TD 3

1 Review Questions

b. On a balanced growth path, all variables grow at the same rate.

▼ *Answer*

False: Recall the definition of Balanced growth path at page 19 on you lecture notes:

> *Definition: A balanced growth path is a trajectory such that all variables grow at a constant rate.*

Translated in mathematical terms, we have that all the variables x_i in our model must have $g_{x_i} = k_i$, where k_i is a constant. However, it is not specified that all k_i must be equal! Each variable can grow at its own, constant rate. We have an example in the exercise below, where different variables of interest grows at different rates.

c. The Solow model needs to assume technological change to check the stylised Kaldor facts of growth.

▼ *Answer*

True: Consider as an example Kaldor fact 1:

> *Kaldor fact 1: Labour productivity has grown at a sustained rate.*

If we do not have technology, labour productivity does not grow in the steady state. In fact, if we do not have technology and we are in a steady state then $g_k = g_K - g_L = 0$, since $y = f(k)$, if k does not grow then also y will not grow. You can check this on page 19 of your lecture notes, but the next exercises constitute a clear example of why this is true. Introducing a technological shift which grows at rate g makes growth positive.

Question: Try to argue the same thing by considering Kaldor fact 2 about capital per worker.

d. The Solow model predicts convergence of all economies in the world to the same GDP per capita.

▼ *Answer*

False: The Solow model can be interpreted as a machine that takes as an input exogenous parameter $(n, \delta, etc...)$ and tells you what happens to the economy. To a different set of exogeneous parameters we obtain a different prediction. Consider as an example the economies at points a. and b. of exercise 3 of this TD(1), the growth predictions are completely different.

4 Problem - The Solow model with natural resources

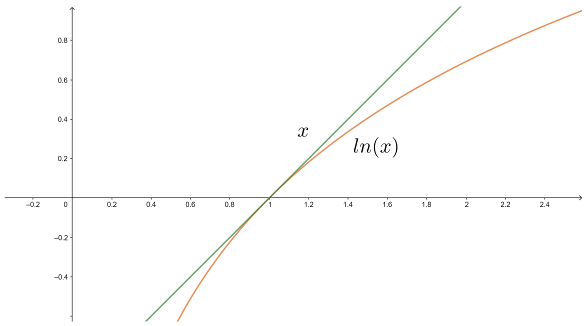
The aim of this exercise is to get you used with growth rates calculations. It is in some sense less interesting from an intuitive point of view, but we will be able to link it to exercise 2 in TD2.

In this problem we have quite a lot of data. The production here is affected by three variables, capital K_t, labour L_t and a natural resource Z_t. We also have capital augmenting technology A_t. The function is the following

$$Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$$

The law of motion of capital is the standard one $\Delta K_t = K_{t+1} - K_t = (1-\delta)K_t + I_t$ where $I_t = sY_t$. Technology, grows at an exogenously fixed rate $A_{t+1} = (1+\gamma)A_t$. Also the stock of natural resources grows at an exogenous fixed rate $Z_{t+1} = (1+\epsilon)Z_t$. Labour also grows, as we already saw $L_{t+1} = (1+n)L_t$. Throughout the problem we will use the following useful approximation:

$$log\left(\frac{X_{t+1}}{X_t}\right) \approx \frac{X_{t+1}}{X_t} - 1$$



a. In this problem use g_x to denote the growth rate of the variable x (for example $g_y = log\left(\frac{Y_{t+1}-Y_t}{Y_t}\right)$) From the definitions, write g_A, g_L and g_Z.

Let's use the definition and the approximation we are given. We start from g_A.

$$\begin{aligned}(1+\gamma)A_t &= A_{t+1} \\ (1+\gamma) &= \frac{A_{t+1}}{A_t} \\ \gamma &= \frac{A_{t+1}}{A_t} - 1 \approx log\left(\frac{A_{t+1}}{A_t}\right) \\ \gamma &= g_A\end{aligned}$$

We can perform the same calculations to see that $g_L = n$ and $g_Z = \epsilon$.
Question: Try to find g_L and g_Z as an exercise.

b. Compute g_Y in terms of $\alpha, \beta, g_A, g_K, g_Z$, and g_L.

This seems like a daunting task, so let's divide this computation by steps.

First, we must identify the variable of which we want to compute the growth rate. In this case we have from the text $Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$.

Second, we use the explicit expression of growth rates to understand how its growth rate is composed. Since we have that $g_Y = log\left(\frac{Y_{t+1}}{Y_t}\right)$, we first have to compute $\left(\frac{Y_{t+1}}{Y_t}\right)$.

$$\begin{aligned}\left(\frac{Y_{t+1}}{Y_t}\right) &= \frac{(A_{t+1}K_{t+1})^{\alpha}\,(L_{t+1})^{1-\alpha-\beta}\,Z_{t+1}^{\beta}}{(A_tK_t)^{\alpha}\,(L_t)^{1-\alpha-\beta}\,Z_t^{\beta}} \\ &= \left(\frac{A_{t+1}}{A_t}\frac{K_{t+1}}{K_t}\right)^{\alpha}\left(\frac{L_{t+1}}{L_t}\right)^{1-\alpha-\beta}\left(\frac{Z_{t+1}}{Z_t}\right)^{\beta}\end{aligned}$$

Third, we take logs, so that we have a direct expression for g_Y.

$$\begin{aligned}log\left(\frac{Y_{t+1}}{Y_t}\right) &= log\left[\left(\frac{A_{t+1}}{A_t}\frac{K_{t+1}}{K_t}\right)^{\alpha}\left(\frac{L_{t+1}}{L_t}\right)^{1-\alpha-\beta}\left(\frac{Z_{t+1}}{Z_t}\right)^{\beta}\right] \\ &= \alpha\left(log\left(\frac{A_{t+1}}{A_t}\right)+log\left(\frac{K_{t+1}}{K_t}\right)\right)+(1-\alpha-\beta)\left(log\left(\frac{L_{t+1}}{L_t}\right)\right)+\beta\left(log\left(\frac{Z_{t+1}}{Z_t}\right)\right) \\ &= \alpha\,(g_A+g_K)+(1-\alpha-\beta)\,(g_L)+\beta\,(g_Z) \\ g_Y &= \alpha\,(\gamma+g_K)+(1-\alpha-\beta)\,(n)+\beta\,(\epsilon)\end{aligned}$$

We have exactly g_Y in terms of $\alpha, \beta, g_A, g_K, g_Z$, and g_L.

c. Compute g_K in terms of δ, s and $\frac{Y_t}{K_t}$.

Exactly as before, we exploit the definition of growth rate and what we know about K_t. The law of motion of capital is always the same.

$$K_{t+1} - K_t = \Delta K_t = sF(A_t, K_t, L_t, Z_t) - \delta K_t$$

We elaborate a little bit on this expression to put it in a form that is convenient to us. First, we divide by K_t to explicitly have the growth rate.

$$\begin{aligned}\frac{K_{t+1}-K_t}{K_t} &= \frac{sF(A_t,K_t,L_t,Z_t)-\delta K_t}{K_t} \\ \frac{K_{t+1}}{K_t}-1 &= s\frac{Y_t}{K_t}-\delta \\ log\left(\frac{K_{t+1}}{K_t}\right) &\approx s\frac{Y_t}{K_t}-\delta \qquad \text{(By the approximation given in the text)} \\ g_K &\approx s\frac{Y_t}{K_t}-\delta\end{aligned}$$

We managed to find an expression of g_K in terms of δ, s and $\frac{Y_t}{K_t}$.

d. Argue why, along a balanced growth path, $\frac{Y_t}{K_t}$ must be constant. Then argue why $g_Y = g_K$.

Recall the definition of a balance growth path: all the variables must grow at a constant rate! This means, in order, that K_t must grow at a constant rate, that g_K must be equal to a constant, and that $s\frac{Y_t}{K_t} - \delta$ must be constant. We know that s and δ are indeed constant, but, if we are not on a balance growth path $\frac{Y_t}{K_t}$ evolve with time. Therefore, $\frac{Y_t}{K_t}$ for g_K to be constant, so that K grows at a constant rate.

As for the second question, the answer is only one step ahead of the previous reasoning. In order for the ratio $\frac{Y_t}{K_t}$ to be constant, the two variables must grow at the same rate in each time t. If, as an example, K_t grows quicker than Y_t, the ratio will not be constant in time, therefore $g_Y = g_K$. More precisely, if $\frac{Y_t}{K_t}$ grows at a constant rate, it means that $g_{\frac{Y_t}{K_t}} = 0$. By exploiting the rules of growth rates:

$$g_{\frac{Y_t}{K_t}} = 0 \Leftrightarrow g_Y - g_K = 0 \Leftrightarrow g_Y = g_K$$

Which is what we wanted to prove.

e. Using your answers to earlier parts of the problem, solve for g_Y in terms of $\alpha, \beta, \gamma, \epsilon$ and n.

From point b. we have that $g_Y = \alpha\left(\gamma + g_K\right) + \left(1 - \alpha - \beta\right)(n) + \beta\left(\epsilon\right)$, while form point d. we know that along a balanced growth path $g_Y = g_K$. By substituting the second condition into the first on we obtain:

$$\begin{aligned} g_Y &= \alpha\left(\gamma + g_K\right) + \left(1 - \alpha - \beta\right)(n) + \beta\left(\epsilon\right) \\ &= \alpha\gamma + \alpha g_Y + \left(1 - \alpha - \beta\right)(n) + \beta\left(\epsilon\right) \\ g_Y\left(1 - \alpha\right) &= \alpha\gamma + \left(1 - \alpha - \beta\right)(n) + \beta\left(\epsilon\right) \\ g_Y &= \frac{\alpha\gamma + \left(1 - \alpha - \beta\right)(n) + \beta\left(\epsilon\right)}{1 - \alpha} \end{aligned}$$

Which gives us g_Y in terms of $\alpha, \beta, \gamma, \epsilon$ and n.

f. What is the condition for $g_{\frac{Y}{L}}$ to be positive along a balanced growth path? Interpret.

To answer this question we have first to compute the quantity of interest. The rule is always the same:

$$\begin{aligned} g_{\frac{Y}{L}} &= g_Y - g_L \\ &= \frac{\alpha\gamma + \left(1 - \alpha - \beta\right)(n) + \beta\left(\epsilon\right)}{1 - \alpha} - n \\ &= \frac{\alpha\gamma + \left(1 - \alpha - \beta\right)(n) + \beta\left(\epsilon\right) - n(1 - \alpha)}{1 - \alpha} \\ &= \frac{\alpha\gamma + n - \alpha n - \beta n + \beta\left(\epsilon\right) - n + \alpha n}{1 - \alpha} \\ g_{\frac{Y}{L}} &= \frac{\alpha\gamma - \beta n + \beta\left(\epsilon\right)}{1 - \alpha} \end{aligned}$$

Now we are ready to evaluate when this expression is positive. First, we know that the denominator is always positive, as $\alpha < 1$. Therefore, the whole fraction is positive when the numerator is positive.

$$g_{\frac{Y}{L}} > 0 \Leftrightarrow \alpha\gamma - \beta n + \beta\epsilon > 0 \Leftrightarrow \alpha\gamma + \beta\epsilon > \beta n$$

Before interpreting this result, we must understand what $g_{\frac{Y}{L}}$ indicates. It is the growth rate of what we usually denote y, production in per capita terms. So, asking when $g_{\frac{Y}{L}}$ is positive is the same as asking: "when does production in per capita terms has a positive growth rate?". Hopefully this interpretation of the question helps us understand this condition. There are three factors that affect consumption per capita.

$$\underbrace{\alpha\gamma}_{\text{Technological growth}} + \underbrace{\beta\epsilon}_{\text{Natural Resource Growth}} > \underbrace{\beta n}_{\text{Population growth}}$$

The first two increases product per capita, while the third one decreases it. Therefore, growth will be positive when the sum of the first two is higher than the third.

Question: Can you guess what is the role of α and β exactly?

2 Solow-Swan with non-renewable resources (from TD2!)

This problem is tightly related to the previous one but it has less computations and more intuition. In particular, its focus is to study the employment of renewable and non-renewable resources and its sustainability.

We have the same production function and growth rates as before.

$$Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$$

However, here we specify how Z is composed. We can split natural resources in renewable R and non-renewable N. The rate of exploitation of N is r, therefore we have that $Z = R + rN$.

a. Assume that initially, the economy is in a balanced growth path (BGP) where the stock of renewable resources is stable and where there are no non-renewable resources at all. What is the growth rate of y? Interpret in what conditions we get a positive rate of growth for y. Knowing what we can anticipate about the rate of technology and population growth in the 21^{st} century, should we expect y to grow or not in the coming decades ?

We already have the growth rate of $y = \frac{Y}{L}$ on a balanced growth path from the previous exercise.

$$g_y = g_{\frac{Y}{L}} = \frac{\alpha\gamma - \beta n + \beta(\epsilon)}{1-\alpha}$$

However, in this case there are no non-renewable resources $N = 0$ and the stock of renewable resources R is stable, which implies that it is not growing. Since $Z = R + rN$ and $N = 0$ we have that here $Z = R$. The growth rate of Z was ϵ in the previous exercise, but since here R does not grow, Z does not grow either, as it is composed by R only. This translates into $\epsilon = 0$. The new growth rate is therefore:

$$g_y = \frac{\alpha\gamma - \beta n}{1-\alpha}$$

The evaluation of its sign is the same as before. In particular, we know that $1-\alpha$ is always positive, therefore the sign of the numerator is the significant one. The whole fraction is positive when the numerator is positive:

$$g_y > 0 \Leftrightarrow \alpha\gamma > \beta n$$

We have 4 factors that affect this inequality:

1. α captures the relative importance of capital in the production function. Intuitively, if capital is relatively more important there are more chances that per capita growth is positive, as it is directly affected by technological progress;
2. γ is the rate of growth of technology. Of course, the more technology improves the more likely is that growth per capita increases;
3. β is α counterpart for natural resources, it measures its relative importance in the production. Since these do not grow, if they are less important than growth will be positive despite the fact that the stock is fixed;
4. n represents the growth rate of population. Intuitively, if population increases the growth per capita decreases, as there are more mouths to feed.

Since we know that n is quite low while γ is high, if the premises of this model are true then we would be sure to enjoy positive growth in the future.

b. Assume that at time t (when the economy was previously on the balanced growth path), a new source of non-renewable resource of size N_t is discovered. Each ensuing period, $r\%$ of the resource stock is used in production, such that its stock goes progressively to zero in the long run. If $R = 1$, $N_t = 20$ and $r = 0.1$, what is the growth rate of the supply of resource Z before t? Right after t? In the very long run?

Before answering this question we have to compute the growth rate of Z_t. However, in this case the expression of interest is a sum, therefore we can not use the rules of product and ratios of growth rate. The idea is to subtract Z_{t-1} to Z_t in order to find the Δ. The calculations to attain the growth rate are the following (I omit t in the computations):

$$\begin{aligned}
Z_t &= R_t + rN_t \\
Z_t - Z_{t-1} &= R_t - R_{t-1} + r(N_t - N_{t-1}) \\
\Delta Z &= \Delta R + r\Delta N \\
\frac{\Delta Z}{Z} &= \frac{R}{R}\frac{\Delta R}{Z} + r\frac{N}{N}\frac{\Delta N}{Z} \\
\frac{\Delta Z}{Z} &= \frac{R}{Z}\frac{\Delta R}{R} + r\frac{N}{Z}\frac{\Delta N}{N} \\
g_Z &= \frac{R}{Z}g_R + r\frac{N}{Z}g_N \\
g_{Z,t} &= \frac{R}{R+rN_t}g_R + r\frac{N_t}{R+rN_t}g_{N,t}
\end{aligned}$$

By substituting the numbers we have we obtain the growth rate when the new non-renewable resource is discovered, at time t. Remember that R does not grow ($g_R = 0)$ and that Z has a negative growth of -0.1 We have:

$$
\begin{aligned}
g_{Z,t} &= \frac{R}{R+rN} g_R + r\frac{N}{R+rN} g_N \\
g_{Z,t} &= \frac{1}{1+(0.1)(20)}(0) + \frac{(0.1)(20)}{1+(0.1)(20)}(-0.1) \\
g_{Z,t} &= \frac{(0.1)(20)}{1+(0.1)(20)}(-0.1) = -6.\bar{6}\% = -\frac{1}{15}
\end{aligned}
$$

As for $g_{Z,\tau}$ for $\tau < t$, we have that $g_{Z,\tau} = 0$, as $N_t = 0$ and R does not grow, exactly as we had in the previous point. Instead, when $\tau \to \infty$ the growth rate also goes to zero. This is due to the fact that Z has a negative growth, and therefore after it is completely exploited it will not grow (negatively) anymore.