4️⃣

TD 4

1 Review questions

a. In an ecosystem, the natural growth of a renewable resource is an increasing function of the amount of this resource.

▼ *Answer*

False: As an example, in class, you considered a logistic growth, captured by the equation $\tau(S_t) = rS_t\left(1 - \frac{S_t}{K}\right)$. As you can see, when $S_t \to K$ then $\tau(S_t) \to 0$. Therefore, it is not true that if S_t increases then its growth also increases.

b. An improvement in extractive technology always increases fish production if fishing is free.

▼ *Answer*

False: In our model the total production of fish when there is free entry is given by the following:

$$H_F = B_F \alpha S_F = \left(1 - \frac{c}{p\alpha K}\right) \frac{r}{\alpha} \frac{c}{p}$$

As you can see, we have an α at the denominator with a minus sign (positive effect on H_F), but we also have an α at the denominator with a plus sign (negative effect on H_F). Hence, the total effect is ambiguous.

2 Solow-Swan with non-renewable resources

b. Assume that at time t (when the economy was previously on the balanced growth path), a new source of non-renewable resource of size N_t is discovered. Each ensuing period, $r\%$ of the resource stock is used in production, such that its stock goes progressively to zero in the long run. If $R = 1$, $N_t = 20$ and $r = 0.1$, what is the growth rate of the supply of resource Z before t ? Right after t? In the very long run?

Before answering this question we have to compute the growth rate of Z_t. However, in this case the expression of interest is a sum, therefore we can not use the rules of product and ratios of growth rate. The idea is to subtract Z_{t-1} to Z_t in order to find the Δ. The calculations to attain the growth rate are the following (I omit t in the computations):

$$\begin{aligned}
Z_t &= R_t + rN_t \\
Z_t - Z_{t-1} &= R_t - R_{t-1} + r(N_t - N_{t-1}) \\
\Delta Z &= \Delta R + r\Delta N \\
\frac{\Delta Z}{Z} &= \frac{R}{R}\frac{\Delta R}{Z} + r\frac{N}{N}\frac{\Delta N}{Z} \\
\frac{\Delta Z}{Z} &= \frac{R}{Z}\frac{\Delta R}{R} + r\frac{N}{Z}\frac{\Delta N}{N} \\
g_Z &= \frac{R}{Z}g_R + r\frac{N}{Z}g_N \\
g_{Z,t} &= \frac{R}{R + rN_t}g_R + r\frac{N_t}{R + rN_t}g_{N,t}
\end{aligned}$$

By substituting the numbers we have we obtain the growth rate when the new non-renewable resource is discovered, at time t. Remember that R does not grow ($g_R = 0$) and that Z has a negative growth of -0.1 We have:

$$g_{Z,t} = \frac{R}{R+rN}g_R + r\frac{N}{R+rN}g_N$$
$$g_{Z,t} = \frac{1}{1+(0.1)(20)}(0) + \frac{(0.1)(20)}{1+(0.1)(20)}(-0.1)$$
$$g_{Z,t} = \frac{(0.1)(20)}{1+(0.1)(20)}(-0.1) = -6.\bar{6}\% = -\frac{1}{15}$$

As for $g_{Z,\tau}$ for $\tau < t$, we have that $g_{Z,\tau} = 0$, as $N_t = 0$ and R does not grow, exactly as we had in the previous point. Instead, when $\tau \to \infty$ the growth rate also goes to zero. This is due to the fact that Z has a negative growth, and therefore after it is completely exploited it will not grow (negatively) anymore.

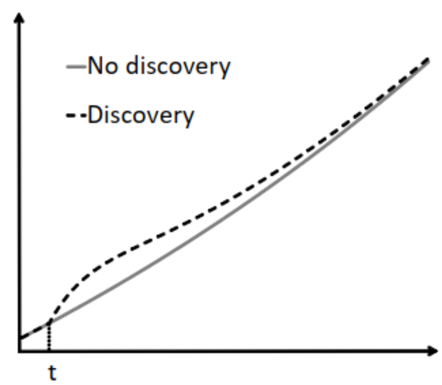
For the last points the professor run an analysis with particular values of parameters $n = 0.01$, $g = 0.015$, $\delta = 0.02$, $s = 0.3$, $\alpha = 0.3$, $\beta = 0.2$, $R = 1$, $N_t = 20$, $r = 0.1$. Finally you see the model at work! The graphs in the text of the exercise are the plot of three time series: $\frac{Y}{K}$, $\frac{Y}{L}$ and $\frac{K}{L}$. The continuous line describe an economy in a balanced growth path with no non-renewable resources, while the dashed line depicts a scenario where non-renewable resources are discovered at time t. The main of the following points is to connect the graphs to ratios.

c. Which graph is $\frac{K}{L}$? Explain in words what happens at time t and in the ensuing periods.

Let's try to find a general way of answering these kind of questions. The variables involved are K, L and Y. The question is: how are these variables affected by an increase in N? In order to answer we need to know the dependencies that all these variables have with N. As an example, L is only determined by its growth, we start from L_0 and then we get L_1 based on how big n is. Therefore, L is not directly affected by N. The same hold for K, its growth is given by its growth rate g_K, and not directly by N. Hence, K does not jump either. Instead, $Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$ where $Z_t = R + rN_t$. A jump in N causes a jump in Y. This analysis offers a first insight to answer this question. In fact, we must consider the three ratios $\frac{K}{L}$, $\frac{Y}{K}$ and $\frac{Y}{L}$. Since of these three the only ratio that does not jump is $\frac{K}{L}$ we are sure that the right graph is *C*, as there is a smooth evolution of the dashed line.

To understand what happens here recall that $g_K = s\frac{Y_t}{K_t} - \delta$. Since Y increases, as we elaborated before, the growth rate of K increases, so K increases more then what it would without N. However, N will go to zero slowly, which means that the accumulation of capital K slowly go back to its original path.

C



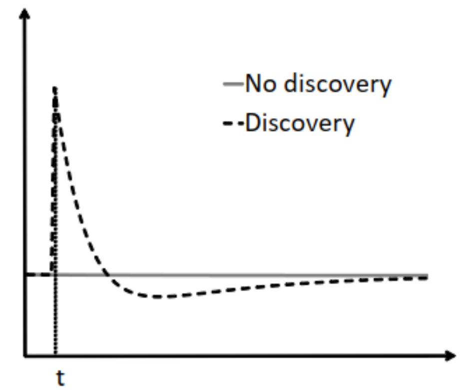
d. Which graph is $\frac{Y}{K}$? Explain in words what happens at time t and in the ensuing periods.

First step done, now we have to distinguish between $\frac{Y}{L}$ and $\frac{Y}{K}$. The difference between the two graphs we are left with is that on one of the the balanced growth path is constant (horizontal line). Therefore, we have to answer the question: which ration between $\frac{Y}{K}$ and $\frac{Y}{L}$ should be fixed without the increase in N? Well, we now that the growth rate of L is exogenously given and it is $n > 0$, so it is impossible that L will be fixed. Also Y and K will grow, but at which rate? We know from the previous TD that $g_{\frac{Y_t}{K_t}} = 0 \Leftrightarrow g_Y - g_K = 0 \Leftrightarrow g_Y = g_K$, therefore $\frac{Y}{K}$ is constant and identified by a constant. Hence, it is represented by graph *B*. Here two things happen. First, as we saw before, Y jumps.

Instead, K does not jumps immediately, but smoothly increases. Therefore at time t $\frac{Y}{K}$ will steadily increase.

However, the push of Y is only big at time t, after that the growth rate of Y will slowly return normal. Instead, the growth rate of K will increase, and the increase of K will offset the increase of Y. This explains why the dotted line goes below the horizontal line. After some time also the growth of K returns normal, and the dotted line returns on the old balanced growth path.

B

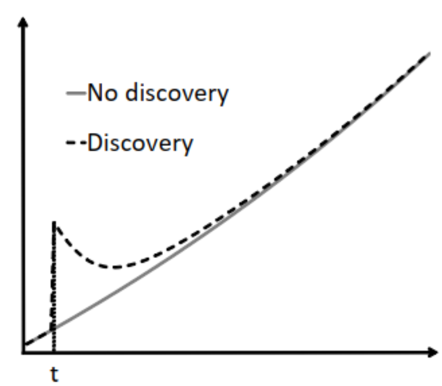


e. Which graph is $\frac{Y}{L}$? Explain in words what happens at time t and in the ensuing periods.

We are only left with one ratio and graph *A*, so the answer is easy here, but we must understand also what is going on. The jump is always given by the steady increase of Y, as in the previous points.

However, L will be always increasing with the same rate, in contrast to K in the previous graph. Nevertheless, the ratio immediately starts to decrease after t, slowly reaching the old balanced growth path. This is due to the fact that the stock of natural resources is depleted with a

A



rate way higher $(-6.\bar{6}\%)$ than the rate at which productivity increases (1.5%).

> “We are in the beginning of mass extinction, and all you can talk about is money and fairy tales of eternal economic growth”- Greta Thunberg at the United Nations Climate Action Summit, September 23, 2019

f. There is little doubt that a mass extinction is going on. However, this widespread idea of sustained economic growth being a myth is open to debate. Expanding on the model, can we comment on it.

This is more of a philosophical question, so fell free to answer whatever you think that is relevant and consistent with the model (it is true that many answer could be consistent with what we have here, but there are also many things that are not!). What this little exercise can tell us is that there is no need of non-renewable resources for an economy to grow, as long as other rates $(n, g...)$ are positive. However, a sudden increase in N also brings to a jump in Y. Maybe this model could also suggest that we need a fixed amount of renewable resources R, even if it does not grow. Therefore, according to this model, it might be not a very good idea to destroy forests.

3 The Dynamics of a fish population with threshold

One of the problems that the fishing model has is that the only circumstance in which there is an extinction of fishes (or natural resources in general) is when the starting stock is equal to 0. Of course, this is counterfactual with reality as we go from a state in which there is a positive amount of resources to a state in which they are extinct. The aim of this exercise is to augment to model by assuming that when the stock of fishes goes below a threshold T then it is destined to converge to 0. I think it is an interesting exercise, it helps interpreting some real world facts.

a. Find the values of $S(t) > 0$ for which the fish stock does not grow naturally.

As always, we first need to understan what the question is asking. When does the fish stock grows? When its growth rate is different then 0. If the growth rate is equal to zero then the stock will not grow. The question is asking ask to determine for which values of $S(t)$ the growth rate is equal to 0. The growth rate is:

$$N(S(t)) = r(S(t) - T)\left(1 - \frac{S(t)}{K}\right)$$

Since $r > 0$ and the expression is a product, we have that $N(S(t)) = 0$ when one of the two element of the product is zero.

$$N(S(t)) = 0 \Leftrightarrow \begin{cases} S(t) - T = 0 \\ 1 - \frac{S(t)}{K} = 0 \end{cases} \Leftrightarrow \begin{cases} S(t) = T \\ S(t) = K \end{cases}$$

Hence, fishes will not grow when their stock is exactly equal to their maximum capacity K and when the stock is equal to the minimum treshold T. Notice that this is clear also from the plot of the growth rate.

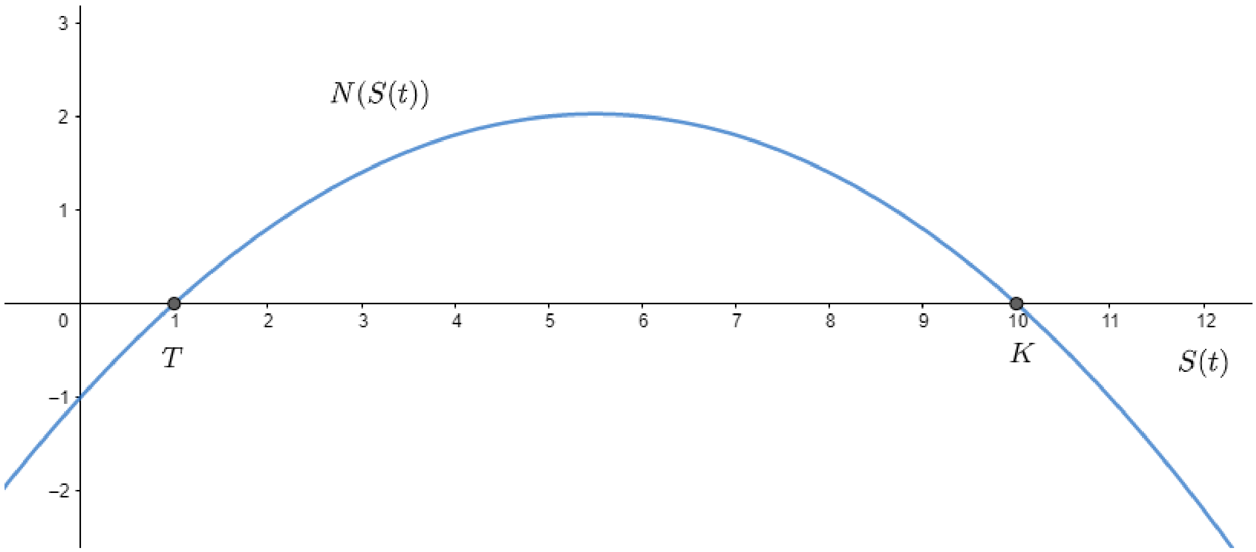


Figure 1: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$.

b. Of these values, which are stable, which are not?

First of all, what does stable mean in this context? When speaking about steady states (growth rates equal to 0), we say that a steady state is stable if a small perturbation of the system from a steady steady returns to the previous point autonomously. In this case, a steady state is stable if by slightly increasing or decreasing the stock of fishes from $S(t) = T$ or $S(t) = K$ we then return to the previous steady state or the system evolves in a different direction.

Checking for stability seems a daunting task, but if you have the graph it becomes easier. Here I will show you to check for stability with both a graphical and a mathematical technique. Let's start from the graphical one. Consider the steady state $S(t) = T$, what happens if we move slightly on the right (e.g. $S(t) = T + \epsilon$) ? You see that $N(S(T) + \epsilon)$ is positive. Hence, the stock will continue to grow and will become significantly different with respect to $S(T)$. In the same way, if we perturb the stock in the other direction ($S(T) - \epsilon$) we can see that $N(S(T) - \epsilon)$ is negative, the stock will become lower and lower.

This steady state is not stable. If we perturb it slightly it will not return to its original point. Now consider the second steady state, when $S(t) = K$. If we perturb it by moving slightly on the right $(S(K) + \epsilon)$ we see that $N(S(K) + \epsilon))$ is negative, therefore the stock of fish will decrease until it returns to its stable value $S(t) = K$. The same happens when you perturb the stock in the other direction, we have that $N(S(K) - \epsilon))$ is positive, which will make the stock increase until it is stable in $S(t) = K$. We concluded that $S(t) = T$ is not stable while $S(t) = K$ is stable. The graph below captures this reasoning pattern.

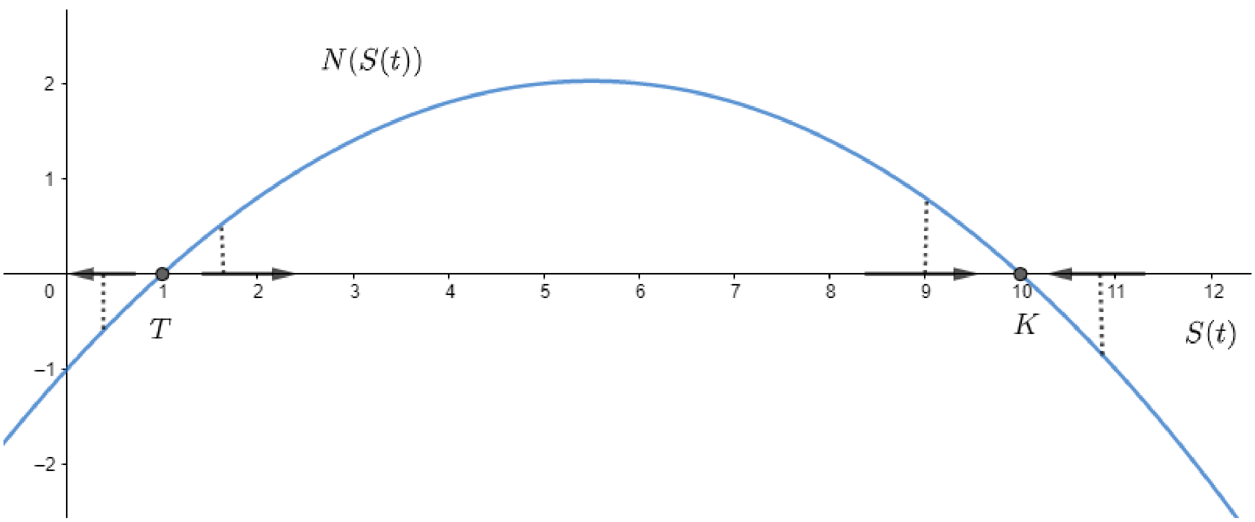


Figure 2: Stability of steady states.

If you don't like this graphical reasoning, there is also the math way. This perspective amounts to taking the derivative of $N(S(t))$ with respect to $S(t)$, which measures the change in growth by a change in stock of fish. By evaluating the derivative in the two steady states we can check its sign. If the sign of the derivative is positive, this means that a positive variation will be magnified even more, and therefore the steady state is not stable. If the sign is negative, it means that after a positive variation the stock will decrease and return to its original value. This would mean that the steady state is stable. Let's start taking the derivative. It might look a little bit scary to the the derivative with respect to $S(t)$, but you just have to consider the entire expression as a single variable and derive it with the rules you know (you can look at TD1 for an explanation on how to derive products).

$$\begin{aligned}\frac{\partial N(S(t))}{\partial S(t)} &= r\left[\left(1-\frac{S(t)}{K}\right)+(S(t)-T)\left(-\frac{1}{K}\right)\right] \\ &= r\left[1-\frac{S(t)}{K}-\frac{S(t)}{K}+\frac{T}{K}\right] \\ N'(S(t)) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right]\end{aligned}$$

We can now evaluate the derivative in the two points of interest. Recall that $r > 0$.

$$\begin{aligned}N'(T) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right] \\ &= r\left[1-\frac{2T}{K}+\frac{T}{K}\right] \\ &= r\underbrace{\left[1-\underbrace{\frac{T}{K}}_{<1}\right]}_{>0} > 0\end{aligned}$$

This result confirms our graphical analysis. Since the derivative at T is greater than 0, this means that a positive perturbation of $S(t)$ at T will increase the stock even more, and therefore will push the system far from the original state. On the contrary, for K we have:

$$
\begin{aligned}
N'(K) &= r\left[1 - \frac{2K}{K} + \frac{T}{K}\right] \\
&= r\left[1 - 2 + \frac{T}{K}\right] \\
&= r\underbrace{\left[\underbrace{\frac{T}{K}}_{<1} - 1\right]}_{<0} < 0
\end{aligned}
$$

Which again goes in the same direction as the graphical intuition. If we positively perturb the steady state at K, the stock of fish will decrease until we reach the previous state again.

c. What is the natural growth of the fish population at t if $S(t) = 0$? Is it also an equilibrium?

To answer this question we just need to evaluate the growth rate in the point $S(t) = 0$.

$$
N(0) = r(0 - T)(1 - 0) = -rT
$$

We should have expected this result, as we know that T is a threshold for the fish to grow and $S(t) = 0 < T$. Since the computed growth rate is negative and since the stock can not go lower than 0, we conclude that $S(t) = 0$ is also a steady state. Notice that $S(t)$ is the point in which the growth rate crosses the y axis, as shown in the picture below.

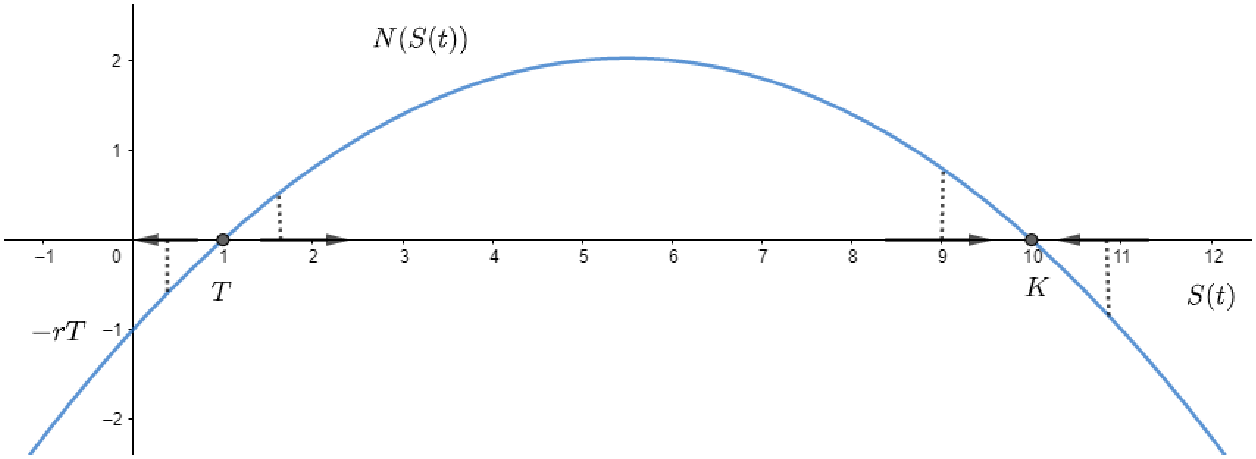


Figure 3: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Notice that $-rT = -1(1) = -1$, where the growth rate is negative and the stock is 0.

d. What is the maximum number of fish that can be caught per unit of time such that the fish population is constant? This is also called the maximum sustained yield. What is the fish stock $S(t)$ at this value?

To answer this question we must ask when the growth rate of fishes is the highest. This would allow us to capture the maximum number of fishes every time t and then obtain for $t + \epsilon$ the greatest amount of growth so that we can always maximise our catches. Hence, we must maximise the growth rate of fishes with respect to the stock. We already have the derivative. To check for the maximum we need to find the value of $S(t)$ for which the derivative is equal to 0.

$$\begin{aligned}
\frac{\partial N(S(t))}{\partial S(t)} = 0 \Leftrightarrow r\left[\left(1 - \frac{S(t)}{K}\right) + (S(t) - T)\left(-\frac{1}{K}\right)\right] = 0 \\
\Leftrightarrow \cancel{r}\left[1 - \frac{2S(t)}{K} + \frac{T}{K}\right] = 0 \\
\Leftrightarrow K - 2S(t) + T = 0 \\
\Leftrightarrow S(t) = \frac{K + T}{2}
\end{aligned}$$

Now that we have the stock of fishes that maximises growth we can ask by how much fishes grow for this value of the stock. Of course, to answer this question we just need to plug the value we just found in the growth rate.

$$
\begin{aligned}
N\left(\frac{K+T}{2}\right) &= r\left(\frac{K+T}{2}-T\right)\left(1-\frac{K+T}{2K}\right) \\
&= r\left(\frac{K+T-2T}{2}\right)\left(\frac{2K-K-T}{2K}\right) \\
&= r\left(\frac{K-T}{2}\right)\left(\frac{K-T}{2K}\right) \\
&= r\frac{(K-T)^2}{4K}
\end{aligned}
$$

This expression tells us by how much the fish grows at the optimal stock.

e. Graph the fish growth function $S(t)$. Place all the elements previously computed on the graph.

We already did a big part of the graph, the one below has also the answers to the last question.

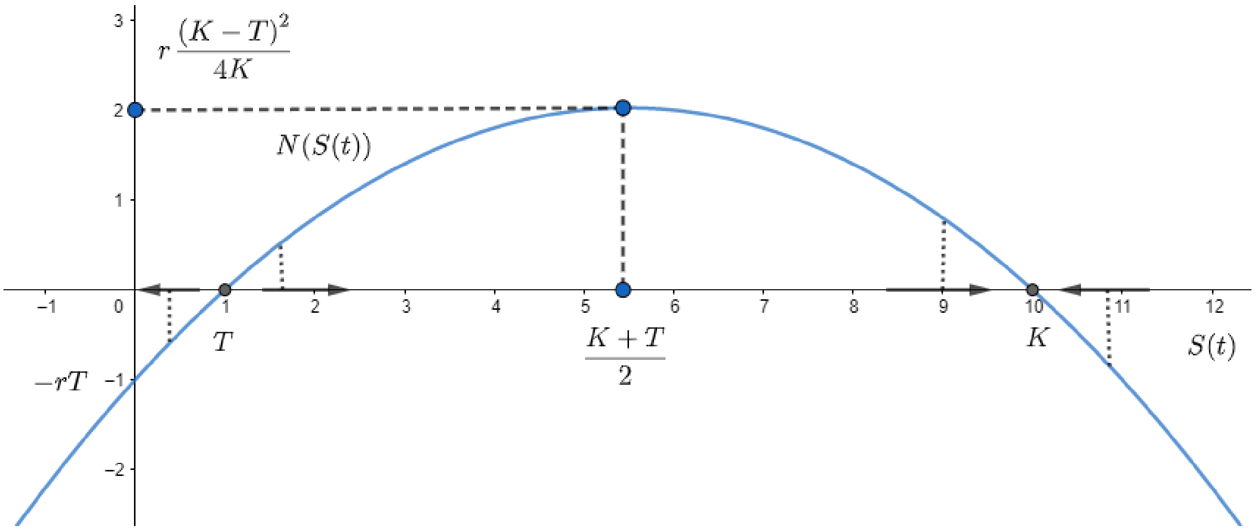


Figure 4: Graph of the growth rate of fishes for $T=1$, $K=10$ and $r=1$. Here $S^*(t)=\frac{10+1}{2}=5.5$ and $N(S^*(t))=\frac{9^2}{40}\approx 2$.