Logistic Growth Model

Modelling of population dynamics.

Let t be time and N the population at time t.

Note the unit for t can e.g. be days, months or years

The unit N can be number of individuals or biomass (

How to model:

The simplest hypothesis concerning the variation of N is that the rate of change of N is proportional to the current value of N

Initial condition

$$\frac{dN}{dt} = rN, \quad N(t=0) = N_0$$

The constant r is called the rate of growth or decline. This is the same type off differential equation as we saw for the radioactive decay example.

$$\frac{dN}{N} = r\,dt \quad | \text{Method of separation of variables}$$

$$\int \frac{dN}{N} = \int r\,dt$$

$$\ln N = rt + c$$

$$e^{\ln N} = e^{rt+c}$$

$$e^{\ln N} = e^{rt} \overbrace{e^{c}}^{C_1}$$

$$N(t) = C_1 e^{rt}$$

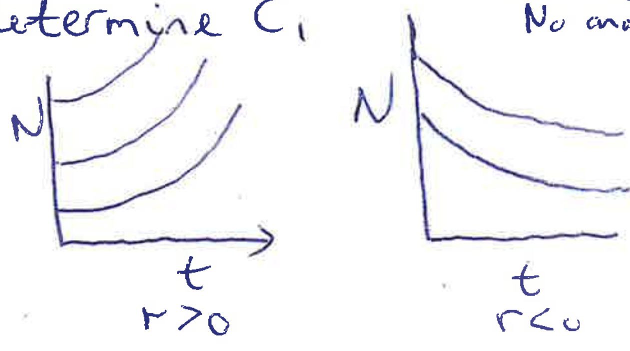
Then use $N(0) = N_0$ to determine C_1

$$N_0 = C_1 e^{r0} = C_1$$

$$N(t) = N_0 e^{rt}$$

Exponential growth or decay

Different solution curves depending on N_0 and r



However we know that a population can not grow exponentially forever.

E.g if N becomes to large there can be lack of food resources or diseases can spread more easily.

To take into account that the growth rate actually depends on the population, we replace the constant r by a function $f(N)$

$$\frac{dN}{dt} = f(N) \cdot N$$

We want to choose $f(N)$ such that:

1) $f(N) \approx r$ when N is small
2) $f(N)$ decreases as N grows lager
3) $f(N) < 0$ when N is sufficiently large (decline)

A simple function that fits these requirements are:

$$f(N) = r - aN$$

decline term

where $a > 0$

growth term

Using this we get

$$\frac{dN}{dt} = (r - aN)N, \quad N(0) = N_0$$

This is called the Verhulst equation or the logistic equation. It is a first order non linear differential equation.

This is often rewritten as

$$\frac{dN}{dt} = r\left(1 - \frac{a}{r}N\right)N$$

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N \quad \text{where } K = \frac{r}{a}$$

K has same unit as N

It is a parameter used when analysing the model.

For this model, it is actually possible to derive an exact solution.
Only the result is shown here:

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0) e^{-rt}}$$

Analysis of Model

Let us try to analyse the model by looking directly at the differential equation:

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N$$

← By plotting the right hand side we can study how the value of $\frac{dN}{dt}$ changes as N varies.
When $\frac{dN}{dt} > 0$, the population grows, and $\frac{dN}{dt} < 0$ corresponds to decline.

$$G(N) = r\left(1 - \frac{N}{K}\right)N$$

$G(N) = 0$ for $N = 0$ & $N = K$

zero points (nullpkt for grafen)

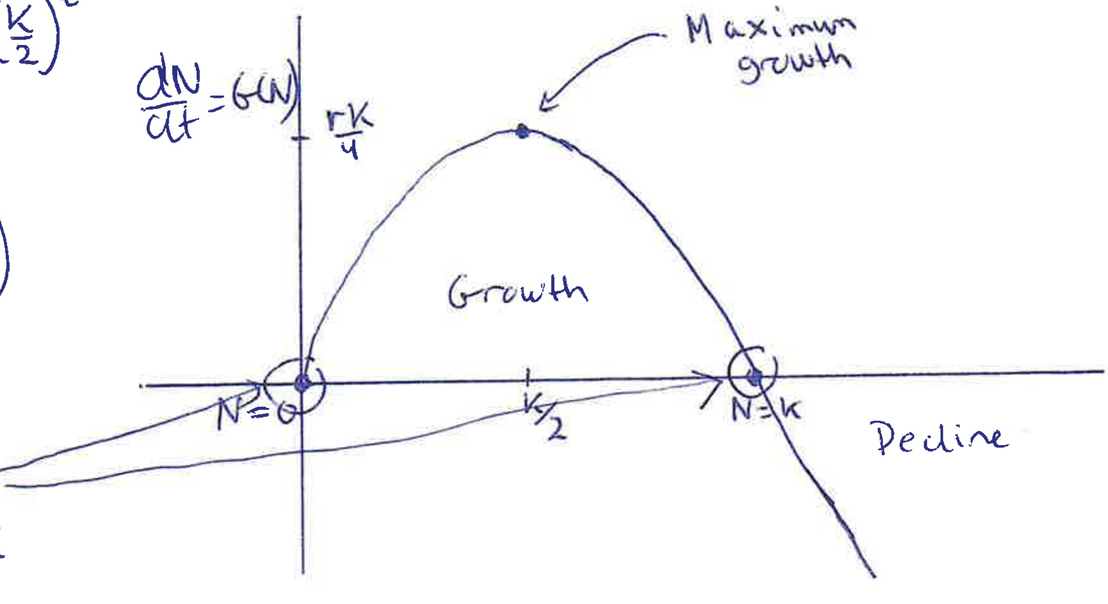
$G(N) = rN - r\frac{N^2}{K}$. G is a second order polynoma and we will use the derivative set to zero to pick out the maximum point.

$G'(N) = r - \frac{2rN}{K} = 0 \Rightarrow N = \frac{K}{2}$. Insert this in $G(N)$

$$G\left(\frac{K}{2}\right) = \frac{rK}{2} - \frac{r}{K}\left(\frac{K}{2}\right)^2$$

$$= \frac{rK}{2} - \frac{rK}{4} = \frac{rK}{4}$$

Max point: $\left(\frac{K}{2}, \frac{rK}{4}\right)$



These points are interesting, especially $N = K$

Critical Points/Equilibrium Points

$$\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)N \qquad N(t=0)=N_0$$

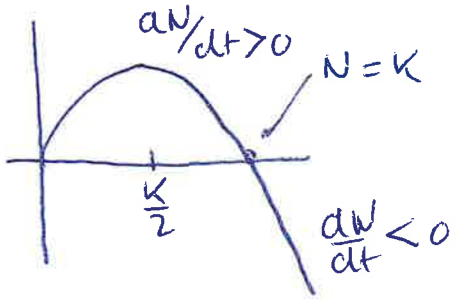
$\frac{dN}{dt}=0$ gives us the critical points

These occur for $N=0$ and $N=K$

These represent constant solutions

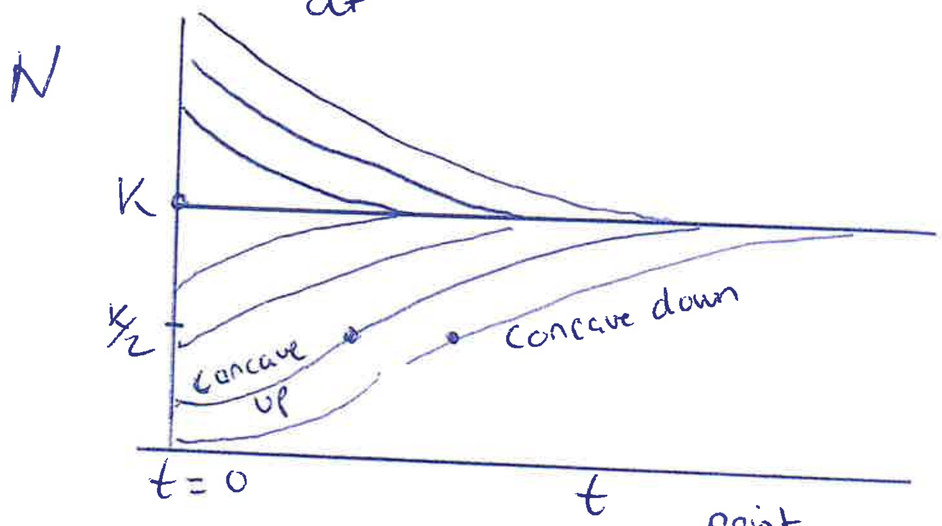
$N=0$ or $N=K$ will remain the same for all times.

Let us check $N=K$



if $N<K \rightarrow \frac{dN}{dt}>0 \Rightarrow N$ will grow towards $N=K$

if $N>K \rightarrow \frac{dN}{dt}<0 \Rightarrow N$ will decrease towards $N=K$



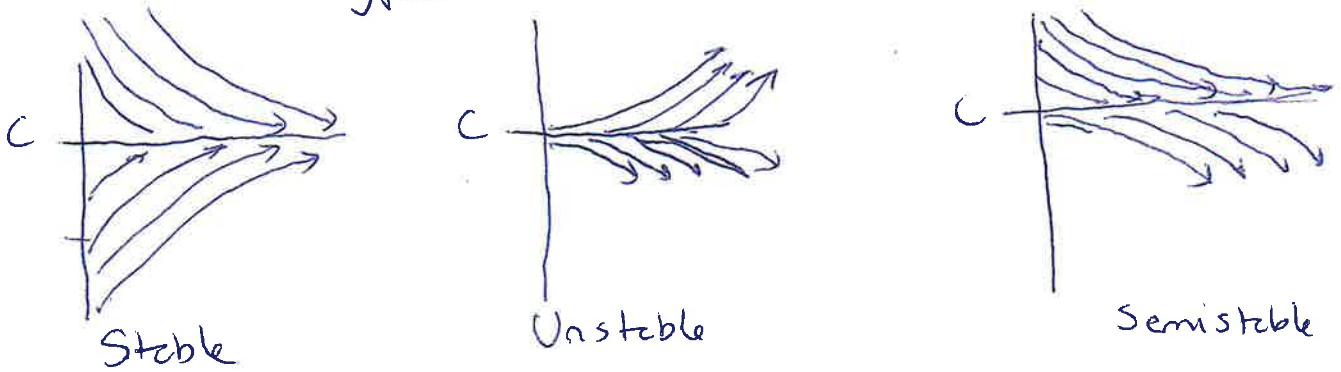
We call this a stable critical point since when $N>K$ or $N<K$, the solution will approach $N=K$.

for $0<N<K/2 \rightarrow \frac{dN}{dt}$ increasing $\rightarrow$ concave up

for $K/2<N<K \rightarrow \frac{dN}{dt}$ decreasing $\rightarrow$ concave down

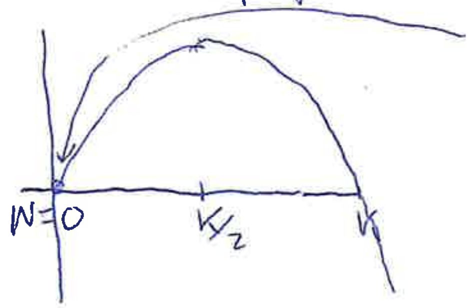
$N=K/2$ represent an inflection point for the $N(t)$ graph.

Types of critical point/equilibrium points



$N = K$ is a stable point

But $N = 0$ is unstable. This is because if N_0 is slightly above 0, the population will start to grow.



The derivative $\frac{dN}{dt}$ is positive as long as $N > N_0$ even if N is very small.

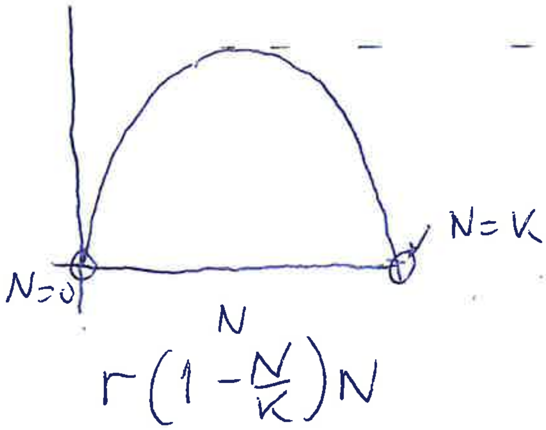
Fishing/Harvesting the Population

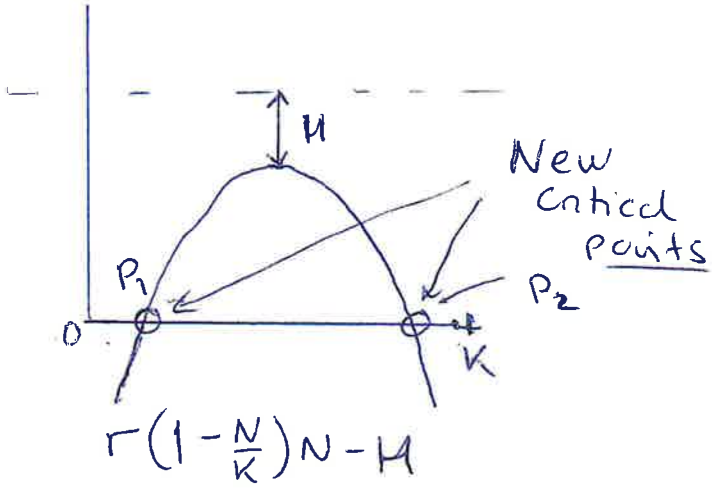
Let us assume that we harvest with a certain amount per time.

Let H be the harvesting rate. It has the same unit as $\frac{dN}{dt}$. The model then becomes:

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N - H \quad , H > 0$$

Harvesting

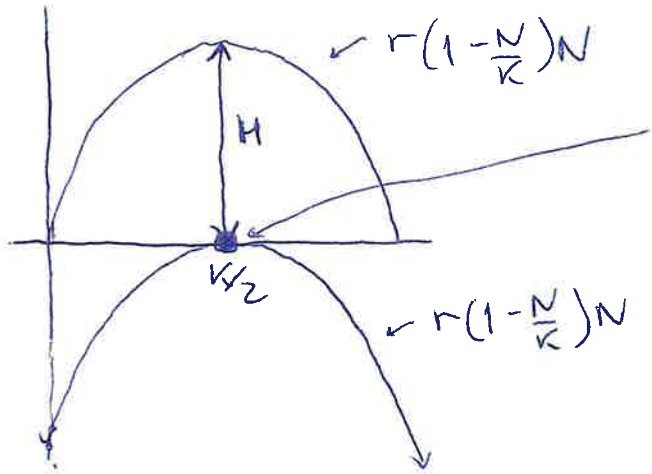




The introduction of H will shift the graph of the right hand side of the differential equation downwards. New critical points will emerge

P_1 is unstable since $\frac{dN}{dt}$ is positive for $N > P_1$ and $\frac{dN}{dt} < 0$ for $N < P_1$

P_2 is stable since $\frac{dN}{dt} < 0$ for $N > P_2$ and $\frac{dN}{dt} > 0$ for $N < P_2$



Here H is so large that only one point of the graph has $dN/dt = 0$. This is a critical point.

This is a semistable point. If $N_0 > K/2$, N will be reduced towards K/2 as time increases. But if $N_0 < K/2$, the ~~populat~~ population will go extinct.

And if H increases more, the whole graph will become negative → extinction for all populations

Model to be used in project

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N$$

We will look at a fish population where N is the biomass in kg. K = 10000 kg
time t unit is months
$r = 0.6/12$ per month (month^{-1})

Download the notebook:

$\boxed{\text{LogisticGrowthBasis}}$

This can be used as a starting point. Here only the exact solution has been implemented in the simulator.