

# Fundamentals of Machine learning for and with engineering applications

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1 Statistics

2 Univariate statistics

3 Distributions

4 Linear Regression

## Definition

Statistics is the science of acquiring and utilizing data

- It comprises tools for data collection, summarization, and interpretation.
- The aim is identifying the underlying structure, trends, and relationships inherent in the data.
- Is it all statistics then? Yes.
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# Data properties

Before we talk about machine learning, we need to refresh some terminology.

## Population

The universe of all possible outcomes and events.

## Sample

A finite subset extracted from the population.

## Exhaustivity

The samples covered the population spectra.

## Representativity

The population is properly described by the samples.



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- Variety different types of data sources with different length and scale.
- Frequency of data generation

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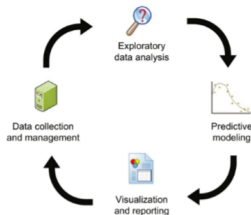
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# Sampling

Samples shall have no bias (to be randomly selected). If not, the bias has to be corrected for.

## Cycle of data

- 1 Data is collected
- 2 Checked upon
- 3 Some modelling
- 4 Analysis and visualization

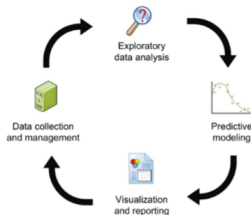


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# Data quality

- 1 Data has to be acquired and integrated
- 2 Data are passed to a quality analysis and control
- 3 Data cleaning, consistency check. Most of time goes here





## Main tasks:

- 1 Hunt for redundancy
  - 2 Reduce dimensionality
  - 3 AnOmAlles removal
- Descriptive modeling (unsupervised learning)
  - Predictive modeling (supervised learning)
  - The model can be used to guide data acquisition (risky!)

# Visualization and reporting

- The data has to be condensed into a visualization to provide input for decisions.
- Depending on the goal, very very different visualizations are possible.
- Use a model to indicate what is undersampled or oversampled.

Summarizing and visualizing data as a starting point for more analysis later on.

- Computing summary statistics (e.g. means and variance)
- Determining conditional probabilities of cause+effect relationships
- Calculating correlation and rank correlation coefficient between two variables
- Visualizing univariate, bivariate and multivariate data
- ...

Summarizing and visualizing data as a starting point for more analysis later on.

- ...
- Estimating probability coverage levels for different distributions
- Analyzing behaviour of normal distributions
- Calculating confidence interval and sampling distribution for the mean
- Testing for significance of difference in means
- Comparing two different distributions for statistical equivalence
- Developing a nonparametric regression model from given data
- Reducing data dimensionality
- Grouping data

# Random Variables

- A random variable is a real valued function that assigns a value to each outcome in the sample space
- A random variable (RV) can be either discrete or continuous
  - Discrete RV
  - Continuous RV

- The probability mass function (PMF),  $P$ , of a discrete RV,  $X$ , denotes the probability that the RV is equal to a specified value,  $a$ .

$$p(a) = p(X = a)$$

- The cumulative distribution function (CDF),  $F$ , denotes the sum

$$F(a) = P(X \leq a) = \sum_0^a f(x)dx$$

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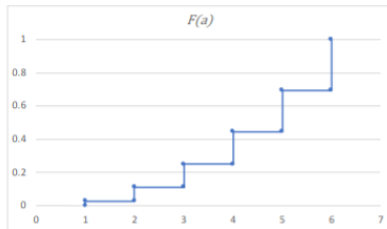
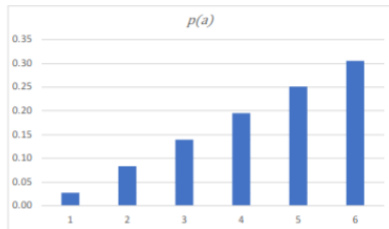
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# Random Variables

$a$	1	2	3	4	5	6
$p(a)$	1/36	3/36	5/36	7/36	9/36	11/36
$F(a)$	1/36	4/36	9/36	16/36	25/36	1



- What are the effective sampling strategies? (Wind turbine example)
- Solar Panels to determine the efficiency of the source (Usage patterns, energy production forecast)
- Drilling (penetration rate)
- Corrosion extension
- Concrete Rigidity
- etc



# Wind turbine example example

Turbine	Height	X	Y	Wind Speed	Air Density	Temperature	Power Output	Rotor Diameter	Hub Height	Air Pressure	Turbulence Intensity
WT-1	80	752.1	3945	7.5	1.225	15	1500	82	80	1013	0.1
WT-1	80	752.2	3945	8	1.223	15	1600	82	80	1012	0.12
WT-1	80	752.3	3945	7.8	1.224	16	1550	82	80	1013	0.11
WT-2	90	753.5	3946	6.5	1.226	14	1400	85	90	1012	0.15
WT-2	90	753.6	3946	7	1.225	14	1500	85	90	1011	0.13
WT-2	90	753.7	3946	7.2	1.227	14	1520	85	90	1012	0.14

# Sampling approaches

## Experimental design

Grid, parallel, series.

## Sampling without replacement

SPR (single point representation).

## Sampling with replacement

The number of the members of the population does not change.

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1 Statistics

2 **Univariate statistics**

3 Distributions

4 Linear Regression

# Univariate statistics

- Easy to displaying data:
  - histogram
  - frequency plots
  - cumulative
- Measures of Location
  - Mean, median, mode
  - Quartiles, Percentiles, Quantiles
- Measure of Dispersion (Spread)
  - Standard deviation (sd)
  - Sariance (Var) or coefficient of variation
- Measures of shape
  - Skewness, modality

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# Histograms

- Task 1: make a histogram from a 2d random distribution
- Task 2: make a 2d heat map from a 2d random distribution

# Frequency plots and Histograms

Given a set of data

- 1 Look for min and max values
- 2 Divide the range of values into a number of sensible class intervals (bins)
- 3 Count
- 4 Make a frequency table (or percentage)
- 5 Plot (see jupyter notebook)

Does this histogram represent uncertainty?

No. It shows variability, but it can be used to quantify uncertainty.

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# Class widths

- Class widths (bin sizes) are usually **CONSTANT**
  - the height of each bar is proportional to the number of values in it
- If class width are **VARIABLE**
  - the **AREA** of each bar is proportional to the number of values in it
- For small samples, the shape of the histogram can be very sensitive to the number and definition of the class intervals

## Exercise

Plot a histogram from different random number distributions and bin sizes.

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Plot a histogram from different random number distributions and bin sizes.

# Cumulative Histogram

- Cumulative frequency
- Each data point can be plotted individually
- It helps to read quantiles and compare distributions
- Practice with your jupyter notebook



# Measure of Location: Central Tendency, MEAN

$$m_x = \langle x \rangle = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Each point weighted equally by  $\frac{1}{n}$  (assumption)

- Every element in the data set contributes to the value of the mean
- An average provides a common measure for comparing one set of data to another
- The mean is influenced by the extreme values in the data set
- The mean may not be an actual element of the dataset
- The sum of all deviations from the mean is zero, and the sum of squared deviations is minimized when those deviations are measured from the mean

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# Means

- Arithmetic
  - Mean of raw data

$$\frac{1}{n} \sum_{i=1}^n x_i$$

- Geometric
  - $n^{th}$  root of product

$$\left( \prod_{i=1}^n x_i \right)^{\frac{1}{n}}$$

- Geometric

\* Mean of logarithms

$$\exp \left( \frac{1}{n} \sum_{i=1}^n \ln(x_i) \right)$$

- Harmonic
  - Mean of inverses

$$\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{x_i} \right)^{-1}$$

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# Median

```
if n is odd:  
    median = x[(n+1)/2]  
else:  
    median = x[n/2] + x[(n/2)+1]
```

- On a cumulative density plot, the value of the x-axis that corresponds to 50 % of the y-axis
- Not influenced by extreme values
- May not be contained in the dataset (if n is even)
- For a perfectly symmetrical dataset, means = median

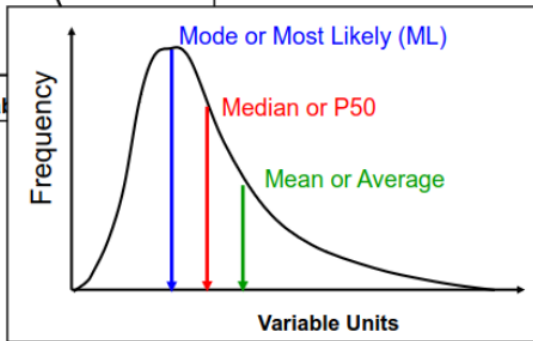
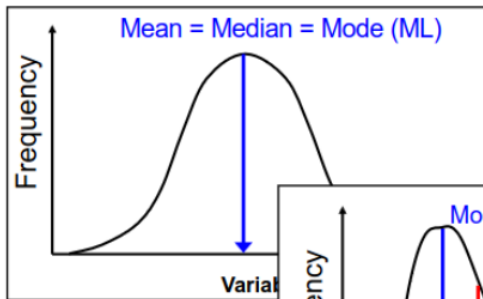
- The mode is the most frequently occurring data element

or the most likely or most probable value (for a pmf)

- A data set may have more than one mode and it thus called multimodal
- A mode is always a data element in the set
- For a perfectly symmetrical dataset,  $\text{means} = \text{median} = \text{mode}$



# Distribution Descriptors



# Quantiles

## Quartiles

The data split into quarters.

## Deciles

The data are split into tenths. The fifth decile is also the median.

## Percentiles

The data are split into hundredths. P10, P25, P50, P75 and P90 are the most commonly used.

## Quantiles

A generalization of splitting data into any fraction

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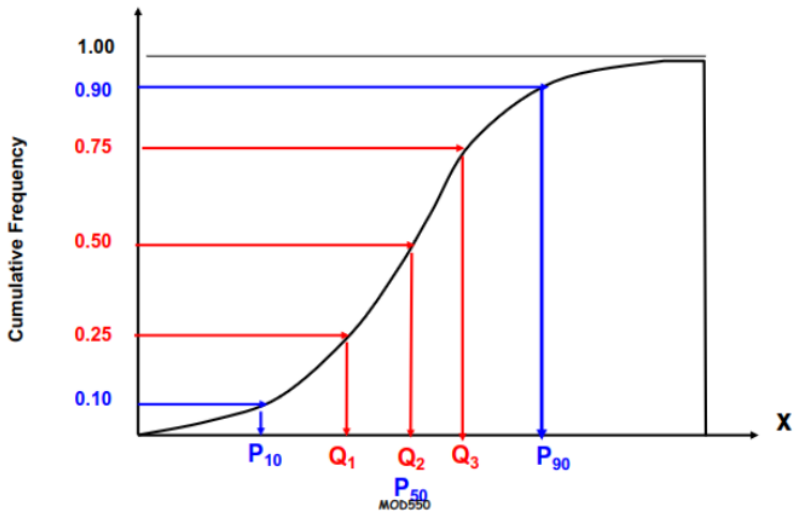
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# Distribution Descriptors



# Dispersion (Spread)

Range

$$R = \text{maximum} - \text{minimum}$$

Inter-quantile Range

$$\text{IQR} = Q3 - Q1$$

Mean Deviation from the Mean

$$\text{MD} = \sum_{i=1}^n (x_i - \bar{x}) / n$$

Mean Absolute Deviation

$$\text{MAD} = \sum_{i=1}^n |x_i - \bar{x}| / n$$

# Variance

The variance is the average of squared differences between the sample data points and their mean

## Variance

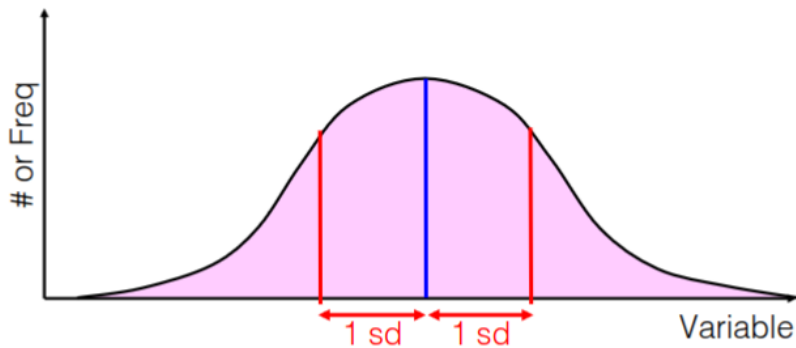
$$s_x^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

## Standard Deviation (SD)

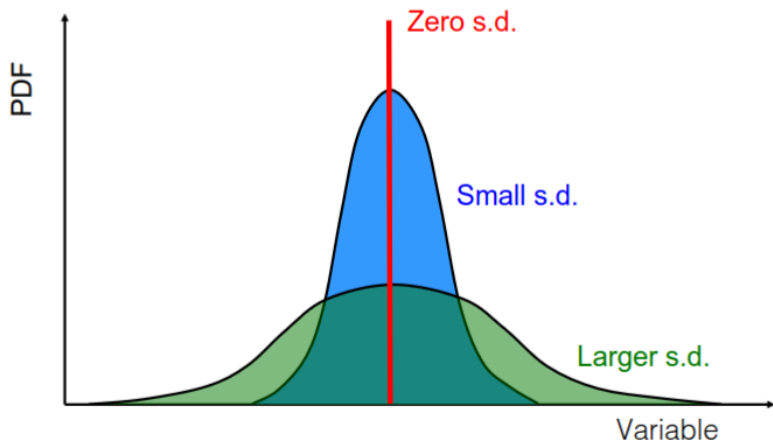
$$s_x = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$$



# Standard Deviation



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# Measures of dispersion

Standard Deviation (SD)

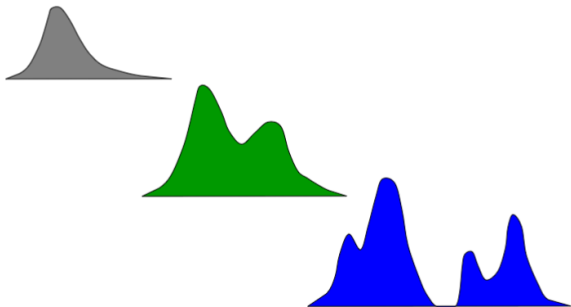
$$SE_x = \frac{s_x}{\sqrt{n}}$$

Coefficient of Variability

$$CV = \frac{s_x}{\bar{x}}$$

# Modality

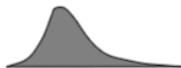
- Unimodal
- Bimodal
- Polymodal



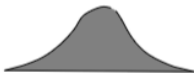
It measures the symmetry in a distribution

$$Sk = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^3}{s^3}$$

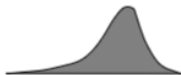
Positive - Values clustered toward the lower end



Zero – Symmetric distribution



Negative - Values clustered toward the higher end



A bit out of fashion with ML

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## Distribution

means of expressing uncertainty or variability

## Models

- Uniform: useful when only upper and lower bounds are known
- Triangular: useful when estimates of min, max, mode [P10, P50, P90] are available
- Normal: symmetric model of random errors or unbiased uncertainties with mean and standard deviation specified
  - very common for observed data
  - additive processes tend to be normal as a result of the Central Limit Theorem
- log normal comes from multiplicative uncertainties with mean and standard deviation specified

# Uniform Distribution

- The uniform distribution is useful as a rough model for representing low states of knowledge when only the upper and lower bounds are known.
- All possible values within the specified maximum and minimum values are equally likely ( $b=\max$ ,  $a=\min$ ):
- It can express maximum uncertainty

PDF:  $f(x) =$

$$\frac{1}{b-a}, a \leq x \leq b$$

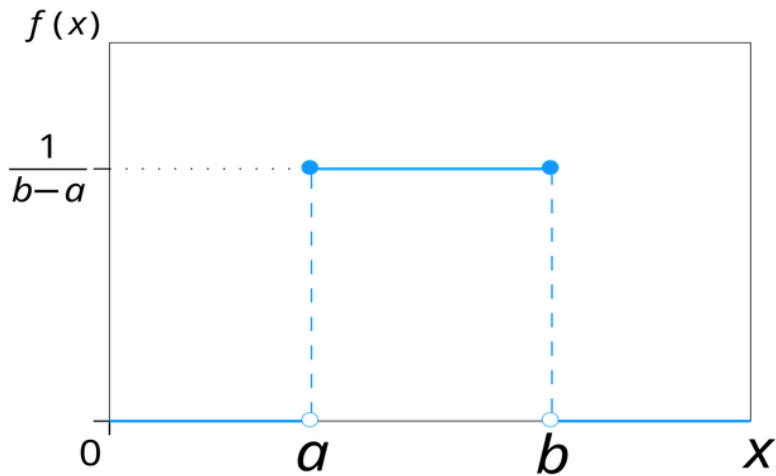
CDF:  $F(x) =$

$$\frac{x-a}{b-a}$$

Notation:  $X \sim U(a, b)$



# Uniform Distribution



# Triangular distribution

- The triangular distribution can be used for modeling situations, where non extremal (central) values are more likely than the upper and lower bounds.
- Take min, mode and max as inputs. Typically on the basis of subjective judgement:

PDF:  $f(x) =$

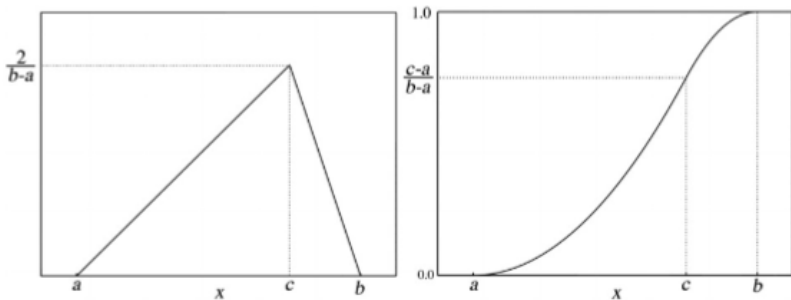
$$\frac{2(x-a)}{(b-a)(c-a)}; \text{ if } a \leq x \leq c$$
$$\frac{2(b-x)}{(b-a)(c-a)}; \text{ if } c \leq x \leq b$$

CDF:  $F(x) =$

$$\frac{(x-a)^2}{(b-a)(c-a)}; \text{ if } a \leq x \leq c$$
$$1 - \frac{(b-x)^2}{(b-a)(c-a)}; \text{ if } c \leq x \leq b$$

# Triangular Distribution

Notation:  $X \sim T(a, b, c)$



It can be symmetric or asymmetric

# Normal Distribution

- The normal distribution ('bell curve' or Gaussian) for modeling unbiased uncertainties and random errors of the additive kind of symmetrical distributions of many material processes and phenomena.
- A commonly cited rational for assuming normal distribution is the central limit theorem, which states that the sum of independent observations asymptotically approaches a normal distribution regardless of the shape of the underlying distributions(s=

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left\{ -\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 \right\}; \quad -\infty \leq x \leq \infty$$

CDF:  $F(x) =$

has no closed form solution but is often presented using the complementary error function solution

# Normal Distribution

Notation:  $X \sim G(\mu, \sigma)$

It is a Symmetric distribution around the mean

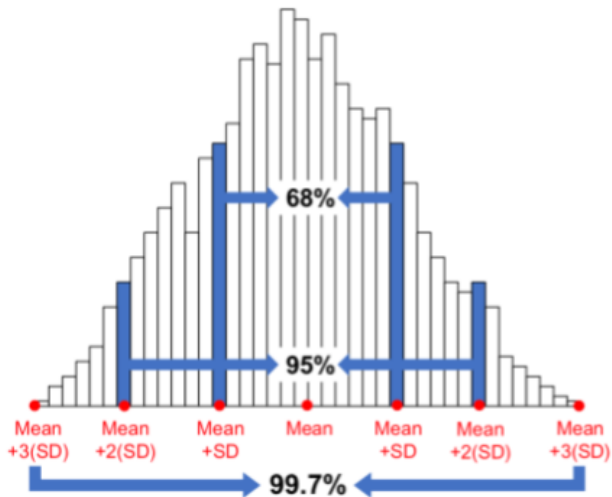
$\mu$  is the mean,  $\sigma$  is the standard deviation

$\mu \pm \sigma$  : 68.3% *probability*

$\mu \pm 2\sigma$  : 95.4% *probability*

$\mu \pm 3\sigma$  : 99.7% *probability*

# Normal Distribution

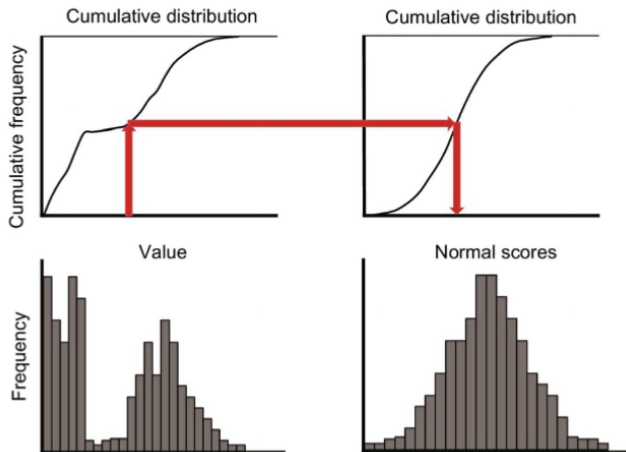


# Data transformations

- Often, it is useful to transform a sample distribution into the space of an equivalent normal distribution, where many statistical operations can be easily performed and visualized
- The approach involves a rank-preserving one-to-one transformation.
- Transforming the data so that their distribution matches a prescribed (target) distribution.
- Sometimes we must transform the data...

# Normal Score Transformation

- 1 From data to cumulative distribution.
- 2 From cumulative distribution and map back.



Match Quantiles



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# A linear model

Considering a univariate case, we have:

$$q = f(x)$$

which relates the **independent variable**  $x$  to the **true dependent variable**  $q$ .

Assuming a linear model

$$q = \beta_0 + \beta_1 x$$

where  $\beta_i$  are the arbitrary selected coefficients.

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# Model set-up

For a given  $x$  we do not know the true response  $q$ , only the measurement  $y_i$  for experiment  $i$ .

We have that:

$$y_i = q_i + \epsilon_i$$

NOTE: do not proceed if you do not fully understand this equation.

which is

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Discussion point

Does  $\epsilon_i$  matter? And why so?

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## Estimated model parameters

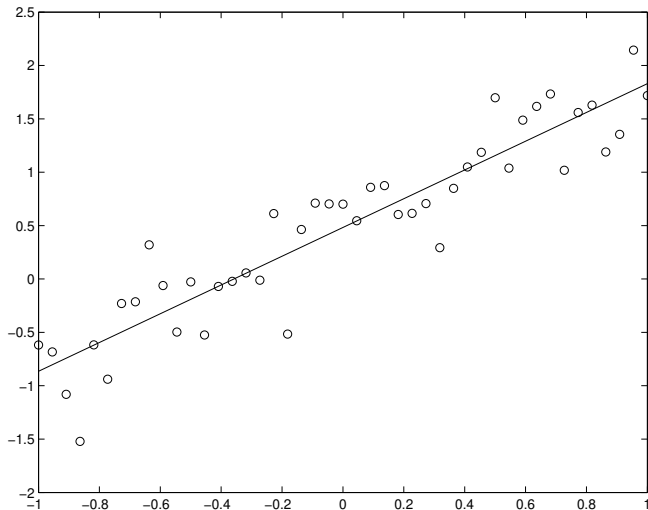
The model parameters  $\beta_0, \beta_1$  are unknown, but they can be **estimated**. To distinguish estimates from true model parameters we call them  $b_0, b_1$ . These estimates are calculated such that the model

$$\hat{y} = b_0 + b_1x$$

fits the  $n$  different experimental observations as well as possible.

# Linear model example

We would like to find the TRUTH (What it it?)



# Python Example

```
import numpy as np
import matplotlib.pyplot as plt

def generate_linear_data(n_random_points, noise=16):
    x = np.random.rand(n_random_points) * 10

    # Make 'perfect' data
    true_slope, true_intercept = 2, 5
    y = true_slope * x + true_intercept

    # Add noise
    y += np.random.randn(n_random_points)*noise

    return x, y, true_slope, true_intercept

# Use the function to generate data
x, y, true_slope, true_intercept = generate_linear_data(
    n_random_points=166,
    noise=3)

# Plot all
plt.plot(x, true_slope*x + true_intercept,
         color='red', label='Truth Line')
plt.scatter(x, y, color='blue', label='Data Points')
plt.show()
```

Does a linear model mean only straight lines (or hyperplanes in general)?

That answer to this is *no*. different. In general for a model  $f$  to be defined linear, it has to be linear with respect to the unknown parameters  $\beta_0, \dots, \beta_n$ . The general linear model is

$$q = \beta_0 + \beta_1 f_1(x_1) + \beta_2 f_2(x_2) + \dots + \beta_n f_n(x_n)$$

where  $f_i(x_i)$  may be non-linear functions. It is the **form** of the equation which makes it linear, i.e. that  $f_i(x_i)$  does not depend on the parameters  $\beta_i$ .

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Consider the following model example - is it linear?

$$q = \beta_0 + \beta_1 x_1^2 + \beta_2 x_2^{-1} + \beta_3 \log x_3$$

The answer is yes because by simple substitution it is possible to convert this formula into a linear form.

With  $h_1 = x_1^2$ ,  $h_2 = x_2^{-1}$  and  $h_3 = \log x_3$ , then we can formulate the new model:

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# Curvilinear models

A special class of linear models which we will investigate later are those which are expressed in terms of *polynomials* (here in only 1D):

$$q = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_m x^m = \sum_{i=0}^n \beta_i x^i$$

For n-D case there are a wide range of interaction terms and combinations, that can still be converted to standard linear form.

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But there are many other models that cannot be substituted to such a form. For instance:

$$q = \beta_0 + \log(x - \beta_1)$$

No substitution can transform this equation to the linear form.

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where  $\hat{y}$  is the estimated y-value from the approximate model that has been generated from a set of measurements  $(x_i, y_i)$ . We aim to find the  $b_i$  parameters such that the regression line fits the observed data as well as possible.

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Thus, we find the linear regression coefficients by *minimising*

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Regression!

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Before to continue, let's make sure to have all the main elements clear:

- 1 Splitting the data in dependent and independent variables
- 2 Assumption of a linear model between them
- 3 Recognise the difference between the truth and the estimation
- 4 Aiming to *minimize* the residuals

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# Regression

To minimize the sum of the square residuals, we can try to solve the following equations:

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Skipping the math (but you are more than welcome to try), here are the results:

$$\begin{aligned}b_0 &= \bar{y} - b_1\bar{x} \\ b_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}\end{aligned}$$

Straightforward derivation becomes very cumbersome for multiple variables. Thus, a different approach must be used. Yet, it is important to understand that there is an analytical solution (even if not all the time).

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# Many variable equation

The general equation would look like:

$$\hat{y} = b_0 + b_1x_1 + b_2x_2 + \cdots + b_nx_n = \sum_{j=0}^m x_{ij}b_j$$

where we will have to solve all the  $n + 1$  equations (called the **normal equations**) of the form:

$$\frac{\partial R}{\partial b_j} = 0 \quad \forall j \in [0, n]$$

Is there a way for us to simplify this?

We can use vector and matrix algebra.

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# Regression via Matrix operation

Remember that

$$R = \sum_{i=1}^N (y_i - \hat{y}_i)^2$$

then we can define the vector  $\mathbf{e}$ :

$$\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$$

thus

$$\mathbf{e}^T = [(y_1 - \hat{y}_1) \ (y_2 - \hat{y}_2) \cdots \ (y_N - \hat{y}_N)]$$

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From the following equation:

$$\hat{y}_i = b_0 + \sum_{j=1}^m x_{ij} b_j = \sum_{j=0}^m x_{ij} b_j$$

where  $x_{i0} = 1$

we make the matrix equation:

$$\hat{\mathbf{y}} = \mathbf{X}\mathbf{b}$$

where the first column in  $\mathbf{X}$  consists of ones only.

$$\begin{aligned} R &= \mathbf{e}^T \mathbf{e} \\ &= (\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}}) \\ &= (\mathbf{y} - \mathbf{X}\mathbf{b})^T (\mathbf{y} - \mathbf{X}\mathbf{b}) \\ &= (\mathbf{y}^T - \mathbf{b}^T \mathbf{X}^T) (\mathbf{y} - \mathbf{X}\mathbf{b}) \\ &= \mathbf{y}^T \mathbf{y} - \mathbf{y}^T \mathbf{X}\mathbf{b} - \mathbf{b}^T \mathbf{X}^T \mathbf{y} \\ &+ \mathbf{b}^T \mathbf{X}^T \mathbf{X}\mathbf{b} \end{aligned}$$

All the parts of this equation are scalar values. This means e.g. that

$$\mathbf{y}^T \mathbf{X}\mathbf{b} = \mathbf{b}^T \mathbf{X}^T \mathbf{y}$$

This gives

$$R = \mathbf{y}^T \mathbf{y} - 2\mathbf{y}^T \mathbf{X}\mathbf{b} + \mathbf{b}^T \mathbf{X}^T \mathbf{X}\mathbf{b}$$

But how can we now compute  $\frac{\partial R}{\partial b_j}$  more efficiently in matrix form?

**Vector differentiation !** Let

$$y = a^T x = a_1 x_1 + \cdots + a_n x_n$$

If

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a$$

and  $y = x^T a$ , then:

$$\frac{\partial y}{\partial x} = a$$

In general, when  $y = \mathbf{x}^T \mathbf{A} \mathbf{x}$ , then

$$\frac{\partial y}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

if  $\mathbf{A}$  is symmetric

(check *Matrix calculus* for more properties)

We can use this to compute

$$\frac{\partial R}{\partial \mathbf{b}}$$

## General solution

We have from above:

$$\begin{aligned} R &= y^T y - 2y^T Xb \\ &\quad + b^T X^T Xb \end{aligned}$$

Vector differentiation gives

$$\frac{\partial R}{\partial b} = 0 - 2X^T y + 2X^T Xb = 0$$

Solving this for  $b$  we get:

$$\begin{aligned} X^T Xb &= X^T y \\ (X^T X)^{-1} X^T Xb &= (X^T X)^{-1} X^T y \\ b &= (X^T X)^{-1} X^T y \end{aligned}$$

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Previous equation make the solution of MLR rather obvious!

When we have a matrix of y-variables **Y**:

$$B = (X^T X)^{-1} X^T Y$$

in the equation:

$$Y = XB.$$

These equations give us the **multiple linear regression (MLR)** solution.

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