Assignment

Homelab HL3

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Musical Acoustics

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Modeling techniques

Exercise 1

The aim of the first exercise is to model through *finite difference method* a string of a piano when struck by the hammer. In particular, it is required to compute the simulation for a C2 string whose fundamental frequency is $f_0 = 65.4\ Hz$.

a) Hammer-string interaction

To simulate how the hammer and the string interact, a basic version of the piano string mechanism, shown in figure 1, has been considered. This model neglects the action of the damper to the string and this leads to a simplification of the equations used to describe the hammer-string interaction.

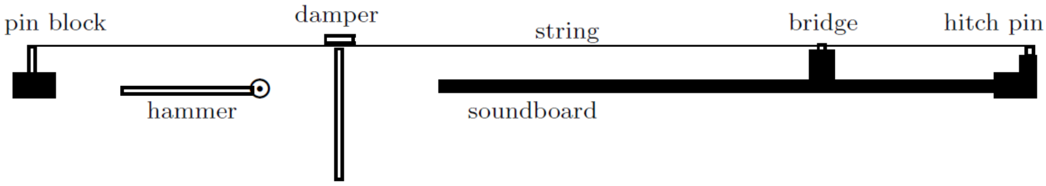

Figure 1: Hammer-string mechanism

Hammer characterization

The hammer is composed of an hard-wood core of graduated sizes covered by two layers of felt with different thicknesses depending on the note range. The hammer response is influenced by the velocity of its striking action. Due to the properties of the felt, an higher velocity causes the hammer to harden and consequently it solicits higher harmonics. Experimental measurements show that this behaviour is highly non-linear and therefore it is possible to consider the hammer as a lumped mass connected to a nonlinear spring which represents the felt. Hence, the force F_H due to the felt can be described by the following power law:

$$F_H = K\xi^p \tag{1}$$

where K is the felt stiffness, $\xi(t)$ is a time dependent function which describes the felt compression and p is the stiffness exponent which describes how the stiffness changes with the force. Furthermore, the function $\xi(t)$ can be written as $\xi(t) = |\eta(t) - y(x_0, t)|$ where $\eta(t)$ is the hammer displacement with respect to the string equilibrium position and $y(x_0, t)$ is the vertical displacement of the string in the striking point x_0 at time instant t.
Finally, the hammer-string interaction is described by the following differential equation:

$$M_H \frac{d^2\eta}{dt^2} = -F_H(t) - b_H \frac{d\nu}{dt} \tag{2}$$

where M_H is the hammer mass and $F_H(t)$ is the occurring interaction force.

When considering the excitation of the whole hammer on the string a force density needs to be considered. It is assumed that the force density term does not propagate along the string, so that the time and space dependences can be separated as follows:

$$f(x, x_0, t) = f_H(t) \cdot g(x, x_0) \tag{3}$$

where the function $f_H(t)$ refers to the time history of the hammer force $F_H(t)$ on the string, while the space function $g(x_0, x)$ is a dimensionless function which describes the force density over space. In particular, in the simulation which follows below, the function $g(x_0, x)$ is an *Hanning window* with width w and centered in a, the relative hammer striking position.

Finally, to comprehensively describe the interaction between the hammer and the string, it is also necessary to evaluate the duration of contact between them. The influence of the hammer on the movement of the string persists until the subsequent condition holds true:

$$\eta(t) < y(x_0, t) \tag{4}$$

The hammer parameters considered for the simulation are the following:

- $M_H = 4.9 \cdot 10^{-3}$ $[kg]$ mass of the Hammer;
- $V_{H0} = 2.5$ $[m/s]$ hammer initial velocity;
- $K = 4 \cdot 10^8$ hammer felt stiffness;
- $w = 0.2$ $[m]$ width of the window $g(x, x_0)$;
- $a = 0.12$ relative striking position;
- $b_H = 1 \cdot 10^{-4}$ air damping coefficient.

String characterization

Regarding the string behaviour, it is necessary to develop a string model that replicates real-world conditions by taking into account losses and stiffness into the equation of motion which is given by the following expression:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - k^2 \frac{\partial^4 y}{\partial x^4} - 2b_1 \frac{\partial y}{\partial t} + 2b_2 \frac{\partial^3 y}{\partial x^2 \partial t} + \rho^{-1} f(x, x_0, t) \tag{5}$$

where the fourth order term derives from the analysis of the stiff string behaviour and k is the defined as *string stiffness coefficient*. As for the term $-2b_1 \frac{\partial y}{\partial t} + 2b_2 \frac{\partial^3 y}{\partial x^2 \partial t}$, it introduces frequency dependent losses being b_1 the *air damping coefficient* of the string and b_2 the *string internal friction coefficient*. Finally, the term $\rho^{-1} f(x, x_0, t)$ represents the acceleration imposed to the string by the hammer where ρ is the *linear mass density* of the string and $f(x, x_0, t)$ is the force density function introduced previously.

The partial differential equation is associated with this set of boundary conditions:

- Boundary conditions at the bridge:

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{k^2}\frac{\partial y}{\partial y} + \frac{\zeta_b c}{k^2}\frac{\partial y}{\partial t}; \qquad \zeta_b = \frac{R_b}{\rho c}; \qquad \left.\frac{\partial^2 y}{\partial x^2}\right|_{x_M, t_n} = 0; \tag{6}$$

  where ζ_b is the normalized bridge's impedance R_b, and x_M is the coordinate at the bridge end where the string cannot move.

- Boundary conditions at the left hinged end:

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{k^2}\frac{\partial y}{\partial y} + \frac{\zeta_l c}{k^2}\frac{\partial y}{\partial t}; \qquad \zeta_l = \frac{R_l}{\rho c}; \qquad \left.\frac{\partial^2 y}{\partial x^2}\right|_{x_0, t_n} = 0; \tag{7}$$

  where ζ_l is the normalized left end's impedance R_l and x_0 is the coordinate of the left end where, also in this case, the string actually cannot move.

Also for the case of the string it is necessary to introduce and evaluate some parameters which are useful for the simulation:

- $f_1 = 65.4$ $[Hz]$ fundamental frequency of the string;
- $L = 1.92$ $[m]$ length of the string;
- $M_s = 35 \cdot 10^{-3}$ $[kg]$ mass of the string;
- $\rho = 18.2 \cdot 10^{-3}$ $[kg/m]$ linear mass density;
- $b_1 = 0.5$ $[s^{-1}]$ air damping coefficient for the string;
- $b_2 = 6.25 \cdot 10^{-9}$ $[s]$ internal friction coefficient;
- $k = \epsilon = 7.5 \cdot 10^{-6}$ string stiffness coefficient;
- $\zeta_l = \frac{R_l}{\rho c} = 1 \cdot 10^{20}$ left end normalized impedance
- $\zeta_b = \frac{R_b}{\rho c} = 1 \cdot 10^{3}$ normalized bridge's impedance
- *Tension*: it is computed tuning the string to the fundamental frequency of the string; the formula from which the tension can be derived knowing the fundamental frequency is given by $T = 4L^2 \rho f_1^2 = 1149.7$ $[N]$
- *Propagation velocity along the string*: It can be evaluated as $c = \sqrt{\frac{T}{\rho}} = 251.136$ $[m/s]$

In the next page the finite differences technique which approximates the string behaviour is introduced.

b) Finite difference modeling

The *finite difference method* is employed to discretize in space and time the partial differential equations governing the system's physical behaviour. As a consequence, this numerical method allows for the computation of the string's displacement at each time sample by subdividing the string into multiple spatial samples.

Sampling parameters

In order to implement the FD method, firstly it is necessary to define the *sampling frequency* f_s and the corresponding *sampling period* T_s:

$$f_s = 44100\ [Hz] \rightarrow T_s = \frac{1}{f_s} = 2,26 \cdot 10^{-5}\ [s] \tag{8}$$

Moreover, since the *Nyquist Frequency* is given by $F_{nyq} = \frac{F_s}{2} = 22050\ [Hz]$, the anti-aliasing condition is then satisfied being $f_1 = 65.4\ [Hz] < F_{nyq}$.

It is now needed to subdivide the string in M finite small segments of length X_s, such that $L = MX_s$. In order for the spatial stability to be guaranteed, the spatial step must be limited. Defining the *Courant number* as $\lambda = \frac{cT_s}{X_s}$, it is possible to apply the Courant-Friedrichs-Lewy condition (CFL) to obtain the maximum number of spatial steps M_{max} as follows:

$$X_s \leq cT_s \quad \rightarrow \quad M_{max} = \left\lfloor \frac{L}{cT_s} \right\rfloor = 337 \tag{9}$$

However, as outlined in the provided paper (Chaigne-Askenfelt), ensuring stability during computation requires limiting the maximum number of spatial samples according to the following condition:

$$M_{max} = \left\lfloor \sqrt{\frac{-1 + \sqrt{1 + 16k\gamma^2}}{8\epsilon}} \right\rfloor = 217 \tag{10}$$

using this last value, dispersion and instability of the solutions are avoided and the CFL condition is still satisfied being $\lambda < 1$.

Also the hammer force function $F_H(t)$ and $g(x, x_0)$ are discretized in time and space obtaining $F_H(n)$ and $g(m, m_0)$. The latter one is the spatial window centered in m_0 which represents the spatial sample corresponding to the striking position and it can be defined as follows: $m_0 = \lfloor \frac{aL}{X_s} \rfloor = 26$ where $a = 0.12$ is the relative striking position defined in the hammer parameters. As a consequence, the length in samples of the window is given by $w_s = \lfloor \frac{w}{X_s} \rfloor = 22$, where $w = 0.2\ [m]$ is the length of the window in meters.

FD coefficients

After all the equations and the parameters for the problem have been defined, the string's behaviour can be implemented through the *finite differences* technique. In particular, the *difference equation* describing the transverse displacement of the stiff and lossy string, including the hammer-string interaction, is given by:

$$y_m^{n+1} = a_1(y_{m+2}^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n \quad (11)$$

with $F_m^n = F_H(n) g(m, m_0)$.

The difference equation coefficient $a_i (i = 1, 2, ..., 5, F)$ can be computed according to the analytical formulas listed in the table below:

a_1	a_2	a_3	a_4	a_5	a_F
$\frac{-\lambda^2\mu}{1+b_1T_s}$	$\frac{\lambda^2+4\lambda^2\mu+\nu}{1+b_1T_s}$	$\frac{2-2\lambda^2-6\lambda^2\mu-2\nu}{1+b_1T_s}$	$\frac{-1+b_1T_s+2\nu}{1+b_1T_s}$	$\frac{-\nu}{1+b_1T_s}$	$\frac{T^2/\rho}{1+b_1T_s}$

where the parameter μ and ν are defined as follows:

$$\mu = \frac{k^2}{c^2 X_s^2}; \qquad \nu = \frac{2b_2T_s}{X_s^2} \quad (12)$$

It is now possible to compute the coefficients that allow for the definition of the hammer displacement in the FD scheme. In particular, the expression used to determine the hammer position is given by:

$$\eta(n+1) = d_1\eta(n) + d_2\eta(n-1) + d_F F_H(n) \quad (13)$$

where the coefficients d_i can be computed according to the following table:

d_1	d_2	d_F
$\frac{2}{1+b_1T_s/2M_H}$	$\frac{-1+b_1T_s/2M_H}{1+b_1T_s/2M_H}$	$\frac{-T^2/M_H}{1+b_1T_s/2M_H}$

Also the *boundary conditions* need to be discretized through finite difference analysis. In particular, the wave difference equation (11) is valid only for the spatial samples from $m \geq 2$ to $m \leq M-2$. This being said, different boundary conditions are defined for $m = 0, 1, M-1, M$ where the first two samples represents the left end, while the last two ones are related to the right end.

- Boundary conditions for the hinged string's end (left end)
  -For the spatial sample $m = 0$:

$$y_m^{n+1} = b_{L_1} y_m^n + b_{L_2} y_{m+1}^n + b_{L_3} y_{m+2}^n + b_{L_4} y_m^{n-1} + b_{L_F} F_m^n \quad (14)$$

where all the coefficients b_{Li} can be evaluated according to the following table:

b_{L1}	b_{L2}	b_{L3}	b_{L4}	b_{LF}
$\frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T_s+\zeta_l\lambda}$	$\frac{4\lambda^2\mu+2\lambda^2}{1+b_1T_s+\zeta_l\lambda}$	$\frac{-2\lambda^2\mu}{1+b_1T_s+\zeta_l\lambda}$	$\frac{-1+b_1T_s+\zeta_l\lambda}{1+b_1T_s+\zeta_l\lambda}$	$\frac{T^2/\rho}{1+b_1T_s+\zeta_l\lambda}$

-For the spatial sample m=1:

$$y_m^{n+1} = a_1(y_{m+2}^n - y_m^n + 2y_{m-1}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n \quad (15)$$

- Boundary conditions for the right string's end (bridge)
  -For the spatial sample $m = M - 1$:

$$y_m^{n+1} = a_1(2y_{m+1}^n - y_m^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n \quad (16)$$

  -For the spatial sample $m = M$:

$$y_m^{n+1} = b_{R_1} y_m^n + b_{R_2} y_{m-1}^n + b_{R_3} y_{m-2}^n + b_{R_4} y_m^{n-1} + b_{R_F} F_m^n \quad (17)$$

  where all the coefficients b_{Ri} can be evaluated according to the following table:

b_{R1}	b_{R2}	b_{R3}	b_{R4}	b_{RF}
$\frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T_s+\zeta_b\lambda}$	$\frac{4\lambda^2\mu+2\lambda^2}{1+b_1T_s+\zeta_b\lambda}$	$\frac{-2\lambda^2\mu}{1+b_1T_s+\zeta_b\lambda}$	$\frac{-1+b_1T_s+\zeta_b\lambda}{1+b_1T_s+\zeta_b\lambda}$	$\frac{T^2/\rho}{1+b_1T_s+\zeta_b\lambda}$

- Initial conditions at the time sample $n = 0$
  The string and the hammer are at rest and therefore:

$$y_m^0 = 0 \qquad \eta(0) = 0 \qquad F_H(0) = 0 \quad (18)$$

- Initial conditions at the time sample $n = 1$
  The hammer hits the string at the spatial sample m_0 with force:

$$F_H = K|\eta^n - y(m_0, n)|^p, \qquad \eta^n = V_{H_0} T_s \quad (19)$$

- Initial conditions at the time sample $n = 2$
  String displacement and the hammer position will follow these relationships respectively:

$$y_m^n = y_{m-1}^n + y_{m+1}^n - y_{m-1}^n + \frac{T_s^2 M F_H g(m)}{M_S}, \quad (20)$$

$$\eta^n = 2\eta^{n-1} - \eta^{n-2} - \frac{T_s^2 F_H}{M_H} \quad (21)$$

  For the remaining time samples, the relative position of the hammer and the string must be evaluated in order to verify either the hammer is exerting a force on the string or not. In particular, if $\eta^n < y_{m_0}^n$ i.e. the hammer is not touching the string, then no force will be applied, $F_H = 0$. On the other hand, if $\eta^n \geq y_{m_0}^n$ the force applied to the string will follow equation (19).

c) Numerical simulation results

Once the finite difference method has been implemented, the values of the displacement of the string are simulated for eight seconds. The results obtained through the numerical simulation are shown in the figures below.

In particular, in figure (2) six frames of the string displacement during time have been plotted in order to observe the time behaviour of the string.

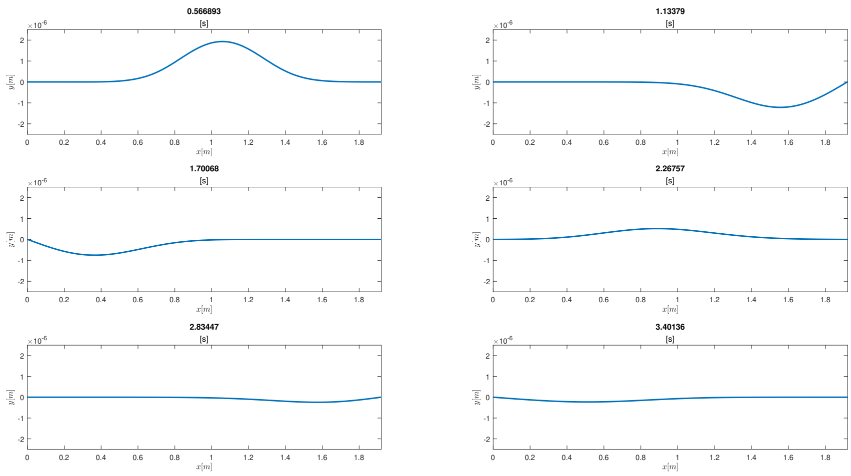

Figure 2: Vertical displacement of the string for various time instants.

Finally, the string's displacement has been averaged over 12 spatial samples. Such portion has been centered in a specular position with respect to the hammer's striking position.

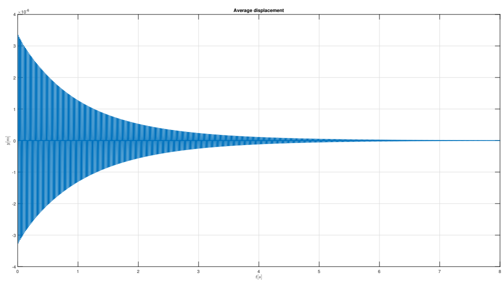

Figure 3: Average displacement of the string over time.

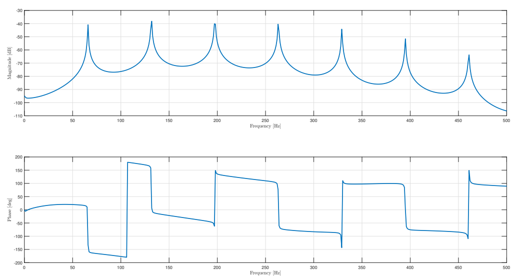

Figure 4: Spectral content of the average displacement of the string.

The spectral content of the signal, obtained by employing the *Fast Fourier Transform* algorithm, can be observed in figure (4) and it can be noticed that the fundamental frequency is the one expected, i.e. $f = 65.4\ Hz$.

Exercise 2

The purpose of this second exercise is to study the behaviour of an acoustic guitar adopting as models different kinds of equivalent electric circuits.

Guitar modeling

Ignoring the motion of the back plate and of the ribs, it is possible to model a guitar as a *two mass vibrating system* represented in the figure on the side. The vibrating strings apply a force $F(t)$ to the top plate, whose mass and stiffness are represented by m_p and k_p. A second piston of mass m_h represents the mass of air in the soundhole, and the volume V of enclosed air acts as the second spring. By exploiting the electro-mechano-acoustical analogies, in particular the *impedance analogy*, it is possible to obtain the equivalent electric circuit of the two mass model. This circuit is characterized by only two resonance frequencies. In order to build a model which gives more accurate results it is possible to consider also the vibration of the back plate, that means having three coupled simple oscillators with three resulting resonance frequencies. This being said, it is evident that increasing the number of oscillator, the model is able to characterize the instrument also for high frequency values.

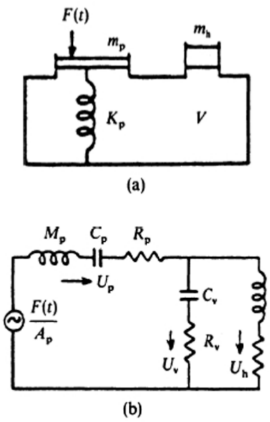

Figure 5: Two mass model of a guitar and its electric equivalent

Twenty resonances without string model

As a consequence, in order to study 20 resonances of the guitar, the two mass model has been extended adding a bank of 20 RLC branches as shown in figure (6). The series L1, C1, R1 relates to the resonance of the top plate, while R2, R3, C2, and L2 are associated with the resonance of the sound box.

In this first simulation, the force applied to the top plate has been obtained through the use of a signal controlled voltage generator which is fed with a time decaying square wave of frequency $f_F = 300Hz$; the output of the system is the velocity of the top plate, the sound radiator of the guitar, which is repre-

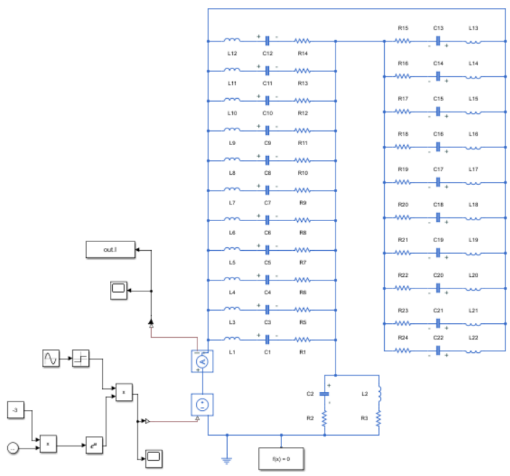

Figure 6: Refined circuit model of the acoustic guitar without string.

sented by the current flowing out the voltage generator. The signal is then measured through an amperometer, imported in MATLAB and the time plot is visible in figure (7).

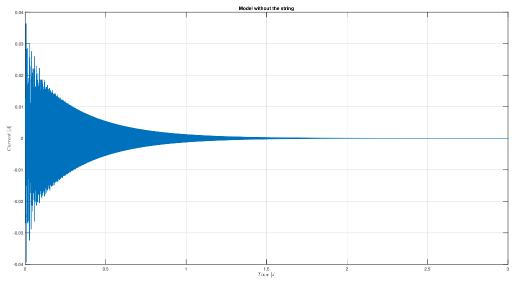

Figure 7: Simulation's resulting current signal in time.

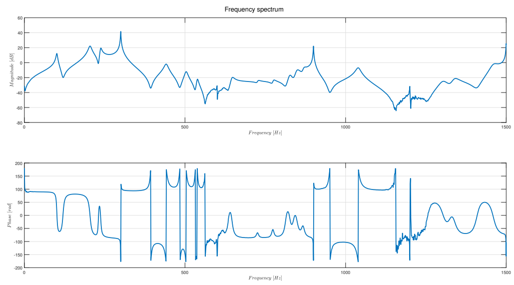

Figure 8: Spectral content of the computed signal.

Twenty resonances with string model

Now, another component is added to the simulation: the model of a real plucked string is consider. The string taken into account is the E4 string whose fundamental frequency is $f_{E4} = 329.63\ Hz$. In order to simulate it, it is needed to model the standing wave which occurs in fixed-fixed strings and the initial deformation due to the plucking action. Since the standing waves are caused by reflections, the latter ones are taken into account through the use of a transmission line. A travelling wave into a transmission line can be reflected and/or transmitted according to the relation between the characteristic impedance and the load impedance. The transmission line is fed on both sides with two symmetrical triangular current pulses which correspond to two propagating D'Alembert solutions for the transverse wave propagation on the string. Their duration is linked to half of the string fundamental frequency of vibration.

The initial excitement of the string is given by the model of the plucking of the string at one fifth (1/5) of its length. In order to do it, the transmission line delay has been set to 1.515 $[ms]$, that corresponds to half of the period of a wave with frequency f_{E4}.
Finally, the new circuit configuration and the obtained results are shown in the next figures:

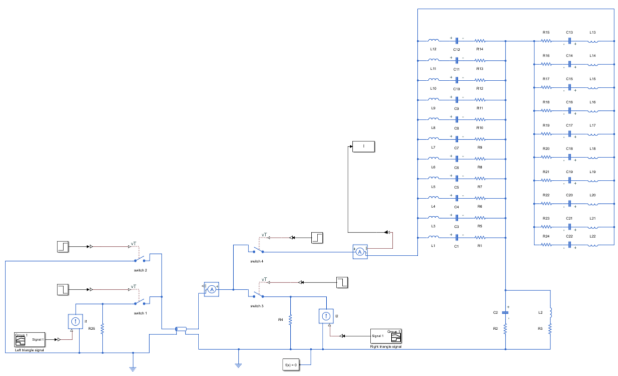

Figure 9: Refined circuit model of the acoustic guitar.

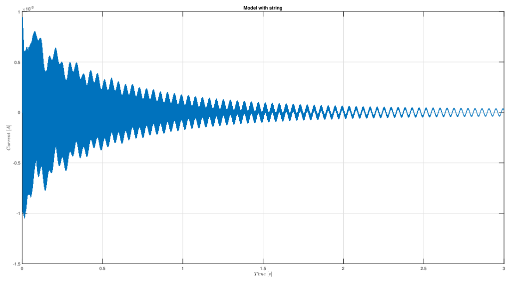

Figure 10: Simulation's resulting current signal in time.

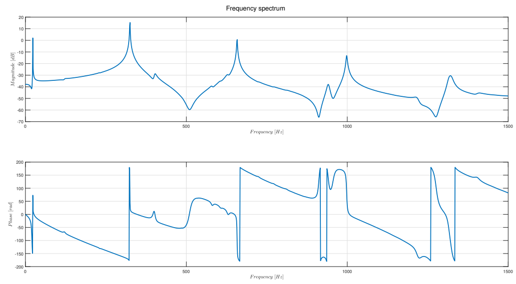

Figure 11: Spectral content of the computed signal.

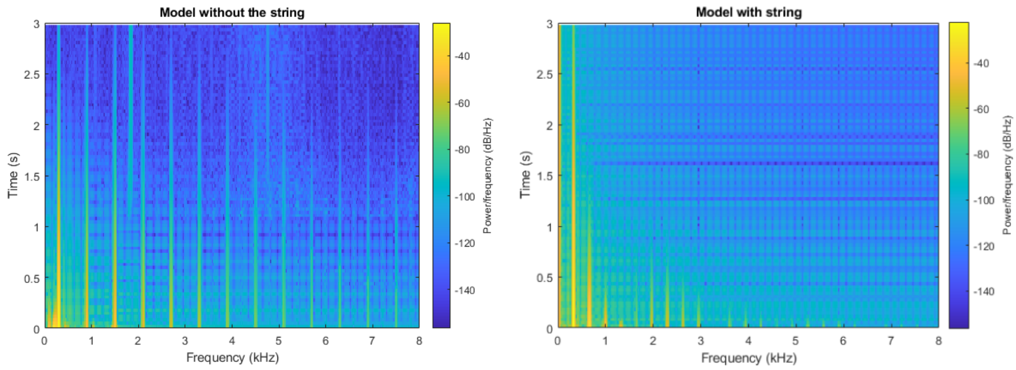

Figure 12: Spectrograms obtained from the two models.

The resulting signal shows a much more accurate approximation obtained from the refined model. This is particularly evident in figure (12) where higher frequencies spectral components decay in time. It is also important to mention that, comparing figure (11) with figure (8), the harmonics are more evenly spaced and clear. Furthermore, the new model accounts for the attenuation of harmonics multiple of the plucking position g.