MOD300-2024

Project – Modelling with Ordinary Differential Equations

(Fårenesvannet – 2008)

Sustainable harvesting??

Introduction

The aim of the project is to get experience with modelling using a system described by an ordinary differential equation. We will only consider initial value problems where the starting point is known.

The project will focus on all aspects of the modelling process i.e. 1) How to formulate the equations, 2) How to numerically solve the equations 3) How to apply simulation models and discuss the results.

The project is closely linked to what will be taught in the lectures.

The focus is on studying a model for population dynamics. This type of model can be used for studying the future growth and decline of a given species. The subject here will be a fish population.

The numerical methods that will be used for solving the ordinary differential equation is the Euler method and the fourth order Runge-Kutta method. The description of these methods is e.g., given in [1] page 246-263. This is the same book as used in the course Numerisk Modellering 1 (MAF 310) which is a course that you most likely have taken. See also Chapter 7 in [2].

Note that we will go through code examples using the methods in class.

What to deliver?:

The end results should be one report addressing the different questions posed. Follow the format shown in the example project (Covid 19) for how to write the report (abstract, introduction, different exercises, conclusion and discussion, self-reflections, references).

Here one should remark:

- Full names of all group members must be in the final report. It is sufficient that only one of the members delivers the report on CANVAS.
- The report must be a Jupyter notebook where text and codes are integrated (use code cells and markdown cells)
- For tasks where one must perform analytical calculations, perform calculations on paper, scan it and include it in the notebook. (e.g. edit – insert image in a markdown cell). But note that it is possible to make formulas in Jupyter notebook.
- Deliver both notebook and pdf of the final report.

Modelling of Population Dynamics

In the following, we will consider a population model that takes into account that the growth rate actually depends on the size of the population. If the population is small, it will increase while if it is too large, it will decrease. This is often referred to as the logistic growth model [3]. Let t be time and N be a measurement of the population in terms of numbers or biomass. The constant r is the intrinsic growth rate while a >0 is a constant related to the decline in population. The model reads:

$$\frac{dN}{dt} = (r - aN)N, \quad N(0) = N_0$$

This is often rewritten as:

$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N,\ N(0) = N_0$ where K = r/a. K has same unit as N.

As shown in [3], it is in this case possible to derive an exact solution to this nonlinear ordinary differential equation by using the method of separation of variables.

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rt}}$$

We will consider a fish population in a lake. N will represent the biomass in kg and we will consider initial values for N_0 in the range 0 – 20000 kg. The time unit will be months. K = 10000 kg. The intrinsic growth rate is 0.6 per year or with the chosen time unit: 0.6/12 per month. One can start with a timeframe of 15 years or more for studying how the population evolves in time.

Exercise 1.0 – Literature Review

Start out by performing a literature review on models for population dynamics. Provide examples of models and areas where they have been applied. Remember to include references.

Exercise 1.1 - How is decline in population incorporated in the model?

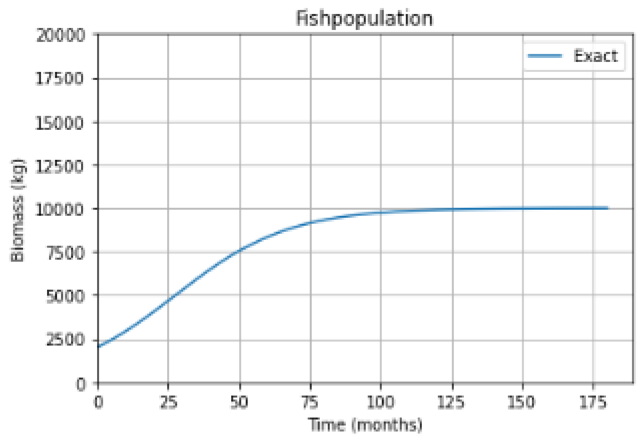
What is the purpose of trying to model the decline in population as $-aN$ in the logistic growth model?

Exercise 1.2 – Exact Solution vs Numerical Approximations.

In the figure below, a code is shown that uses the exact solution to simulate forward in time how the population will evolve. Use this code framework as basis and implement the Euler method and the fourth order Runge Kutta method. It can be beneficial to use functions here.

Then set the initial biomass N_0 to 20000 kg. Vary the timestep using the values 18, 12, 6 and 1 months and compare the results that is provided by the two numerical methods and how they approach the exact solution. What can be observed and why? Discuss this in the report and illustrate this with appropriate plots. What is the recommended timestep for each method? The code you have developed must be included in the report notebook.

```python
import matplotlib.pyplot as plt
import numpy as np
import math

timestep = 1 #  Use month as timeunit.

starttime = 0
endtime = 12*15  # Number of months that shall be simulated

K = 10000    # K = 10000 kg
r = 0.6/12   # per month

N0  = 2000 # Initial  biomass in kg

T1=[]
YExact=[]

t = starttime
y = N0


YExact.append(y)
T1.append(t)


while (t<endtime):
    t = t+timestep


    y=N0*K/(N0+(K-N0)*math.exp(-r*t))  # Exact solution
    T1.append(t)
    YExact.append(y)


plt.plot(T1,YExact)
plt.title('Fishpopulation')
plt.xlabel('Time (months)')
plt.ylabel('Biomass (kg)')
plt.ylim(ymin=0,ymax =20000)
plt.xlim(xmin=0)
plt.grid(True)
plt.legend(['Exact'])
plt.show()
```



Exercise 1.3 - Play with the Model

Set the timestep to 1 month. Use the following values for N_0: 0 kg, 1 kg, 2000 kg, 10000 kg, 15000 kg. Simulate 25 years ahead. Discuss what you observe and illustrate this in the report with appropriate plots!

Exercise 1.4 - Analysis of Model

The differential equation was given as:

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N$$

Plot the right-hand side of this equation as function of N to study how dN/dt changes as function of N. Include this in the report. What happens when N = 0 and N = K? What does these points represent? Discuss this also in relation to what you saw from the previous simulations.

For which N value will dN/dt have its largest value and what will this value be? (Show this analytically and check expressions with numbers and the graph). Also identify from the previous time simulations where we can see the effect of this and what does this represent?

Exercise 1.5 – Harvesting the Fish Population

We will now introduce a term H in our original model to study the effect when we take out a certain number of fish every month.

$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N - H,\ N(0) = N_0$ where K = r/a. K has same unit as N. H has unit kg/month.

Implement this in your previous simulation framework where you use the Euler method or the fourth order Runge-Kutta method with an appropriate timestep. Test the following values: H = 50, 100, 120, 150 kg/month. Use both $N_0 = 5000\ kg, N_0 = 15000\ kg.$

Present and discuss what you observe. Choose appropriate plots for illustration. Also remember to include this code in the report notebook.

Exercise 1.6 –Determining the Maximum Sustainable Yield (MSY)

There is a critical limit for how much we can harvest before the population collapses independent on how large it was initially.

Show that this critical limit which we name Maximum Sustainable Yield is defined by $H_{MSY} = \frac{rK}{4} = 125\frac{kg}{month}$.

Exercise 1.7 – Equilibrium Points for the Case where H < rK/4 & Sustainable Harvesting

Show analytically that there are two equilibrium points for the case where we harvest at a rate H that is lower than the maximum sustainable yield. Determine the stability of these. Also evaluate and visualize how the value of these equilibrium points vary depending on the value of H.

Based on this, how can we say that the size of the initial population is also important for determining if a population will survive or collapse? This you can also demonstrate by appropriate simulations and plots using e.g. H = 100 kg/month.

Exercise 1.8 – Harvesting at Maximum Sustainable Yield H_{MSY}

Assume that we harvest at maximum sustainable yield. Use simulations to investigate under which conditions it is possible to avoid a collapse of the population. What can we say about equilibrium point/points of the system in this case? Use appropriate plots for illustration.

Do you think it is a good idea in practice to harvest at this rate? Give some reasons for your answers.

Exercise 1.9 - Overfishing and Collapse

If the harvest rate is larger than H_{MSY}, we have a situation called overfishing. Try H = 132 kg/month with N_0 = 15000 kg. Try to explain how the decrease in population evolves in time and relate this to the change in DN/Dt. Why can it be difficult to detect that we are in an overfishing situation if you for instance compare this to a situation where H = 125 kg/month? If we still use H = 132 kg/month, will there be other values for N_0 that will give a different type of decrease in the population.

Finally, try to answer why it is very important to act fast when we approach the collapse of the population. Do we have examples where a very important fish population has collapsed and what was the experiences there?

Remember to use plots from simulations to support your discussions.

Exercise 1.10 – Management and Intervention

There are signs of reduction in the fish population and one suspect that overfishing is taking place. But one does not want to close the fishery completely since it will have large social and economic consequences.

We assume H_{MSY} = 125 kg/month. Use simulations in combination with mathematical analysis of the model and other evaluations to give recommendation for how much one should reduce the catch below H_{MSY} for various conditions. Also evaluate how long time it will take for the population to recover again for different scenarios.

Remember to use plots from simulations to support your discussions.

References

1. Kiusalaas, J. 2013. Numerical Methods in Engineering with Python 3. https://bibsys-ur.alma.exlibrisgroup.com/leganto/readinglist/citation/6166275760002208?auth=SAML
2. Hiorth, A. 2022. Modeling and Computational Engineering
3. Boyce,W.E. and DiPrima, R.C. 1986. Elementary Differential Equations and Boundary Value Problems. Fourth Edition. John Wiley & Sons