TD 4

1 Review questions

a. In an ecosystem, the natural growth of a renewable resource is an increasing function of the amount of this resource.

▼ *Answer*

False: As an example, in class, you considered a logistic growth, captured by the equation $\tau(S_t) = rS_t\left(1 - \frac{S_t}{K}\right)$. As you can see, when $S_t \to K$ then $\tau(S_t) \to 0$. Therefore, it is not true that if S_t increases then its growth also increases.

b. An improvement in extractive technology always increases fish production if fishing is free.

▼ *Answer*

False: In our model the total production of fish when there is free entry is given by the following:

$$H_F = B_F \alpha S_F = \left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha}\frac{c}{p}$$

As you can see, we have an α at the denominator with a minus sign (positive effect on H_F), but we also have an α at the denominator with a plus sign (negative effect on H_F). Hence, the total effect is ambiguous.

2 Solow-Swan with non-renewable resources

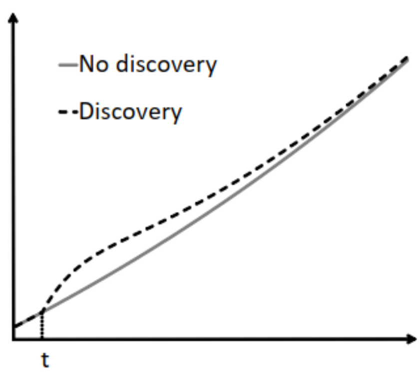
For the last exercise the professor run an analysis with particular values of parameters $n = 0.01$, $g = 0.015$, $\delta = 0.02$, $s = 0.3$, $\alpha = 0.3$, $\beta = 0.2$, $R = 1$, $N_t = 20$, $r = 0.1$. Finally you see the model at work! The graphs in the text of the exercise are the plot of three time series: $\frac{Y}{K}$, $\frac{Y}{L}$ and $\frac{K}{L}$. The continuous line describe an economy in a balanced growth path with no non-renewable resources, while the dashed line depicts a scenario where non-renewable resources are discovered at time t. The main of the following points is to connect the graphs to ratios.

c. Which graph is $\frac{K}{L}$? Explain in words what happens at time t and in the ensuing periods.

Let's try to find a general way of answering these kind of questions. The variables involved are K, L and Y. The question is: how are these variables affected by an increase in N? In order to answer we need to know the dependencies that all these variables have with N. As an example, L is only determined by its growth, we start from L_0 and then we get L_1 based on how big n is. Therefore, L is not directly affected by N. The same hold for K, its growth is given by its growth rate g_K, and not directly by N. Hence, K does not jump either. Instead, $Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$ where $Z_t = R + rN_t$. A jump in N causes a jump in Y. This analysis offers a first insight to answer this question. In fact, we must consider the three ratios $\frac{K}{L}, \frac{Y}{K}$ and $\frac{Y}{L}$. Since of these three the only ratio that does not jump is $\frac{K}{L}$ we are sure that the right graph is C, as there is a smooth evolution of the dashed line.

To understand what happens here recall that $g_K = s\frac{Y_t}{K_t} - \delta$. Since Y increases, as we elaborated before, the growth rate of K increases, so K increases more then what it would without N. However, N will go to zero slowly, which means that the accumulation of capital K slowly go back to its original path.

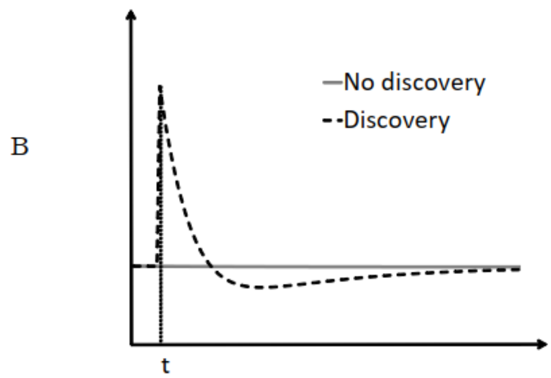
C



d. Which graph is $\frac{Y}{K}$? Explain in words what happens at time t and in the ensuing periods.

First step done, now we have to distinguish between $\frac{Y}{L}$ and $\frac{Y}{K}$. The difference between the two graphs we are left with is that on one of the the balanced growth path is constant (horizontal line). Therefore, we have to answer the question: which ration between $\frac{Y}{K}$ and $\frac{Y}{L}$ should be fixed without the increase in N? Well, we now that the growth rate of L is exogenously given and it is $n > 0$, so it is impossible that L will be fixed. Also Y and K will grow, but at which rate? We know from the previous TD that $g_{\frac{Y_t}{K_t}} = 0 \Leftrightarrow g_Y - g_K = 0 \Leftrightarrow g_Y = g_K$, therefore $\frac{Y}{K}$ is constant and identified by a constant. Hence, it is represented by graph *B*. Here two things happen. First, as we saw before, Y jumps. Instead, K does not jumps immediately, but smoothly increases. Therefore at time t $\frac{Y}{K}$ will steadily increase.

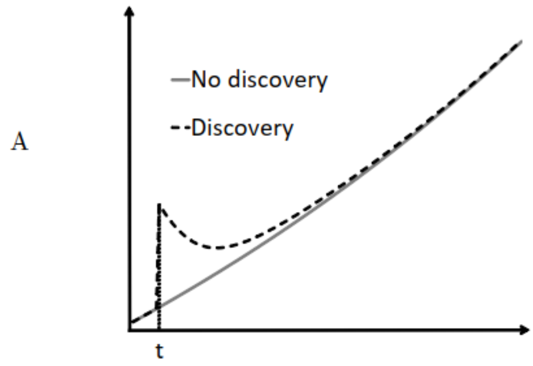
However, the push of Y is only big at time t, after that the growth rate of Y will slowly return normal. Instead, the growth rate of K will increase, and the increase of K will offset the increase of Y. This explains why the dotted line goes below the horizontal line. After some time also the growth of K returns normal, and the dotted line returns on the old balanced growth path.



e. Which graph is $\frac{Y}{L}$? Explain in words what happens at time t and in the ensuing periods.

We are only left with one ratio and graph A, so the answer is easy here, but we must understand also what is going on. The jump is always given by the steady increase of Y, as in the previous points.

However, L will be always increasing with the same rate, in contrast to K in the previous graph. Nevertheless, the ratio immediately starts to decrease after t, slowly reaching the old balanced growth path. This is due to the fact that the stock of natural resources is depleted with a rate way higher $(-6.\bar{6}\%)$ than the rate at which productivity increases (1.5%).



> "We are in the beginning of mass extinction, and all you can talk about is money and fairy tales of eternal economic growth"- Greta Thunberg at the United Nations Climate Action Summit, September 23, 2019

f. There is little doubt that a mass extinction is going on. However, this widespread idea of sustained economic growth being a myth is open to debate. Expanding on the model, can we comment on it.

This is more of a philosophical question, so fell free to answer whatever you think that is relevant and consistent with the model (it is true that many answer could be consistent with what we have here, but there are also many things that are not!). What this little exercise can tell us is that there is no need of non-renewable resources for an economy to grow, as long as other rates $(n, g...)$ are positive. However, a sudden increase in N also brings to a jump in Y. Maybe this model could also suggest that we need a fixed amount of renewable resources R, even if it does not grow. Therefore, according to this model, it might be not a very good idea to destroy forests.

2 The baby boom (from last year midterm!)

This exercise is not significantly different to exercises 2 and 3 of TD1! You should already know how to do the computations and the graph of the first questions. The second part is a little bit more difficult and it requires you to translate what is happening in mathematics. We have $F(k_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. Investment is, as always, $I = sY = sF(K_t, L_t)$. The law of motion of capital is $\Delta K_t = K_{t+1} - K_t = I - \delta K_t$. The work population grows at rate $n = \frac{L_{t+1} - L_t}{L_t} = 0.05$. We also have that $\delta = 0.05$, $s = 0.1$ and $\alpha = 0.5$.

Remark: In these solutions I spell out every step just for you to understand, as you can see from the professor's solutions it is not needed that you specify each step as I do. However, for sure it can not hurt you.

a. On the balanced growth path, k and y are stable. Compute their numerical values.

The answer to this question requires no more than performing the computations we have been doing in the previous TDs and substituting numbers. First, we have to convert everything in per capita terms. The production function becomes:

$$
\begin{aligned}
\frac{1}{L_t}F(K_t, L_t) &= F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}\right) \\
&= \left(\frac{K_t}{L_t}\right)^{\alpha}\left(\frac{L_t}{L_t}\right)^{1-\alpha} \\
f(k_t) &= k_t^{\alpha}
\end{aligned}
$$

You should recognise that we have been using this function a lot! Is the classical Cobb-Douglas. To compute the numerical value of k in the balanced growth path we can rely on the condition under which this variable is indeed on such path, which means that its growth rate is equal to zero. Hence, we must first compute its growth rate.

$$
\begin{aligned}
\frac{\Delta k_t}{k_t} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\
&= \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n \\
&= \frac{s\frac{1}{L_t}F(K_t, L_t) - \delta\frac{1}{L_t}K_t}{\frac{1}{L_t}K_t} - n \\
\frac{\Delta k_t}{k_t} &= \frac{sf(k_t) - \delta k_t}{k_t} - n \\
\Delta k_t &= sf(k_t) - \delta k_t - nk_t \\
&= sf(k_t) - (\delta + n)k_t
\end{aligned}
$$

The condition for being in steady state its growth rate equal to 0, thus, we check what this condition implies in this exercise.

$$
\Delta k_t = 0 \Leftrightarrow sf(k_t^*) - (\delta + n)k_t^* = 0 \Leftrightarrow s(k^*)^{\alpha} = (\delta + n)k_t^*
$$

By substituting the numbers we are given in the text we obtain that:

$$
k^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{0.05 + 0.05}{0.1}\right)^{-2} = 1
$$

Since $y^* = f(k^*)$ we obtain:

$$y^* = (k^*)^\alpha = 1^{1/2} = 1$$

b. Is the savings rate s is at its golden rule value? If not, what should be the golden-rule savings rate?

From problem 2 of TD 1 you may recall that the golden rule savings rate is $s = \alpha = 0.5$, while in this case we have that $s = 0.1 \neq 0.5$! However, let's try to prove it again. There are many ways to do it. The first one is to express consumption as $c^* = (1-s)f(k^*)$ and compute the s that maximises it in steady state. The steps to do this are detailed in the solution of TD 1. A different method could be to check the k which comes from the golden rule of savings and derive the s for which we get that k. The steps are the following. First, express consumption only as a function of capital:

$$\begin{aligned}
c^* &= (1-s)y^* \\
&= (1-s)f(k^*) \\
&= f(k^*) - sf(k^*) \\
&= f(k^*) - (\delta+n)k^*
\end{aligned}$$

Where the last step come from the steady state condition we derived in the previous point. We then obtain the k that maximises consumption by solving a maximisation problem:

$$\begin{aligned}
\frac{\partial c^*}{\partial k^*} = 0 &\Rightarrow f'(k^*) - (\delta+n) = 0 \\
&\Rightarrow \alpha(k^*)^{\alpha-1} = (\delta+n) \\
&\Rightarrow k^* = \left(\frac{\delta+n}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
&\Rightarrow k^* = \left(\frac{0.05+0.05}{0.5}\right)^{\frac{1}{0.5-1}} = 25
\end{aligned}$$

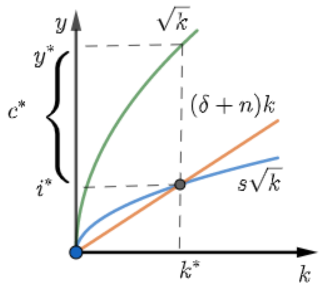
Which is quite different from the $k^* = 1$ we got before! What is the s that rationalises this result? From the expression we derived in the previous point we have:

$$
\begin{aligned}
k^* &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \\
25 &= \left(\frac{0.05+0.05}{s}\right)^{\frac{1}{0.5-1}} \\
25 &= \left(\frac{0.1}{s}\right)^{-2} \\
25 &= \left(\frac{s}{0.1}\right)^{2} \\
5 &= \left(\frac{s}{0.1}\right) \\
0.5 &= s \neq 0.1
\end{aligned}
$$

Which indeed gives us $\alpha = s$. Remember that $i^* = sy^* = 0.1(1) = 0.1$, moreover $c^* = y^* - i^* = 1 - 0.1 = 0.9$.

c. Make a plot like the ones made in class with k on the horizontal axis and y on the vertical axis. Draw $f(k) = k^{0.5}$. Draw $sf(k)$. Draw $(n+\delta)k$. Indicate k^* and y^*.

The graph is ecxactly the one we did in exrcise 3 odf TD 1! I'll report the same picutre I put there. The production function is $f(k_t) = (k_t)^{\frac{1}{2}} = \sqrt{k_t}$, while $\delta = n = 0.05$, $y^* = 1$ and $k^* = 1$ (you may want to substitute the numbers in your graph).



Graph from exercise 3 TD1.

1. $\sqrt{k}$
2. $(\delta+n)k_t$
3. $s\sqrt{k}$

d. After World War 2, many American soldiers fighting in Europe came back home and made lots of babies. Imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. What is the numerical value of the growth rate of k_t just after the doubling? (If you don't have a calculator, you can leave a mathematical expression as is (as long as it just involves numbers, no variables).)

We are asked to compute the growth rate of k_t at time t, when the populatoin doubles. We have to perform the exact computations we did in the previous points, but instead of having L_t, we have $2L_t$. We start from the production function:

$$\begin{aligned}\frac{1}{2L_t}F(K_t, 2L_t) &= F\left(\frac{K_t}{2L_t}, \frac{2L_t}{2L_t}\right) \\ &= \left(\frac{K_t}{2L_t}\right)^{\alpha}\left(\frac{2L_t}{2L_t}\right)^{1-\alpha} \\ f(k_t) &= \left(\frac{k_t}{2}\right)^{\alpha} = \bar{k}_t^{\alpha}\end{aligned}$$

Now we have to compute the growth rate. Again the calculations follow the same logic as before:

$$\begin{aligned}\frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\ &= \frac{sF(K_t, 2L_t) - \delta K_t}{K_t} - n \\ &= \frac{s\frac{1}{2L_t}F(K_t, 2L_t) - \delta\frac{1}{2L_t}K_t}{\frac{1}{2L_t}K_t} - n \\ \frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{sf(k_t) - \delta\frac{k_t}{2}}{\frac{k_t}{2}} - n \\ \Delta k_t &= sf(k_t) - \delta\frac{k_t}{2} - n\frac{k_t}{2} \\ &= sf(k_t) - (\delta + n)\frac{k_t}{2}\end{aligned}$$

By expliciting the production function we obtain:

$$\Delta k_t = s\left(\frac{k_t}{2}\right)^{\alpha} - (\delta+n)\frac{k_t}{2} = 0.1\left(\frac{1}{2}\right)^{0.5} - (0.05+0.05)\frac{1}{2} = 0.0207$$

However, we must find the growth rate, not only the Δ:

$$\frac{\Delta k_t}{\frac{k_t}{2}} = \frac{0.0207}{\frac{1}{2}} = 0.0414$$

e. As previously, imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. Also, since these young people want to make babies as fast as possible, at the same moment, the growth rate jumps from $n = 0.05$ to $n = 0.15$. Reproduce the graph in c. and show how the long run equilibrium will change. Indicate what happens to k just after soldiers get back (at time t) and where the economy converges in the long run (indicate numerical values).

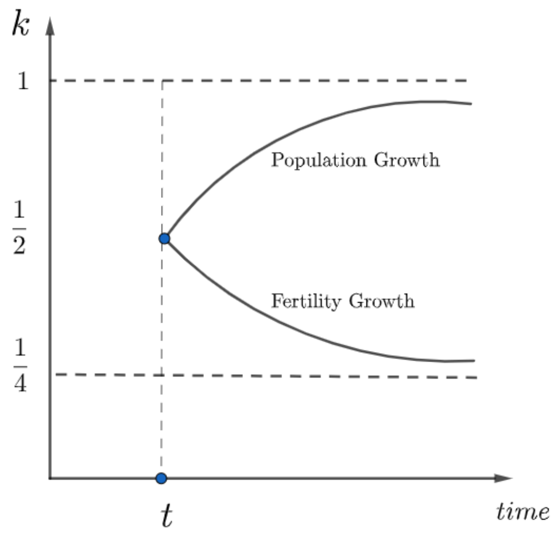
Before showing the graph we need to compute the new k^* and y^* . As always, we set the law of motion equal to zero:

$$\begin{aligned}\Delta k_t &= 0 \\ &\Leftrightarrow s\left(\frac{k_t^*}{2}\right)^{\alpha} - (\delta+n)\frac{k_t^*}{2} = 0 \\ &\Leftrightarrow \left(\frac{k_t^*}{2}\right)^{\alpha-1} = \frac{\delta+n}{s} \\ &\Leftrightarrow \left(\frac{k_t^*}{2}\right) = \left(\frac{0.05+0.015}{0.1}\right)^{\frac{1}{0.5-1}} = \frac{1}{4}\end{aligned}$$

As for y^* we have that $y^* = f(k^*) = \left(\frac{k_t^*}{2}\right)^{0.5} = \left(\frac{1}{4}\right)^{0.5} = \frac{1}{2}$. I do not include the graph here again as it is the same graph as before with different numbers.

f. Plot k over time around the period t (hence make a graph with time on the horizontal axis and k on the vertical axis). On the same graph, show how k adjusts after t in the situation described in d. and in the situation described in e. Make sure to show where k is converging in each case.

At time t the input in the production function is immediately halved due to the increase in population. If there is no fertility growth after a while k will return on its original balanced growth path. In the case of fertility growth, instead, it will converge to $\frac{1}{4}$.



g. In the context of this model, are American workers (those always in the US) better off before time t (before the influx of workers), some time after time t if fertility does not change d. some time after t or if fertility does change e.? Rank the three situations from best to worst and justify briefly.

This question just amounts to compare the y^* in different circumstances. In $\tau < t$ we have $y^*_\tau = 1$, when fertility increase $(n = 0.15)$ occurs at time t we have that the new steady state gdp is $y^* = \frac{1}{2}$, while without the increase in n but after the increase in population we are outside the growth path leading to $y^* = 1$, and therefore we are slightly below this value. Therefore, the best situation is the one before the shock, then we have the increase in population without the increase in n and lastly the worse situation is when we also observe a fertility rate increase.