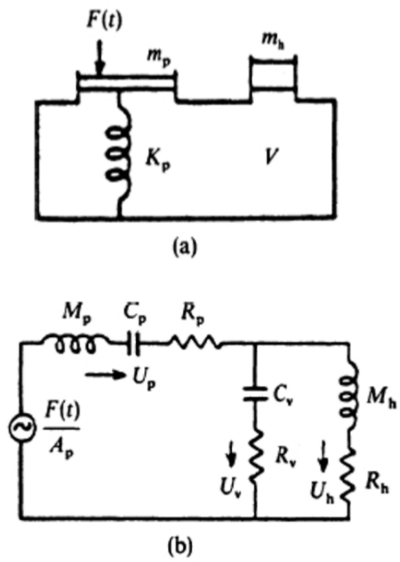

Figure 1: Two mass model of a guitar and its electric equivalent

- $M_p = \frac{m_p}{A_p^2} = 236,42$ $[Kg/m^4]$ inertance of the top plate.
- $M_h = \frac{m_h}{A_h^2} = 13,04$ $[Kg/m^4]$ inertance of air in the soundhole.
- $C_p = \frac{A_p^2}{K_p} = 1,4710^{-9}$ $[N/m^5]$ compliance of the top plate.
- $C_v = \frac{V}{\rho c^2} = 1,2110^{-7}$ $[N/m^5]$ compliance of the enclosed air volume.
- $R_p = 32$ $[Nm/kg/s]$ represents the radiational and mechanical losses of the top plate.
- $R_h = 30$ $[N/m]$ accounts for the radiation from the soundhole.
- $R_v \approx 0$ is due to air losses in the cavity.

It's important to underline that studying the acoustic behaviour employing this electrical equivalent is valid only under the *lumped element condition*. This assumption requires the wavelength associated to the frequencies of study to be much greater than the characteristic dimension of the instrument, i.e. $\lambda >> L$. However, this condition is satisfied by one of the hypothesis of the *two mass model* according to which the top plate is considered as a rigid element, i.e. all the points of the plate move with the same phase. This being said, the results of this study will be accurate only for frequency values close to the frequency of the fundamental mode of the top plate.

The evaluation of the bridge impedance has been carried out in Matlab using Simulink plug-in. Once the circuit has been reconstructed, the input voltage signal has been set to be impulse-like and it has been obtained by subtracting two steps distant $\frac{1}{f_s}$ seconds one from the other. f_s is the chosen sampling frequency and it has been set to be equal to the standard audio file sampling frequency of $44100\ Hz$. Successively, a current sensor has been connected in series to the rest of the circuit with the purpose to measure the flux-type quantity that characterizes it. Finally, the bridge impedance is calculated as the ratio between the *Fourier transform* of the input voltage signal and that one of the current sensor signal.

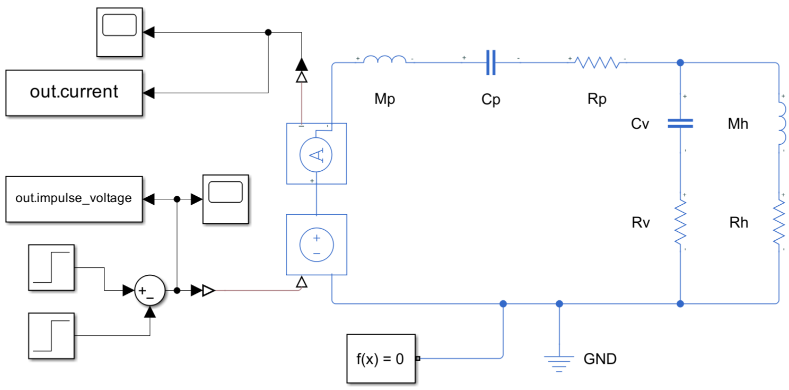

Figure 2: Electric analog implemented in Simulink.

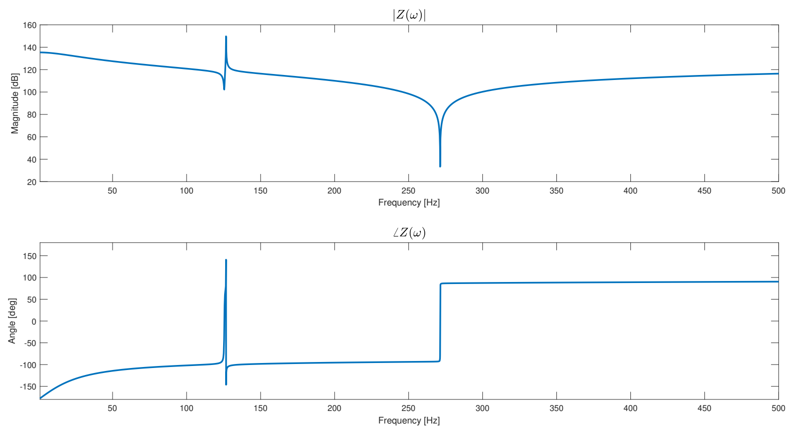

Figure 3: Magnitude and phase of the bridge impedance.

From the plot depicted in the figure above, it is evident that the impedance is characterized by two resonances (impedance minima) and an antiresonance (impedance maximum) in between. In particular, the first resonance occurs at $f_1 \approx 125.5\ Hz$ and the second one at $f_2 \approx 271.3\ Hz$: at the lower resonance, air flows out of the soundhole in phase with the inward moving top plate; at the upper resonance, air moves into the soundhole when the plate moves inward. Finally, the antiresonance occurs at $f \approx 126\ Hz$ and it is given by the Helmholtz resonance of the soundbox.

b) Transfer function from the plucking point to the bridge

The behavior of a vibrating string with a plucked excitation can be described in terms of two traveling waves propagating along the string in opposite directions and reflecting back at the string terminations. To carry the traveling-wave solution into the "digital domain" it is necessary to sample the traveling-wave amplitudes at intervals of T seconds, corresponding to a sampling rate $f_s = 1/T$ samples per second. In addition to time sampling, it is necessary also a spatial sampling whose fundamental sampling variable is the spatial sampling interval X which corresponds to the distance the wave propagates in one temporal sampling interval T. Since the whole wave moves left or right by one spatial sample each time sample, simulation only requires digital delay lines.

As a consequence, the propagation of the two traveling waves are modeled by two delay lines, one for the forward propagation while the other one for the backward one. The delay lines are connected via reflection filters $R_f(z)$ and $R_b(z)$ which represent the nut and the bridge respectively. These filters produce phase inversion and slight damping and, for this reason, they have been modeled as $R_f(z) = R_b(z) = -0.99$.

The computation has been developed in the *Z-transform* domain, in which the delay lines are modeled as Z^{-M}, where M is the delay-line length in samples. The length of the delay lines controls the frequency of oscillation, and consequently the pitch of the output signal. In particular, for a string plucked in a point whose distance from the nut is βL_0, where $0 < \beta < 1$ and $L = 0.65\ m$ is string length, the samples needed for the propagation to travel from the plucking point to the nut is given by $N_{left} = floor(\beta M)$; while the number of samples needed for modeling the propagation toward the bridge is given by the complementary of N_{left}, i.e. $N_{right} = M - N_{left}$.

The resulting model is shown in figure (4): the input signal can be of any wave-variable type, such as displacement, velocity, acceleration. In the case under analysis, acceleration has been selected as the input wave variable, since then an ideal pluck corresponds to an impulse. The final relation between the input acceleration at a given position of the string and the output transverse force at the bridge is given by the following transfer function:

$$H_{E,B}(\omega) = \frac{1}{2}[1 + H_{E_2R_1}(\omega)]\frac{H_{E_1R_1}(\omega)}{1 - H_{loop}(\omega)}\frac{Z(\omega)}{\omega}[1 - R_b(\omega)] \tag{1}$$

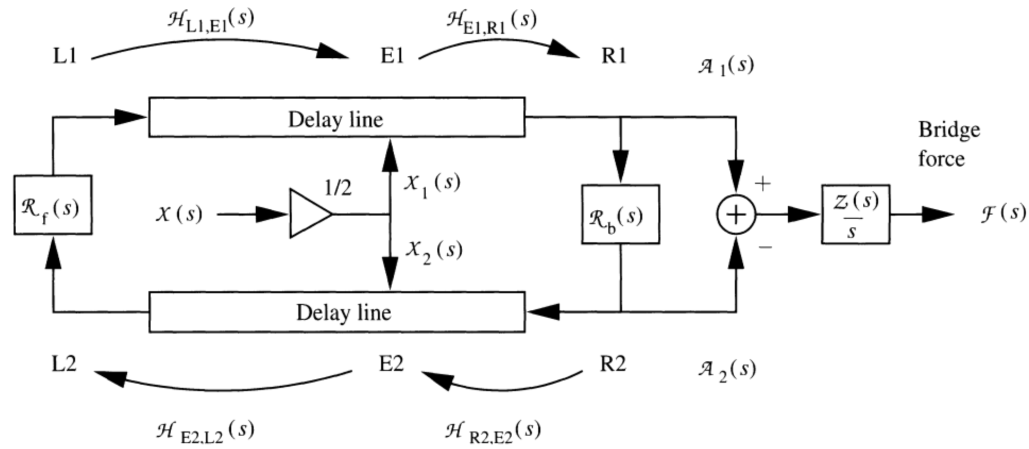

Figure 4: Dual delay-line waveguide model for an acoustic guitar (output at the bridge).

Looking at the block diagram in the figure above, it is possible to evaluate the expressions which characterize the filters H_{E_2,R_1}, H_{E_1,R_1}, H_{loop}:

- $H_{E_2,R_1} = z^{-N_{left}} \cdot (-R_f) \cdot z^{-N_{left}} \cdot z^{-N_{right}}$
- $H_{E_1,R_1} = z^{-N_{right}}$
- $H_{loop} = (-R_b) \cdot (-R_f) \cdot z^{-2N_{right}} \cdot z^{-2N_{left}}$

By evaluating the equation (1) for the six strings of a standard tuned guitar, assuming that they are plucked at $L/5$, the results in figure (5) are obtained. Note that the bridge impedance $Z(s)$ has been normalized before the calculation.

As can be seen from the graphs, the transfer function of each string is characterized by resonance frequencies which are multiple of the string's fundamental one. Furthermore, it can be noticed that the bridge impedance contribution depicted in figure (3) can be clearly distinguished in these transfer functions. Moreover, observing the graph related to the $E2$ string (82.41 Hz), it can be noticed that the amplitude of the fifth resonance is more attenuated if compared to the others. This occurs because when a string is plucked at $1/g$ of its length, modes g, $2g$, $3g$... are not present.

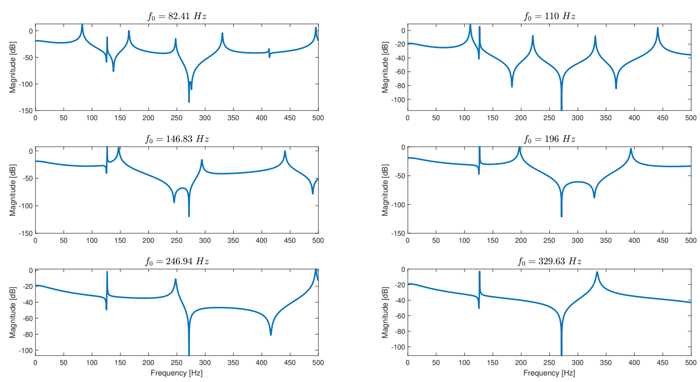

Figure 5: Transfer function $H_{E,B}(\omega)$ for the six strings in standard tuning.

c) Time domain response

It is now needed to compute the time domain response of the system to a pluck of amplitude $0.003\ m$ happening at one fifth of the length of the string. The relationship between the input and the output of the system is defined by the transfer function $H_{E,B}(\omega)$:

$$H_{E,B}(\omega) = \frac{F(\omega)}{X(\omega)} \tag{2}$$

where $X(\omega)$ is the chosen input wave variable, i.e. acceleration and $F(\omega)$ is the output transverse force at the bridge. Given this expression, the easiest way to compute the time response $F(t)$ is to convolve in the time domain a unitary acceleration impulse $\delta(t)$ (representing an ideal pluck) with the function $h_{E,B}(t)$. According to the convolution theorem, convolving in the time domain is equivalent to a multiplication in the frequency domain. Therefore, a unit impulse has to be defined in the time domain, converted to the frequency domain by means of the *Fast Fourier Transform*, multiplied by the transfer function $H_{E,B}(\omega)$ and finally the result has to be anti-transformed to obtain the response in the time domain.

Despite that, in order to evaluate the magnitude of the impulse and therefore the one of the response, a numerical approach can be adopted. The input triangular wave associated with the ideal pluck can be modeled as follows:

$$y(x,t) = \sum_{n=1}^{N} \frac{2h}{n^2\pi^2} \frac{L^2}{d(L-d)}\ sin\left(\frac{dn\pi}{L}\right)\ sin\left(\frac{n\pi}{L}x\right)\ cos\left(2\pi\frac{nc}{2L}t\right) \tag{3}$$

Given this expression it is possible to plot both the shape of the string at the initial time instant (figure 6) and the displacement of the excitation point during time (figure 7). These results have been computed for a plucking position of $\frac{L}{5}$, amplitude of the plucking equal to $0.003\ m$ and length of the string $L = 0.65\ m$ for the Martin D28.

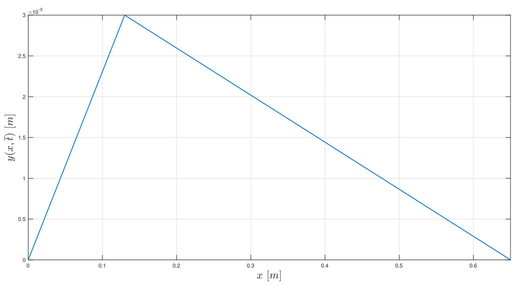

Figure 6: Shape of the string at the plucking instant t_0: $y(x,0)$

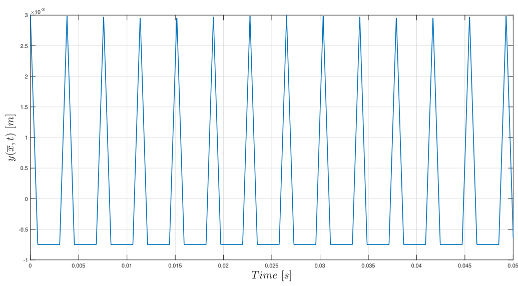

Figure 7: Motion of the plucking point during time: $y(L/5,t)$

Once this has been computed for a number of harmonics $N = 1000$ it is possible to derive the time dependent function $y(L/5,t)$ in order to obtain the velocity and, after the second derivation, the acceleration. Under a good approximation the impulse has been built as an ideal one by setting the amplitude of the first sample of a zeros array to be equal to the velocity amplitude. In fact, the derivative of a step function (in this case the velocity) is a *Dirac's delta* with the same amplitude: $\frac{d}{dt}(Au(t)) = A\delta(t)$.

By evaluating the convolution between this impulse and the function $h_{E,B}(t)$ the following time responses have been obtained:

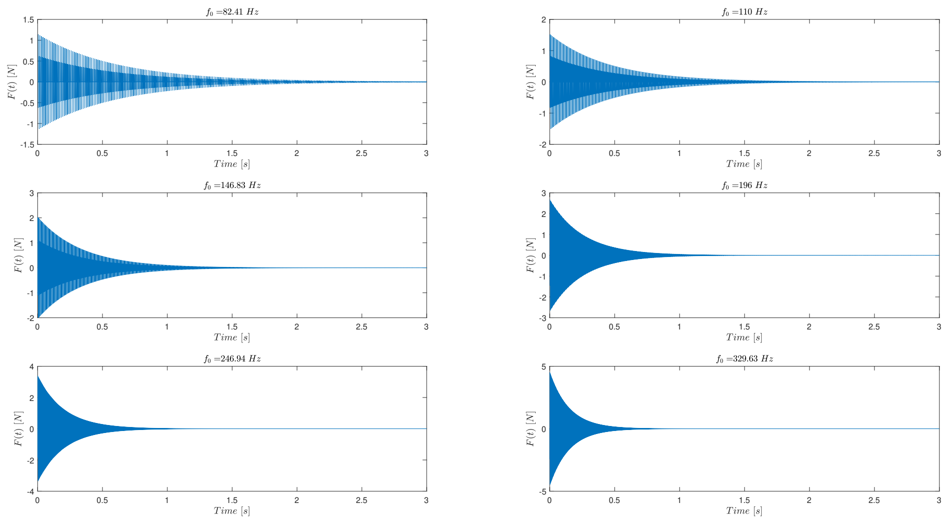

Figure 8: Time response $F(t)$ for an ideal pluck of amplitude 3 mm at $\frac{L}{5}$ for the six strings of a Martin D28.

The decay time and amplitude of the force is coherent with what is expected for the various strings of the guitar. For example, it is evident how the decay time is longer for lower frequency strings. In this case, the compound decay due to a normal plucking is not observable because the adopted mathematical model takes into account only the transverse component F_T of the force at the bridge and not the longitudinal one F_L. This is also the reason for the short time duration of the response as it can be clearly seen from the figure below.

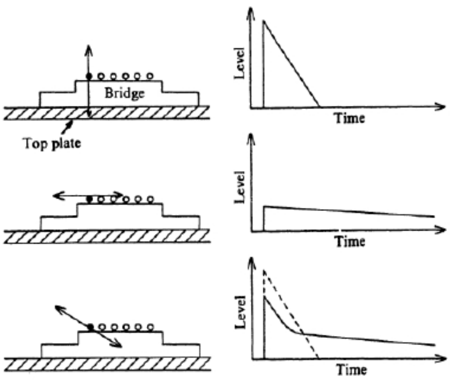

Figure 9: Time decay of the two force components and of a normal plucking

Furthermore, even if dealing with forces, it is possible, by listening to the response, to notice that the result is actually tuned with the fundamental frequency of the string.