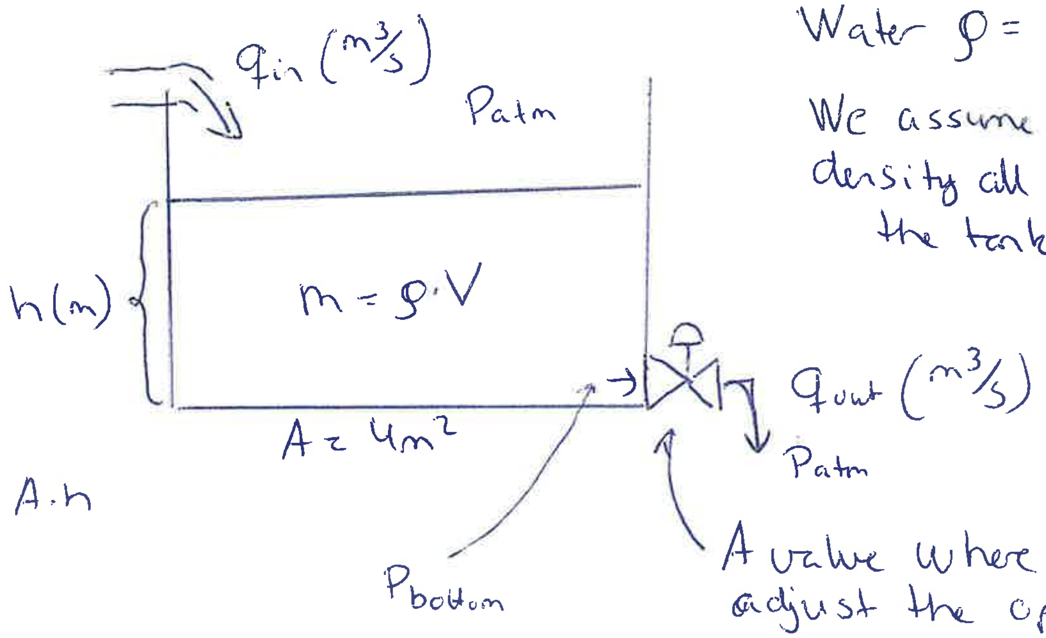
# Water Tank Model




Water $\rho = 1000\ kg/m^3$

We assume constant density all places in the tank.

$t$ – s
$m \sim kg$
$V \sim m^3$
$P \sim$ pascal
(SI units)

$V = A \cdot h$

A valve where we can adjust the opening from ($z = 0$ to $z = 1$)
closed — fully open.

Note that the flowrate ~~in~~ across the valve will depend on the differential pressure across the valve $P_V = P_{bottom} - P_{ATM}$

if $q_{in} > q_{out} \rightarrow h$ will increase
if $q_{in} < q_{out} \rightarrow h$ will decrease
if $q_{in} = q_{out} \rightarrow h$ will be constant.

In order to derive a model for how the fluid level $h$ will change in time, we will use mass conservation:

$$\boxed{\text{Change in mass in tank (kg/s)} = \text{inflow (kg/s)} - \text{outflow (kg/s)}}$$

massrate (kg/s) = density ($kg/m^3$) × flowrate ($m^3/s$)
$\rho$ — $q$

Initial mass of water in tank.

$$\rightarrow \frac{dm}{dt} = \rho q_{in} - \rho q_{out}, \quad m(0) = m_0$$

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$$\frac{dm}{dt} = \rho q_{in} - \rho q_{out} \quad , \; m(0) = m_0$$

Note $m = \rho \cdot V = \rho \cdot A \cdot h$ where $A$ is the area of the bottom of the tank.

$$\frac{dm}{dt} = \frac{d(\rho A \cdot h)}{dt} = \rho \cdot A \frac{dh}{dt}$$

$$\rho A \frac{dh}{dt} = \rho q_{in} - \rho q_{out} \quad , \; h(0) = h_0$$

We can delete $\rho$ and also divide by $A$.

$$\frac{dh}{dt} = \frac{1}{A}(q_{in} - q_{out}) \, , \; h(0) = h_0$$

We will now model the flow through the valve in more detail.

We will use a valve equation that links the volumetric rate across the valve with the differential pressure across the valve, valve opening $z \in [0,1]$ and other charactcteristics.

The equation reads:

$$q_{out} = K_v(z)\sqrt{\frac{P_v}{G}}$$

$P_v$ is differential pressure across valve

$G$ is fluid density relative to water

In this case: $\frac{1000\,kg/m^3}{1000\,kg/m^3} = 1$

We will assume a so called linear valve characteristic $K_v(z) = kz$ where $k$ is a constant and $z$ is the valve opening $z \in [0,1]$

($k = 0{,}002$ in the project)

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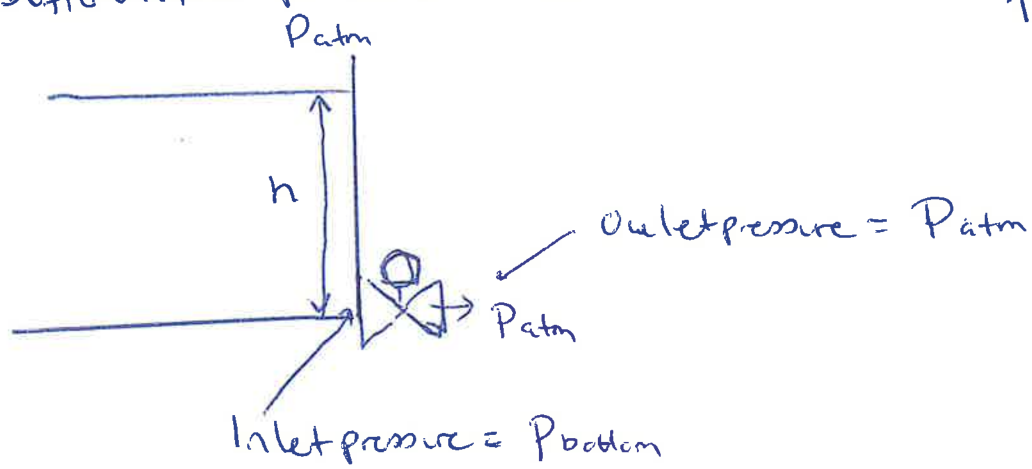
Differential pressure across the valve:

$P_{atm} = 101325\ Pa$
$1\ bar = 100000\ Pa$
$P_{atm} \approx 1\ bar$




$$P_{bottom} = P_{atm} + \rho g h$$
$$g = 9.81\ m/s^2$$

Differential pressure = Inletpressure − Outletpressure

$$P_v = \cancel{P_{atm}} + \rho g h - \cancel{P_{atm}} = \rho g h.$$

Note: The differential pressure will increase with increasing h and decrease with h reduced.
Hence the flow out will depend on the water level h.

The valve equation now becomes:

$$q_{out} = K_v(z) \cdot \sqrt{\frac{P_v}{G}} = kz\sqrt{\rho g h} = k\sqrt{\rho} \cdot z \cdot \sqrt{g \cdot h}$$
$$= k_{eff} \cdot z \cdot \sqrt{g \cdot h} \quad \text{where } k_{eff} = k \cdot \sqrt{\rho}$$

The final differential equation becomes:

$$\frac{dh}{dt} = \frac{1}{A}\left(q_{in} - k_{eff} \cdot z\sqrt{g \cdot h}\right), \quad h(0) = h_0$$

(first order nonlinear)

Note that both $q_{in}$ and $z$ can be changed as function of time.

rate in

Valve opening

# Water Tank Model

$$\frac{dh}{dt} = \frac{1}{A}\left(q_{in} - k_{eff} \cdot z\sqrt{gh}\right), \quad h(0) = h_0$$

Define right hand side as $f(h, q_{in}, z) = \frac{1}{A}\left(q_{in} - k_{eff} \cdot z\sqrt{gh}\right)$

We will now use the Euler method to simulate the water tank forward in time. Use time step $\Delta t = 1\,\text{sec}$.

$$h_{new} = h_{old} + \Delta t \times f(h_{old}, q_{in,old}, z_{old}) \quad \text{or}$$

$$h_{k+1} = h_k + \Delta t \times f(h_k, q_{in,k}, z_k)$$

Tank data:

$A = 4\,m^2$
$h_0 = 2\,m \rightarrow$ fluid level at $t = 0$
$\rho = 1000\,kg/m^3$
$k = 0.002$
$g = 9.81\,m/s^2$
$q_{in} = 0.03333\,m^3/s$
$z = 0.12$ (12% open)

Download: Mod300Vanntankbasisoppgave.ipynb

Study how the numerical solution for the water tank has been implemented. ~~This code can be modified to perform the tasks in the project.~~

# Control Engineering & PI Controller

A PID (proportional, integral and derivative) controller can be used to regulate different systems/processes where one wants to control a certain variable to stay at a fixed level.

Examples:
1. frequency of electric grid when having instable resources like solar and wind
2. Control pressure down in a well
3. Temperature regulation in a process

Here we want to control the fluid level in the tank. The level we want to keep is called the setpoint $h_{setpoint}$. The process variable that we want to bring to the setpoint is $h$.

The process disturbance variable that will affect the level in the tank is flowrate in $q_{in}$

The control variable that is used to bring back $h$ to the setpoint is in this case the valve opening $z$

Example:
1) Assume $h = h_{setpoint}$ (where it should be)
2) $q_{in}$ is increased
3) This leads to increase in $h$
4) Valve opening $z$ must be increased to increase $q_{out}$
5) $h$ is brought back to setpoint.

Have a look at the figure on page 6 in the pdf note about WaterTank and PI controller

# Water Tank & PI controller for regulating the fluid level.



→ Note that the setpoint can also be changed during the process (not only $q_{in}$)

→ Note that PI controllers are often used on real measured process data. But here the water tank simulator replaces the real process. I.e the simulated $h(t)$ becomes our process variable that we want to bring to the setpoint.

Define the error (deviation) at time level $t_{k+1}$ between simulated fluid level $h_{k+1}$ and the setpoint $h_{setpoint}$:

$$e(t_{k+1}) = h_{k+1} - h_{setpoint}$$

A time discrete PI controller for the new valve opening $Z$ at time $t_{k+1}$ is given by the following formula:

$$\underbrace{Z(t_{k+1})}_{\text{new opening}} = \underbrace{Z(t_k)}_{\text{old opening}} + K_p\big(e(t_{k+1}) - e(t_k)\big) + K_i\, e(t_{k+1})\cdot \Delta t$$

$K_p$ ~ proportional gain parameter ($K_p = 3.0$ / given in this case)
$K_i$ ~ integral gain parameter ($K_i = 0.3$ / given in this case)

We choose $\Delta t = 1\,s$, same as for the Euler method in this case.

Note that $K_p$, $K_i$ are parameters that must be tuned and depends on what kind of process one wants to control. There are/exist different algorithms for this but we dont go into depth of this.