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Rewriting a higher order differential equation as a system of first order differential equations:

An ordinary differential equation of order n is written as: $y^{(n)} = f(x, y, y', \quad y^{(n-1)})$

with some initial conditions: $y(a) = \alpha_0, y'(a) = \alpha_1 \quad , y^{(n-1)}(a) = \alpha_{n-1}$

$y'' = -0.1y' - x$, $y(0) = 0$, $y'(0) = 1$ (Second order, linear diff. equation)

$y^{(4)} = 4y'' \cdot \sqrt{1-y^2}$ (forth order, non linear diff equation)

We can rewrite a general nth order differential equation as a system of first order equations:

$$y^{(n)} = f(x, y, y', \quad y^{(n-1)}), \; y(a) = \alpha_0, y'(a) = \alpha_1 \quad y^{(n-1)}(a) = \alpha_{n-1}$$

We introduce some new variables:

$$Y_0 = y, \; Y_1 = y', \; Y_2 = y'' \quad , Y_{n-1} = y^{(n-1)}$$

Then the equivalent first order equations become:

$Y_0' = Y_1$ (from $Y_0' = y' = Y_1$)

$Y_1' = Y_2$ (from $Y_1' = y'' = Y_2$)

$Y_2' = Y_3$ etc

$\vdots$

$Y_{n-1}' = f(x, Y_0, Y_1 \quad , Y_{n-1})$

The initial conditions become:

$Y_0(a) = \alpha_0$

$Y_1(a) = \alpha_1$

$\vdots$

$Y_{(n-1)}(a) = \alpha_{n-1}$

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This can be written in more compact form:

$$\bar{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}, \quad \bar{F}(x,\bar{y}) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ f(x,\bar{y}) \end{bmatrix}, \quad \bar{y}(a) = \bar{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$$

$$\boxed{\bar{y}' = \bar{F}(x,\bar{y}), \quad \bar{y}(a) = \bar{\alpha}}$$

Example 7.2

$$y'' = -0.1y' - x, \quad y(0) = 0, \quad y'(0) = 1$$

New variables: $y_0 = y$, $y_1 = y'$ $\rightarrow$ $y_0(0) = 0$, $y'(0) = y_1(0) = 1$

$\downarrow$

$$y_0' = y' = y_1 \quad \rightarrow \quad y_0' = y_1$$

$$y_1' = y'' = -0.1y' - x = -0.1y_1 - x \quad \rightarrow \quad y_1' = -0.1y_1 - x$$

$$\bar{y}' = \bar{F}(x,\bar{y}), \quad \bar{y}(a) = \bar{\alpha}$$

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}' = \begin{bmatrix} y_1 \\ -0.1y_1 - x \end{bmatrix}, \quad \bar{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We can then apply the Euler method or the fourth order Runge Kutta method directly on the system of first order equations. The code which is described in the book is made flexible to handle this

12/9-23 (3) → Exercise

Convert the following differential equations into first order equations of the form $\bar{y}' = F(x, \bar{y})$

1) $y''y - xy' - 2y^2 = 0$

2) $y^{(4)} - 4y'' \cdot \sqrt{1-y^2} = 0$

Solution 1) $y'' \cdot y - xy' - 2y^2 = 0$

$$y'' = \frac{xy' + 2y^2}{y}$$

New variables $y_0 = y$, $y_1 = y'$,

$$y_0' = y_1$$
$$y_1' = y'' = \frac{xy' + 2y^2}{y} = \frac{xy_1 + 2y_0^2}{y_0}$$

$\bar{y}' = \bar{F}(x, \bar{y})$

$$\bar{y}' = \begin{bmatrix} y_0' \\ y_1' \end{bmatrix}, \quad \bar{F}(x, \bar{y}) = \begin{bmatrix} y_1 \\ \frac{xy_1 + 2y_0^2}{y_0} \end{bmatrix}$$

Solution 2) $y^{(4)} - 4y'' \cdot \sqrt{1-y^2} = 0$

$$y^{(4)} = 4y'' \cdot \sqrt{1-y^2}$$

$y_0 = y$, $y_1 = y'$, $y_2 = y''$, $y_3 = y'''$

$$y_0' = y_1$$
$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = 4y'' \cdot \sqrt{1-y^2} = 4y_2 \cdot \sqrt{1-y_0^2}$$

$$\bar{y}' = \bar{F}(x, \bar{y}) \qquad \bar{y}' = \begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \end{bmatrix}, \quad \bar{F}(x, \bar{y}) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 4y_2\sqrt{1-y_0^2} \end{bmatrix}$$

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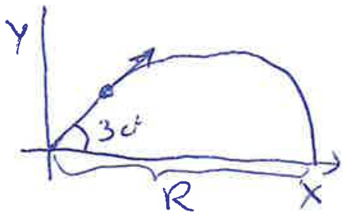
What if we have several higher order differential equations?

(Exercise 13 in book)



$\downarrow g$

$$\ddot{x} = -\frac{C_D}{m}\dot{x}V^{1/2}, \quad \ddot{y} = -\frac{C_D}{m}\dot{y}\cdot V^{1/2} + g$$

$$V = \sqrt{\dot{x}^2 + \dot{y}^2}$$

Note $\dot{x} = \frac{dx}{dt}$

(Newtons notation)

The idea is then to form a system of four first order differential equations. Each second order differential equation leads to two first order differential equations.

We will come back to this later.

We will now study the numerical framework provided in the book that can be used to solve systems of first order differential equations using either the Euler method or the fourth order Runge Kutta method.

Download the Jupyter Notebooks:

$\boxed{\text{FlexibleEulerMethod} \\ \text{FlexibleRungeKutta4}}$

We will now study how these are implemented.

The problem that is solved is:

$y'' = -0.1y' - x, \; y(0) = 1, \; y'(0) = 1$ which is

rewritten as $\bar{y}' = \bar{F}(x, \bar{y})$, $\bar{y}(0) = \bar{\alpha}$

$$\bar{y}' = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \; \bar{F}(x,\bar{y}) = \begin{bmatrix} y_1 \\ -0.1y_1 - x \end{bmatrix}, \; \bar{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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This problem has an exact solution

$$y = 100x - 5x^2 + 990(e^{-0.1x} - 1)$$

which we will compare against.

We will solve from $x = 0$ to $x = 2.0$

$h = 0.05$ for Euler method

$h = 0.2$ for 4th order Runge Kutta method

In book this is example 7.2 and 7.4