# Assignment

Homework HW1

**Enrico Dalla Mora - 10937966**
**Michele Murciano - 10883559**

Musical Acoustics

POLITECNICO
MILANO 1863

October 2, 2023

# Characterization of a resonator

## a) Natural frequency $\omega_0$

The typical model adopted to characterize a resonator consists of a mass $m$ attached to a spring with characteristic stiffness $K$ and to a damper characterized by a resistance $R$. It can be schematized as shown in figure (1).

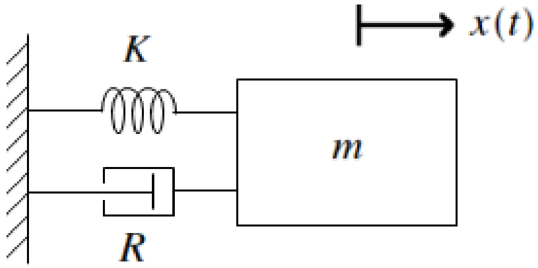


Figure 1: Resonator model.

The natural frequency is computed assuming the system without damping. Consequently, the equation of motion is given by:

$$m\ddot{x} + Kx = 0, \tag{1}$$

which can be rewritten as:

$$\ddot{x} + \omega_0^2 x = 0, \qquad where \quad \omega_0 = \sqrt{\frac{K}{m}}. \tag{2}$$

Given the mass $m = 0.1\ Kg$ and the stiffness of the spring $K = 2.53 * 10^4 N/m$, the natural frequency of the system can be computed as follows:

$$\omega_0 = \sqrt{\frac{K}{m}} = 502.99\ \frac{rad}{s} \quad \Rightarrow \quad f_0 = \frac{\omega_0}{2\pi} = 80.05\ Hz. \tag{3}$$

## b) Decay time $\tau$

Frictional losses must be taken into account in order to solve this problem. These kind of losses are proportional to the velocity of displacement and they are modelled by the damper in the so-called "Mass-spring-damper" model shown in figure (1). The equation of motion which governs the damped case is reported below:

$$m\ddot{x} + R\dot{x} + Kx = 0, \tag{4}$$

which can be rewritten as:

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0, \qquad where \quad \alpha = \frac{R}{2m}. \tag{5}$$

Assuming a solution of the type:

$$\tilde{x} = \tilde{A}e^{\gamma t} \tag{6}$$

evaluating its first and second time derivatives and substituting them in the equation (5), the following characteristic equation can be obtained:

$$\gamma^2 + 2\alpha\gamma + \omega_0^2 = 0 \tag{7}$$

solving for $\gamma$ gives:

$$\gamma = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d \qquad where \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{8}$$

As a consequence, the solution for the second order homogeneous differential equation (5) is given by:

$$x(t) = e^{-\alpha t}Acos(\omega_d t + \phi) \tag{9}$$

It can be noticed that in presence of damping, the oscillation gradually decreases in amplitude and the system reaches a stable equilibrium position over time.
The time decay factor $\tau$ is the time quantity that measures how quickly the amplitude is decreased about a factor of 1/e.

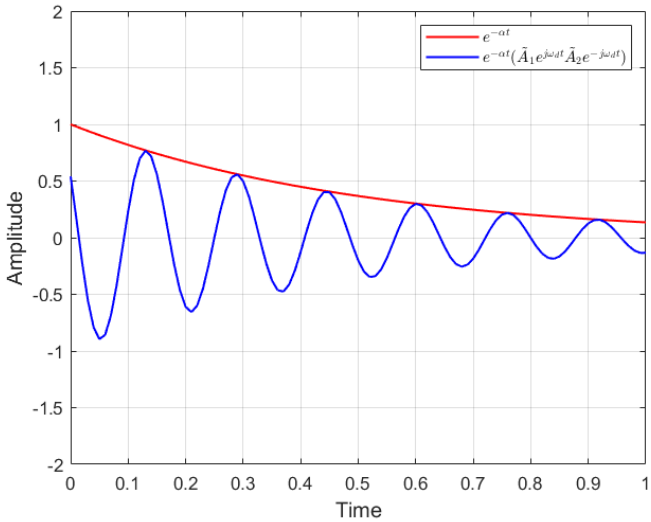


Figure 2: Time response of a generic damped resonator and its exponential component.

Denoting the initial amplitude of oscillation with $X_0$ and the amplitude at -5dB with $X_{-5}$ (where $X_{-5} = X_0 \cdot e^{-\frac{0.576}{\tau}}$ ), the decay time can be computed as follows:

$$10 \cdot log_{10}\left(\frac{X_{-5}}{X_0}\right) = -5dB \quad \Rightarrow \quad \frac{X_{-5}}{X_0} = 10^{-\frac{1}{2}} \quad \Rightarrow \quad \frac{X_0 \cdot e^{-\frac{0.576}{\tau}}}{X_0} = 10^{-\frac{1}{2}} \quad \Rightarrow \quad e^{-\frac{0.576}{\tau}} = 10^{-\frac{1}{2}} \tag{10}$$

$$\tau = -\frac{0.576}{ln(10^{-\frac{1}{2}})} = 0.5003\ s \tag{11}$$

Finally, this result can be used to compute the parameter $\alpha$ since it is defined as:

$$\alpha = \frac{1}{\tau} = 1.9985 \tag{12}$$

Being $\alpha << \omega_0$ the system is considered to be **lightly damped** and therefore $\omega_d \approx \omega_0$.

## c) Quality factor Q

The Q factor (or merit factor) is a qualitative/quantitative measure that describes how narrow (in terms of frequency range) and how high (in terms of amplitude) the resonance peak is. It is an adimensional parameter which can be computed using the following expression which relates the central frequency $\omega_0$ of the system with its bandwidth which is represented by $2\alpha$:

$$Q = \frac{\omega_0}{2\alpha} \tag{13}$$

Substituting $\alpha$ and $\omega_0$ in the above equation, the following result can be obtained:

$$Q \approx 126 \tag{14}$$

Higher is Q, narrower is the resonance peak; lower is Q, larger is the resonance peak (the energy is spread over a larger frequency range). This feature can be observed both in the system's admittance module (reported in figure 3) and in the system's receptance module.

## d) Resistance R associated to the system

The resistance R represents the level of damping or resistance to motion in the system. In the case of a forced damped resonator, the parameter R also corresponds to the real part of the impedance $Z(\omega)$. It can be computed starting from $\tau$:

$$\tau = \frac{1}{\alpha} = \frac{2m}{R} \quad \Rightarrow \quad R = \frac{2m}{\tau} = 0.3997\ \frac{Ns}{m} \tag{15}$$

## e) -3 dB bandwidth

It can be demonstrated that the magnitude of the Receptance (defined by the ratio Displacement/Force in the frequency domain) decreases by 3dB with respect to its maximum value when $\omega = \omega_d \pm \alpha$. As a consequence, the -3dB bandwidth is given by:

$$2\alpha = 3.997\ Hz \tag{16}$$

## f) Admittance $Y(\omega)$

The Admittance $Y(\omega)$ is one of the direct (output/input) frequency response functions used to analyze the behaviour of a forced system in the frequency domain. In particular, it is defined as the ratio between velocity $\tilde{V}(\omega)$ and the input force $\tilde{F}(\omega)$ and it is the inverse function of the impedance $Z(\omega)$ which is given by:

$$Z(\omega) = \frac{\tilde{F}(\omega)}{\tilde{V}(\omega)} = \frac{\tilde{F}(\omega)}{j\omega\tilde{X}(\omega)} = R + jX = R + j(\omega m - K/\omega) \tag{17}$$

Therefore the admittance $Y(\omega)$ can be computed as follows:

$$Y(\omega) = \frac{1}{Z(\omega)} = \frac{\tilde{V}(\omega)}{\tilde{F}(\omega)} \tag{18}$$

As it can be seen, it is a complex function of $\omega$ and therefore the magnitude and phase plots can be computed. The results are shown in the figure below:

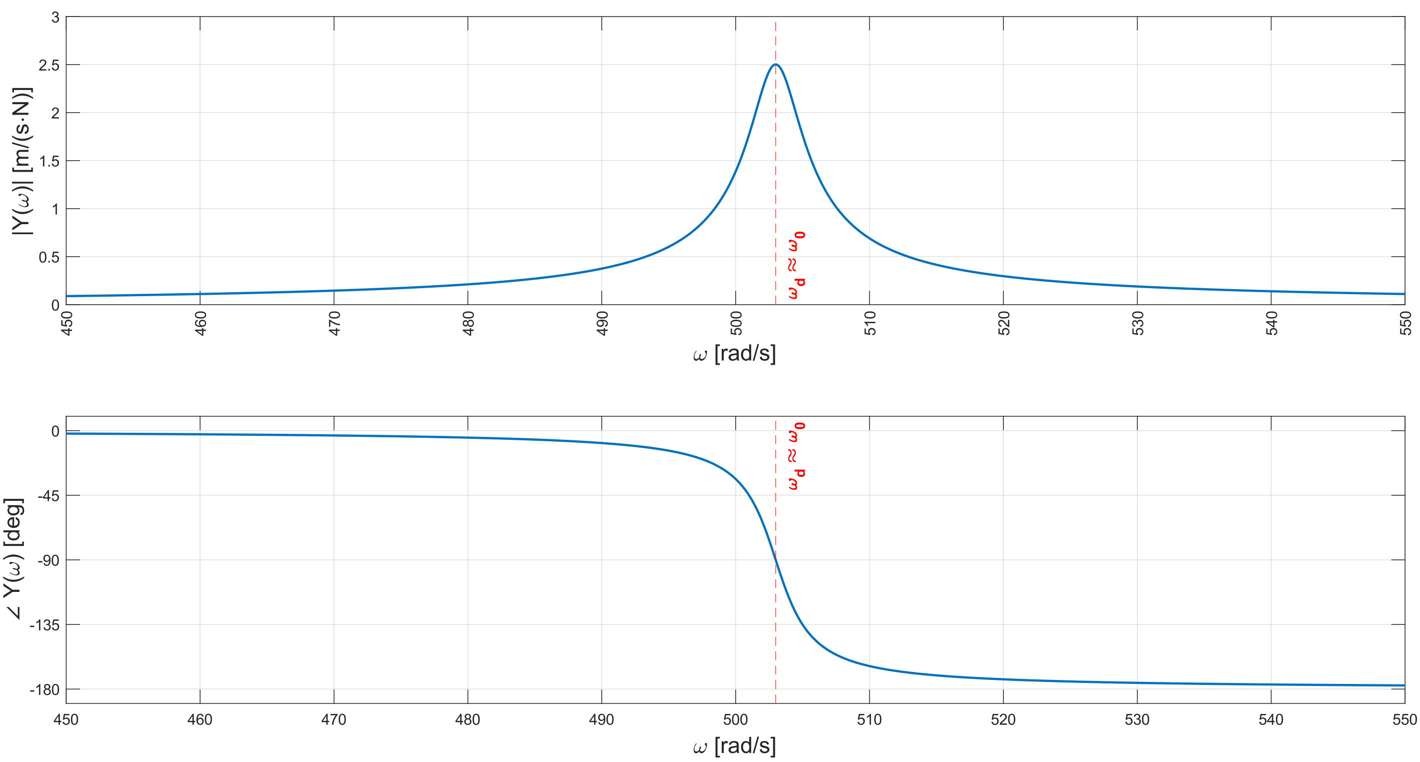


Figure 3: Magnitude and phase of the admittance $Y(\omega)$.

# g) Time response to an harmonic excitation with varying frequency

The complete time response of the system to an harmonic excitation $F(t) = F_0 sin(2\pi f_I)$ with varying frequency $f_I$ is given by the sum of the free response and the steady-state one:

$$x(t) = x_h(t) + x_p(t) \tag{19}$$

where $x_h(t)$ is in the form of equation (9) and the steady-state response $x_p(t)$ is given by:

$$x_p(t) = \frac{F_0}{\Omega|Z(\Omega)|} sin(\Omega t + \angle Z(\Omega)) \tag{20}$$

Therefore the complete time response results as:

$$x(t) = e^{-\alpha t}[Acos(\omega_d t + \phi)] + \frac{F_0}{\Omega|Z(\Omega)|} sin(\Omega t + \angle Z(\Omega)) \tag{21}$$

where $\Omega = 2\pi f_I$ and $f_I = [60, 80, 100, 120, 140, 160]\ Hz$.

In order to fully characterize the response, the constants A and $\phi$ must be determined starting from initial conditions. Being the system at rest before the force is applied in $t = 0$, the following initial conditions must be imposed:

$$\begin{cases} x(t = 0^-) = 0\ [m] \\ \dot{x}(t = 0^-) = 0\ [\frac{m}{s}] \end{cases} \tag{22}$$

$$\begin{cases} x(t = 0^-) = 0 = Acos(\phi) + \frac{F_0}{\Omega|Z(\Omega)|} sin(\angle Z(\Omega)) \\ \dot{x}(t = 0^-) = 0 = -\alpha Acos(\phi) - \omega_d Asin(\phi) + \frac{F_0}{|Z(\Omega)|} cos(\angle Z(\Omega)) \end{cases} \tag{23}$$

Solving the previous system of equations for $A$ and $\phi$ allows to compute the complete time responses reported in figures (4) and (5). The first graph highlights how the resonance frequency impacts the response of the system. In particular, when the system is excited at the resonance frequency ($f_0 \approx 80Hz$), it will respond with a magnitude much higher than in the case of other excitation frequencies, accordingly with the estimated $Q$ factor and admittance (figure 3). Furthermore the same figure underlines how the system requires many oscillations before reaching a steady-state condition. Figure (5), on the other hand, shows the effect of the so-called **transient** which is a particular behaviour caused by the discontinuity represented by the application of the input force. It can be noticed that the transient is characterized by the presence of two frequency components: the natural frequency and the frequency of the input force. The first contribution will die out as soon as the steady state condition is reached and only the frequency component of the forcing term will remain (the speed with which this phenomenon happens depends on the damping of the system). It is also noticeable that beating effects are more evident for frequencies closer to the natural one ($f_1 = 60\ Hz$, $f_3 = 100\ Hz$) while the oscillation builds up to the steady-state solution without any beat when $f \approx f_0$.

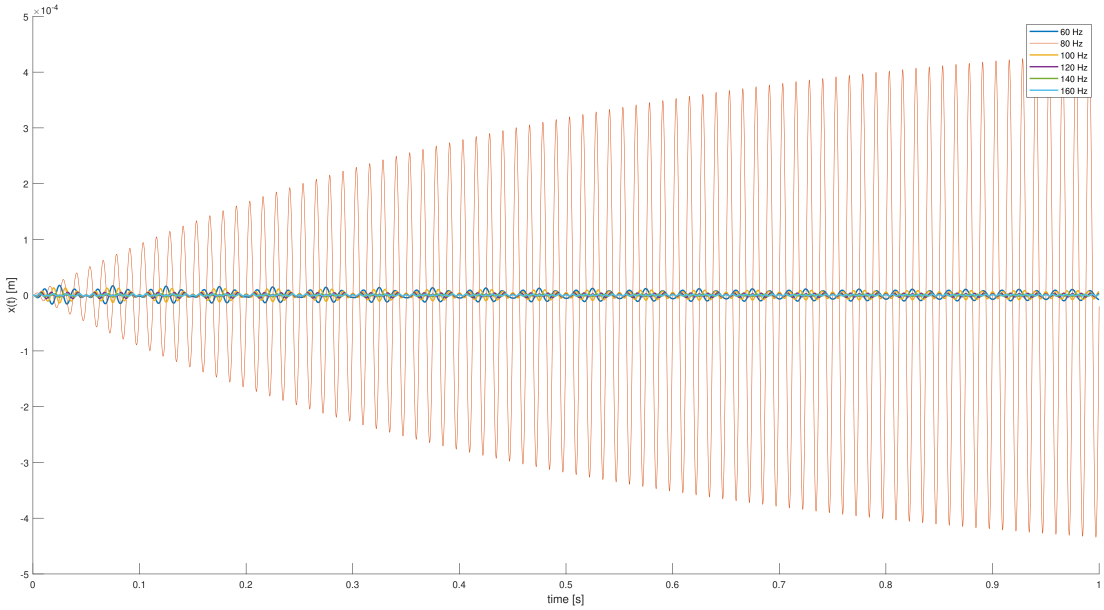


Figure 4: Complete time response for different excitation force frequencies.

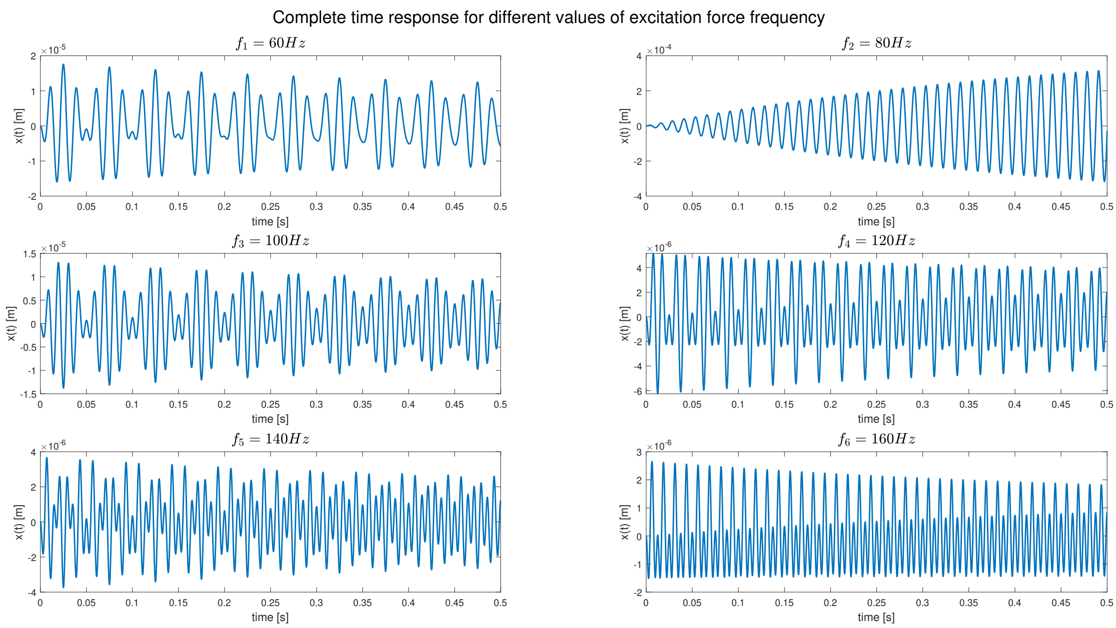


Figure 5: Complete time response with different amplitude scales.