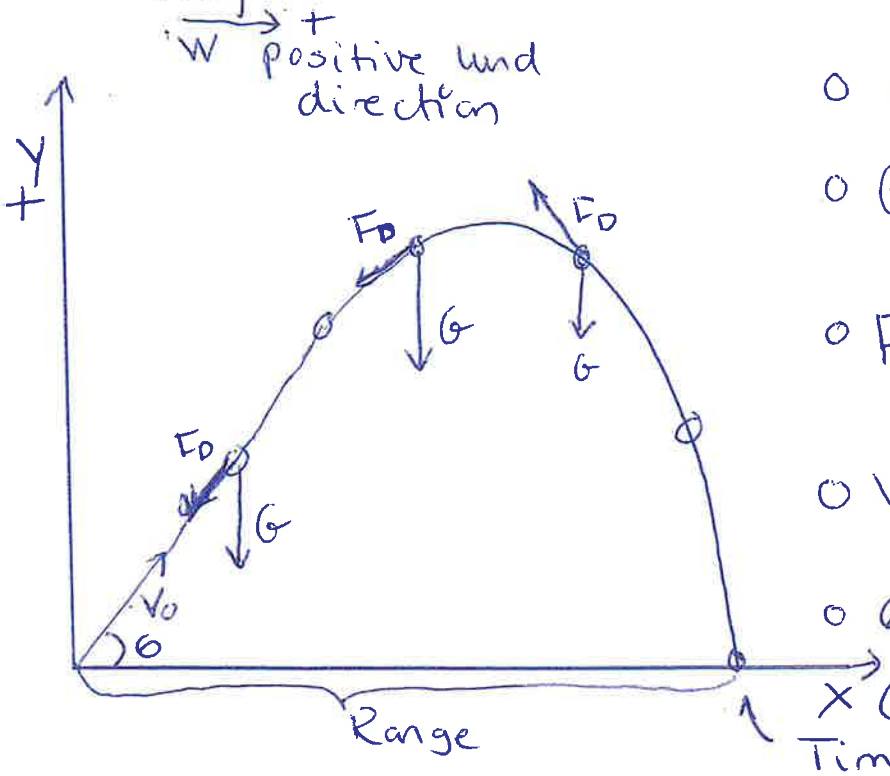
Projectile Motion in 2 Dimensions



- projectile with mass m (kg)
- $G = mg$ (N) (gravitational force)
- F_D – Drag force acting on projectile (N)
- V_0 – Initial velocity (m/s)
- θ – angle with horizontal
- W – wind speed (m

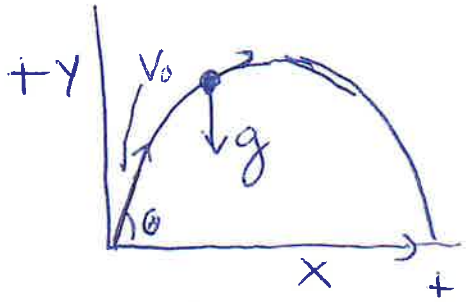
Time of flight is when object hits ground

In the physics course first year, you studied this kind of motion but where one neglected the drag force (air resistance) F_D and ~~eventual~~ effect of wind.

We will first repeat the equations describing the projectile motion when we neglect drag and wind.

We can look upon V_0, θ as input parameters and the range and time of flight as output parameters/results

2 D projectile motion with no drag and Wind



We apply N2 law and set up the equations for X and y direction separately

(Only gravity works)

$$m \cdot a_x = \sum F_x = 0 \qquad m a_y = \sum F_y = -mg$$

$$\cancel{m} \cdot \ddot{x} = 0 \qquad / \ \cancel{m} \cdot \ddot{y} = -\cancel{m} g$$

$\ddot{x} = a_x$
$\dot{x} = V_x$
$\ddot{y} = a_y$
$\dot{y} = V_y$

1) $\ddot{x} = 0$
2) $\ddot{y} = -g$

} Two second order differential equations. We need two initial conditions for each.

We know that $x(t=0) = 0$ & $y(t=0) = 0$

$V_{0x} = V_0 \cos\theta$, $V_{0y} = V_0 \cdot \sin\theta$ which gives us:
$\dot{x}(t=0) = V_0 \cos\theta$, $\dot{y}(t=0) = V_0 \cdot \sin\theta$.

1) $\ddot{x} = 0$ & $x(0) = 0$ + $\dot{x}(0) = V_0 \cdot \cos\theta$

Use initial condit. constant in x direction

$$\int \ddot{x} = \int 0 \rightarrow \dot{x} = C_1 \rightarrow \dot{x} = V_0 \cos\theta \quad (V_x = V_0 \cos\theta)$$

Integrate again $\int \dot{x} = \int V_0 \cos\theta \rightarrow x = V_0 \cos\theta \cdot t + C_2$

Use $x(0) = 0 \rightarrow C_2 = 0$

2) $\ddot{y} = -g$ & $y(0) = 0$, $\dot{y}(0) = V_0 \sin\theta$

Use initial condition

$$\int \ddot{y} = \int g \rightarrow \dot{y} = -gt + C_1, \quad \dot{y}(0) = V_0 \sin\theta \rightarrow \dot{y} = -gt + V_0 \sin\theta$$

$C_1 = V_0 \sin\theta$

$\dot{y} = V_0 \sin\theta - gt$

$V_y = V_0 \sin\theta - gt$

Integrate again

$$\int \dot{y} = \int V_0 \sin\theta - gt$$

$$y = V_0 \sin\theta \cdot t - \frac{1}{2} g t^2 + C_2$$, Use $y(0) = 0$ to determine C_2

$$y(0) = 0 - 0 + C_2 = 0 \Rightarrow C_2 = 0$$

2D projectile motion with no drag

1 $V_x = \dot{X} = V_0 \cos\theta$

2 $V_y = \dot{y} = V_0 \sin\theta - gt$

3 $X(t) = V_0 \cos\theta \cdot t$

4 $y(t) = V_0 \sin\theta \cdot t - \frac{1}{2} g t^2$

$\Big\}$ position as function of t

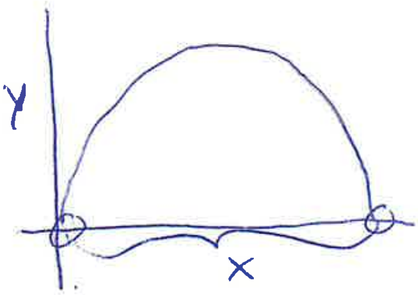
From 3) $t = \dfrac{x}{V_0 \cos\theta}$ → insert this in 4)

position as function of x

$$y(x) = \frac{V_0 \sin\theta}{V_0 \cos\theta} \cdot x - \frac{g}{2 V_0^2 \cos^2\theta} \cdot x^2$$

$$y(x) = \tan\theta \cdot x - \frac{g}{2 V_0^2 \cos^2\theta} \cdot x^2$$

Second degree polynoma
Parabolic form of projectile movement →



To find the Range, one can find the root of $y(x)$ by setting $y(x) = 0$

It can be shown that Range ~~si~~ is given by

$$x = \frac{2V_0^2 \tan\theta (\cos)^2}{g} = \frac{V_0^2 \sin(2\theta)}{g}$$

Time of flight can then be found using 3)

$$X = V_0 \cos\theta \cdot t$$

$$t = \frac{\text{Range}}{V_0 \cdot \cos\theta}$$

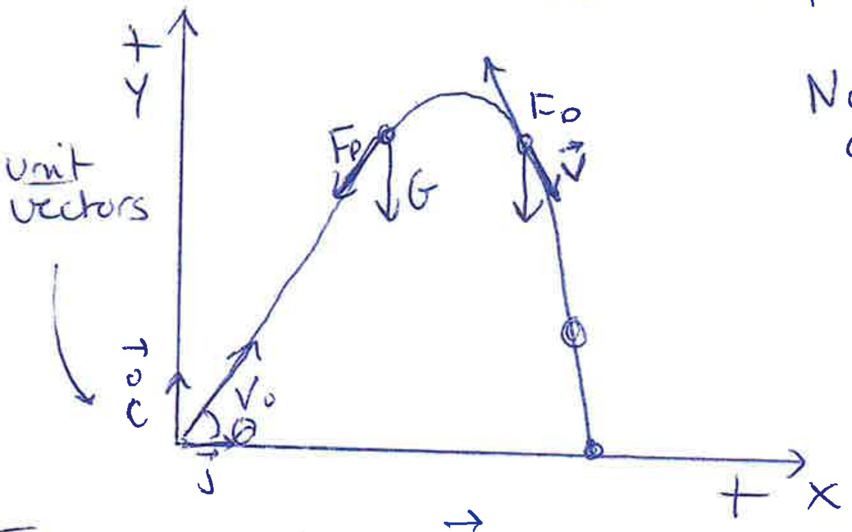
Note that the mass of the projectile has nothing to say here.

The case with no drag could be solved analytically.

The shape of the curve is parabolic.

2D projectile movement with drag and wind

$\rightarrow$ + positive direction for wind w



Note $\vec{F}_D$ works in the opposite direction as $\vec{v}$

The drag force $\vec{F}_D$ can have different modelling forms. Here we assume the drag force is proportional to velocity squared. $|\vec{F}_D| = C_D V^2$. The drag coefficient is a number that must be given.

Note that the force $\vec{F}_D$ is a vector that changes direction

The outline of the model will again be N2 law in vector form:

$$m\vec{a} = \vec{F}_D - m\vec{g} \qquad \vec{a} = a_x \vec{i} + a_y \vec{j} = \ddot{x}\vec{i} + \ddot{y}\vec{j}$$

Again we look upon forces acting i x and y direction separately.

+ We have the same initial conditions

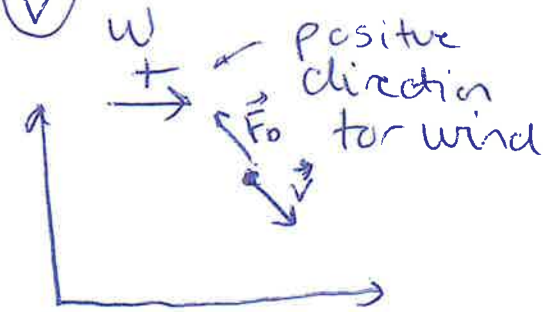
$$x(0) = 0 \quad \dot{x}(0) = V_0 \cos\theta$$
$$y(0) = 0 \quad \dot{y}(0) = V_0 \sin\theta$$

2D projectile movement with drag and wind

We define $\vec{i}$ and $\vec{j}$ as unit vectors along x and y axis

If there is no wind $\vec{F}_0 = -C_0 V^2 \left(\frac{\vec{V}}{V}\right)$ unit vector defining the direction

$\vec{V} = \dot{x}\vec{i} + \dot{y}\vec{j}$ $\quad V = \sqrt{(\dot{x})^2 + (\dot{y})^2}$



But if we have wind w

$\vec{V} = (\dot{x} - w)\vec{i} + \dot{y}\vec{j}$ where $V = \sqrt{(\dot{x}-w)^2 + (\dot{y})^2}$

Note there must be a minus here if positive direction is chosen along the positive x axis.

Tail wind (e.g. +10 m/s) will reduce the relative velocity
Medwind
in x direction between the projectile and the air

$$\vec{F}_0 = -C_0 V^2 \cdot \frac{\vec{V}}{V} = -C_0 V \cdot \vec{V} = -C_0 V\left((\dot{x}-w)\vec{i} + \dot{y}\vec{j}\right)$$

$$m\vec{a} = \vec{F}_0 - m\vec{g}$$

$$m(\ddot{x}\vec{i} + \ddot{y}\vec{j}) = -C_0 V\left((\dot{x}-w)\vec{i} + \dot{y}\vec{j}\right) - mg\vec{j}$$

$$m\ddot{x} = -C_0 V(\dot{x}-w) \rightarrow \ddot{x} = -\frac{C_0 V}{m}(\dot{x}-w) \qquad x(0) = 0,\ \dot{x}(0) = V_0 \cos$$

$$m\ddot{y} = -C_0 V \cdot \dot{y} - mg \rightarrow \ddot{y} = -\frac{C_0 V}{m}\dot{y} - g \qquad y(0) = 0,\ \dot{y}(0) = V_0 \sin$$

$$V = \sqrt{(\dot{x}-w)^2 + \dot{y}^2}$$

We have two coupled second order differential equation.

Exercise:
Rewrite these to a set of 4 first order differential equations

Solution

$y_0 = y,\ y_1 = \dot{y},\ y_2 = x,\ y_3 = \dot{x}$

$$\dot{y}_0 = y_1$$

$$\dot{y}_1 = \ddot{y} = -\frac{C_0 V}{m}\dot{y} - g = -\frac{C_0 \sqrt{(\dot{x}-w)^2+\dot{y}^2}}{m}\cdot\dot{y} - g$$

$$= -\frac{C_0}{m}\cdot\sqrt{(y_3-w)^2+y_1^2}\cdot y_1 - g$$

$$\dot{y}_2 = \dot{x} = y_3$$

$$\dot{y}_3 = \ddot{x} = -\frac{C_0 V}{m}(\dot{x}-w) = -\frac{C_0}{m}\sqrt{(\dot{x}-w)^2+\dot{y}^2}\cdot(\dot{x}-w)$$

$$= -\frac{C_0}{m}\cdot\sqrt{(y_3-w)^2+y_1^2}\cdot(y_3-w)$$

$$\downarrow$$

$$\dot{y}_0 = y_1 \qquad y_0(0) = 0$$

$$\dot{y}_1 = -\frac{C_0}{m}\cdot y_1\cdot\sqrt{y_1^2+(y_3-w)^2} - g \qquad y_1(0) = V_0\cdot\sin\theta$$

$$\dot{y}_2 = y_3 \qquad y_2(0) = 0$$

$$\dot{y}_3 = -\frac{C_0}{m}(y_3-w)\cdot\sqrt{y_1^2+(y_3-w)^2} \qquad y_3(0) = V_0\cos\theta$$

$$\dot{\bar{y}} = \vec{F}(x,\bar{y}) \qquad \bar{y}(0) = \bar{\alpha}$$

$$\dot{\bar{y}} = \begin{bmatrix} \dot{y}_0 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix},\quad \vec{F}(x,\bar{y}) = \begin{bmatrix} y_1 \\ -\frac{C_0}{m} y_1 \sqrt{y_1^2+(y_3-w)^2} - g \\ y_3 \\ -\frac{C_0}{m}(y_3-w)\cdot\sqrt{y_1^2+(y_3-w)^2} \end{bmatrix},\quad \bar{\alpha} = \begin{bmatrix} 0 \\ V_0\sin\theta \\ 0 \\ V_0\cos\theta \end{bmatrix}$$

Simulator for 2D projectile movement with drag and wind!

Download the simulator in the notebook

$\boxed{\text{Ballistic.ipynb}}$

We are now back to the flexible numerical framework that was provided in the book.

We will have a look at the following:

1) Where do we set simulation time and time step?
2) Where do we set initial speed V_0 and angle θ 

3) Where do we specify the initial conditions

Let $m = 2kg$, $C_D = 0.01$, $w = 0$

Important!

4) How do we specify the F function?
5) What does the function exact_solution do?
6) What does the function printSoln do?
7) What does the function findRange do?
8) What kind of plots are shown?

Let $m = 2\,kg$, $C_d = 0{,}01$, $W = 0$, $V_0 = 80\,m/s$, $\theta = 60$ degrees

1) Simulate and compare with the solution when we have no drag. What are the main differences in results?

Answer,

1) Shorter range, shorter time of flight
2) The height vs time is no longer a symmetric parabola
3) Velocity i x direction no longer constant
4) The velocity profiles V_x, V_y, $V = \sqrt{V_x^2 + V_y^2}$ are quite different.

What happens if you set $C_D = 0$?

What can this tell us?

2) Let $m = 2$, $C_d = 0.01$, $V_0 = 80\,m/s$, $\theta = 60$ degrees.

Compare zero wind with headwind $= -20\,m/s$ and tailwind $= +20$

motvind · medvind

What are the main differences?

$w = -20 \rightarrow$ time of flight 9.121, range $= 50.76$

$w = 0 \rightarrow$ time of flight $9.554\,s$, range $= 172.4$

$w = +20 \rightarrow$ time of flight $9.92\,s$, range $= 299.2$

~~Ra~~ Range is heavily affected while time of flight not.

~~If an~~

Compare height vs Range for $\pm 20\,m/s$ wind!

Explain what happens!

Later in the course when you work with the exercises related to regression & machine learning you will need results from these simulations.

Note that V_0, θ and eventually wind will be the input parameters

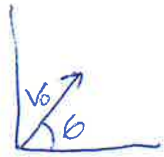
The output will typically be Range & Time of Flight

We will now look at a code that runs many simulations at the same time, the input is randomized and the result are written to a data file

Simulator for generating random data for 2D projectile motion with drag & wind:

Download the notebook: $\boxed{\text{RandomBallisticData.ipynb}}$

We will study the following:



1) How we introduce a loop to run the simulator several times:

2) How we open a file to write V_0, θ, TimeofFlight, Range for each simulation (+ close it)

3) How we randomize V_0 and θ for each simulation to produce a different result

4) Where do we specify the wind w now & how has this altered the functions?

5) Study the resulting file (ballistiskedata.txt)

Note that the plots etc will only show the last simulation. Could have been removed, but kept in case we only want to run 1 simulation.

Exercise: Try to randomize the wind and also include this when writing to a file. I.e include this third input parameter as the third column in the file ballistiskedata.txt

Note, the data file can then be used for training of an ANN (neural network) / machine learning.