# Assignment

Homework HW1


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Musical Acoustics

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# Plate characterization and string plate interaction

## a) Propagation speed of quasi-longitudinal and longitudinal waves

In general, *longitudinal waves* are defined as waves where the particle motion is in the same direction of the propagating wave. This results is the presence of alternate compression and rarefaction regions along the medium. Nevertheless, in the particular case of a thin plate (characterized by a thickness much smaller than the other two dimensions) the motion of particles might not be purely longitudinal but exhibits some transverse components as an effect of the Poisson's phenomenon; in this case the waves are named *quasi-longitudinal waves*. The velocity of propagation of quasi-longitudinal waves in a thin plate is given by the following expression:

$$C_L = \sqrt{\frac{E}{\rho(1-\nu^2)}} \quad \Rightarrow \quad C_L = 5.36 \cdot 10^3 \ \frac{m}{s} \tag{1}$$

While the velocity of propagation of purely longitudinal waves is given by:

$$C_L' = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \quad \Rightarrow \quad C_L' = 6.19 \cdot 10^3 \ \frac{m}{s} \tag{2}$$

Purely longitudinal waves are not that interesting from a musical acoustics standpoint because particles move back and forth along the longitudinal direction without causing any air displacement.

## b) Propagation speed of bending waves

Most sound radiated by vibrating structures is caused by the propagation of *bending waves* which deforms a structure transversely, so that they generate acoustic waves in the surrounding air. Bending waves propagating in a thin plate are described by the following partial differential equation:

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} + \frac{Eh^2}{12\rho(1-\nu^2)}\Delta^4 z(x,y,t) = 0 \tag{3}$$

Assuming an harmonic solution of the type $z(x,y,t) = Z(x,y)e^{j\omega t}$ and substituting it into the equation of motion, the following result is obtained:

$$\Delta^4 Z - k^4 Z = 0 \tag{4}$$

where $k^2 = \frac{\sqrt{12}\omega}{C_L h}$ is the wavenumber from whose expression the speed of propagation of bending waves in a thin plate can be evaluated:

$$k^2 = \frac{\omega^2}{v^2} \quad \Rightarrow \quad v^2 = \frac{\omega^2}{k^2} \quad \Rightarrow \quad v(f) = \sqrt{\frac{2\pi}{\sqrt{12}} f h C_L} = \sqrt{1.8 f h C_L} \tag{5}$$

It can be noticed that the speed of bending waves in a thin plate depends not only on the elastic properties and density (respectively $E$,$\nu$ and $\rho$ in the expression of $C_L$) of the structural material it travels through, but also on the thickness $h$ of the plate. Furthermore, the dependency of the propagation speed on frequency, causes bending wave to be **dispersive** (nonlinear relationship between the propagation velocity $v$ and the frequency $f$). This dependency can be observed in the following plot:

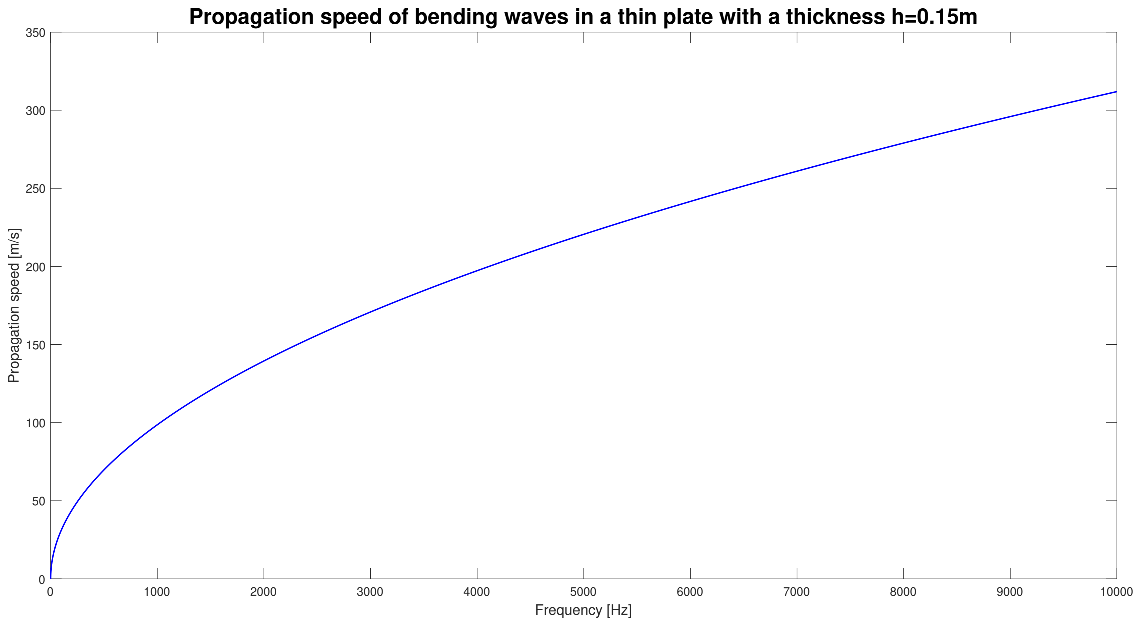



Figure 1: Propagation speed of the bending waves as a function of the frequency.

## c) Modal frequencies of the first six bending modes

The mode $(0,0)$ is the fundamental bending mode of a square thin plate with side $a$ and the corresponding fundamental frequency $f_B(0,0)$ is given by the following expression:

$$f_{B(0,0)} = 1.654 \frac{C_L h}{a^2} = 394.2581\ Hz \tag{6}$$

In order to compute the next five modal frequencies, it is useful to look at this table:

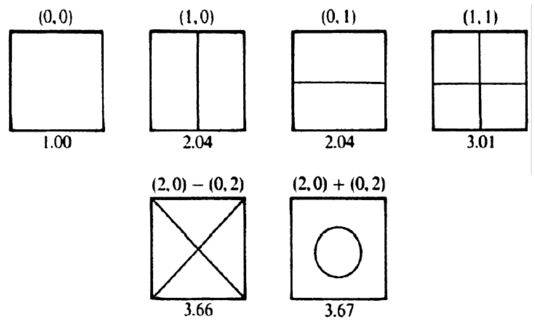



Figure 2: First six bending mode shapes in a square plate with clamped edges.

The number below each modal shape represents the ratio between the frequency of the specific mode and the fundamental frequency $f_{(0,0)}$. As a consequence, the other modal frequencies of the analyzed square plate can be evaluated as follows:

$$\begin{aligned}
f_{B(1,0)} &= 2.04 \cdot f_{B(0,0)} = 804.286\ Hz \\
f_{B(0,1)} &= 2.04 \cdot f_{B(0,0)} = 804.286\ Hz \\
f_{B(1,1)} &= 3.01 \cdot f_{B(0,0)} = 1186.716\ Hz \\
f_{B(2,0)-(0,2)} &= 3.66 \cdot f_{B(0,0)} = 1442.984\ Hz \\
f_{B(2,0)+(0,2)} &= 3.67 \cdot f_{B(0,0)} = 1446.927\ Hz
\end{aligned}$$

It can be immediately noticed that the modal frequencies are not harmonically spaced (they are not integer multiples of the fundamental one). As it can be observed, the two different combinations of the modes $(2, 0)$ and $(0, 2)$ give rise to *non-degenerate modes* which are characterized by modal frequencies that are different from the ones of the original combining modes.

## d) Sitka spruce

Since this wooden plate is realized using the quarter-cut scheme, *longitudinal* and *radial* axis lie on the plate, while *tangential* axis is in the direction of its thickness.

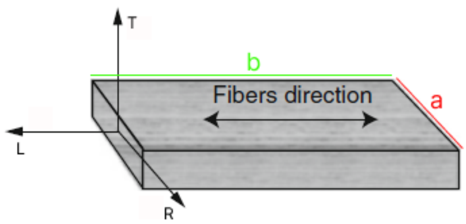



Figure 3: Reference system in a wooden plate according to the quarter-cut technique.

*X-mode* and *ring-mode* are used by luthiers to tune the top and back plates of a violin before assembling them. In order to observe these two vibrational modes in a rectangular wooden plate, the aspect ratio $\frac{L_L}{L_R}$ must satisfy the following relationship:

$$\frac{L_L}{L_R} = (\frac{E_L}{E_R})^{\frac{1}{4}} \tag{7}$$

In particular, for a Sitka spruce $\frac{L_L}{L_R} = 1.9$ is needed. Therefore the length $b = L_L$ of the side parallel to the fibers is given by:

$$b = 1.9 \cdot a = 1.9 \cdot 0.15\ m = 0.286\ m \tag{8}$$

# e) String tuning

Let's now tune the fundamental frequency $f_s(1)$ of a string of length $L_s$ with the one of the aluminum square plate $f_{B(0,0)}$ (evaluated in section 3) to which the string is attached. In particular it is needed to compute the tension $T_s$ for which this condition is satisfied. In order to do so the following relation valid for ideal strings fixed at both ends has to be employed:

$$f_s(1) = \frac{c}{2L_s} = f_{B(0,0)} = 394.2581\ Hz \quad \Rightarrow \quad \sqrt{\frac{T_s}{\mu_s}} \cdot \frac{1}{2L_s} = 394.2581\ Hz \tag{9}$$

where $\mu_s$ is the linear density of the ideal string and it is given by the product between the volumetric density of the string $\rho_s$ and the area $A_s = \pi r^2 = 3.8 \cdot 10^{-6}\ m^2$ of its cross section:

$$\mu_s = \rho_s A_s = 0.019 \frac{kg}{m}$$

Therefore the tension $T_s$ can be evaluated as follows:

$$T_s = (f_s(1) \cdot 2L_s)^2 \cdot \mu_s = 2393.05\ N \tag{10}$$

# f) String and plate coupling

The characterization of the coupling between string and plate is a very important task because it allows instrument makers to correctly design each single component in order to reach the desired results. For the evaluation of the modal frequencies of the string-board system in the coupled configuration, it is first needed to determine if the coupling is either *strong* or *weak*. The arising of one of these two conditions will depend on the characteristics of the two structures and on the frequency. In particular, strong coupling occurs when:

$$\frac{m}{n^2 M} > \frac{\pi^2}{4Q_B^2} \tag{11}$$

where $m = \rho_s \pi r^2 L$ is the string's mass, $M = \rho_B h a^2$ is the board's mass, $n$ is the number of the considered string's mode and $Q_B$ is the board's merit factor which can be considered constant for all the resonances of the plate. If the above condition is not satisfied, weak coupling will occur.

When a string is strongly coupled with another vibrating system such as a soundboard which has resonances of its own, looking at the impedance of the vibrating system, it can be noticed that below each of its resonances the impedance is negative and the soundboard behaves like a spring decreasing the vibrational frequency of the string; conversely, above each resonance the impedance is positive so the soundboard behaves like a mass increasing the vibrational frequency of the string. Consequently, the final result is that the plate's resonances tend to split the string's resonance into two new resonance frequencies, one above and one below the original string's resonance.
Considering modal frequencies up to the fourth one for the string and to the sixth one for the board, the coupling condition always results as *strong*. The following graph illustrates how the frequencies of the string-board system will behave in *strong coupling* condition:

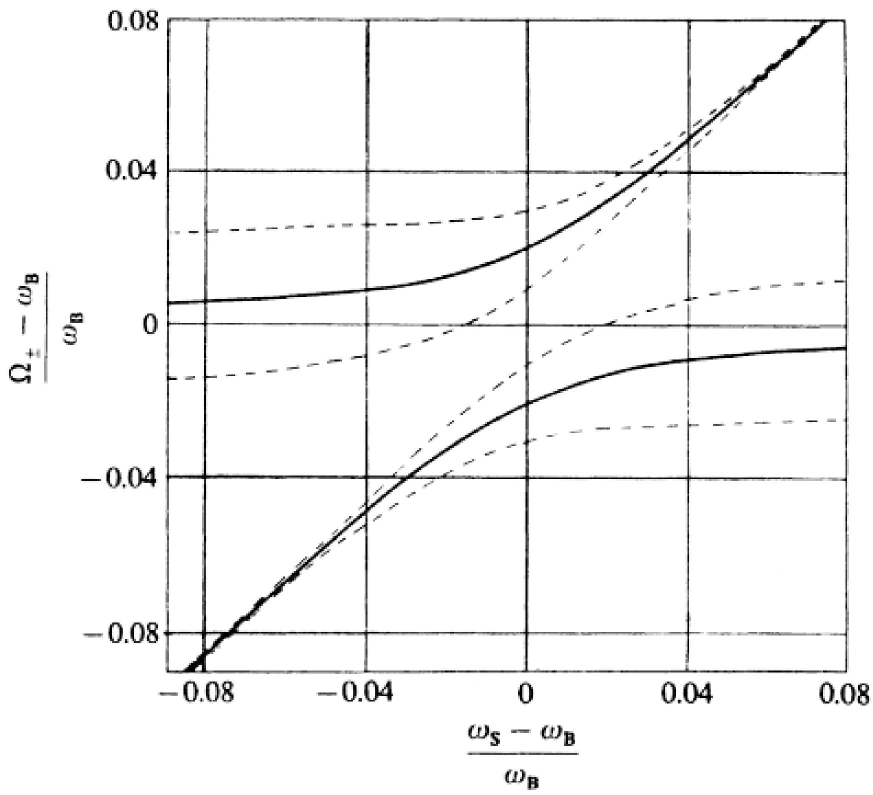



Figure 4: Typical dispersion curves for the normal mode frequencies of a string that is strongly coupled to a soundboard.

In particular, the coupled system will be characterized by two new resonance frequencies determined by the relative distance between the original string and board frequencies $\frac{\omega_s - \omega_b}{\omega_b}$.
While the modal frequencies for the board have already been calculated, the string ones can be obtained as integer multiples of the fundamental one.
In the case of the two systems having the same resonance frequency $\omega_0$ (mode 1 of the string and mode (0,0) of the board), the normal modes are split symmetrically around $\omega_0$

and their damping factor is $2Q_B$. In particular they'll be at $\pm 2\%$ from $\omega_0$. For the other modal frequencies the $\frac{\omega_s - \omega_b}{\omega_b}$ ratio is no longer equal to zero and, observing the graph in figure (4), the results reported in the following table have been obtained:

| | $\frac{\omega_s-\omega_B}{\omega_B}$ | $\frac{\Omega_{c+}-\omega_B}{\omega_B}$ | $\frac{\Omega_{c-}-\omega_B}{\omega_B}$ |
|---|---|---|---|
| $f_{B(0,0)}$, $f_s(1)$ | 0 | 0.02 | -0.02 |
| $f_{B(1,0)}$, $f_s(2)$ | -0.0196 | 0.015 | -0.03 |
| $f_{B(0,1)}$, $f_s(2)$ | -0.0196 | 0.015 | -0.03 |
| $f_{B(1,1)}$, $f_s(3)$ | -0.0033 | 0.02 | -0.02 |
| $f_{B(2,0)-(0,2)}$, $f_s(4)$ | 0.0929 | - | - |
| $f_{B(2,0)+(0,2)}$, $f_s(4)$ | 0.0899 | - | - |

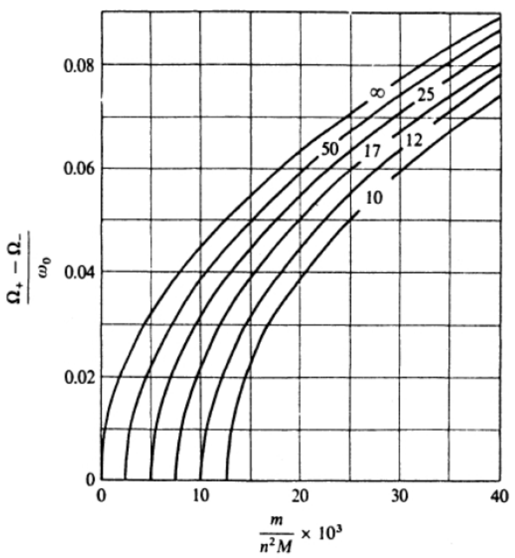
Table 1: String-board relative frequency ratios and effect on system resonance frequencies.

In the case of the *x-mode* $f_{B(2,0)-(0,2)}$ coupled with the fourth modal frequency of the string $f_s(4)$ the ratio $\frac{\omega_s - \omega_b}{\omega_b}$ exceeds the limit (0.08) provided in the graph (4) and therefore the coupling effect is negligible. The same is valid for the *ring-mode* $f_{B(2,0)+(0,2)}$ with the same string modal frequency. The final results for the new resonance frequencies of the coupled system are reported in the table below:

| | $f_{c+}[Hz]$ | $f_{c-}$ $[Hz]$ |
|---|---|---|
| $f_{B(0,0)}$, $f_s(1)$ | 402.14 | 386.37 |
| $f_{B(1,0)}$, $f_s(2)$ | 800.34 | 776.69 |
| $f_{B(0,1)}$, $f_s(2)$ | 800.34 | 776.69 |
| $f_{B(1,1)}$, $f_s(3)$ | 1210.45 | 1162.98 |

Table 2: Resulting system frequencies for different modes combinations.

Nonetheless, it must be mentioned that an alternative procedure exists for assessing the eigenfrequencies of the coupled system but only when the string's resonance frequency matches that one of the soundboard. In this case, it can be also useful to observe this plot:




It represents the relative separation between the new vibrational frequencies ($\Omega_+$ and $\Omega_-$) as a function of the ratio $m/(n^2M)$ and for different values of the merit factor $Q_B$.
In the context of the analyzed coupled system, since $m/(n^2M) \cdot 10^3 = 140.8$, it is possible to evaluate $\Omega_+$ and $\Omega_-$ by extrapolating a linear relationship from the curve corresponding to $Q_B = 25$.