Solutions Manual

Topics in Macro 1 (Fall 2020)

Enrico Mattia Salonia

TD 1

1 Review Questions

a. Inequality of income across countries was already very wide at the end of the medieval period.

▼ *Answer*

False: It started widening after the industrial revolution. This is due to the fact that before the revolution growth was almost null around the world. The production was very close to subsistence level and there were no investments. The technological progress of the revolution was the catalyst of growth which developed differently and caused inequalities among different countries.

2 The Golden Rule for Savings

The aim of this problem is to get you used with the basic calculations of the Solow model. This exercise is a particular case in which we have a Cobb-Douglas production function, population growth and no technology. Therefore we have $n > 0$ and $g = 0$ and the production function $f(k) = k^\alpha$ with $0 < \alpha < 1$.

Question: Can you get what is the original production function $F(K_t, L_t)$?

a. Express the steady-state level of consumption c^* as a function of k^* and the exogenous parameters n, δ and α (but not s).

First, we have to recall how we express consumption in this model. What we consume is what we produce minus what we invest, therefore $c = y - sy$. However, production is a function of capital $y = f(k)$. We are also reminded that $sf(k^*) = (\delta + n)k^*$. This is key to express c as a function of the exogenous variables and k, but not s:

$$\begin{aligned} c^* &= (1-s)y^* \\ &= (1-s)f(k^*) \\ &= f(k^*) - (\delta + n)k^* \quad \text{By substituting the steady state condition} \\ &= (k^*)^\alpha - (\delta + n)k^* \end{aligned}$$

Here we have consumption as a function of k^*, n, δ and α.

b. Use the result to the previous question to find the optimum level of k^* from the point of view of consumption.

This question is asking us to maximise consumption with respect to capital. In fact, we want to answer the question: what is the level of k^* that gives me the maximum c^*? The first order condition of this problem amounts to set the first derivative to be equal to 0, as we do in the following:

$$\frac{\partial c^*}{\partial k^*} = 0 \Rightarrow \alpha(k^*)^{\alpha-1} - (\delta + n) = 0 \Rightarrow k^* = \left(\frac{\delta + n}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{1}$$

Remember that you can express this quantity in a different way by playing with the exponent!

Question: what about the second order condition?

c. Express the steady-state level of consumption c^* as a function of the exogenous parameters only (n, δ, α, also including s).

We have to perform the same operation as before, but without including k^* in the expression for consumption. Therefore, we must first find k^* as a function of the exogenous variables only, to be able to substitute it in the expression for consumption. We start from the fact that $sf(k^*) = (\delta + n)k^*$:

$$
\begin{aligned}
sf(k^*) &= (\delta + n)k^* \\
\frac{1}{k^*}(k^*)^\alpha &= \frac{\delta + n}{s} \\
(k^*)^{-1}(k^*)^\alpha &= \frac{\delta + n}{s} \\
(k^*)^{\alpha - 1} &= \frac{\delta + n}{s} \\
(k^*)^{\frac{\alpha-1}{\alpha-1}} &= \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha-1}} \\
k^* &= \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha-1}}
\end{aligned}
\tag{2}
$$

We can now substitute k^* in the expression for c^*, as it is expressed only as a function of exogenous variables:

$$
\begin{aligned}
c^* &= (1 - s)f(k^*) \\
&= (1 - s)\left(\frac{\delta + n}{s}\right)^{\frac{\alpha}{\alpha-1}}
\end{aligned}
$$

Here we have c^* which only depends on s, n, δ and α.

d. Use the result to the previous question to find the optimum level of the savings rate s from the point of view of consumption.

We have to perform the exact same operation as before, but instead of maximising for k^* we do it for s, but first we rewrite c^* to be able to take the derivative easily. We know in fact that $\left(\frac{\delta+n}{s}\right)^{-a} = \left(\frac{s}{\delta+n}\right)^a$. In our case the new fraction is therefore $\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$. We also isolate the variable with respect to which we must take the derivative (only to make the computations easier, there is no need to do this if you are comfortable with derivatives):

$$c^* = (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} = (1-s)(s)^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

Now we are ready to solve the maximisation problem (here I provided the calculations in a very detailed way, if you are comfortable with calculus you do not need to write everything as I do here). First, I recall the rules for deriving a product. In general we have the following:

$$\frac{\partial f(x)g(x)}{\partial x} = f'(x)g(x) + f(x)g'(x)$$

In our case the two functions are $f(s) = (1-s)$ and $g(s) = s^{\frac{\alpha}{1-\alpha}}$. Moreover, everything is multiplied by the constant $\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$ which does not affect the derivative. First, we compute the derivative for each of the function that compose the product:

$$g'(s) = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}-1} = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}$$

$$f'(s) = -1$$

Then, by applying the general rule:

$$\begin{aligned}\frac{\partial[f(s)g(s)]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}}{\partial s} &= \frac{\partial[f(s)g(s)]}{\partial s}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \\ &= \left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

By setting the derivative equal to zero, we can get rid of the constant (equivalent to divide each side of the equality by the constant itself).

$$\left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\cancel{\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}} = 0$$

We are therefore left with the following:

$$
\begin{aligned}
\frac{\partial c^*}{\partial s} = 0 \Rightarrow & \left[-1 s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha} s^{\frac{\alpha}{1-\alpha}} \frac{1}{s} \right] = 0 \\
\Leftrightarrow & \left[-1 \cancel{s^{\frac{\alpha}{1-\alpha}}} + (1-s)\frac{\alpha}{1-\alpha} \cancel{s^{\frac{\alpha}{1-\alpha}}} \frac{1}{s} \right] = 0 \\
\Leftrightarrow & \left[-1 + (1-s)\frac{\alpha}{1-\alpha} \frac{1}{s} \right] = 0 \\
\Leftrightarrow & (1-s)\frac{\alpha}{1-\alpha}\frac{1}{s} = 1 \\
\Leftrightarrow & (1-s)\alpha = (1-\alpha)s \\
\Leftrightarrow & \alpha = s
\end{aligned}
$$

Therefore, the s that maximise consumption is exactly α, the exponent of the production function. This is more or less intuitive, the more your function is relatively productive, as captured by α, the more you save.

Question: Again, what about the second order conditions for this problem?

e. Comment the results obtained to questions 2 and 4: how are they related?

Consider the expressions (1) and (2) that we computed in points **b**. and **c**.

$$k_1^* = \left(\frac{\delta + n}{\alpha}\right)^{\frac{1}{\alpha - 1}} \tag{1}$$

$$k_2^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} \tag{2}$$

Next, consider the result $\alpha = s$ obtained in point **d**.

$$\alpha = s \tag{3}$$

By combining (1) or (2) with (3) (substituting s or α in one of the two expressions) you find that $k_1^* = k_2^*$! In fact, if

$s \neq \alpha$ then we would have that two different expressions maximise consumption, which is not possible if we only have one maximum in steady state, as in this case.

TD 2

1 Review Questions

e. At the steady state, investments are equal to what is lost to depreciation, population growth and technological progress.

▼ Answer

TRUE: At the steady state we must have that capital per unit of effective labour must not grow. From the results of the class we have

$$\Delta\tilde{k} = sf\left(\tilde{k}\right) - (\delta + n + g)\tilde{k}$$

Remember that here $\tilde{k} = \frac{K}{AL}$. By setting $\Delta\tilde{k} = 0$ we get:

$$sf\left(\tilde{k}^*\right) = (\delta + n + g)\tilde{k}^*$$

Which exactly means that investments $\tilde{i}^*$ are equal to the loss in capital due to depreciation, population growth and technological progress.

f. The golden rule of savings states that in steady state, capital should be barely productive enough to compensate for depreciation, population growth and technological progress.

▼ Answer

TRUE: The golden rule of savings tells us what is the optimal $\tilde{k}$ (or s, in the previous TD we found a relation between these two problems) to maximise steady state consumption $\tilde{c}^*$. To work it out we need to recall how to express consumption. We consume what we produce minus what we invest, therefore $\tilde{c}^* = (1-s)\tilde{y}^* = (1-s)f\left(\tilde{k}^*\right)$. By the result in the previous question we can rewrite this expression:

$$\begin{aligned}\tilde{c}^* &= (1-s)f\left(\tilde{k}^*\right)\\ &= f\left(\tilde{k}^*\right) - sf\left(\tilde{k}^*\right)\\ &= f\left(\tilde{k}^*\right) - (\delta+n+g)\tilde{k}^*\end{aligned}$$

What is the capital that maximises $\tilde{c}^*$? We have to solve a standard optimisation problem.

$$\underset{\tilde{k}^*}{max} \quad \tilde{c}^* \Rightarrow \underset{\tilde{k}^*}{max} \quad f\left(\tilde{k}^*\right) - (\delta+n+g)\tilde{k}^*$$

$$\frac{\partial \tilde{c}^*}{\partial \tilde{k}^*} = 0 \Rightarrow f'\left(\tilde{k}^*\right) - (\delta+n+g) = 0 \Rightarrow f'\left(\tilde{k}^*\right) = \delta+n+g$$

Which exactly tells us that the marginal productivity of capital must exactly offset the loss due to depreciation, population growth and technological progress (remember that $f'\left(\tilde{k}^*\right)$ tells us how much production increases after an infinitesimal change in $\tilde{k}^*$, namely the marginal product).

3 Exercise - Convergence Towards the steady state

The aim of this exercise is to understand the role of assumptions in the Solow - Swan model. It may be tempting to read assumptions once and then forget about them, but they are of crucial importance in these and in all the other models in economics (science and reasoning in general).

Our production function for the first point is $F(K_t, L_t) = (K_t L_t)^{\frac{1}{2}} = \sqrt{K_t L_t}$. The saving rate is s, population growth rate is $\frac{\Delta L_t}{L_t} = n$ and capital depreciates at rate δ.

a. Represent graphically in the (k, y) plane the dynamics of the Solow model with population growth and no technological change.

The first step is to transform all the variables in per capita quantities. We perform this step because we are interested in the steady state of capital per worker, and not in its absolute value.

1. $k_t = \frac{K_t}{L_t}$
2. $y_t = \frac{Y_t}{L_t}$
3. $c_t = \frac{C_t}{L_t}$
4. $i_t = \frac{I_T}{L_t}$

As for the production function, we perform the same changes by dividing with L_t and exploiting constant returns to scale (*CRS*).

$$\begin{aligned} \frac{1}{L_t}F(K_t, L_t) &= \left(\frac{K_t}{L_t}\frac{\cancel{L_t}}{\cancel{L_t}}\right)^{\frac{1}{2}} \\ &= \left(\frac{K_t}{L_t}\right)^{\frac{1}{2}} \\ f(k_t) &= (k_t)^{\frac{1}{2}} \end{aligned}$$

To check for the dynamics of the model we need the *law of motion of capital,* as capital is the principal state (endogenous) variable which determines what happens in the economy as time changes. The law of motion tells us how capital evolves in time. We ask ourselves the question: "if at time t I have capital k_t (per capita), how much capital k_{t+1} do I have in the next period? To answer this question we need to know what is the relation between these two variables.

On the one hand, we have the saved resources that we can use in the next period $sy = sf(k_t)$, on the other hand, the capital we have depreciates at rate δ, and therefore we lose δk. Moreover, in this formulation of the model we also have population growth. An increase in population decreases the amount of capital per capita. These are the two important factor that affects the dynamics of capital. To find the law of motion we rely on the rules of growth rates of products and ratios, in particular:

$$\frac{\Delta k_t}{k_t} = \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \qquad \text{Check page 17 of the lecture notes to verify that this is true}$$

We already know that $\frac{\Delta L_t}{L_t} = n$. Moreover, we know from the standard Solow model that $\Delta K_t = sF(K_t, L_t) - \delta K_t$. Therefore,

$$\frac{k_{t+1}-k_t}{k_t} = \frac{\Delta k_t}{k_t} = \frac{sF(K_t,L_t)-\delta K_t}{K_t} - n$$
$$\frac{\Delta k_t}{k_t} = \frac{\frac{1}{L_t}(sF(K_t,L_t)-\delta K_t)}{\frac{1}{L_t}K_t} - n$$
$$\frac{\Delta k_t}{k_t} = \frac{sf(k_t)-\delta k_t}{k_t} - n$$

By multiplying on the right and on the left by k_t we get the final expression:

$$\Delta k_t = \underbrace{sf(k_t)}_{\text{Saved resources}} - \underbrace{\delta k_t}_{\text{Depreciation loss}} - \underbrace{nk_t}_{\text{Population growth loss}}$$

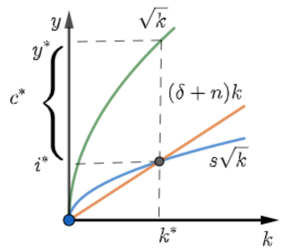
$$\Delta k_t = sf(k_t) - (\delta+n)k_t$$

In the steady state variables do not change over time, therefore the capital per capita will be stable, which means that $\Delta k_t = 0$.

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) = (\delta+n)k_t^*$$

Investment exactly offsets the loss due to depreciation and population growth.

In our case $f(k_t) = (k_t)^{\frac{1}{2}} = \sqrt{k}$, therefore the three elements of our graph are:



Graph1: Solow Model with Population Growth

1. $\sqrt{k}$
2. $(\delta+n)k_t$
3. $s\sqrt{k}$

As always we have that $y^* = f(k^*)$, $i^* = s(fk^*)$ and $c^* = f(k^*) - sf(k^*) = y^* - i^*$, as shown in the graph.

b. Same question for the AK production function $F(K_t, L_t) = AK_t$, assuming $sA > \delta + n$. Does k^* exist in this case?

For this point I offer a different path to reach the solution with respect to the one you will receive from the professor. I think this way is more in line with the standard method, but you choose which you prefer.

To answer this question we proceed as we did in the previous point, but of course, we have to take into account the different production function and the technological change. First, let's express F as a function of per capita capital. We perform the same computation as before.

$$\frac{1}{L_t}F(K_t, L_t) = \left(A\frac{K_t}{L_t}\right)$$
$$f(k_t) = Ak_t$$

Notice that, contrary to what we do in the lecture notes, the exercise asks to draw the graph in the space (k, y) and not $\left(\tilde{k}, \tilde{y}\right)$, that's why we do not divide by units of effective labour AL. What if we had $\tilde{k} = \frac{K_t}{AL}$? Due growth rates rules:

$$\begin{aligned}\frac{\Delta\tilde{k}}{\tilde{k}} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} - \frac{\Delta A}{A} \\ &= \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n - g \\ \Delta\tilde{k} &= sf\left(\tilde{k}\right) - (\delta + n + g)\tilde{k}\end{aligned}$$

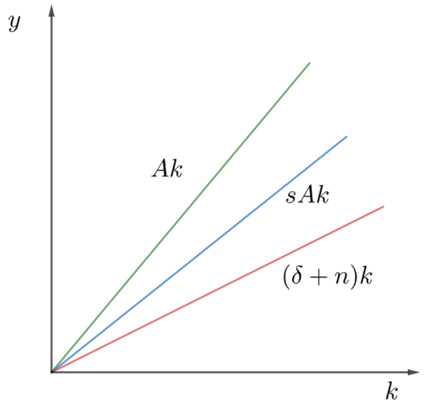
This is also why the law of motion of capital is the same as the one in the previous point, we still have the following:

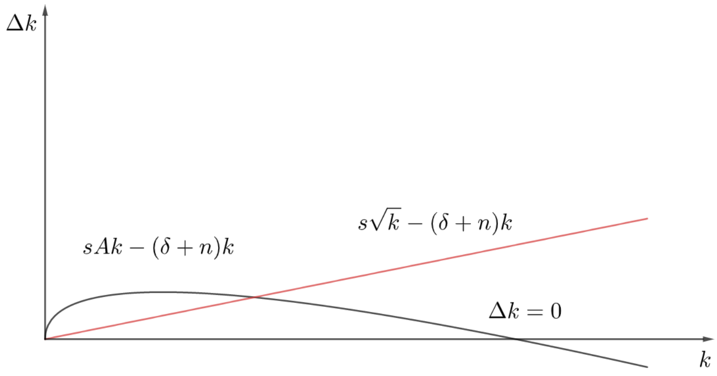
$$\begin{aligned}\Delta k_t &= sf(k_t) - (\delta + n)k_t \\ &= sAk_t - (\delta + n)k_t\end{aligned}$$

So, does k^* exists in this case? The steady state condition is always the same:

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) = (\delta + n)k_t^* \Leftrightarrow sA\cancel{k_t^*} = (\delta + n)\cancel{k_t^*} \Leftrightarrow sA = (\delta + n)$$

However, in this model, this can never be true as we have $sA > \delta + n$! So the answer is no, there exists no steady state capital k^* (notice however that $k^* = 0$ is a solution, in fact, the condition $sAk_t^* = (\delta + n)k_t^*$ is respected in this case). The reason is also apparent from the graph. Inspired by a question of one of your classmates (and the mistake in the text) I also plotted a graph of the two economies we just studied in the $(k, \Delta k)$ space.





Graph 2 and 3: Solow model with technological change and linear production function. *Question*: Can you guess what point the intersection between the curve and the x-axis is?

The problem here is that the saving rate and the technological change outset the decrease of capital per capita due to depreciation and population growth. Therefore, the increase in capital will always be greater than its loss. Its growth will never stop, it will continue to increase in every time t. This example shows why assumptions are a fundamental ingredient of the model and not something we use just for convenience and that we can forget by putting them below the carpet. We will discuss this in the following point.

Question: Do you think the results would have been different if we checked capital per units of effective labour $\left(\tilde{k} = \frac{K_t}{AL_t}\right)$?

c. Make a list of the properties that the AK function does not satisfy with regards to the Solow model. Which one explains the previous result?

In the Solow model, we have three assumptions on the production function and three extra assumptions that are denominated *Inada Conditions*.

1. $F(K) > 0$ if $K > 0$. This condition just ensures that our production is positive if we use a positive amount of capital;
2. $\frac{\partial F(K)}{\partial K} > 0$. This condition tells us that we have a *positive marginal product*, that is, increasing capital always increases production;

3. $\frac{\partial^2 F(K)}{\partial K^2} < 0$. This condition is a crucial one in this exercise. It ensures that the marginal product is decreasing in K. This means that the n^{th+1} unit of capital K will increase the production less than the n^{th} one.

Then we have the *Inada Conditions.*

1. $F(0) = 0$. This only imposes that you can not have a positive production by employing zero capital;
2. $\lim\limits_{K \to 0} \frac{\partial F(K)}{\partial K} = \infty$. This assumption just tells us that a little bit of capital is infinitely productive, as we go from 0 production to positive production;
3. $\lim\limits_{K \to \infty} \frac{\partial F(K)}{\partial K} = 0$. This assumption is also broken in this exercise. It ensures that employing an infinite amount of capital is not convenient, as the marginal product will eventually reach 0 so that using capital will not be productive at all and therefore will be wasted.

Let's check that $F(K_t, L_t) = AK_t$ does not satisfy assumption 3 and Inada condition 3. We have

$$\frac{\partial F(K)}{\partial K} = A > 0$$

$$\frac{\partial^2 F(K)}{\partial K^2} = 0 \geq 0$$

$$\lim_{K \to \infty} \frac{\partial F(K)}{\partial K} = A \neq 0$$

Therefore, as we noticed before, capital is too productive and its increase due to production always offset its loss due to depreciation and population growth.

Question: Check that the production function in the first part of the exercise indeed satisfies all these assumptions.

TD 3

1 Review Questions

b. On a balanced growth path, all variables grow at the same rate.

▼ *Answer*

False: Recall the definition of Balanced growth path at page 19 on you lecture notes:

> *Definition: A balanced growth path is a trajectory such that all variables grow at a constant rate.*

Translated in mathematical terms, we have that all the variables x_i in our model must have $g_{x_i} = k_i$, where k_i is a constant. However, it is not specified that all k_i must be equal! Each variable can grow at its own, constant rate. We have an example in the exercise below, where different variables of interest grows at different rates.

c. The Solow model needs to assume technological change to check the stylised Kaldor facts of growth.

▼ *Answer*

True: Consider as an example Kaldor fact 1:

> *Kaldor fact 1: Labour productivity has grown at a sustained rate.*

If we do not have technology, labour productivity does not grow in the steady state. In fact, if we do not have technology and we are in a steady state then $g_k = g_K - g_L = 0$, since $y = f(k)$, if k does not grow then also y will not grow. You can check this on page 19 of your lecture notes, but the next exercises constitute a clear example of why this is true. Introducing a technological shift which grows at rate g makes growth positive.

Question: Try to argue the same thing by considering Kaldor fact 2 about capital per worker.

d. The Solow model predicts convergence of all economies in the world to the same GDP per capita.

▼ *Answer*

False: The Solow model can be interpreted as a machine that takes as an input exogenous parameter $(n, \delta, etc...)$ and tells you what happens to the economy. To a different set of exogeneous parameters we obtain a different prediction. Consider as an example the economies at points a. and b. of exercise 3 of this TD(1), the growth predictions are completely different.

4 Problem - The Solow model with natural resources

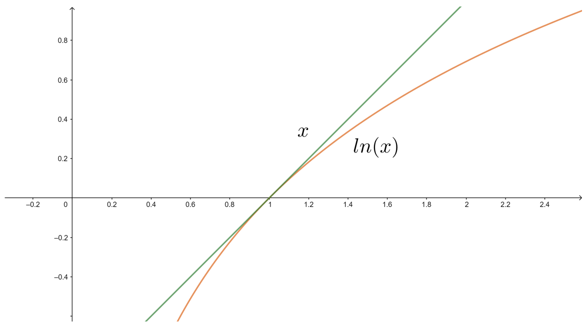
The aim of this exercise is to get you used with growth rates calculations. It is in some sense less interesting from an intuitive point of view, but we will be able to link it to exercise 2 in TD2.

In this problem we have quite a lot of data. The production here is affected by three variables, capital K_t, labour L_t and a natural resource Z_t. We also have capital augmenting technology A_t. The function is the following

$$Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$$

The law of motion of capital is the standard one $\Delta K_t = K_{t+1} - K_t = (1-\delta)K_t + I_t$ where $I_t = sY_t$. Technology, grows at an exogenously fixed rate $A_{t+1} = (1+\gamma)A_t$. Also the stock of natural resources grows at an exogenous fixed rate $Z_{t+1} = (1+\epsilon)Z_t$. Labour also grows, as we already saw $L_{t+1} = (1+n)L_t$. Throughout the problem we will use the following useful approximation:

$$log\left(\frac{X_{t+1}}{X_t}\right) \approx \frac{X_{t+1}}{X_t} - 1$$



a. In this problem use g_x to denote the growth rate of the variable x (for example $g_y = log\left(\frac{Y_{t+1}-Y_t}{Y_t}\right)$) From the definitions, write g_A, g_L and g_Z.

Let's use the definition and the approximation we are given. We start from g_A.

$$\begin{aligned}(1+\gamma)A_t &= A_{t+1}\\ (1+\gamma) &= \frac{A_{t+1}}{A_t}\\ \gamma &= \frac{A_{t+1}}{A_t} - 1 \approx log\left(\frac{A_{t+1}}{A_t}\right)\\ \gamma &= g_A\end{aligned}$$

We can perform the same calculations to see that $g_L = n$ and $g_Z = \epsilon$.
Question: Try to find g_L and g_Z as an exercise.

b. Compute g_Y in terms of $\alpha, \beta, g_A, g_K, g_Z$, and g_L.

This seems like a daunting task, so let's divide this computation by steps.

First, we must identify the variable of which we want to compute the growth rate. In this case we have from the text $Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$.

Second, we use the explicit expression of growth rates to understand how its growth rate is composed. Since we have that $g_Y = log\left(\frac{Y_{t+1}}{Y_t}\right)$, we first have to compute $\left(\frac{Y_{t+1}}{Y_t}\right)$.

$$\begin{aligned}\left(\frac{Y_{t+1}}{Y_t}\right) &= \frac{\left(A_{t+1}K_{t+1}\right)^{\alpha}\left(L_{t+1}\right)^{1-\alpha-\beta} Z_{t+1}^{\beta}}{\left(A_tK_t\right)^{\alpha}\left(L_t\right)^{1-\alpha-\beta} Z_t^{\beta}} \\ &= \left(\frac{A_{t+1}}{A_t}\frac{K_{t+1}}{K_t}\right)^{\alpha}\left(\frac{L_{t+1}}{L_t}\right)^{1-\alpha-\beta}\left(\frac{Z_{t+1}}{Z_t}\right)^{\beta}\end{aligned}$$

Third, we take logs, so that we have a direct expression for g_Y.

$$\begin{aligned}log\left(\frac{Y_{t+1}}{Y_t}\right) &= log\left[\left(\frac{A_{t+1}}{A_t}\frac{K_{t+1}}{K_t}\right)^{\alpha}\left(\frac{L_{t+1}}{L_t}\right)^{1-\alpha-\beta}\left(\frac{Z_{t+1}}{Z_t}\right)^{\beta}\right] \\ &= \alpha\left(log\left(\frac{A_{t+1}}{A_t}\right)+log\left(\frac{K_{t+1}}{K_t}\right)\right)+(1-\alpha-\beta)\left(log\left(\frac{L_{t+1}}{L_t}\right)\right)+\beta\left(log\left(\frac{Z_{t+1}}{Z_t}\right)\right) \\ &= \alpha\left(g_A+g_K\right)+(1-\alpha-\beta)\left(g_L\right)+\beta\left(g_Z\right) \\ g_Y &= \alpha\left(\gamma+g_K\right)+(1-\alpha-\beta)\left(n\right)+\beta\left(\epsilon\right)\end{aligned}$$

We have exactly g_Y in terms of $\alpha, \beta, g_A, g_K, g_Z$, and g_L.

c. Compute g_K in terms of δ, s and $\frac{Y_t}{K_t}$.

Exactly as before, we exploit the definition of growth rate and what we know about K_t. The law of motion of capital is always the same.

$$K_{t+1} - K_t = \Delta K_t = sF(A_t, K_t, L_t, Z_t) - \delta K_t$$

We elaborate a little bit on this expression to put it in a form that is convenient to us. First, we divide by K_t to explicitly have the growth rate.

$$\begin{aligned}\frac{K_{t+1}-K_t}{K_t} &= \frac{sF(A_t, K_t, L_t, Z_t) - \delta K_t}{K_t} \\ \frac{K_{t+1}}{K_t} - 1 &= s\frac{Y_t}{K_t} - \delta \\ log\left(\frac{K_{t+1}}{K_t}\right) &\approx s\frac{Y_t}{K_t} - \delta \qquad \text{(By the approximation given in the text)} \\ g_K &\approx s\frac{Y_t}{K_t} - \delta\end{aligned}$$

We managed to find an expression of g_K in terms of δ, s and $\frac{Y_t}{K_t}$.

d. Argue why, along a balanced growth path, $\frac{Y_t}{K_t}$ must be constant. Then argue why $g_Y = g_K$.

Recall the definition of a balance growth path: all the variables must grow at a constant rate! This means, in order, that K_t must grow at a constant rate, that g_K must be equal to a constant, and that $s\frac{Y_t}{K_t} - \delta$ must be constant. We know that s and δ are indeed constant, but, if we are not on a balance growth path $\frac{Y_t}{K_t}$ evolve with time. Therefore, $\frac{Y_t}{K_t}$ for g_K to be constant, so that K grows at a constant rate.

As for the second question, the answer is only one step ahead of the previous reasoning. In order for the ratio $\frac{Y_t}{K_t}$ to be constant, the two variables must grow at the same rate in each time t. If, as an example, K_t grows quicker than Y_t, the ratio will not be constant in time, therefore $g_Y = g_K$. More precisely, if $\frac{Y_t}{K_t}$ grows at a constant rate, it means that $g_{\frac{Y_t}{K_t}} = 0$. By exploiting the rules of growth rates:

$$g_{\frac{Y_t}{K_t}} = 0 \Leftrightarrow g_Y - g_K = 0 \Leftrightarrow g_Y = g_K$$

Which is what we wanted to prove.

e. Using your answers to earlier parts of the problem, solve for g_Y in terms of $\alpha, \beta, \gamma, \epsilon$ and n.

From point b. we have that $g_Y = \alpha\left(\gamma + g_K\right) + \left(1-\alpha-\beta\right)\left(n\right) + \beta\left(\epsilon\right)$, while form point d. we know that along a balanced growth path $g_Y = g_K$. By substituting the second condition into the first on we obtain:

$$\begin{aligned} g_Y &= \alpha\left(\gamma + g_K\right) + \left(1-\alpha-\beta\right)\left(n\right) + \beta\left(\epsilon\right) \\ &= \alpha\gamma + \alpha g_Y + \left(1-\alpha-\beta\right)\left(n\right) + \beta\left(\epsilon\right) \\ g_Y\left(1-\alpha\right) &= \alpha\gamma + \left(1-\alpha-\beta\right)\left(n\right) + \beta\left(\epsilon\right) \\ g_Y &= \frac{\alpha\gamma + \left(1-\alpha-\beta\right)\left(n\right) + \beta\left(\epsilon\right)}{1-\alpha} \end{aligned}$$

Which gives us g_Y in terms of $\alpha, \beta, \gamma, \epsilon$ and n.

f. What is the condition for $g_{\frac{Y}{L}}$ to be positive along a balanced growth path? Interpret.

To answer this question we have first to compute the quantity of interest. The rule is always the same:

$$\begin{aligned} g_{\frac{Y}{L}} &= g_Y - g_L \\ &= \frac{\alpha\gamma + \left(1-\alpha-\beta\right)\left(n\right) + \beta\left(\epsilon\right)}{1-\alpha} - n \\ &= \frac{\alpha\gamma + \left(1-\alpha-\beta\right)\left(n\right) + \beta\left(\epsilon\right) - n(1-\alpha)}{1-\alpha} \\ &= \frac{\alpha\gamma + n - \alpha n - \beta n + \beta\left(\epsilon\right) - n + \alpha n}{1-\alpha} \\ g_{\frac{Y}{L}} &= \frac{\alpha\gamma - \beta n + \beta\left(\epsilon\right)}{1-\alpha} \end{aligned}$$

Now we are ready to evaluate when this expression is positive. First, we know that the denominator is always positive, as $\alpha < 1$. Therefore, the whole fraction is positive when the numerator is positive.

$$g_{\frac{Y}{L}} > 0 \Leftrightarrow \alpha\gamma - \beta n + \beta\epsilon > 0 \Leftrightarrow \alpha\gamma + \beta\epsilon > \beta n$$

Before interpreting this result, we must understand what $g_{\frac{Y}{L}}$ indicates. It is the growth rate of what we usually denote y, production in per capita terms. So, asking when $g_{\frac{Y}{L}}$ is positive is the same as asking: "when does production in per capita terms has a positive growth rate?". Hopefully this interpretation of the question helps us understand this condition. There are three factors that affect consumption per capita.

$$\underbrace{\alpha\gamma}_{\text{Technological growth}} + \underbrace{\beta\epsilon}_{\text{Natural Resource Growth}} > \underbrace{\beta n}_{\text{Population growth}}$$

The first two increases product per capita, while the third one decreases it. Therefore, growth will be positive when the sum of the first two is higher than the third.

Question: Can you guess what is the role of α and β exactly?

2 Solow-Swan with non-renewable resources (from TD2!)

This problem is tightly related to the previous one but it has less computations and more intuition. In particular, its focus is to study the employment of renewable and non-renewable resources and its sustainability.

We have the same production function and growth rates as before.

$$Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$$

However, here we specify how Z is composed. We can split natural resources in renewable R and non-renewable N. The rate of exploitation of N is r, therefore we have that $Z = R + rN$.

a. Assume that initially, the economy is in a balanced growth path (BGP) where the stock of renewable resources is stable and where there are no non-renewable resources at all. What is the growth rate of y? Interpret in what conditions we get a positive rate of growth for y. Knowing what we can anticipate about the rate of technology and population growth in the 21^{st} century, should we expect y to grow or not in the coming decades ?

We already have the growth rate of $y = \frac{Y}{L}$ on a balanced growth path from the previous exercise.

$$g_y = g_{\frac{Y}{L}} = \frac{\alpha\gamma - \beta n + \beta(\epsilon)}{1-\alpha}$$

However, in this case there are no non-renewable resources $N = 0$ and the stock of renewable resources R is stable, which implies that it is not growing. Since $Z = R + rN$ and $N = 0$ we have that here $Z = R$. The growth rate of Z was ϵ in the previous exercise, but since here R does not grow, Z does not grow either, as it is composed by R only. This translates into $\epsilon = 0$. The new growth rate is therefore:

$$g_y = \frac{\alpha\gamma - \beta n}{1-\alpha}$$

The evaluation of its sign is the same as before. In particular, we know that $1-\alpha$ is always positive, therefore the sign of the numerator is the significant one. The whole fraction is positive when the numerator is positive:

$$g_y > 0 \Leftrightarrow \alpha\gamma > \beta n$$

We have 4 factors that affect this inequality:

1. α captures the relative importance of capital in the production function. Intuitively, if capital is relatively more important there are more chances that per capita growth is positive, as it is directly affected by technological progress;
2. γ is the rate of growth of technology. Of course, the more technology improves the more likely is that growth per capita increases;
3. β is α counterpart for natural resources, it measures its relative importance in the production. Since these do not grow, if they are less important than growth will be positive despite the fact that the stock is fixed;
4. n represents the growth rate of population. Intuitively, if population increases the growth per capita decreases, as there are more mouths to feed.

Since we know that n is quite low while γ is high, if the premises of this model are true then we would be sure to enjoy positive growth in the future.

b. Assume that at time t (when the economy was previously on the balanced growth path), a new source of non-renewable resource of size N_t is discovered. Each ensuing period, $r\%$ of the resource stock is used in production, such that its stock goes progressively to zero in the long run. If $R=1$, $N_t=20$ and $r=0.1$, what is the growth rate of the supply of resource Z before t? Right after t? In the very long run?

Before answering this question we have to compute the growth rate of Z_t. However, in this case the expression of interest is a sum, therefore we can not use the rules of product and ratios of growth rate. The idea is to subtract Z_{t-1} to Z_t in order to find the Δ. The calculations to attain the growth rate are the following (I omit t in the computations):

$$\begin{aligned}
Z_t &= R_t + rN_t \\
Z_t - Z_{t-1} &= R_t - R_{t-1} + r(N_t - N_{t-1}) \\
\Delta Z &= \Delta R + r\Delta N \\
\frac{\Delta Z}{Z} &= \frac{R}{R}\frac{\Delta R}{Z} + r\frac{N}{N}\frac{\Delta N}{Z} \\
\frac{\Delta Z}{Z} &= \frac{R}{Z}\frac{\Delta R}{R} + r\frac{N}{Z}\frac{\Delta N}{N} \\
g_Z &= \frac{R}{Z}g_R + r\frac{N}{Z}g_N \\
g_{Z,t} &= \frac{R}{R+rN_t}g_R + r\frac{N_t}{R+rN_t}g_{N,t}
\end{aligned}$$

By substituting the numbers we have we obtain the growth rate when the new non-renewable resource is discovered, at time t. Remember that R does not grow ($g_R = 0)$ and that Z has a negative growth of -0.1 We have:

$$g_{Z,t} = \frac{R}{R + rN} g_R + r \frac{N}{R + rN} g_N$$
$$g_{Z,t} = \frac{1}{1 + (0.1)(20)} (0) + \frac{(0.1)(20)}{1 + (0.1)(20)} (-0.1)$$
$$g_{Z,t} = \frac{(0.1)(20)}{1 + (0.1)(20)} (-0.1) = -6.\bar{6}\% = -\frac{1}{15}$$

As for $g_{Z,\tau}$ for $\tau < t$, we have that $g_{Z,\tau} = 0$, as $N_t = 0$ and R does not grow, exactly as we had in the previous point. Instead, when $\tau \to \infty$ the growth rate also goes to zero. This is due to the fact that Z has a negative growth, and therefore after it is completely exploited it will not grow (negatively) anymore.

4

TD 4

1 Review questions

a. In an ecosystem, the natural growth of a renewable resource is an increasing function of the amount of this resource.

▼ *Answer*

False: As an example, in class, you considered a logistic growth, captured by the equation $\tau(S_t) = rS_t\left(1 - \frac{S_t}{K}\right)$. As you can see, when $S_t \rightarrow K$ then $\tau(S_t) \rightarrow 0$. Therefore, it is not true that if S_t increases then its growth also increases.

b. An improvement in extractive technology always increases fish production if fishing is free.

▼ *Answer*

False: In our model the total production of fish when there is free entry is given by the following:

$$H_F = B_F \alpha S_F = \left(1 - \frac{c}{p\alpha K}\right) \frac{r}{\alpha} \frac{c}{p}$$

As you can see, we have an α at the denominator with a minus sign (positive effect on H_F), but we also have an α at the denominator with a plus sign (negative effect on H_F). Hence, the total effect is ambiguous.

2 Solow-Swan with non-renewable resources

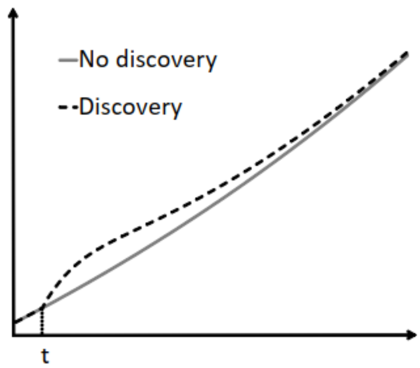
For the last exercise the professor run an analysis with particular values of parameters $n = 0.01$, $g = 0.015$, $\delta = 0.02$, $s = 0.3$, $\alpha = 0.3$, $\beta = 0.2$, $R = 1$, $N_t = 20$, $r = 0.1$. Finally you see the model at work! The graphs in the text of the exercise are the plot of three time series: $\frac{Y}{K}$, $\frac{Y}{L}$ and $\frac{K}{L}$. The continuous line describe an economy in a balanced growth path with no non-renewable resources, while the dashed line depicts a scenario where non-renewable resources are discovered at time t. The main of the following points is to connect the graphs to ratios.

c. Which graph is $\frac{K}{L}$? Explain in words what happens at time t and in the ensuing periods.

Let's try to find a general way of answering these kind of questions. The variables involved are K, L and Y. The question is: how are these variables affected by an increase in N? In order to answer we need to know the dependencies that all these variables have with N. As an example, L is only determined by its growth, we start from L_0 and then we get L_1 based on how big n is. Therefore, L is not directly affected by N. The same hold for K, its growth is given by its growth rate g_K, and not directly by N. Hence, K does not jump either. Instead, $Y_t = (A_t K_t)^{\alpha} (L_t)^{1-\alpha-\beta} Z_t^{\beta}$ where $Z_t = R + rN_t$. A jump in N causes a jump in Y. This analysis offers a first insight to answer this question. In fact, we must consider the three ratios $\frac{K}{L}, \frac{Y}{K}$ and $\frac{Y}{L}$. Since of these three the only ratio that does not jump is $\frac{K}{L}$ we are sure that the right graph is C, as there is a smooth evolution of the dashed line.

To understand what happens here recall that $g_K = s\frac{Y_t}{K_t} - \delta$. Since Y increases, as we elaborated before, the growth rate of K increases, so K increases more then what it would without N. However, N will go to zero slowly, which means that the accumulation of capital K slowly go back to its original path.

C

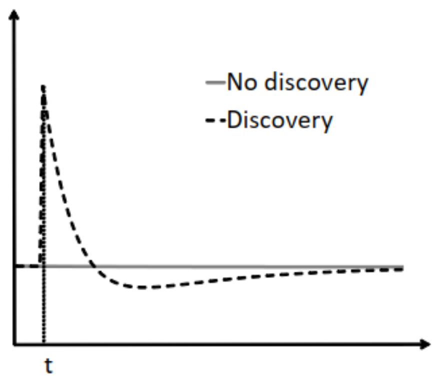


d. Which graph is $\frac{Y}{K}$? Explain in words what happens at time t and in the ensuing periods.

First step done, now we have to distinguish between $\frac{Y}{L}$ and $\frac{Y}{K}$. The difference between the two graphs we are left with is that on one of the the balanced growth path is constant (horizontal line). Therefore, we have to answer the question: which ration between $\frac{Y}{K}$ and $\frac{Y}{L}$ should be fixed without the increase in N? Well, we now that the growth rate of L is exogenously given and it is $n > 0$, so it is impossible that L will be fixed. Also Y and K will grow, but at which rate? We know from the previous TD that $g_{\frac{Y_t}{K_t}} = 0 \Leftrightarrow g_Y - g_K = 0 \Leftrightarrow g_Y = g_K$, therefore $\frac{Y}{K}$ is constant and identified by a constant. Hence, it is represented by graph *B*. Here two things happen. First, as we saw before, Y jumps. Instead, K does not jumps immediately, but smoothly increases. Therefore at time t $\frac{Y}{K}$ will steadily increase.

However, the push of Y is only big at time t, after that the growth rate of Y will slowly return normal. Instead, the growth rate of K will increase, and the increase of K will offset the increase of Y. This explains why the dotted line goes below the horizontal line. After some time also the growth of K returns normal, and the dotted line returns on the old balanced growth path.

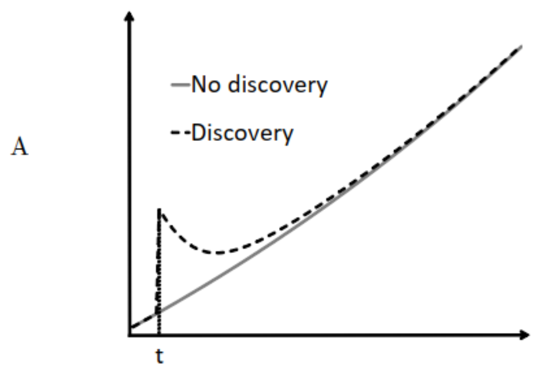
B



e. Which graph is $\frac{Y}{L}$? Explain in words what happens at time t and in the ensuing periods.

We are only left with one ratio and graph *A*, so the answer is easy here, but we must understand also what is going on. The jump is always given by the steady increase of Y, as in the previous points.

However, L will be always increasing with the same rate, in contrast to K in the previous graph. Nevertheless, the ratio immediately starts to decrease after t, slowly reaching the old balanced growth path. This is due to the fact that the stock of natural resources is depleted with a rate way higher $(-6.\bar{6}\%)$ than the rate at which productivity increases (1.5%).



> “We are in the beginning of mass extinction, and all you can talk about is money and fairy tales of eternal economic growth”- Greta Thunberg at the United Nations Climate Action Summit, September 23, 2019

f. There is little doubt that a mass extinction is going on. However, this widespread idea of sustained economic growth being a myth is open to debate. Expanding on the model, can we comment on it.

This is more of a philosophical question, so fell free to answer whatever you think that is relevant and consistent with the model (it is true that many answer could be consistent with what we have here, but there are also many things that are not!). What this little exercise can tell us is that there is no need of non-renewable resources for an economy to grow, as long as other rates $(n, g...)$ are positive. However, a sudden increase in N also brings to a jump in Y. Maybe this model could also suggest that we need a fixed amount of renewable resources R, even if it does not grow. Therefore, according to this model, it might be not a very good idea to destroy forests.

2 The baby boom (from last year midterm!)

This exercise is not significantly different to exercises 2 and 3 of TD1! You should already know how to do the computations and the graph of the first questions. The second part is a little bit more difficult and it requires you to translate what is happening in mathematics. We have $F(k_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. Investment is, as always, $I = sY = sF(K_t, L_t)$. The law of motion of capital is $\Delta K_t = K_{t+1} - K_t = I - \delta K_t$. The work population grows at rate $n = \frac{L_{t+1} - L_t}{L_t} = 0.05$. We also have that $\delta = 0.05$, $s = 0.1$ and $\alpha = 0.5$.

Remark: In these solutions I spell out every step just for you to understand, as you can see from the professor's solutions it is not needed that you specify each step as I do. However, for sure it can not hurt you.

a. On the balanced growth path, k and y are stable. Compute their numerical values.

The answer to this question requires no more than performing the computations we have been doing in the previous TDs and substituting numbers. First, we have to convert everything in per capita terms. The production function becomes:

$$
\begin{aligned}
\frac{1}{L_t}F(K_t, L_t) &= F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}\right) \\
&= \left(\frac{K_t}{L_t}\right)^{\alpha}\left(\frac{L_t}{L_t}\right)^{1-\alpha} \\
f(k_t) &= k_t^{\alpha}
\end{aligned}
$$

You should recognise that we have been using this function a lot! Is the classical Cobb-Douglas. To compute the numerical value of k in the balanced growth path we can rely on the condition under which this variable is indeed on such path, which means that its growth rate is equal to zero. Hence, we must first compute its growth rate.

$$
\begin{aligned}
\frac{\Delta k_t}{k_t} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\
&= \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n \\
&= \frac{s\frac{1}{L_t}F(K_t, L_t) - \delta\frac{1}{L_t}K_t}{\frac{1}{L_t}K_t} - n \\
\frac{\Delta k_t}{k_t} &= \frac{sf(k_t) - \delta k_t}{k_t} - n \\
\Delta k_t &= sf(k_t) - \delta k_t - nk_t \\
&= sf(k_t) - (\delta + n)k_t
\end{aligned}
$$

The condition for being in steady state its growth rate equal to 0, thus, we check what this condition implies in this exercise.

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) - (\delta + n)k_t^* = 0 \Leftrightarrow s(k^*)^{\alpha} = (\delta + n)k_t^*$$

By substituting the numbers we are given in the text we obtain that:

$$k^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{0.05 + 0.05}{0.1}\right)^{-2} = 1$$

Since $y^* = f(k^*)$ we obtain:

$$y^* = (k^*)^\alpha = 1^{1/2} = 1$$

b. Is the savings rate s is at its golden rule value? If not, what should be the golden-rule savings rate?

From problem 2 of TD 1 you may recall that the golden rule savings rate is $s = \alpha = 0.5$, while in this case we have that $s = 0.1 \neq 0.5$! However, let's try to prove it again. There are many ways to do it. The first one is to express consumption as $c^* = (1-s)f(k^*)$ and compute the s that maximises it in steady state. The steps to do this are detailed in the solution of TD 1. A different method could be to check the k which comes from the golden rule of savings and derive the s for which we get that k. The steps are the following. First, express consumption only as a function of capital:

$$\begin{aligned} c^* &= (1-s)y^* \\ &= (1-s)f(k^*) \\ &= f(k^*) - sf(k^*) \\ &= f(k^*) - (\delta+n)k^* \end{aligned}$$

Where the last step come from the steady state condition we derived in the previous point. We then obtain the k that maximises consumption by solving a maximisation problem:

$$\begin{aligned} \frac{\partial c^*}{\partial k^*} = 0 &\Rightarrow f'(k^*) - (\delta+n) = 0 \\ &\Rightarrow \alpha(k^*)^{\alpha-1} = (\delta+n) \\ &\Rightarrow k^* = \left(\frac{\delta+n}{\alpha}\right)^{\frac{1}{\alpha-1}} \\ &\Rightarrow k^* = \left(\frac{0.05+0.05}{0.5}\right)^{\frac{1}{0.5-1}} = 25 \end{aligned}$$

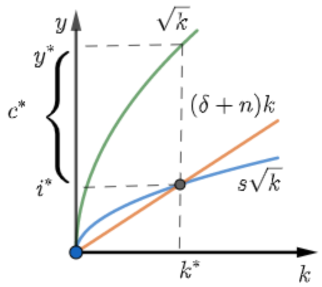
Which is quite different from the $k^* = 1$ we got before! What is the s that rationalises this result? From the expression we derived in the previous point we have:

$$
\begin{aligned}
k^* &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \\
25 &= \left(\frac{0.05+0.05}{s}\right)^{\frac{1}{0.5-1}} \\
25 &= \left(\frac{0.1}{s}\right)^{-2} \\
25 &= \left(\frac{s}{0.1}\right)^{2} \\
5 &= \left(\frac{s}{0.1}\right) \\
0.5 &= s \neq 0.1
\end{aligned}
$$

Which indeed gives us $\alpha = s$. Remember that $i^* = sy^* = 0.1(1) = 0.1$, moreover $c^* = y^* - i^* = 1 - 0.1 = 0.9$.

c. Make a plot like the ones made in class with k on the horizontal axis and y on the vertical axis. Draw $f(k) = k^{0.5}$. Draw $sf(k)$. Draw $(n+\delta)k$. Indicate k^* and y^*.

The graph is ecxactly the one we did in exrcise 3 odf TD 1! I'll report the same picutre I put there. The production function is $f(k_t) = (k_t)^{\frac{1}{2}} = \sqrt{k_t}$, while $\delta = n = 0.05$, $y^* = 1$ and $k^* = 1$ (you may want to substitute the numbers in your graph).



Graph from exercise 3 TD1.

1. $\sqrt{k}$
2. $(\delta + n)k_t$
3. $s\sqrt{k}$

d. After World War 2, many American soldiers fighting in Europe came back home and made lots of babies. Imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. What is the numerical value of the growth rate of k_t just after the doubling? (If you don't have a calculator, you can leave a mathematical expression as is (as long as it just involves numbers, no variables).)

We are asked to compute the growth rate of k_t at time t, when the populatoin doubles. We have to perform the exact computations we did in the previous points, but instead of having L_t, we have $2L_t$. We start from the production function:

$$\begin{aligned}
\frac{1}{2L_t}F(K_t, 2L_t) &= F\left(\frac{K_t}{2L_t}, \frac{2L_t}{2L_t}\right) \\
&= \left(\frac{K_t}{2L_t}\right)^{\alpha}\left(\frac{2L_t}{2L_t}\right)^{1-\alpha} \\
f(k_t) &= \left(\frac{k_t}{2}\right)^{\alpha} = \bar{k}_t^{\alpha}
\end{aligned}$$

Now we have to compute the growth rate. Again the calculations follow the same logic as before:

$$\begin{aligned}
\frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \\
&= \frac{sF(K_t, 2L_t) - \delta K_t}{K_t} - n \\
&= \frac{s\frac{1}{2L_t}F(K_t, 2L_t) - \delta\frac{1}{2L_t}K_t}{\frac{1}{2L_t}K_t} - n \\
\frac{\Delta k_t}{\frac{k_t}{2}} &= \frac{sf(k_t) - \delta\frac{k_t}{2}}{\frac{k_t}{2}} - n \\
\Delta k_t &= sf(k_t) - \delta\frac{k_t}{2} - n\frac{k_t}{2} \\
&= sf(k_t) - (\delta + n)\frac{k_t}{2}
\end{aligned}$$

By expliciting the production function we obtain:

$$\Delta k_t = s\left(\frac{k_t}{2}\right)^{\alpha} - (\delta + n)\frac{k_t}{2} = 0.1\left(\frac{1}{2}\right)^{0.5} - (0.05 + 0.05)\frac{1}{2} = 0.0207$$

However, we must find the growth rate, not only the Δ:

$$\frac{\Delta k_t}{\frac{k_t}{2}} = \frac{0.0207}{\frac{1}{2}} = 0.0414$$

e. As previously, imagine that during World War 2, the American economy is on the balanced growth path. Then when troops come home at time t, the population L doubles. Also, since these young people want to make babies as fast as possible, at the same moment, the growth rate jumps from $n = 0.05$ to $n = 0.15$. Reproduce the graph in c. and show how the long run equilibrium will change. Indicate what happens to k just after soldiers get back (at time t) and where the economy converges in the long run (indicate numerical values).

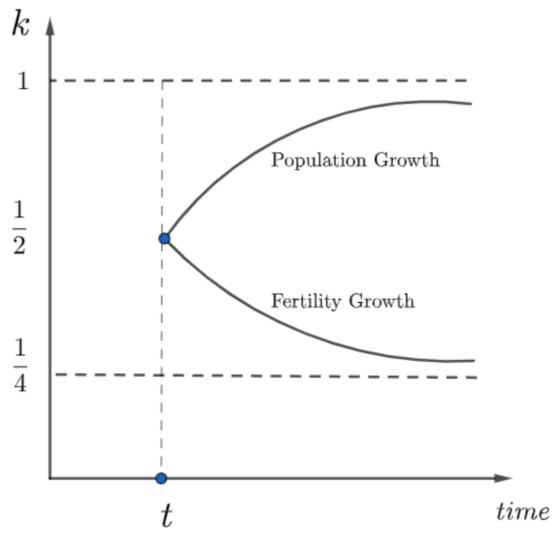
Before showing the graph we need to compute the new k^* and y^*. As always, we set the law of motion equal to zero:

$$\begin{aligned}\Delta k_t &= 0\\ &\Leftrightarrow s\left(\frac{k_t^*}{2}\right)^{\alpha} - (\delta + n)\frac{k_t^*}{2} = 0\\ &\Leftrightarrow \left(\frac{k_t^*}{2}\right)^{\alpha-1} = \frac{\delta + n}{s}\\ &\Leftrightarrow \left(\frac{k_t^*}{2}\right) = \left(\frac{0.05 + 0.015}{0.1}\right)^{\frac{1}{0.5-1}} = \frac{1}{4}\end{aligned}$$

As for y^* we have that $y^* = f(k^*) = \left(\frac{k_t^*}{2}\right)^{0.5} = \left(\frac{1}{4}\right)^{0.5} = \frac{1}{2}$. I do not include the graph here again as it is the same graph as before with different numbers.

f. Plot k over time around the period t (hence make a graph with time on the horizontal axis and k on the vertical axis). On the same graph, show how k adjusts after t in the situation described in d. and in the situation described in e. Make sure to show where k is converging in each case.

At time t the input in the production function is immediately halved due to the increase in population. If there is no fertility growth after a while k will return on its original balanced growth path. In the case of fertility growth, instead, it will converge to $\frac{1}{4}$.



g. In the context of this model, are American workers (those always in the US) better off before time t (before the influx of workers), some time after time t if fertility does not change d. some time after t or if fertility does change e.? Rank the three situations from best to worst and justify briefly.

This question just amounts to compare the y^* in different circumstances. In $\tau < t$ we have $y^*_\tau = 1$, when fertility increase $(n = 0.15)$ occurs at time t we have that the new steady state gdp is $y^* = \frac{1}{2}$, while without the increase in n but after the increase in population we are outside the growth path leading to $y^* = 1$, and therefore we are slightly below this value. Therefore, the best situation is the one before the shock, then we have the increase in population without the increase in n and lastly the worse situation is when we also observe a fertility rate increase.

TD 5

1 Review Questions

c. An improvement in extractive technology always increases fish production if fishing is socially optimal.

▼ *Answer*

True: From you notes (page 9) you can see that the expression for the total harvest in the social planner solution is $H_O^* = \frac{rK}{4}\left(1-\left(\frac{c}{p\alpha K}\right)^2\right)$. You can see from the expression that an increase in α will increase total harvest even without taking derivatives.

d. An improvement in extractive technology is always a bad thing from an environmental point of view.

▼ *Answer*

True: We can see it from the equilibrium expression of the stock of natural resources. We have that $S^* = \frac{c}{p\alpha}$. If α increase then the stock of natural resources decreases.

3 The Dynamics of a fish population with threshold

One of the problems that the fishing model has is that the only circumstance in which there is an extinction of fishes (or natural resources in general) is when the starting stock is equal to 0. Of course, this is counterfactual with reality as we go from a state in which there is a positive amount of resources to a state in which they are extinct. The aim of this exercise is to augment to model by assuming that when the stock of fishes goes below a threshold T then it is destined to converge to 0. I think it is an interesting exercise, it helps interpreting some real world facts.

a. Find the values of $S(t) > 0$ for which the fish stock does not grow naturally.

As always, we first need to understan what the question is asking. When does the fish stock grows? When its growth rate is different then 0. If the growth rate is equal to zero then the stock will not grow. The question is asking ask to determine for which values of $S(t)$ the growth rate is equal to 0. The growth rate is:

$$N(S(t)) = r(S(t) - T)\left(1 - \frac{S(t)}{K}\right)$$

Since $r > 0$ and the expression is a product, we have that $N(S(t)) = 0$ when one of the two element of the product is zero.

$$N(S(t)) = 0 \Leftrightarrow \begin{cases} S(t) - T = 0 \\ 1 - \frac{S(t)}{K} = 0 \end{cases} \Leftrightarrow \begin{cases} S(t) = T \\ S(t) = K \end{cases}$$

Hence, fishes will not grow when their stock is exactly equal to their maximum capacity K and when the stock is equal to the minimum treshold T. Notice that this is clear also from the plot of the growth rate.

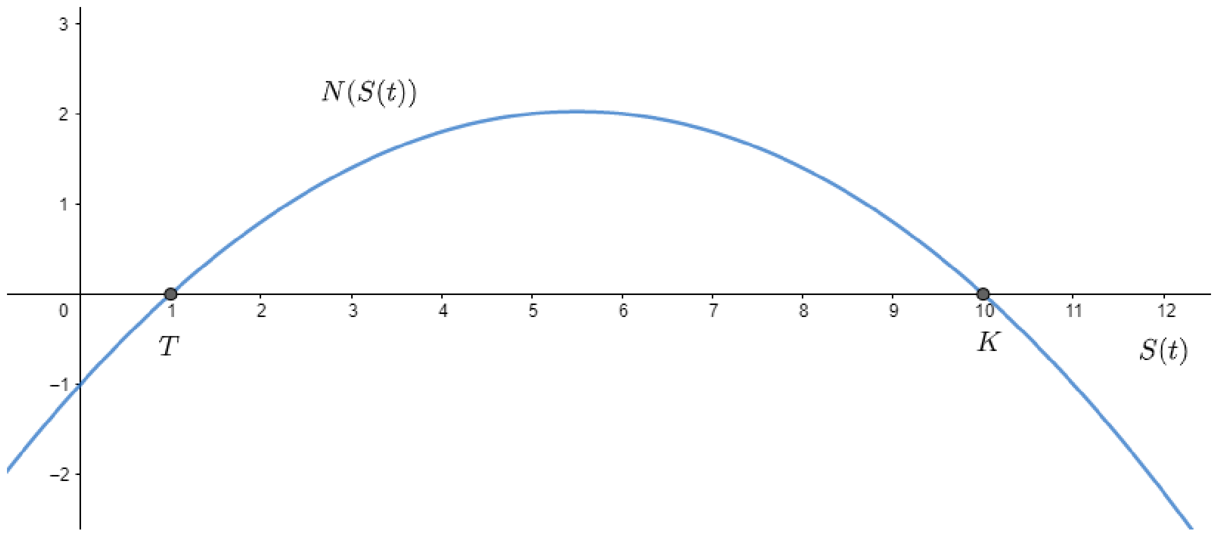


Figure 1: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$.

b. Of these values, which are stable, which are not?

First of all, what does stable mean in this context? When speaking about steady states (growth rates equal to 0), we say that a steady state is stable if a small perturbation of the system from a steady steady returns to the previous point autonomously. In this case, a steady state is stable if by slightly increasing or decreasing the stock of fishes from $S(t) = T$ or $S(t) = K$ we then return to the previous steady state or the system evolves in a different direction.

Checking for stability seems a daunting task, but if you have the graph it becomes easier. Here I will show you to check for stability with both a graphical and a mathematical technique. Let's start from the graphical one. Consider the steady state $S(t) = T$, what happens if we move slightly on the right (e.g. $S(t) = T + \epsilon$) ? You see that $N(S(T) + \epsilon)$ is positive. Hence, the stock will continue to grow and will become significantly different with respect to $S(T)$. In the same way, if we perturb the stock in the other direction ($S(T) - \epsilon$) we can see that $N(S(T) - \epsilon)$ is negative, the stock will become lower and lower. This steady state is not stable. If we perturb it slightly it will not return to its original point. Now consider the second steady state, when $S(t) = K$. If we perturb it by moving slightly on the right (

$S(K)+\epsilon)$ we see that $N(S(K)+\epsilon))$ is negative, therefore the stock of fish will decrease until it returns to its stable value $S(t)=K$. The same happens when you perturb the stock in the other direction, we have that $N(S(K)-\epsilon))$ is positive, which will make the stock increase until it is stable in $S(t)=K$. We concluded that $S(t)=T$ is not stable while $S(t)=K$ is stable. The graph below captures this reasoning pattern.

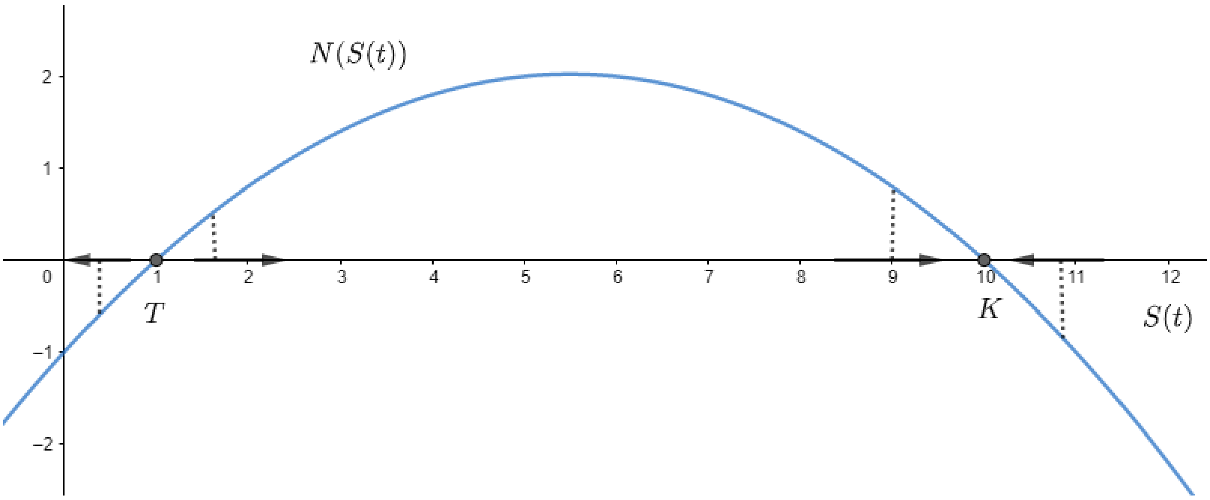


Figure 2: Stability of steady states.

If you don't like this graphical reasoning, there is also the math way. This perspective amounts to taking the derivative of $N(S(t))$ with respect to $S(t)$, which measures the change in growth by a change in stock of fish. By evaluating the derivative in the two steady states we can check its sign. If the sign of the derivative is positive, this means that a positive variation will be magnified even more, and therefore the steady state is not stable. If the sign is negative, it means that after a positive variation the stock will decrease and return to its original value. This would mean that the steady state is stable. Let's start taking the derivative. It might look a little bit scary to the the derivative with respect to $S(t)$, but you just have to consider the entire expression as a single variable and derive it with the rules you know (you can look at TD1 for an explanation on how to derive products).

$$\begin{aligned} \frac{\partial N(S(t))}{\partial S(t)} &= r\left[\left(1-\frac{S(t)}{K}\right)+(S(t)-T)\left(-\frac{1}{K}\right)\right] \\ &= r\left[1-\frac{S(t)}{K}-\frac{S(t)}{K}+\frac{T}{K}\right] \\ N'(S(t)) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right] \end{aligned}$$

We can now evaluate the derivative in the two points of interest. Recall that $r > 0$.

$$\begin{aligned} N'(T) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right] \\ &= r\left[1-\frac{2T}{K}+\frac{T}{K}\right] \\ &= r\underbrace{\left[1-\underbrace{\frac{T}{K}}_{<1}\right]}_{>0} > 0 \end{aligned}$$

This result confirms our graphical analysis. Since the derivative at T is greater than 0, this means that a positive perturbation of $S(t)$ at T will increase the stock even more, and therefore will push the system far from the original state. On the contrary, for K we have:

$$\begin{aligned} N'(K) &= r\left[1-\frac{2K}{K}+\frac{T}{K}\right] \\ &= r\left[1-2+\frac{T}{K}\right] \\ &= r\underbrace{\left[\underbrace{\frac{T}{K}}_{<1}-1\right]}_{<0} < 0 \end{aligned}$$

Which again goes in the same direction as the graphical intuition. If we positively perturb the steady state at K, the stock of fish will decrease until we reach the previous state again.

c. What is the natural growth of the fish population at t if $S(t) = 0$? Is it also an equilibrium?

To answer this question we just need to evaluate the growth rate in the point $S(t) = 0$.

$$N(0) = r(0 - T)(1 - 0) = -rT$$

We should have expected this result, as we know that T is a threshold for the fish to grow and $S(t) = 0 < T$. Since the computed growth rate is negative and since the stock can not go lower than 0, we conclude that $S(t) = 0$ is also a steady state. Notice that $S(t)$ is the point in which the growth rate crosses the y axis, as shown in the picture below.

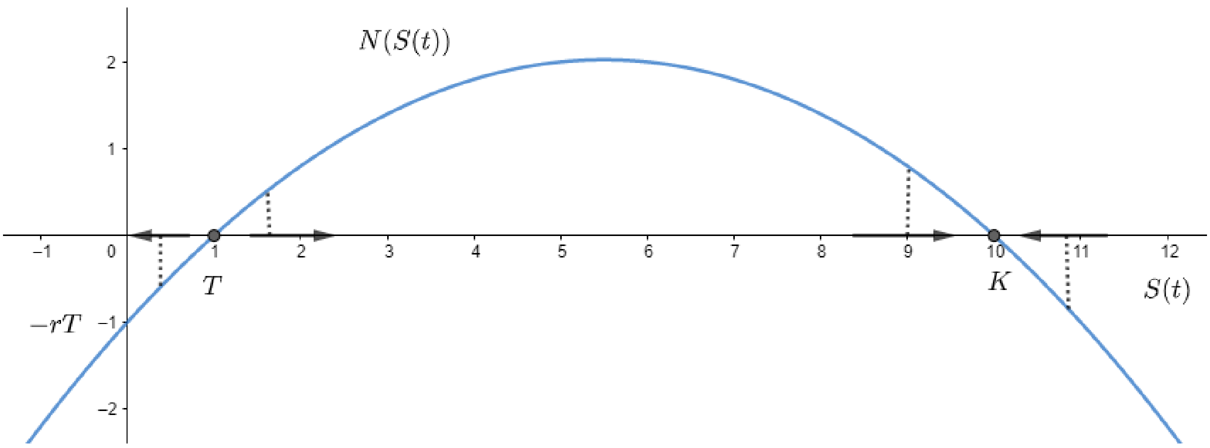


Figure 3: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Notice that $-rT = -1(1) = -1$, where the growth rate is negative and the stock is 0.

d. What is the maximum number of fish that can be caught per unit of time such that the fish population is constant? This

is also called the maximum sustained yield. What is the fish stock $S(t)$ at this value?

To answer this question we must ask when the growth rate of fishes is the highest. This would allow us to capture the maximum number of fishes every time t and then obtain for $t+\epsilon$ the greatest amount of growth so that we can always maximise our catches. Hence, we must maximise the growth rate of fishes with respect to the stock. We already have the derivative. To check for the maximum we need to find the value of $S(t)$ for which the derivative is equal to 0.

$$\begin{aligned}\frac{\partial N(S(t))}{\partial S(t)} = 0 &\Leftrightarrow r\left[\left(1-\frac{S(t)}{K}\right)+(S(t)-T)\left(-\frac{1}{K}\right)\right]=0\\&\Leftrightarrow \cancel{r}\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right]=0\\&\Leftrightarrow K-2S(t)+T=0\\&\Leftrightarrow S(t)=\frac{K+T}{2}\end{aligned}$$

Now that we have the stock of fishes that maximises growth we can ask by how much fishes grow for this value of the stock. Of course, to answer this question we just need to plug the value we just found in the growth rate.

$$\begin{aligned}N\left(\frac{K+T}{2}\right) &= r\left(\frac{K+T}{2}-T\right)\left(1-\frac{K+T}{2K}\right)\\&= r\left(\frac{K+T-2T}{2}\right)\left(\frac{2K-K-T}{2K}\right)\\&= r\left(\frac{K-T}{2}\right)\left(\frac{K-T}{2K}\right)\\&= r\frac{(K-T)^2}{4K}\end{aligned}$$

This expression tells us by how much the fish grows at the optimal stock.

e. Graph the fish growth function $S(t)$. Place all the elements previously computed on the graph.

We already did a big part of the graph, the one below has also the answers to the last question.

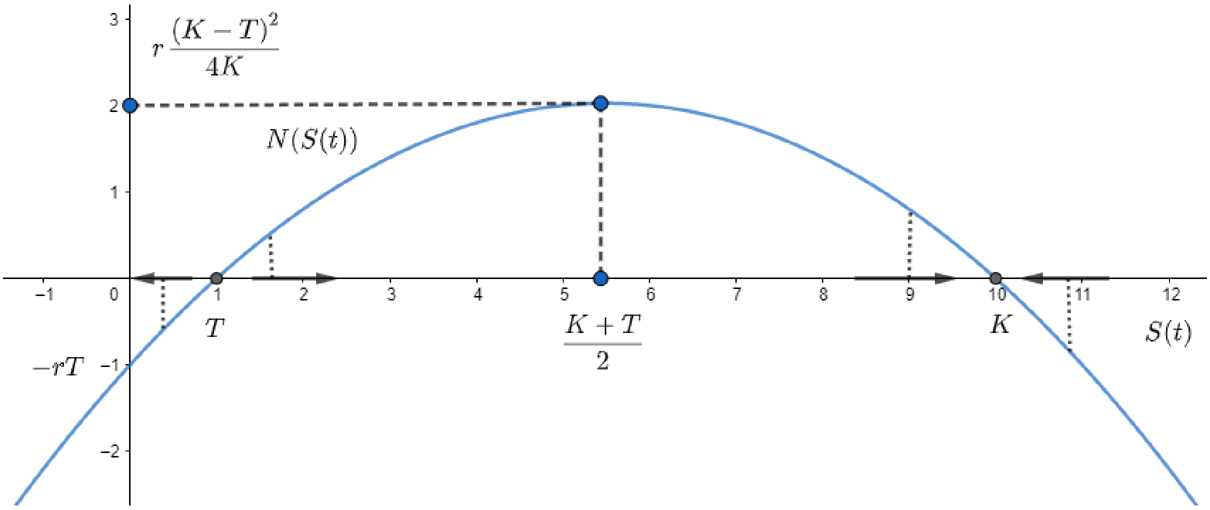


Figure 4: Graph of the growth rate of fishes for $T=1$, $K=10$ and $r=1$. Here $S^*(t)=\frac{10+1}{2}=5.5$ and $N(S^*(t))=\frac{9^2}{40}\approx 2$.

f. If there are B boats catching fishes, their total catches are $H(t)=\alpha BS(t)$. The net growth rate (the law of motion) of the stock is $\dot{S}(t)=N(t)-H(t)$. With B boats in the ocean, what is (are) the steady-state population(s) of fish?

First, notice that $\dot{S}_t$ is just notation for $\frac{\partial S(t)}{\partial t}$ which is the derivative of the stock of fish with respect to time. It is the equivalent of the law of motion of the Solow - Swan growth model, so you should treat it exactly as we did with that model. This observation helps us answering this question. In fact, the steady state population of fish is characterised by setting its growth rate equal to 0, which is the same as saying that $N(t)=H(t)\Leftrightarrow N(t)=\alpha BS(t)$.

$$
\begin{aligned}
\dot{S}_t = 0 &\Leftrightarrow r(S(t)-T)\left(1-\frac{S(t)}{K}\right)-\alpha BS(t)=0 \\
&\Leftrightarrow rS(t)-2T-\frac{rS(t)^2}{K}-\frac{rTS(t)}{K}\alpha BS(t)=0 \\
&\Leftrightarrow -\frac{rS(t)^2}{K}+S(t)\left(r+\frac{rT}{K}-\alpha B\right)-rT=0 \\
&\Leftrightarrow S(t)^2\frac{r}{K}-S(t)\left(r+\frac{rT}{K}-\alpha B\right)+rT=0 \\
&\Leftrightarrow S(t)^2-S(t)\left(K+T-\frac{\alpha BK}{r}\right)+TK=0
\end{aligned}
$$

We have a second order degree equation of which we have to find the roots by the usual formula. We have to solutions that we label S_U and S_S (you will soon see why).

$$
S_U = \frac{K+T-\frac{\alpha BK}{r}-\sqrt{\left(K+T-\frac{\alpha BK}{r}\right)^2-4TK}}{2}
$$

$$
S_S = \frac{K+T-\frac{\alpha BK}{r}+\sqrt{\left(K+T-\frac{\alpha BK}{r}\right)^2-4TK}}{2}
$$

g. Graph the dynamics of the stock with resource extraction and identify the equilibrium population(s) of fish. Show with arrows how population dynamics pushes S to increase or decrease.

Here I put the picture where I added the solutions we computed in the previous point. By performing the same reasoning as before, you can easily see that S_U is unstable while S_S is stable.

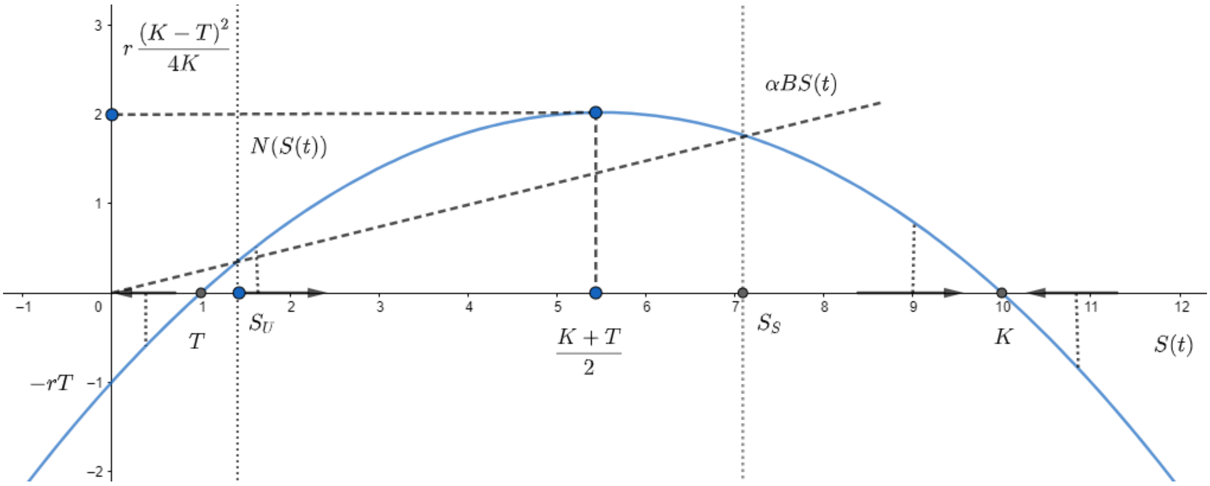


Figure 5: Same graph as before with S_U and S_S. Here I picked $\alpha = \frac{1}{8}$ and $B = 4$.

h. Is there an intensity of fishing (αB) so high that no sustainable fishing is possible? What is it?

To answer this question it is enough to notice that if you increase αB by a lot then the line $\alpha BS(t)$ will not cross $N(S(t))$ anymore, which means in fact that no sustainable fishing is possible. This will happen when there is no solution to the previous second degree equation, that is when the quantity below the square root is negative (imaginary solution). Therefore we just have to check when this condition is satisfied.

$$\left(K + T - \frac{\alpha BK}{r}\right)^2 - 4TK < 0$$

$$\frac{\alpha BK}{r} > K + T - 2\sqrt{KT}$$

$$\alpha B > \frac{K}{r}\left(T + K - 2\sqrt{KT}\right)$$

$$\alpha B > \frac{K}{r}\left(\sqrt{T} - \sqrt{K}\right)^2$$

$$\alpha B > \frac{K}{r}\left(\sqrt{T} - \sqrt{K}\right)^2$$

$$\alpha B > \frac{K}{r}\left(1 - \sqrt{\frac{T}{K}}\right)^2$$

Just as a remark, notice that in the numerical example from which I plotted the graph indeed we have $\alpha B < \frac{K}{r}\left(1-\sqrt{\frac{T}{K}}\right)^2$ and therefore we have the two solutions.

i. The profit from a boat is $\pi(t) = p\alpha S(t) - c$. If there is free entry, fishing boats will enter as long as profits are positive. What is the free market equilibrium value of the stock S_F^* in the steady state.

If boats will continue to enter as long as profits are positive, then they will stop when profits are 0. Therefore, as in class, to find the free market equilibrium value of $S(t)$ we just need to check when this condition is satisfied.

$$\pi(t) = 0 \Leftrightarrow p\alpha S_F^* - c = 0 \Leftrightarrow S_F^* = \frac{c}{p\alpha}$$

However, notice that in class we had $T = 0$, and since $\frac{c}{p\alpha}$ is always weakly greater than 0 we never had any problem. In this case, if $\frac{c}{p\alpha} < T$ the growth is negative and the stock goes to 0.

4 Taxation to obtain optimum resource extraction

This exercise make you compute Pigouvian taxes. These kind of taxes are classical in the economics literature. Their aim is to correct for externalities that affect the market outcome without passing trough the channel of prices. In fact, in the fishing model an increase of boat affect the growth of natural resources in a way that is not transmitted to the market with the price p. Recall that the free market optimal number of boats was $B_F^* = \frac{r}{\alpha}\left(1 - \frac{c}{p\alpha K}\right)$, while from a social planner perspective we should have $B_O^* = \frac{B_F^*}{2}$.

a. Show that the optimal number of boats could be obtained by a tax per boat $t = \frac{p\alpha K - c}{2}$.

There are two ways to answer to this question. The first way, which is the one you will have in the professor's solution, is to ask "which is the value of t such that if boats pay the new cost $c' = c + t$ then $B^*_O = B^*_{F'}$?". This means solving the following equation:

$$B^*_{F'} = \frac{r}{\alpha}\left(1 - \frac{c+t}{p\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B^*_O$$

and realise that $t = \frac{p\alpha K - c}{2}$. Since you have this way already explained in your solution I will show you the second way. I think it is less smart but more algorithmic, in case you do not have the intuition to frame the problem in the way I just exposed.

The second way amounts to perform the same step you did in the class, but the profits are $\pi(t) = \alpha p S(t) - (c+t) = \alpha p S(t) - (c + \frac{p\alpha K - c}{2})$ and realise that the free market equilibrium boats are equal to the social optimum. In optimum we must always have that profits are equal to 0, therefore:

$$\begin{aligned}
\alpha p S^*(B^*) - \left(c + \frac{p\alpha K - c}{2}\right) &= 0 \\
\alpha p K\left(1 - \frac{\alpha B}{r}\right) - \left(c + \frac{p\alpha K - c}{2}\right) &= 0 \\
p\alpha K - \frac{p\alpha K \alpha B}{r} &= c + \frac{p\alpha K - c}{2} \\
1 - \frac{c}{p\alpha K} - \frac{1}{2} + \frac{c}{2p\alpha K} &= \frac{\alpha B}{r} \\
\frac{1}{2} - \frac{c}{2p\alpha K} &= \frac{\alpha B}{r} \\
\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} &= B^*_{F'} = B^*_O
\end{aligned}$$

Which is the result we wanted.

b. Illustrate this tax on the graph of the revenue of the fishing industry.

The change in marginal revenues due to the introduction of the tax change the point in which this line intersect the marginal cost c. You could also interpret it by saying that the new marginal cost is $c + t$ and the optimality condition requires the blue line to intersect with the new marginal cost.

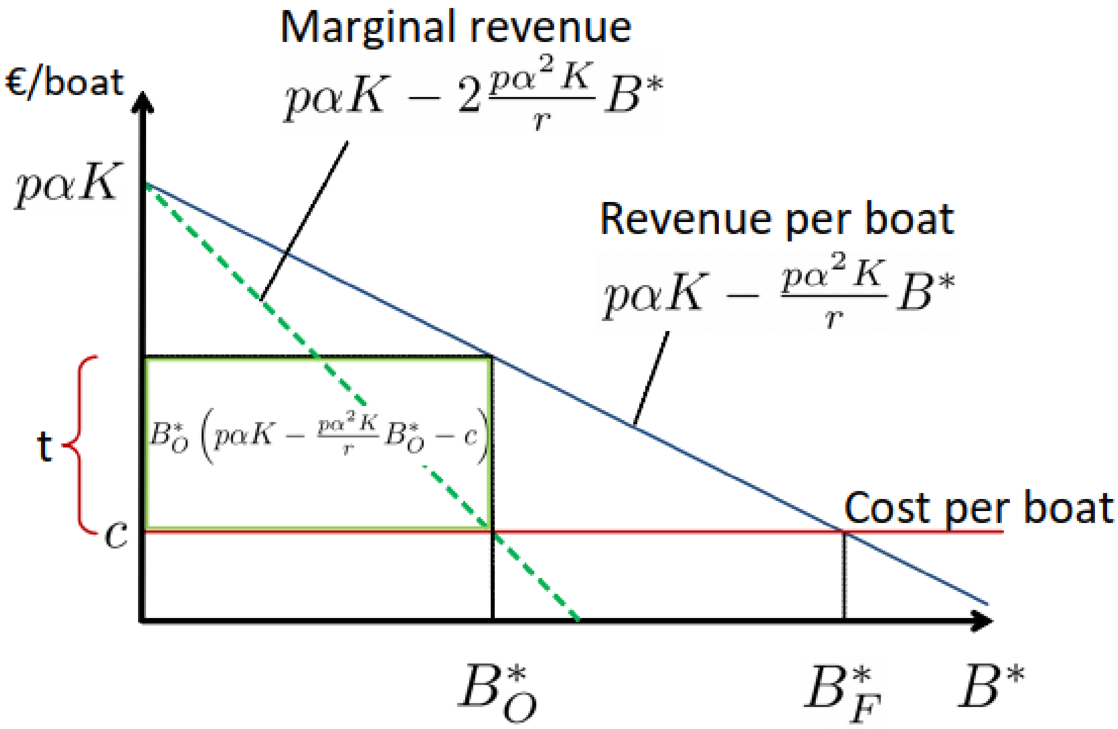


Figure 6: Representation of the fishing industry with a lump-sum tax.

c. Show that an ad valorem tax on fish sales of $\tau = \frac{p\alpha K - c}{p\alpha K + c}$ would achieve the optimum as well.

Exactly as before we can solve this problem in two ways. The first one is to compute the τ proportional tax on p that would make $B_{F'}^* = B_0^*$. This is equivalent to solving the following equation:

$$B_{F'}^* = \frac{r}{\alpha}\left(1 - \frac{c}{p(1-\tau)\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B_O^*$$

You will have the detail of this method in the professor's solution.

The second method follows the same state we did in the previous point. We just change the expression for profits and find the optimal number of boats in free markets $B^*_{F'}$. The new profits are $\pi(t) = p(1-\tau)\alpha S(t) - c = p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(t) - c$ and we must set them equal to 0.

$$
\begin{aligned}
p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(B^*) - c = 0 \\
p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha K\left(1 - \frac{\alpha B}{r}\right) - c = 0 \\
p\left(\frac{2c}{p\alpha K + c}\right)\alpha K\left(1 - \frac{\alpha B}{r}\right) - c = 0 \\
\frac{p\alpha K 2c}{p\alpha K + c} - \frac{p\alpha K 2c\alpha B}{r(p\alpha K + c)} - c = 0 \\
\frac{p\alpha K 2c}{p\alpha K + c} - c = \frac{p\alpha K 2c\alpha B}{r(p\alpha K + c)} \\
1 - \frac{\cancel{c}(p\alpha K + c)}{p\alpha K 2\cancel{c}} = \frac{\alpha B}{r} \\
1 - \frac{1}{2} - \frac{c}{p\alpha K 2} = \frac{\alpha B}{r} \\
\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} = B^*_{F'} = B^*_O
\end{aligned}
$$

Which is again the solution we wanted.

TD 6

2 Review (from TD2)

e. In the model of the Easter Island seen in class, there is a threshold amount of resource such that population is growing above that threshold, and decreasing below that threshold.

▼ *Answer*

True: You can easily see it from the phase diagram. We have a line along which S is fixed, above that line it increases and below it decreases. The expression of that line is $S = \frac{d-b}{\phi\alpha\beta}$, therefore if S is greater than $\frac{d-b}{\phi\alpha\beta}$ the population is growing, otherwise it is decreasing.

f. In the model of the Easter Island seen in class, the net growth of the resource (i.e. net of harvesting by humans) is decreasing in the numbers of humans present in the ecosystem, for a given value of the amount of the resource.

▼ *Answer*

True: The net growth is given by the replenish of the resource minus the harvest, which is the part affected by the number of humans (population).

$$\dot{S}(t) = \underbrace{rS(t)\left(1 - \frac{S(t)}{K}\right)}_{\text{Natural growth}} - \underbrace{\alpha\beta S(t)L(t)}_{\text{Harvest}}$$

We can clearly see that an increase in $L(t)$ decreases the net growth. If you want to be more precise its enough to take the derivative with respect to $L(t)$:

$$\frac{\delta \dot{S}(t)}{\delta L(t)} = -\alpha\beta S(t) < 0$$

What you get is how much growth is smaller after an infinitesimal change in $L(t)$.

Midterm 2020

Remark: Some solutions here are way more elaborate then what was needed to get full points! This is just to make you understand.

2 - Production in the Solow-Swan model

Here we have a Solow-Swan model with population growth $n \in (0, 1)$, no technological growth, where $s \in (0, 1)$ is the saving rate, K is the stock of capital, $k = \frac{K}{L}$ is capital per worker, $F(K, L)$ is the production function with the usual assumptions, $f(k) = \frac{F(K,L)}{L}$ is production per worker, $\delta \in (0, 1)$ is the rate of

capital depreciation, and finally k^* denotes k on the balanced growth path.

In this exercise I put both the general answer and the particular case in which $f(k) = k^\alpha$ to show you that the solutions make sense and to make you visualise them better.

Is $s\frac{\delta f(k)}{\delta k} - \delta - n$ greater than zero, equal to zero or smaller than zero or we cannot say without more information?

▼ **1.** *If $k \to 0$.* In this case $\frac{\delta f(k)}{\delta k} \to \infty$ due to the Inada condition, therefore the left side of the equation is way bigger then the right side and therefore the expression is positive.

▼ **2.** *If $k = k^*$*. A lot of students got this wrong. The standard logic I saw was that since $k = k^*$ we are on the balanced growth path and therefore $sf(k^*) = (\delta - n)k^*$, hence $sf(k^*)' = (\delta + n)$. However, this logic is fallacious. If two functions are equal in one point, which in our case is k^*, then it is not true that their derivative is equal in that point in general. In fact, it is true only if those two functions are equal in every point. This is kind of a technical argument, I understand it is not straightforward, but maybe you will be convinced by the solution. Remember that the derivative of a function tells us how much that function varies after an infinitesimal change in the variable you take the derivative with respect to. Moreover, you know two things. For k a little bit smaller than k^*, say k_- we have that $sf(k_-) > (\delta + n)k_-$ (you can see it from the standard graph). Also, if k is a little bit higher than k^*, call it k_+ it holds that $sf(k_+) < (\delta + n)k_+$. This means that in the process of going from k_- to k_+ the function $(\delta + n)k$ had a higher increase then $sf(k)$, and therefore its derivative $(\delta + n)$ is higher then the derivative $sf'(k)$. Since this holds for all $k_- < k^* < k_+$ then it also holds for k^*.

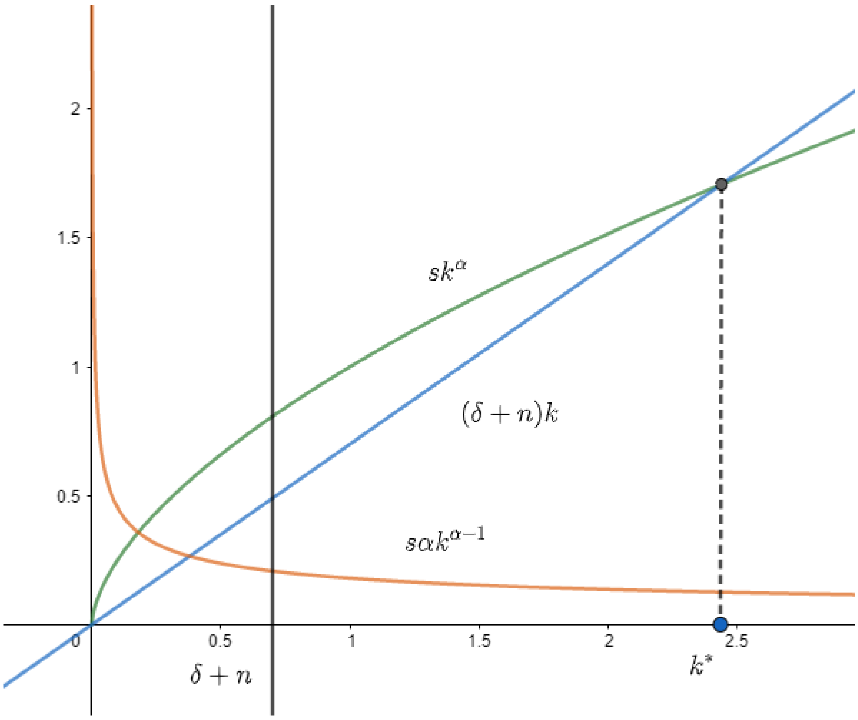


Figure 1: Graph with $f(k) = k^\alpha$, $s = 0.3$, $\alpha = 0.6$, $\delta = 0.3$, $n = 0.4$.

▼ **3.** *If $k \to \infty$.* This is the converse of the point we had before. Due to the Inada conditions $\frac{\delta f(k)}{\delta k} \to 0$ and therefore only the right (negative) side of the equation is left.

▼ **4.** *Is $\frac{\delta f(k)}{\delta k} - \delta - n$ greater than zero, equal to zero or smaller than zero or we cannot say without more information if $k = k^*$? (Note the difference is that here there is no s in the expression.).* I saw two different kind of mistakes in this question. The first one was to assume that we were in the golden rule and therefore $\frac{\delta f(k^*)}{\delta k} = \delta - n$. However, this was not specified by the question. We could have any $s \in (0, 1)$. The other typical mistake was to say that since $s\frac{\delta f(k^*)}{\delta k} = \delta - n$ and $s < 1$ then if you remove it you get $\frac{\delta f(k^*)}{\delta k} > \delta - n$. This logic is faulty, as it does not consider that $\frac{\delta f(k^*)}{\delta k}$ itself depends on s, and therefore it is different for different values of s. Hence, we can not determine the relationship between the two quantities.

3 - A general purpose technology

Consider the production function with general purpose technology $Y = F(A, K, L) = AK^\alpha L^{1-\alpha}$ where $\alpha \in (0, 1)$, K is the capital stock and L is the labor. Assume that $A_{t+1} = (1 + g)A_t$ and $L_{t+1} = (1 + n)L_t$ where g and n are exogenous constants. Capital depreciates at rate δ. You can use the same approximation as in the TD for the growth of some variable X : $g_X = log\left(\frac{X_{t+1}}{X_t}\right)$.

This exercise consists in playing with growth rates, the techniques are the same we used in the TDs.

▼ **1.** *If the growth rate of capital at time t is $g_{K,t}$, compute the growth rate of Y at time t in terms of $g_{K,t}$ and the growth rates of technology and labour.* Most if you got this question right. The idea is to exploit the definition given in the text and do the calculations we did in the TDs. Some of you thought that the α exponent was also on A, probably it was just a misreading of the text. Also, some students took derivatives with respect to time, but here time is discrete!

$$\begin{aligned} g_{Y,t} &\approx log\left(\frac{Y_{t+1}}{Y_t}\right) \\ &= log\left(\frac{A_{t+1}K_{t+1}^{\alpha}L_{t+1}^{1-\alpha}}{A_tK_t^{\alpha}L_t^{1-\alpha}}\right) \\ &= log\left(\frac{A_{t+1}}{A_t}\right) + \alpha log\left(\frac{K_{t+1}}{K_t}\right) + (1-\alpha)log\left(\frac{L_{t+1}}{L_t}\right) \\ &= g + \alpha g_{K,t} + (1-\alpha)n \end{aligned}$$

▼ **2.** *Let's now define general-purpose technology-adjusted labor as $\bar{L} = A^{\frac{1}{1-\alpha}}L$. What is the growth rate of $\bar{L}$?* The procedure for answering this question is the same as before, we just need to use the expression of $\bar{L}$. Most of you got this correct.

$$\begin{aligned} g_{\bar{L}} &= log\left(\frac{A_{t+1}^{\frac{1}{1-\alpha}}L_{t+1}}{A_t^{\frac{1}{1-\alpha}}L_t}\right) \\ &= \frac{1}{1-\alpha}log\left(\frac{A_{t+1}}{A_t}\right) + log\left(\frac{L_{t+1}}{L_t}\right) \\ &= \frac{1}{1-\alpha}g + n \end{aligned}$$

▼ **3.** *What will be the growth of output per worker in the long run?* As always, output per worker is $\frac{Y}{L}$, some of you got confused by the $\bar{L}$ and computed the wrong growth rate. So we have (from now on I omit the logarythm transformation):

$$\frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L} = A\left(\frac{K}{L}\right)^{\alpha}$$

$$g_y = g + \alpha(g_{K,t} - n)$$

▼ **4.** *What will be the growth rate of real wages in the long run?* Real wage is nominal wage times units of technology adjusted labour *over number of workers*. Before starting, notice that First, notice that $g_{\bar{A}} = \frac{g}{1-\alpha}$. We can now compute the growth rate:

$$w_r = \frac{wLA^{\frac{1}{1-\alpha}}}{L} = wA^{\frac{1}{1-\alpha}}$$

$$g_{w_r} = g_w + g_{\bar{A}} = \frac{g}{1-\alpha}$$

▼ **5.** *What will be the growth rate of the real interest rate in the long run?* From the lecture notes you know that r is stable, which implies that it's growth rate is 0 (not generally constant!).

5 - Steady-States

In this exercise you had to deal with a strange production function. We briefly talked about it in one of the TDs, but if you did not remember you had to think carefully to get the answer correct.

▼ **1.** *In the Solow-Swan model with constant population and technology, how many steady-states are there in total if $\frac{\delta F(K)}{dK} = c$ where c is a constant?* Here it was key to understand that the production function is linear. Consider $F(K) = a + cK$, then you have $\frac{\partial F(K)}{\partial K} = c$. We already saw what happens if the production function is linear in TD2. First, there is a steady state in 0. Second, there could be other infinitely many steady states if $\delta = sc$, which is in fact the condition for being in a steady state. The answer to this question should be clear from the graph below.

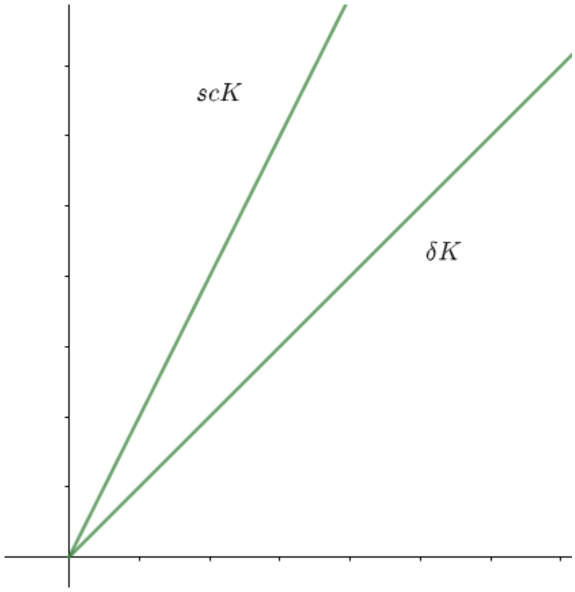


Figure 2: Steady state with linear production function.

▼ **2. (Bonus)** *In the Solow-Swan model with constant population and technology, how many stable steady-states are there in total if $\frac{\delta F(K)}{dK} = c$ where c is a constant? A steady-state is stable if for small deviation around K, the economy returns to K automatically.* This question was a bonus as you did not work with the "stability" concept before. However, is exactly what we did in the previous TD, so you should be a little bit familiar with it by now. Consider the two different cases we studied before. First, it could be that $sc \neq \delta$ and therefore we only have one steady state at 0. Is it stable? What happens if we perturb it a little bit and move on $0 + \epsilon$? It depends. If $sc > \delta$ then there is more investment then depreciation, and K increases more than what is lost due to δ. In this case the steady state is not stable, as we do not return to 0. If instead $sc < \delta$, then after a small increase capital still depreciates at a higher rate then what is saved. Hence, we return back to 0 and the state is stable. In the case $sc = \delta$ any point is a steady state and therefore no state is stable, as if we move a little bit we are already in a new steady state.

6 - An Algal Bloom

Consider the dynamics of the fish population with fishing. At time $t-1$, the stock of fishes is at its long term equilibrium $S^* > 0$. Then, at time t, an algal bloom kills half of the fish population. The bloom lasts a single period. At $t+1$, the algal bloom has ended and the ecosystem is back to where it was before (except that fishes have died of course).

This exercise did not require hard computations, you had to reason about the question and understand more or less intuitively the direction of the answer.

▼ **1.** *At $t+1$, is the net growth of the stock $\Delta S_{t+1} = S_{t+2} - S_{t+1}$ higher, equal or lower than the net growth at $t-2$, or we don't have enough information to tell. Assume that the number of boats have not changed between $t-2$ and $t+1$.* The question here asks to compare ΔS_{t+1} to $\Delta S_{t-2} = S_{t-1} - S_{t-2}$. Since before t we were in steady state, we must have that $\Delta S_{t-2} = 0$ and $S_{t-1} = S_{t-2} = S^*$. Hence, the question becomes: is $\Delta S_{t+1} > 0$? At time t we have the algal bloom, so the steady state population gets halved and we have that $S_t = \frac{S^*}{2}$. At $t+1$, even if the bloom is over, we are not at the steady state as $\frac{S^*}{2} < S^*$. To reach the steady state again the stock of fish must grow positively. For some time periods τ , we will have that $S_{\tau+1} > S_\tau$. We reached the conclusion that $\Delta S_{t+1} > 0$. Not all of you got this correct, I think it may be due to the confusion with the definition of net growth rate.

▼ **2.** *At $t+1$, is the natural growth of the stock $\tau(S_{t+1})$ higher, equal or lower than the natural growth at $t-2$ $\tau(S_{t-2})$, or we don't have enough information to tell. Assume* *that the number of boats have not changed between $t-2$ and $t+1$.* This question asks to compare the levels of the parabola for different values of S_t. In S_{t-2} we were in steady state, but we do not know where! As you can see from the graph, values of $\tau\left(\frac{S^*}{2}\right)$ could be both higher or lower than $\tau(S^*)$.

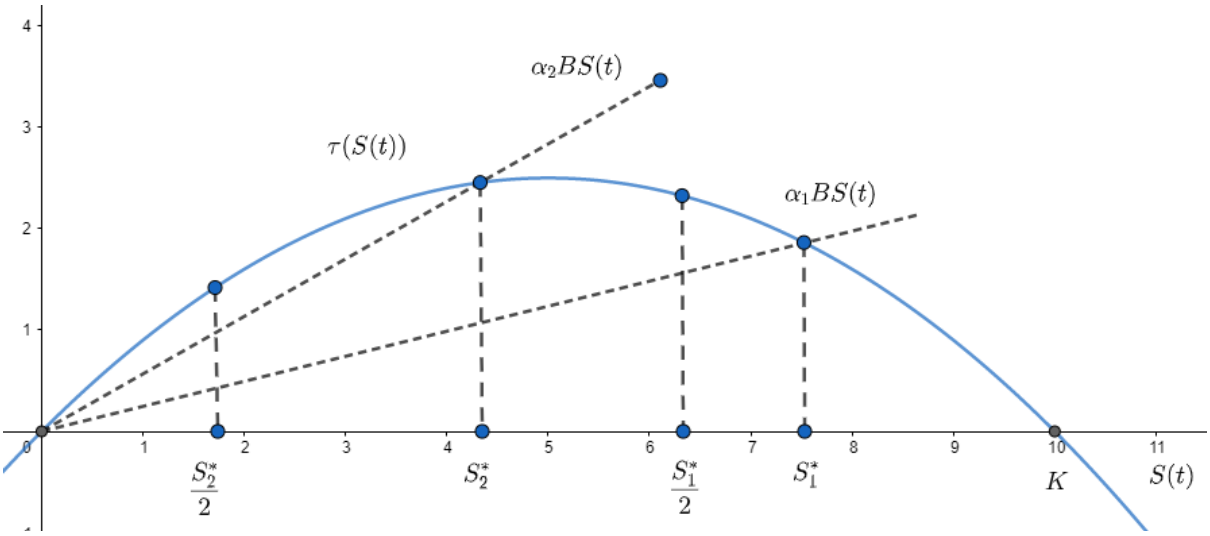


Figure 3: Steady states and growth rate for different fishing intensities.

▼ **3.** *If commercial fishing was very intense, could this bloom cause the population to go extinct?* Here you should consider that fishermans' technology is $\alpha < 1$, therefore they will always be able to catch a fraction of the fish that is lower than the total amount available. If $\alpha \to 1$ there will still be a small fraction ϵ of fishes around. You can see from the graph of the growth rate τ that this is positive at any value greater than 0, so extinction is not possible in this model (does not mean it is not possible in reality!).

5 Equilibrium on Easter island (from TD2)

In this exercise we just have to reason a little bit with the phase diagram to get the answer. I do not want to focus on the calculations which you can find in your notes, therefore I report the fundamental equations here:

$$\dot{S}(t) = \left(r \left(1 - \frac{S(t)}{K} \right) - \alpha\beta L(t) \right) S(t)$$

$$\dot{L}(t) = (b - d + \phi\alpha\beta S(t))\, L(t)$$

To generate the graph we have to find the loci of points in which the two variables are not moving. To do this is enough to set the growth (derivative with respect to time) equal to 0. We obtain the two lines:

$$S^* = K - \frac{K\alpha\beta}{r} L^*$$

$$S^* = \frac{d-b}{\phi\alpha\beta}$$

Here you can find the phase diagrams with these two loci depicted.

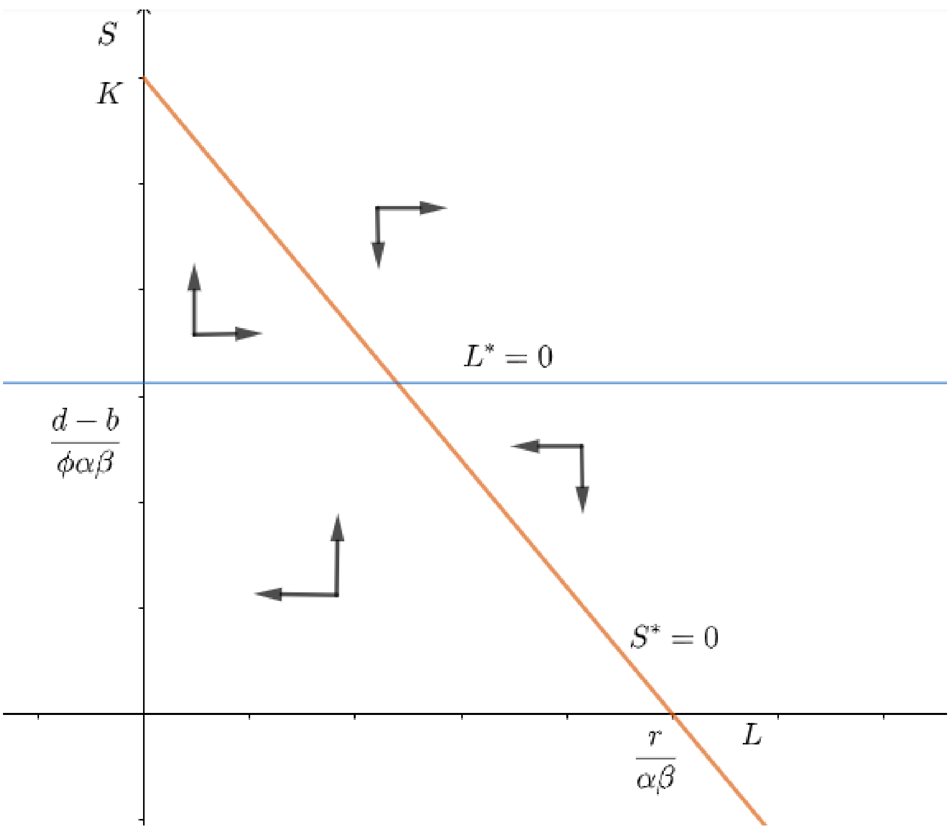


Figure 4: Phase diagram with $r = 0.04$, $\alpha = 0.00001$, $b - d = 0.1$, $\phi = 4$, $\beta = 0.4$, $K = 12000$.

In the rest of the solution I indicate with dashed lines the old loci, while continuous lines represents the new ones after the variable change. We assume we are in the nontrivial internal steady state.

a. What happens if r goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

We can see that r only appears in one of the two equations, namely $S^* = K - \frac{K\alpha\beta}{r}L^*$. Since r is positively affecting its slope, but not the intercept, we have to rotate it counterclockwise. The other locus is not affected.

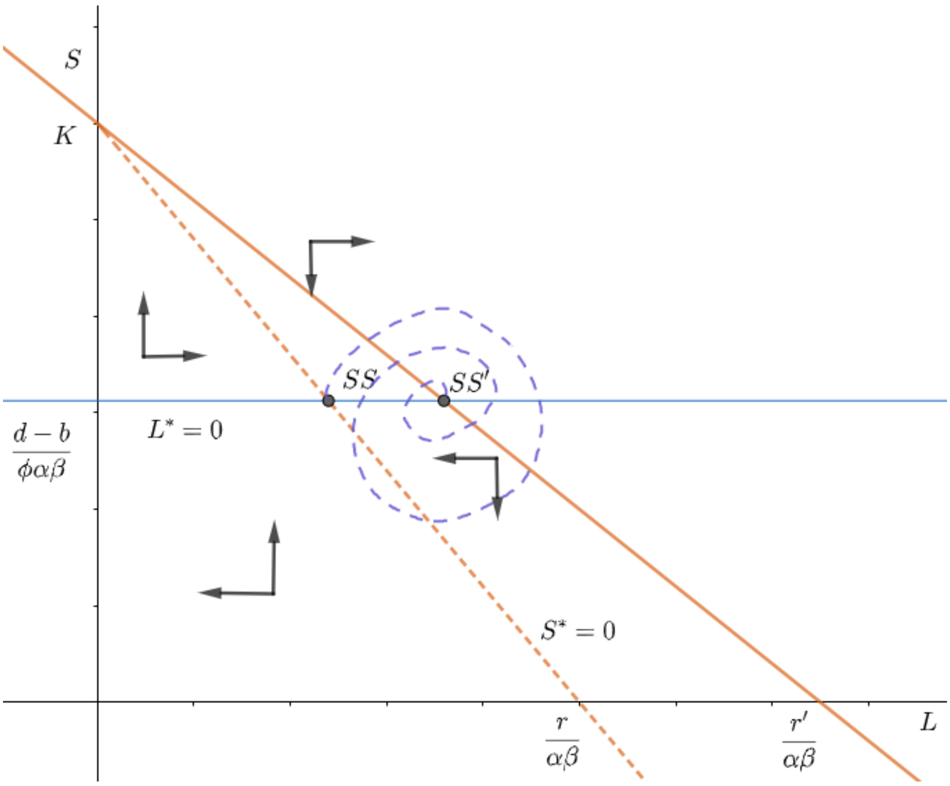


Figure 5: Phase diagram with new $r' = 0.06$.

What we see is that in the new steady state S^* is not affected, but L^* is higher. This is due to the fact that the increase of availability of resources is completely offset by the increase in population, which was possible because of the increase of the regeneration rate.

b. What happens if α goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

The rate α appears again in the intercept of the locus for $\dot{S} = 0$. However, in this case the rotation is clockwise as it is negatively affected by an increase in α. Moreover, α is also in the intercept for $\dot{L} = 0$, which goes down due to their negative relationship.

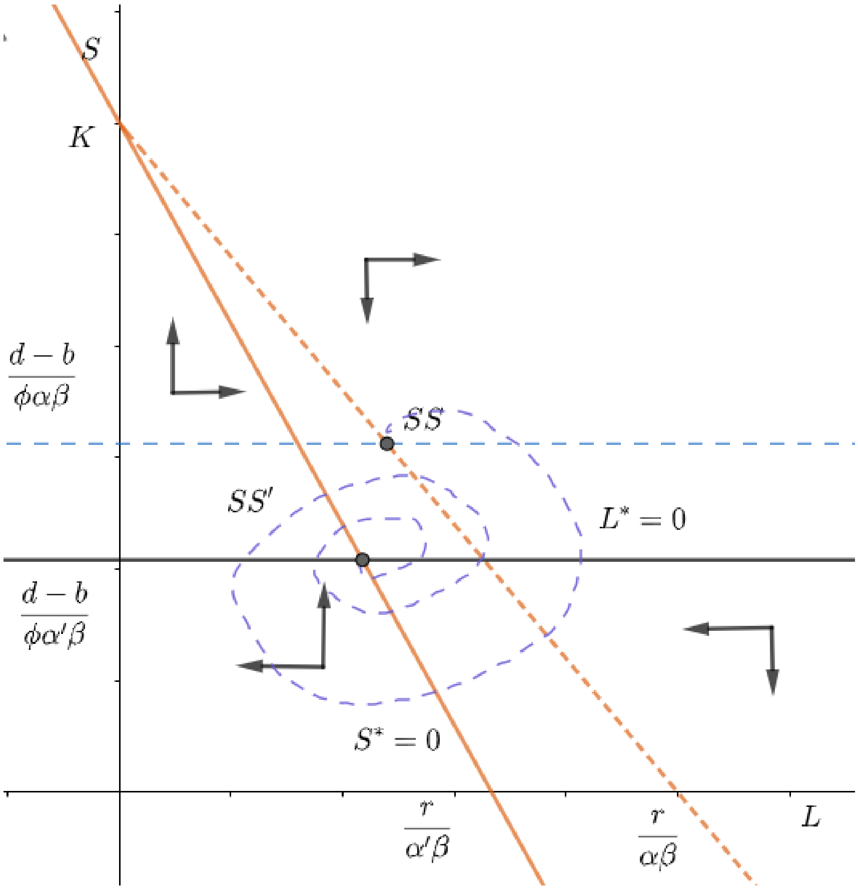


Figure 6: Phase diagram with new $\alpha' = 0.000015$.

In the new steady state we have unambiguously lower S^*, however the impact on L^* depends on the entity of the increase of α. More efficient farming means that we need less workers to keep the amount of resources fixed (clockwise rotation). On the other hand, given the increasy efficiency we need less resources to keep population constant (horizontal line decrease).

c. What happens if K goes down? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

The capacity K only appears in the intercept and slope of $\dot{S}=0$, which then must rotate counterclockwise.

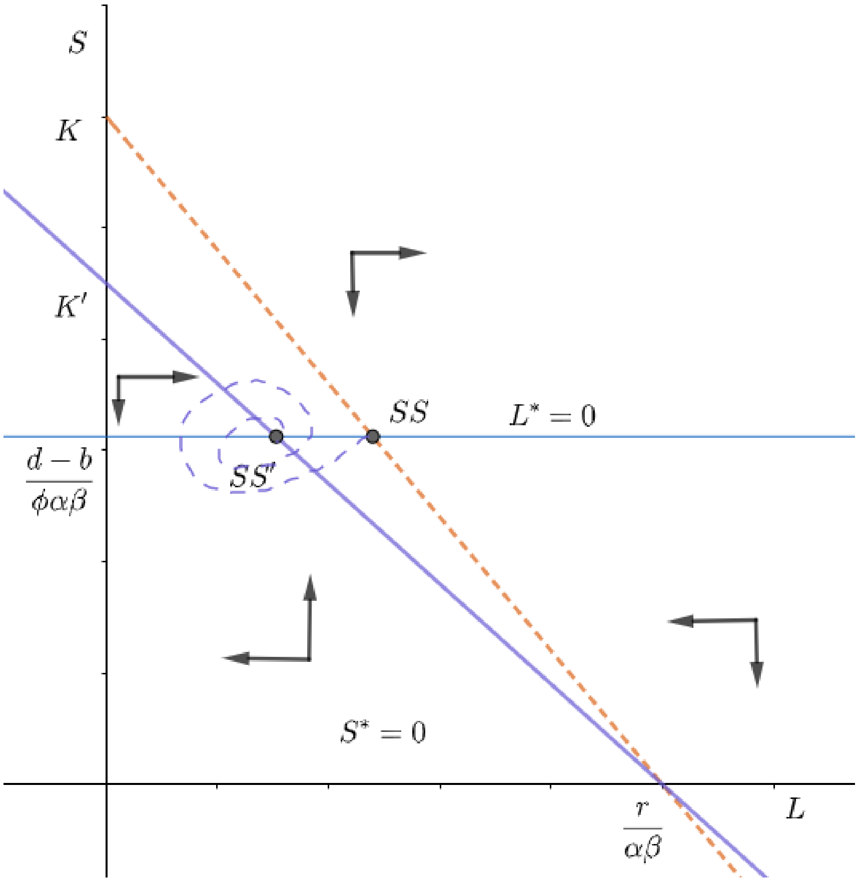


Figure 7: Phase diagram with new $K' = 9000$.

This time S^* is not affected while the impact on L^* is unambiguous. Due to the decrease in maximum capacity, the sotck of resources grows slower. You can see this from the growth rate $\dot{S}$. This implies that L^* is lower than before.

TD 7

1 Review

a. Before the demographic transition, increases in income per capita always caused an increase in the growth rate of population.

▼ *Answer*

True: As you found in your lecture, in the pre industrialised world technological progress and land expansion caused temporary increase in income per capita which in turn increased the growth rate of population.

b. In the contemporary world, an increase in income per capita is associated to a decrease in the growth rate of population.

▼ *Answer*

True: The graph at page 12 of your lecture notes is self explanatory. Higher average income correlates with lower births per woman. This could be due to many reasons, among which we have the fact that woman have now the chance to fully participate in social life.

c. Decreases in the various measures of fertility came after decreases in mortality.

▼ *Answer*

True: You can see from the graph at page 11 of your lecture notes that the fertility rate has decreased from 1950. For sure also the mortality rate has declined from that time.

d. The demographic transition is now over for most of the world population.

▼ *Answer*

True: Quoting from page 11 of your notes "*as of 2017, more than 80% of the world fertility was already at or below replacement rate*".

e. In the model of the Malthusian regime seen in class, an exogenous increase in the birthrate translates into a lower level of steady-state income per capita.

▼ *Answer*

Maybe: It depends, as the birth rate could be affected by exogenous changes of other variables related to it which may in turn increase income per capita. In general if $b(y_t)$ is higher then the statement is true.

2 The Malthusian Regime

This exercise studies a particular example of the model seen in class, with specifications for the primitives of the model. I report them here.

The birth rate is given by:

$$b(y_t) = \alpha_b + \beta_b y_t \tag{1}$$

The mortality rate is:

$$m(y_t) = \alpha_m - \beta_m y_t \tag{2}$$

The production function is:

$$Y(P_t) = \alpha_y + \beta_y P_t \tag{3}$$

Y is the total production (or income), P is the population, y is income per capita. It is assumed that all the population works and that there is no immigration nor emigration. All coefficients of the model are positive. It is further assumed that $\alpha_m > \alpha_b$ and $\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m) > 0$.

a. Discuss equations (1) and (3): how do they relate to the model seen in the lecture?

In your lectures you saw that the birth rate depends positively on income per capita, which translates into $b(y_t)$ with $\frac{\partial b}{\partial y_t} > 0$. This condition is indeed respected in the specific function we have in this exercise, as $\frac{\partial b}{\partial y_t} = \beta_b > 0$.

As for the technology, the condition you had in the lecture were positive marginal product $\frac{\partial Y}{\partial P_t} > 0$ and decreasing returns $\frac{\partial^2 Y}{\partial P_t^2} < 0$. It is easy to check that the first holds while the second does not. In fact, $\frac{\partial Y}{\partial P_t} = \beta_y > 0$ and $\frac{\partial^2 Y}{\partial P_t^2} = 0 \nless 0$. This production function is similar to what you saw in TD2 and the midterm, it is affine! Since it is a line it does not have decreasing returns.

b. Compute the marginal and average productivity of labor. Comment.

To compute average and marginal productivity, we first need productivity. As usual we divide by the population P_t.

$$
\begin{aligned}
y(P_t) &= \frac{Y(P_t)}{P_t} \\
&= \frac{\alpha_y}{P_t} + \beta_y
\end{aligned}
$$

Marginal productivity indicates how much more productive we are by increasing P_t by an infinitesimal amount. By taking derivatives we get:

$$
\frac{\partial y(P_t)}{\partial P_t} = -\frac{\alpha_y}{P_t^2} < 0
$$

Hence, by adding labour we become less productive. As for average productivity we just have to divide by the number of workers

$$\frac{y(P_t)}{P_t} = \frac{\alpha_y}{P_t^2} + \frac{\beta_y}{P_t}$$

Nothing special here, as we increase the number of workers the productivity per worker decreases.

c. Compute the steady-state level of total income, per capita income and population. Show graphically how those steady-state values are determined.

Population stops growing when the mortality rate is equal to the birth rate. We have to set $b(y_t) = m(y_t)$, and we get:

$$\begin{aligned} \alpha_b + \beta_b y^* &= \alpha_m - \beta_m y^* \\ y^*(\beta_b + \beta_m) &= \alpha_m - \alpha_b \\ y^* &= \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} \end{aligned}$$

which is the steady state level of per capita income. We still need to compute the steady-state level of total income and population. Total income is itself a function of the population, so we have to find the steady state of this one first. We can rely on y^* which we just found.

$$
\begin{aligned}
y^* &= \frac{\alpha_y}{P^*} + \beta_y \\
\frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} &= \frac{\alpha_y}{P^*} + \beta_y \\
\frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} - \beta_y &= \frac{\alpha_y}{P^*} \\
P^* \left(\frac{\alpha_m - \alpha_b}{\beta_m + \beta_b} - \beta_y \right) &= \alpha_y \\
P^* \left(\frac{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}{\beta_m + \beta_b} \right) &= \alpha_y \\
P^* &= \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}
\end{aligned}
$$

Now that we have y^* and P^* we are ready to compute the steady state level of total income Y^*. Its expression is given in the text:

$$
\begin{aligned}
Y^* = Y(P^*) &= \alpha_y + \beta_y P^* \\
&= \alpha_y + \beta_y \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
&= \frac{\alpha_y \alpha_m - \alpha_y \alpha_b - \cancel{\alpha_y \beta_y \beta_b} - \cancel{\alpha_y \beta_y \beta_m} + \cancel{\alpha_y \beta_y \beta_b} + \cancel{\alpha_y \beta_y \beta_m}}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
Y^* &= \frac{\alpha_y(\alpha_m - \alpha_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}
\end{aligned}
$$

In the following figure I represented the equilibrium for specific values of the paramenters.

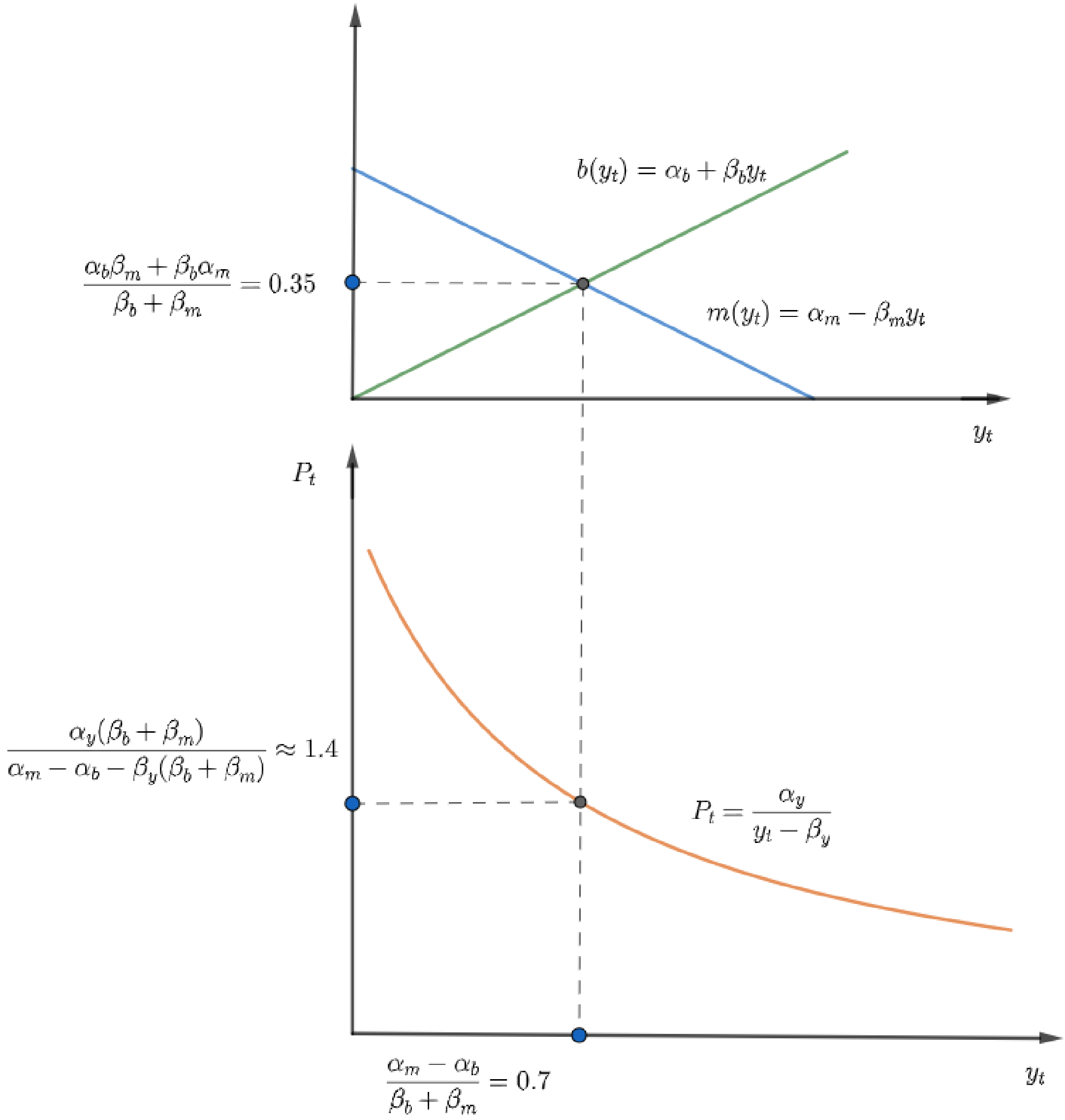


Figure 1: Steady state for $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = 0.7$. Here $Y^* = \alpha_y + \beta_y P^* = 1$.

d. For the rest of the exercise, we assume $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = \alpha > 0$. What are the steady-state levels for this configuration of parameters?

I more or less computed them in the graph, but I assumed a specific value for α. We have again that $Y^* = \alpha_y + \beta_y P^* = 1 + 0P^* = 1$. As for y^* and P^*:

$$y^* = \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} = \frac{\alpha}{1} = \alpha$$

$$P^* = \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} = \frac{1}{\alpha}$$

e. Show that the model dynamics can be summarized by a first-order difference equation in P_t (of the type $P_{t+1} = f(P_t)$, with f some function that you need to find; you can also look for an equation of the type $\Delta P_t = g(P_t)$ with g some function to find, if it is easier for you to do so).

This question is a very involved way of asking: what are the time dynamics of P_t? You know from your lecture notes that $\dot{P} = [b(y_t) - m(y_t)]\,P_t$. However, we are in discrete time here, as the question asks for a difference equation (not differential), therefore in this case $\dot{P}$ is substituted by $P_{t+1} - P_t$. We just have to work out the expression above and plug values for the parameters.

$$\begin{aligned}
P_{t+1} - P_t &= [b(y_t) - m(y_t)]\,P_t \\
&= [\cancel{\alpha_b} + \beta_b y_t - \alpha_m + \beta_m y_t]\,P_t \\
&= [(\beta_b + \beta_m)y_t - \alpha]\,P_t \\
&= [y_t - \alpha]\,P_t \\
&= \left[\frac{\alpha_y}{P_t} + b_y - \alpha\right] P_t \\
&= \left[\frac{1}{P_t} - \alpha\right] P_t \\
P_{t+1} - P_t &= 1 - \alpha P_t \\
P_{t+1} &= P_t(1 - \alpha) + 1
\end{aligned}$$

f. Study the convergence of population to its steady state starting from an initial value of population P_0 close to 0 for the following values of α: (i) $0 < \alpha < 1$, (ii) $\alpha = 1$, (iii) $1 < \alpha < 2$.

This question basically asks you to study the dynamics of population for different values of α. It is more or less about plugging numbers. Let's start from $t = 1$ and see what the dynamics look like. Since $1-\alpha$ is a bit uncomfortable I substitute it with $\gamma = 1-\alpha$.

$$\begin{aligned}
P_1 &= \gamma P_0 + 1 \\
P_2 &= \gamma P_1 + 1 \\
&= (\gamma P_0 + 1)\gamma + 1 \\
&= \gamma^2 P_0 + \gamma + 1 \\
P_3 &= \gamma P_2 + 1 \\
&= (\gamma^2 P_0 + \gamma + 1)\gamma + 1 \\
&= \gamma^3 P_0 + \gamma^2 + \gamma + 1
\end{aligned}$$

You see the pattern. By thinking a little bit we realise that we can express P_t in the following way:

$$P_t(\gamma) = \gamma^t P_0 + \sum_{s=0}^{t-1} \gamma^s$$

For $t \to \infty$, by the rules of power series, we have:

$$\begin{aligned}
P_\infty(\gamma) &= \gamma^\infty P_0 + \sum_{s=0}^{\infty} \gamma^s \\
&= \gamma^\infty P_0 + \frac{1}{1-\gamma}
\end{aligned}$$

We are ready to evaluate the convergence. The following table gives a relationship between $1-\alpha$ and γ.

	α	γ
(i)	$0 < \alpha < 1$	$0 < \gamma < 1$
(ii)	1	0
(iii)	$1 < \alpha < 2$	$-1 < \gamma < 0$

Case (ii) is the easiest. If $\gamma = 0$ then $P_t = 1$ for any t. Population is fixed since the beginning, so in some sense we already converged from the start.

In case (i) we have $0 < \gamma < 1$. If we have no clue we can take one number and see what happens. Let's try $\gamma = 0.5$. We have the following series:

$$
\begin{aligned}
P_1 &= 0.5P_0 + 1 \approx 1 \\
P_2 &= 0.5(1) + 1 = 1.5 \\
P_3 &= 0.5(1.5) + 1 = 1.75 \\
P_4 &= 0.5(1.75) + 1 = 1.875 \\
P_5 &= 1.9375 \\
P_6 &= 1.96875 \\
&\vdots
\end{aligned}
$$

In the following picture you can see the series grahpically:

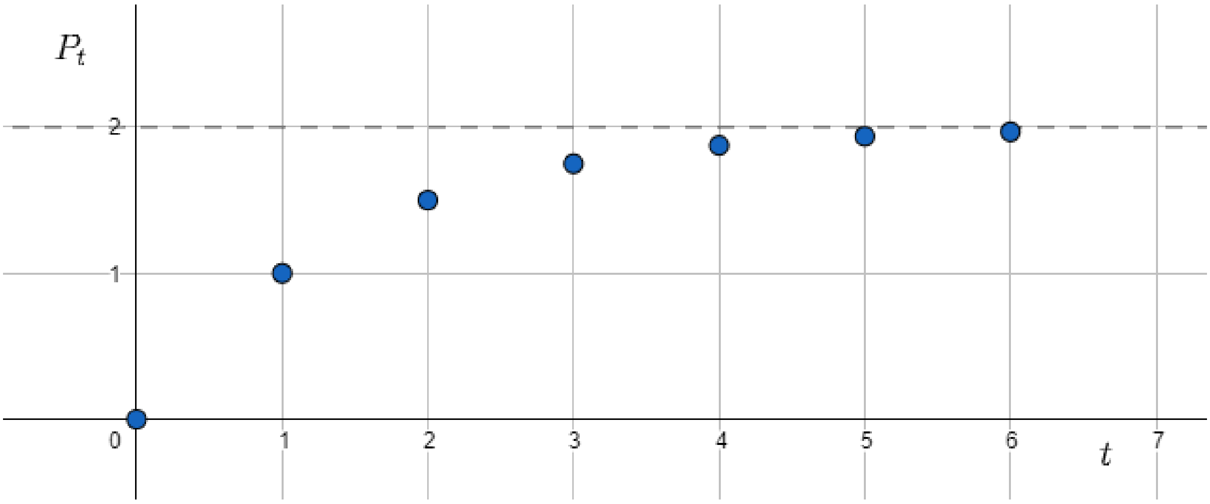


Figure 2: Series with $\gamma = 0.5$.

You can see where we are going. We can immediately compute P_∞ from the expression above:

$$P_\infty(0.5) = 0(P_0) + \frac{1}{1-0.5} = \frac{1}{0.5} = 2$$

What we conclude is that for a value of α between 0 and 1 population grows, slower at each step, and eventually reaches a steady state level.

As for case (iii), with $-1 < \gamma < 0$, we use the same strategy, namely plugging numbers for the specific value $\gamma = -0.5$. The series looks like this:

$$
\begin{aligned}
P_1 &= -0.5P_0 + 1 \approx 1 \\
P_2 &= -0.5(1) + 1 = 0.5 \\
P_3 &= -0.5(0.5) + 1 = 0.75 \\
P_4 &= -0.5(0.75) + 1 = 0.625 \\
P_5 &= 0.6875 \\
P_6 &= 0.65625 \\
P_7 &= 0.671875 \\
&\vdots
\end{aligned}
$$

As before, I plotted the series:

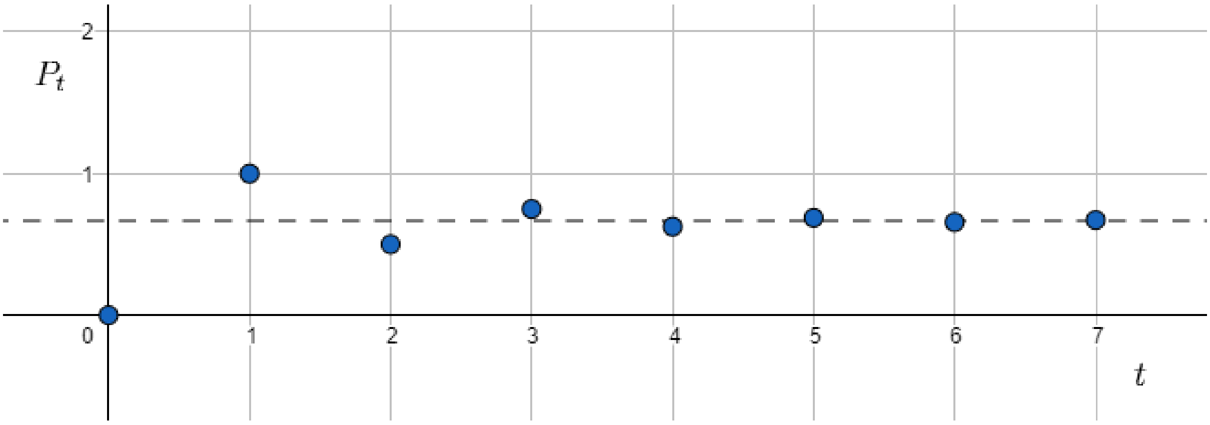


Figure 3: Series with $\gamma = -0.5$.

As you can see the series gere goes up and down, it is not monotonic in its growth. However, we can see where it converges to:

$$P_\infty(-0.5) = 0P_0 + \frac{1}{1-(-0.5)} = \frac{1}{1.5} = 0.\bar{6}$$

Interestingly, notice that the term P_0 has in both case no role in determining the convergence, which is only shaped by the rate γ.

8

TD 8

1 Review

a. In the OLG model of the course, the market for financial contracts without money doesn't exist because the interest rate asked for by the Young is too high.

▼ *Answer*

False: There is now way young can be convinced to transfer the consumption good to the old, as they do not have any guarantee that when they will be old the newborn will do the same. Money allows to overcome the perishability of the good and permits to contract both with the previous and the new generation.

b. When individuals optimally choose their consumption through time, any change in income at any date (or combination of changes in several periods) modifies the consumption choices.

▼ *Answer*

False: As we will soon see what matters in the intertemporal budget constraint is the net present value of resources, not particular levels of income at specific dates. If the net present value of resources does not change (i.e. the income shocks balance out), consumption choices do not change.

c. In the interior solution, if an individual receives more income when old than when young, he or she will always want to borrow when young.

▼ *Answer*

False: As we will see in the following exercise, this depends on preferences for consumption, time preferences as captured by the β, the amount of the net present value of income and the interest rate r. The statement is true when the agent likes "*consumption smoothing*", he likes to consume the same amount in every time period.

d. When there is inflation, the real value of money decreases through time.

▼ *Answer*

True: Simply, the real value of money is its nominal value taking into account inflation. If inflation goes from zero to a positive value then money is nominally less valuable. The real value of money at time t is $\frac{1}{P_t}$. If there is inflation the level of prices increases.

2 Intertemporal choices

This exercise is for you to get acquainted with the basic intertemporal consumption model. We have an individual living two periods $t=1$ and $t=2$, he receives an amount of the consumption good ω_t in each period t, and can save an amount of good s of the consumption good in period 1. The real interest rate received on savings in period 2 is denoted r. The utility of the individual over his or her consumption c_1 and c_2 at both periods is defined as follows:

$$U(c_1,c_2)=u(c_1)+\beta u(c_2)=\frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}+\beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}$$

with discount rate $0\leq\beta$ and intertemporal elasticity of consumption $\sigma\in(0,1)\cup(\infty)$.

a. Write the budget constraint of the individual in both periods.

Intuitively, in time $t=1$ you can not consume more than your income ω_1, while in time $t=2$ you can not consume more than ω_2 plus the amount of savings from period 1. At time 1 you have to decide how to divide your income in consumption and savings, we have:

$$c_1 + s \leq \omega_1$$

At time 2 the individual does not have the problem of saving, he can just consume. What he will get is $\omega_2 + s(1+r)$, therefore we have:

$$c_2 \leq \omega_2 + (1+r)s$$

b. Write the intertemporal budget constraint of the individual.

The intertemporal budget constraint is the constraint that a forward looking (who considers the future) individual faces when he chooses how to consume and save at time $t = 1$. The first step is to recognise that the per period constraints hold with equality. Why? Because there is no reason individuals should waste resources by not consuming or savings (technically, this is because preferences satisfy "*local non-satiation*"). Hence, we have that:

$$c_1 + s = \omega_1$$

$$c_2 = \omega_2 + (1+r)s$$

The two contraints are linked by s. From period 1 we get $s = \omega_1 - c_1$, by substituting s into the budget constraint for period 2 we get:

$$\begin{aligned} c_2 &= \omega_2 + (\omega_1 - c_1)(1+r) \\ \frac{c_2}{1+r} &= \frac{\omega_2}{1+r} + \omega_1 - c_1 \\ c_1 + \frac{c_2}{1+r} &= \underbrace{\omega_1 + \frac{\omega_2}{1+r}}_{\text{Net present value of resources}} \end{aligned}$$

which is in fact the intertemporal budget constraint.

c. Solve the intertemporal consumption problem, assuming an interior solution with $c_1 > 0$, $c_2 > 0$. Write the Euler equation,

that describes the evolution of consumption through time as a function of β and r at optimality. When does consumption rise through time? When does it fall through time? Interpret the role of r and β. What happens when $\sigma \to 0$? What happens when $\sigma \to \infty$?

Let's start from the first part of the question. The intertemporal consumption problem constitutes, from a technical point of view, a constrained maximisation. In fact, the agents would like to solve the following program:

$$\underset{c_1,c_2}{max}\ U(c_1,c_2) = \underset{c_1,c_2}{max}\ \frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} + \beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}$$

The solution to this problem alone would be to consume an infinite amount of c_1 and c_2, of course. However, agents can not do that, as they are constrained by their resources. These are given by the equation we computed before, namely the net present value of income. The problem therefore becomes to maximise the quantity above subject to the constraint. In particular, since the agent needs to choose c_1 and c_2 given β, r, σ, ω_1 and ω_2 the problem he is solving allows him to distribute consumption in time to maximise his utility.

$$\underset{c_1,c_2}{max}\ \frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} + \beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}$$

$$\text{subject to}\ \ c_1 + \frac{c_2}{1+r} = \omega_1 + \frac{\omega_2}{1+r}$$

Luckily enough there is a mathematical technique to solve constrained maximisation problems, the Lagrangian technique. It amounts to construct a new objective function from the old one and the constraint. First, let's express the constraint in a slightly different way, as we want everything to be on one side of the equal sign:

$$\text{subject to}\ \ c_1 + \frac{c_2}{1+r} - \omega_1 - \frac{\omega_2}{1+r} = 0$$

Now, to construct the Lagrangian we just have to place the multiplier λ in front of the left hand side of the constraint expressed in this way, with a minus sign in front, and add the objective function:

$$\underset{c_1,c_2,\lambda}{max}\ \mathcal{L}(c_1,c_2,\lambda) = \underset{c_1,c_2,\lambda}{max}\ \frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} + \beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} - \lambda\left[c_1 + \frac{c_2}{1+r} - \omega_1 - \frac{\omega_2}{1+r}\right]$$

Solving this problem is equivalent to solving the previous problem. We take derivatives with respect to the three variables of interest:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial c_1} = 0 \\ \frac{\partial \mathcal{L}}{\partial c_2} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} \cancel{\frac{\sigma-1}{\sigma}}\frac{c_1^{\frac{\sigma-1}{\sigma}-1}}{\cancel{\frac{\sigma-1}{\sigma}}} - \lambda = 0 \\ \beta\cancel{\frac{\sigma-1}{\sigma}}\frac{c_2^{\frac{\sigma-1}{\sigma}-1}}{\cancel{\frac{\sigma-1}{\sigma}}} - \frac{\lambda}{1+r} = 0 \\ c_1 + \frac{c_2}{1+r} - \omega_1 - \frac{\omega_2}{1+r} = 0 \end{cases} \Rightarrow \begin{cases} c_1^{\frac{\sigma-1}{\sigma}-1} = \lambda & (1) \\ \beta c_2^{\frac{\sigma-1}{\sigma}-1} = \frac{\lambda}{1+r} & (2) \\ c_1 + \frac{c_2}{1+r} = \omega_1 + \frac{\omega_2}{1+r} & (3) \end{cases}$$

As you can see, condition (3) is the constraint itself, this will happen with every Lagrangian you solve. The question asks to find the Euler equation. It is enough to divide condition (2) by condition (1) and rearrange terms. First, notice that $\frac{\sigma-1}{\sigma} - 1 = \frac{\sigma-1-\sigma}{\sigma} = -\frac{1}{\sigma}$ We have the following:

$$\frac{\beta c_2^{\frac{\sigma-1}{\sigma}-1}}{c_1^{\frac{\sigma-1}{\sigma}-1}} = \frac{\frac{\cancel{\lambda}}{1+r}}{\cancel{\lambda}} = \beta\left(\frac{c_2}{c_1}\right)^{\frac{\sigma-1}{\sigma}-1} = \frac{1}{1+r}$$

$$\left(\frac{c_2}{c_1}\right)^{-\frac{1}{\sigma}} = \frac{1}{\beta(1+r)}$$

$$\left(\frac{c_2}{c_1}\right)^{-\frac{1\cdot(-\cancel{\sigma})}{\cancel{\sigma}}} = \left[\frac{1}{\beta(1+r)}\right]^{-\sigma}$$

$$\frac{c_2}{c_1} = [\beta(1+r)]^{\sigma}$$

The ratio of c_2 over c_1 indicates the evolution of consumption trough time, the higher the ratio the higher future consumption will be relative to present consumption. In particular, we can check if consumption rises trough time be looking at the value of the ratio. In fact, if $c_2 > c_1$ then $\frac{c_2}{c_1} > 1 \Rightarrow [\beta(1+r)]^{\sigma} > 1$. This will happen more likely if the discounting factor β is high (the agent care more about the future), if the interest rate r is high (savings pay more) and σ is high (elasticity between consumption today and tomorrow). Of course, if $\frac{c_2}{c_1} < 1 \Rightarrow [\beta(1+r)]^\sigma < 1$, then consumption decreases over time.

To understand what happens for limit values of σ we can look at the marginal rate of substitution between consumption today and tomorrow:

$$\frac{\partial U(c_1, c_2)/\partial c_1}{\partial U(c_1, c_2)/\partial c_2} = \left(\frac{c_2}{c_1}\right)^{-\frac{1}{\sigma}} = \left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}}$$

we have that

$$\lim_{\sigma \to 0} \left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}} = +\infty$$

You may recall that this is the marginal rate of substitution of the Leontief utility function (perfect complements), therefore in this limit case we have that $U(c_1, c_2) = min\{c_1, c_2\}$. There is zero elasticity between consumption today and tomorrow. On the contrary, in the opposite limit case:

$$\lim_{\sigma \to \infty} \left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}} = 1$$

Which is the marginal rate of substitution of a linear utility function (perfect substitute). We have that $U(c_1, c_2) = c_1 + c_2$. The agent has very high elasticity between consumption today and tomorrow, he does not care how he distributes consumption over time.

d. Rewrite the Euler equation when $\sigma = 1$. What is the value of consumption when $\beta = 1$ and $r = 0$? Under the same parameter values, does the individual save ($s > 0$) or borrow ($s < 0$) in the first period? Comment.

When $\sigma = 1$ the Euler equation becomes:

$$\frac{c_2}{c_1} = [\beta(1+r)]$$

We have to evaluate it in $\beta = 1$ and $r = 0$. This is a case in which the individual cares about future consumption tomorrow as much as he cares about consumption today ($\beta = 1$), and savings have no returns ($r = 0$). Can you guess what the consumption pattern will be?

$$\frac{c_2}{c_1} = 1 \Rightarrow c_2 = c_1$$

The agent wants to smooth consumption as more as he can (see question **c.** in the Review!). We can now evaluate the full profile of consumption (as a function of exogenous variables). From condition (3) in the Lagrangian (the budget constraint):

$$\begin{aligned} c_1 + \frac{c_2}{1+r} &= \omega_1 + \frac{\omega_2}{1+r} \\ c_1 + c_1 &= \omega_1 + \omega_2 \\ 2c_1 &= \omega_1 + \omega_2 \\ c_1 = c_2 &= \frac{\omega_1 + \omega_2}{2} \end{aligned}$$

Hence, when does the individual saves? If he has $\omega_1 > \omega_2$ he will save to equalise consumption in the two time periods. The opposite holds when $\omega_1 < \omega_2$, he is forced to borrow. We can see it from the expression for the savings:

$$\begin{aligned} s = \omega_1 - c_1 = \omega_1 - \frac{\omega_1 + \omega_2}{2} \\ = \frac{2\omega_1 - \omega_1 - \omega_2}{2} \\ = \frac{\omega_1 - \omega_2}{2} \end{aligned}$$

When is $s > 0$? Indeed, when $\omega_1 > \omega_2$.

3 A market for time preferences

In this problem, we explore how markets allow people with different attitudes towards intertemporal consumption to use financial markets to their advantage.

Suppose that an economy is made up of two types of households. There are $N_E = 1$ (E stands for elastic) households of the first type. These households have infinitely elastic preferences:

$$U(c_{1E}, c_{2E}) = c_{1E} + \beta c_{2E}$$

There are $N_I = 1$ (I is inelastic) households of the second type. These households have the following utility function (with $0 < \alpha < 1$):

$$\hat{U}(c_{1I}, c_{2I}) = \frac{c_{1I}^{1-\alpha}}{1-\alpha} + \beta \frac{c_{2I}^{1-\alpha}}{1-\alpha}$$

Both types of households have the same discount factor β. Finally, assume that elastic households have real incomes y_1 in period 1 and y_2 in period 2, while the inelastic households only earn a revenue y_I in the first period.

a. Write the intertemporal budget constraint for the type-I consumer.

We perform the same step as before, except for the fact that individual I has no income in period 2, he can consume only savings. We get the two periods budget constraints:

$$c_{1I} + s_I \leq y_I$$

$$c_{2I} \leq (1+r)s_I$$

As usual, we recognise that these two will hold with equality, as the individual does not want to waste resources. By explicating s in the first constraint and substituting it in the second we get the intertemporal budget constraint.

$$s_I = y_I - c_{1I}$$

$$c_{2I} = (1+r)(y_I - c_{1I})$$

$$\frac{c_{2I}}{1+r} = y_I - c_{1I}$$

$$c_{1I} + \frac{c_{2I}}{1+r} = y_I$$

As you can see there is no present value of income in period 2.

b. State the inelastic household's optimization problem and calculate the optimal choices c^*_{1I}, c^*_{2I} and s^*_I, individual I's savings, in terms of the model parameters and r.

The problem is very similar to the one we solved in the previous exercise. In particular, the methodology is identical and relies on the maximisation of the Lagrangian. Also the computational steps are really close with the ones of the exercise above.

$$\max_{c_{1I},c_{2I},\lambda} \mathcal{L}(c_{1I},c_{2I},\lambda) = \max_{c_{1I},c_{2I},\lambda} \frac{c_{1I}^{1-\alpha}}{1-\alpha} + \beta\frac{c_{2I}^{1-\alpha}}{1-\alpha} - \lambda\left[c_{1I} + \frac{c_{2I}}{1+r} - y_I\right]$$

We take derivatives:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial c_{1I}} = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{2I}} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} \cancel{1-\alpha}\frac{c_{1I}^{1-\alpha-1}}{\cancel{1-\alpha}} - \lambda = 0 \\ \beta\cancel{1-\alpha}\frac{c_{2I}^{1-\alpha-1}}{\cancel{1-\alpha}} - \frac{\lambda}{1+r} = 0 \\ c_{1I} + \frac{c_{2I}}{1+r} - y_I = 0 \end{cases} \Rightarrow \begin{cases} c_{1I}^{-\alpha} = \lambda & (1) \\ \beta c_{2I}^{-\alpha} = \frac{\lambda}{1+r} & (2) \\ c_{1I} + \frac{c_{2I}}{1+r} = y_I & (3) \end{cases}$$

As before, condition (3) is the budget constraint. From condition (1) and (2) we can compute the Euler equation:

$$\begin{aligned} \frac{\beta c_{2I}^{-\alpha}}{c_{1I}^{-\alpha}} = \frac{\frac{\cancel{\lambda}}{1+r}}{\cancel{\lambda}} = \beta\left(\frac{c_{2I}}{c_{1I}}\right)^{-\alpha} &= \frac{1}{1+r} \\ \left(\frac{c_{2I}}{c_{1I}}\right)^{\cancel{-\alpha}\cdot\frac{1}{\cancel{-\alpha}}} &= \left[\frac{1}{\beta(1+r)}\right]^{\frac{1}{-\alpha}} \\ \frac{c_{2I}}{c_{1I}} &= [\beta(1+r)]^{\frac{1}{\alpha}} \\ c_{2I} &= [\beta(1+r)]^{\frac{1}{\alpha}}\, c_{1I} \end{aligned}$$

From the Euler equation we can always express c_2 as a function of c_1 (or the contrary), so that thanks to condition (3) (the budget constraint) we can always compute the optimal value of the two.

$$c_{1I} + \frac{[\beta(1+r)]^{\frac{1}{\alpha}}\, c_{1I}}{1+r} = y_I$$

$$c_{1I}\left[1 + \frac{[\beta(1+r)]^{\frac{1}{\alpha}}}{1+r}\right] = y_I$$

$$c_{1I}\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right] = y_I$$

$$c_{1I} = \frac{y_I}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

From the Euler equation we can get the expression for c_{2I}:

$$c_{2I} = \frac{y_I\,[\beta(1+r)]^{\frac{1}{\alpha}}}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

Finally, we get expression of optimal savings from $s_I^* = y_I - c_{1I}$:

$$s_I^* = y_I - \frac{y_I}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

c. Write the intertemporal budget constraint for the type-E consumer.

The steps are the same as usual. I report without commenting, you can see the previous points.

$$c_{1E} + s_E \leq y_1$$

$$c_{2E} \leq y_2 + (1+r)s$$

$$s_E = y_1 - c_{1E}$$

$$c_{2E} = y_2 + (1+r)(y_1 - c_{1E})$$

$$c_{1E} + \frac{c_{2E}}{1+r} = y_1 + \frac{y_2}{1+r}$$

Here, we have the income in period 2, y_2.

TD 9

1 Review questions

f. At the OLG equilibrium with money, the real value of money increases at the same rate as the size of each generation, whatever the changes in aggregate money supply.

▼ Answer

False: The real value of money depends on the money supply and the level of prices. Individuals can not control the money supply, but is it changes it will also affect the real value of money. Therefore, the statement is true only if the supply of money does not change.

g. In the long run, nominal variables are disconnected from real ones.

▼ *Answer*

True: This is an empirical question. Apparently it seems that these two variables are connected. You can check the paper "some monetary facts" by (McCandless & Weber, 1995), there are a lot of interesting things about money. They say that "growth rates of the money supply and the general price level are highly correlated".

3 A market for time preferences

Recall some data from the problem. We are dealing with this E consumer with utility function and constraint as follows:

$$U(c_{1E}, c_{2E}) = c_{1E} + \beta c_{2E}$$

$$c_{1E} + \frac{c_{2E}}{1+r} = y_1 + \frac{y_2}{1+r}$$

d. State the elastic household's optimization problem and calculate the optimal choice of c^*_{1E}, c^*_{2E} and s^*_E, individual E's savings, in terms of the model parameters and r. Interpret the result.

The problem is the same as the one we have been solving until now, optimising an objective function with a constraint. We still use the Lagrangian.

$$\underset{c_{1E},c_{2E},\lambda}{max} \ \mathcal{L}(c_{1E}, c_{2E}, \lambda) = \underset{c_{1E},c_{2E},\lambda}{max} \ c_{1E} + \beta c_{2E} - \lambda \left[c_{1E} + \frac{c_{2E}}{1+r} - y_1 - \frac{y_2}{1+r} \right]$$

The first order conditions are:

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial c_{1E}}=0\\ \frac{\partial\mathcal{L}}{\partial c_{2E}}=0\\ \frac{\partial\mathcal{L}}{\partial\lambda}=0\end{cases}\Rightarrow\begin{cases}1-\lambda=0\\ \beta-\frac{\lambda}{1+r}=0\\ c_{1E}+\frac{c_{2E}}{1+r}-y_1-\frac{y_2}{1+r}=0\end{cases}\Rightarrow\begin{cases}1=\lambda & (1)\\ \beta=\frac{\lambda}{1+r} & (2)\\ c_{1E}+\frac{c_{2E}}{1+r}=y_1+\frac{y_2}{1+r} & (3)\end{cases}$$

From (1) and (2) we get that $\beta(1+r)=1$. This means that the fact that a consumption bundle is the optimal one is only related to exogenous parameters. What does this mean? This utility function is linear! You should remember that with linear utility function we could be in the two corner solutions or any bundle of consumption that satisfy the budget constraint is fine. This depends on the ratio of prices as compared to the marginal rate of substitution, which in this case is $\frac{1}{\beta}$, but what are prices here? We have that $\frac{1}{1+r}$ is the "price" of consuming tomorrow (not consuming today) and the ratio of prices is $(1+r)$.

Hence, we have that when $\frac{1}{\beta}>1+r\Rightarrow\beta(1+r)<1$ we have a corner solution and consume only $c_{1E}=y_1+\frac{y_2}{1+r}$. If $\frac{1}{\beta}<1+r\Rightarrow\beta(1+r)>1$ we have the opposite corner solution and $c_{2E}=y_1+\frac{y_2}{1+r}$. When $\frac{1}{\beta}=1+r\Rightarrow\beta(1+r)=1$ any value will do the job.

e. Why wouldn't a straight application of the Euler Equation work for type-E individuals?

Technically because the utility function is not concave, which is an important condition in an optimisation problem. Economically, it is nice to see it from the standard graph.

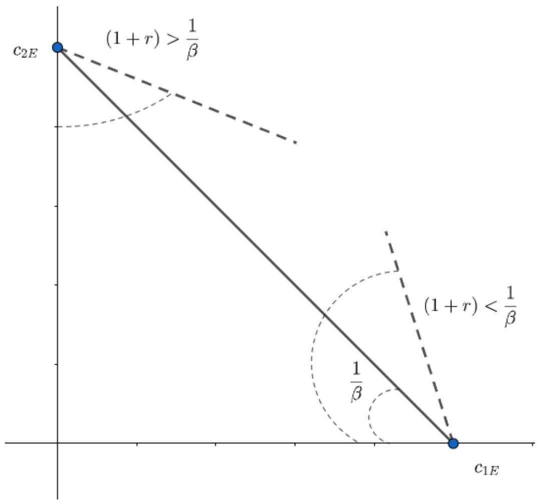


Figure 1: Corner solutions with linear utility.

Whether we are in a corner solution or not is related to the marginal rate of substitution and its magnitude with respect to the ratio of prices.

f. State the market-clearing condition in the bond market. Show that this market will only clear if $\beta(1+r) \leq 1$. Do not find the solution for r though (or if you do, I'm impressed!).

What we study here is a general equilibrium problem in which the two agents are I and E. They are demanding and supplying credit. When does the market of credit clears? Since we have only two individual here, the market clears when what someone is saving is the same as what the other one wants as credit. Recall that I only gets resources in period 1 y_I, while agent E gets y_1 and y_2. This means that agent I can not borrow (how would he repay his debt?), he can also save. Therefore, in the market clearing we must have that his savings are weakly greater than 0, $s_I \geq 0$. In turn, this implies that agent E savings must be weakly lower than zero (he can not give credit) $s_E \leq 0$. First, we exclude the corner solution in which $s_I = 0$. From point **b.** we had

$$\begin{aligned} s_I^* &= y_I - \frac{y_I}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]} \\ &= y_I\left[1 - \frac{1}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}\right] \\ &= y_I\left[\frac{\cancel{1}+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}-\cancel{1}}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}\right] \\ &= y_I\frac{\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}}{1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}} \end{aligned}$$

Which is zero only if $\beta = 0$ or $r = -1$. However, this would mean that $\beta(1+r) = 0 < 1$, which means that agent E would like to borrow, so no market clearing in this case. Hence, we must have $s_I > 0$ and $s_E < 0$ with $s_I = -s_E$. The exact amount is given by the exogenous variables β and the equilibrium price r. Tell me if you are strange enough to be willing to find the equilibrium interest rate! Hint: think about an Edgeworth Box in which the goods are c_1 and c_2 and the prices are as exposed in the previous point.

g. From now on, assume for the sake of simplicity that $\alpha = 1$ (You'll admit that the per-period utility function is then $u(c) = ln(c)$). Also assume that $\beta(1+r) = 1$. Solve for the equilibrium $(c_{1I}^*, c_{2I}^*, c_{1E}^*, c_{2E}^*)$. Which parameters allow this equilibrium to exist?

We can use the same reasoning as we did in the previous exercise to "prove" that $\alpha = 1$ implies a logarithmic function. To find the equilibrium values we look at the results of the previous points for consumption. For individual I we had:

$$c_{1I} = \frac{y_I}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

$$c_{2I} = \frac{y_I \left[\beta(1+r)\right]^{\frac{1}{\alpha}}}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

Substituting $\beta(1+r)=1$ and $\alpha=1$ we get:

$$c_{1I} = \frac{y_I}{1+\beta}$$

$$c_{1I} = \frac{y_I}{1+\beta}$$

The individual chooses to perfectly smooth consumption, due to the values of β and r. Type E individual is exactly indifferent between any consumption bundle, as we saw in point **.d**. Therefore, to find his value of consumption which clear the market we must check when what he consumes in each period plus what individual I consume in each period satisfies the budget constraint (market clearing conditions). Hence, we must have:

$$c_{1I} + c_{1E} = \frac{y_I}{1+\beta} + c_{1E} = y_1 + y_I$$
$$c_{1E} = y_1 + y_I - \frac{y_I}{1+\beta}$$

And the same at period 2:

$$c_{2I} + c_{2E} = \frac{y_I}{1+r} + c_{2E} = y_2$$
$$c_{2E} = y_2 - \frac{y_I}{1+r}$$

Since we have a minus sign, we must also specify that this solution holds when the values of parameters y_2, y_I and r are such that c_{2E} is positive, which answer the last bit of the question.

h. Assume again that $\alpha = 1$ (You'll admit that the per-period utility function is then $u(c) = ln(c)$). Now assume that $\beta(1+r) < 1$. Solve for the equilibrium

$(c^*_{1I}, c^*_{2I}, c^*_{1E}, c^*_{2E})$. Use these solutions to compute $\hat{U}(c^*_{1I}, c^*_{2I})$.

Here we don't have a specific value for $\beta(1+r)$, therefore we must rely on the market clearing condition which we found at point **f.** $s_I = -s_E$. Given that $\beta(1+r) < 1$ individual E only consumes in the first period $c_{1E} = y_1 + \frac{y_2}{1+r}$ and $s_E = y_1 - c_{1E} = -\frac{y_2}{1+r}$. By working out the condition we get:

$$y_I \frac{\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}}{1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}} = -\left(-\frac{y_2}{1+r}\right)$$
$$y_I \frac{\beta}{1+\beta} = \frac{y_2}{1+r}$$
$$r = \frac{y_2}{y_I}\frac{1+\beta}{\beta} - 1$$

To see what this implies, we substitute this expression in the condition $\beta(1+r) < 1$:

$$
\begin{aligned}
\cancel{\beta}(\cancel{1} + \frac{y_2}{y_I}\frac{1+\beta}{\cancel{\beta}} - \cancel{1}) < 1 \\
y_2(1+\beta) < 1
\end{aligned}
$$

which therefore must hold in equilibrium. Since we have the equilibrium r it is enough to substitute it into the consumption decisions. We already now that $c_{2E} = 0$ (since $\beta(1+r) < 1$). Moreover:

$$
\begin{aligned}
c_{1E} &= y_1 + \frac{y_2}{1+r} \\
c_{1E} &= y_1 + \frac{y_2}{\cancel{1} + \frac{y_2}{y_I}\frac{1+\beta}{\beta} - \cancel{1}}
\end{aligned}
$$

For individual I:

$$
\begin{aligned}
c_{1I} &= \frac{y_I}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]} \\
c_{1I} &= \frac{y_I}{1+\beta}
\end{aligned}
$$

$$
\begin{aligned}
c_{2I} &= \frac{y_I\left[\beta(1+r)\right]^{\frac{1}{\alpha}}}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]} \\
c_{2I} &= \frac{y_I\beta(1+r)}{1+\beta} \\
c_{2I} &= \frac{\cancel{y_I}\cancel{\beta}(\cancel{1} + \frac{y_2}{\cancel{y_I}}\frac{\cancel{1+\beta}}{\cancel{\beta}} - \cancel{1})}{\cancel{1+\beta}} \\
c_{2I} &= y_2
\end{aligned}
$$

To compute $\hat{U}(c^*_{1I}, c^*_{2I})$, just substitute the equilibrium values (remember that we had the logarythm):

$$
\hat{U}(c^*_{1I}, c^*_{2I}) = ln\,\frac{y_I}{1+\beta} + \beta ln\, y_2
$$

4 An OLG growth model

In this problem, we shall consider an overlapping-generation (OLG) model with production, capital accumulation, technological progress and population growth. The aim of the problem is to check that an economy represented by this model attains a balanced growth path (BGP) in the long run, and verifies some Kaldor facts on the BGP.

Time is considered discrete and denoted by t. In any period, a closed economy is populated by 2 generations: the young, born in t, and the old, born in $(t-1)$. In what follows, all variables relative to an individual will be denoted with a y superscript if the variable concerns a young individual, and with an o superscript for an old individual; variables will also have a time index that represents the current period (and not the generation as in the lecture).

The total population in time t is

$$N_t = N_t^y + N_t^o$$

with $N_t^y = (1+n)N_t^o = (1+n)N_{t-1}^y$, and $n > 0$ is a constant.

Individuals from the generation born in t seek to maximise their lifetime (or intertemporal) utility, defined as follows:

$$U(c_t^y, c_{t+1}^o) = \gamma ln\ c_t^y + (1-\gamma) ln\ c_{t+1}^o$$

with c denoting individual consumption of a final good of price 1.

It is further assumed that individuals obtain some labor income w_t by working when young, and can save an amount s_t of this income to accumulate capital. In the next period, individuals (who are now old) are paid a real return r_{t+1} on this capital: the capital income which is generated is then fully consumed. Assume full depreciation of capital after one period of production, so that the real interest rate r is also the return on capital net of depreciation. To avoid ambiguities, old people do not work.

The final good (which can be consumed or saved to accumulate capital) is produced according to the following aggregate production function:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

where K_t denotes aggregate capital available for production at the beginning of t, L_t is labor and A_t represents labor-augmenting technology. Assume $0 < \alpha < 1$.

You are given the following laws of motion for K and A:

$$K_{t+1} = I_t$$

$$\Delta A_t = g A_t$$

where $g > 0$ is constant, and I_t denotes aggregate investment.

Part 1 - Individual choices of consumption and savings

a. Prove that the intertemporal budget constraint of this individual is:

$$c_t^y + \frac{c_{t+1}^o}{r_{t+1}} = w_t$$

In this exercise we just have heavier notation, but the steps to get the budget constraints are the same as before. When an individual is young, he gets w_t and he must choose how to divide it between consumption and savings:

$$c_t^y + s_t \leq w_t$$

When old individuals do not have any income, they only have the return of their savings. Notice, however, that here only the return $r_{t+1}s_t$ is available, not the amount saved s_t plus the return. This is due to the full depreciation of capital. The budget constraint for old individuals is therefore:

$$c_{t+1}^o \leq s_t(1 + r_{t+1}) - s_t = r_{t+1}s_t$$

As before, from preferences we understand that agents do not want to waste money, they will use all they have. These constraint will then hold with equality. As usual, we must explicit savings from the first constraint and substitute into the second to find the intertemporal budget constraint.

$$s_t = w_t - c_t^y$$

$$c_{t+1}^o = r_{t+1}(w_t - c_t^y)$$

$$c_t^y + \frac{c_{t+1}^o}{r_{t+1}} = w_t$$

which is the expression given by the question.

b. Show that when the individual chooses optimally consumption and savings, one obtains the following Euler equation:

$$\frac{c_{t+1}^o}{c_y^y} = \frac{1-\gamma}{\gamma} r_{t+1}$$

To find the optimal choice we rely on the Lagrangian procedure. Remember, the Lagrangian is the sum of the objective function (utility) minus λ times the constraint with everything on one side:

$$c_t^y + \frac{c_{t+1}^o}{r_{t+1}} - w_t = 0$$

The Lagrangian is:

$$\max_{c_t^y, c_{t+1}^o, \lambda} \mathcal{L}(c_t^y, c_{t+1}^o, \lambda) = \max_{c_t^y, c_{t+1}^o, \lambda} \gamma ln\, c_t^y + (1-\gamma) ln\, c_{t+1}^o - \lambda \left[c_t^y + \frac{c_{t+1}^o}{r_{t+1}} - w_t \right]$$

Exactly as before, we take derivatives for the first order conditions:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial c_t^y} = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{t+1}^o} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\gamma}{c_t^y} - \lambda = 0 \\ \frac{1-\gamma}{c_{t+1}^o} - \frac{\lambda}{r_{t+1}} = 0 \\ c_t^y + \frac{c_{t+1}^o}{r_{t+1}} - w_t = 0 \end{cases} \Rightarrow \begin{cases} c_t^y = \frac{\gamma}{\lambda} & (1) \\ c_{t+1}^o = \frac{r_{t+1}(1-\gamma)}{\lambda} & (2) \\ c_t^y + \frac{c_{t+1}^o}{r_{t+1}} = w_t & (3) \end{cases}$$

To find the Euler equation dividing condition (2) by condition (1).

$$\frac{c_{t+1}^o}{c_t^y} = \frac{1-\gamma}{\gamma} r_{t+1}$$

c. Compute c_t^y and c_{t+1}^o, for prices w_t and r_{t+1} given.

To find the optimal amounts of consumption, we must use the relation between c_t^y and c_{t+1}^o specified by the Euler equation and substitute it into condition (3), the budget constraint. From Euler:

$$c_t^y = c_{t+1}^o \frac{\gamma}{1-\gamma} \frac{1}{r_{t+1}}$$

Plugging this into (3):

$$\begin{aligned} c_{t+1}^o \frac{\gamma}{1-\gamma} \frac{1}{r_t} + \frac{c_{t+1}^o}{r_{t+1}} &= w_t \\ c_{t+1} \frac{1}{r_{t+1}} \left(\frac{\gamma}{1-\gamma} + 1 \right) &= w_t \\ c_{t+1} \frac{1}{r_{t+1}} \left(\frac{1}{1-\gamma} \right) &= w_t \\ c_{t+1}^o &= w_t r_{t+1} (1-\gamma) \end{aligned}$$

From Euler we can recover c_t^y:

$$\begin{aligned} c_t^y &= c_{t+1}^o \frac{\gamma}{1-\gamma} \frac{1}{r_{t+1}} \\ &= w_t \cancel{r_{t+1}} \cancel{(1-\gamma)} \frac{\gamma}{\cancel{1-\gamma}} \frac{1}{\cancel{r_{t+1}}} \\ c_t^y &= w_t \gamma \end{aligned}$$

Notice that c_t^y is a fraction γ of income, while c_t^o is what remains from that fraction $(1-\gamma)$ multiplied by the interest rate! This is useful to answer the next question.

d. From the previous question, show that individual savings are $s_t = (1-\gamma)w_t$. What does a high value of mean γ in terms of individual preferences? Explain the link with savings.

To compute savings we use the expression we found before (or, if you forget, just think that savings are what you get minus what you consume).

$$
\begin{aligned}
s_t &= w_t - c_t^y \\
&= w_t - \gamma w_t \\
&= w_t(1-\gamma)
\end{aligned}
$$

Notice, in fact, that indeed what you consume when you are old are your savings times the interest rate! The parameter γ identifies your relative preference for present and future consumption. Higher γ means you are more impatient and you care more about consumption today then tomorrow. The contrary holds for lower values of γ.

Part 2 - Production, factor prices and factor shares

Let $\tilde{k}_t = \frac{K_t}{A_t L_t}$. Assume that production is managed by a firm operating in perfectly competitive markets. Note: contrary to class, define the wage bill paid by the firm by wL (not wAL).

e. Prove that $w_t = (1-\alpha)A_t\tilde{k}_t^{\alpha}$ and $r_t = \frac{\alpha}{\tilde{k}_t^{1-\alpha}}$.

To answer this question we must realise that wages and interest rates are costs for the firms. Therefore, they will be determined by the optimal choice about how much work and capital they want to use to produce. This is why they are determined by the optimal choice of the firm. In particular, firms maximise profits, which is production minus costs.

$$
\Pi_t(K_t, L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha} - w_t L_t - r_t K_t
$$

Notice that firms pay r_t as they employ capital when they receive it (contrary to the households that have their returns the time after they save). To maximise profits we must take first order conditions with respect to the production factors:

$$
\begin{cases} \frac{\partial \Pi_t}{\partial K_t} = 0 \\ \frac{\partial \Pi_t}{\partial L_t} = 0 \end{cases} \Rightarrow \begin{cases} \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} - r_t = 0 \\ (1-\alpha)A_t K_t^{\alpha}(A_t L_t)^{-\alpha} - w_t = 0 \end{cases} \Rightarrow \begin{cases} \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} = r_t & (1) \\ (1-\alpha)A_t K_t^{\alpha}(A_t L_t)^{-\alpha} = w_t & (2) \end{cases}
$$

The last step is to find these costs as a function of $\tilde{k}$ as expressed in the text. By rewriting condition (1):

$$
\begin{aligned}
\alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} &= r_t \\
\alpha\left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} &= r_t \\
\frac{\alpha}{\tilde{k}_t^{1-\alpha}} &= r_t
\end{aligned}
$$

Instead, by rewriting condition (2):

$$\begin{aligned}(1-\alpha)A_tK_t^\alpha(A_tL_t)^{-\alpha} &= w_t\\ (1-\alpha)A_t\left(\frac{K_t}{A_tL_t}\right)^\alpha &= w_t\\ (1-\alpha)A_t\tilde{k}_t^\alpha &= w_t\end{aligned}$$

which are the two expressions given by the text.

f. Factor shares are defined as the shares of each factor of production in total income. Show that the factor share of capital is α, and the factor share of labor is $(1-\alpha)$.

This question asks the relative shares of employed capital and labour by firms. For the purpose of this calculation, we should first notice that:

$$w_tL_t = (1-\alpha)A_t\tilde{k}_t^\alpha L_t = (1-\alpha)\frac{A_tL_t}{K_t}\tilde{k}_t^\alpha K_t = (1-\alpha)\tilde{k}_t^{-1}\tilde{k}_t^\alpha K_t = (1-\alpha)\tilde{k}_t^{\alpha-1}K_t = \frac{1-\alpha}{\tilde{k}_t^{1-\alpha}}K_t$$

Now, the quantity we are looking for (for capital), is (you can check that $Y_t = r_tK_t + w_tL_t$):

$$\frac{r_tK_t}{r_tK_t+w_tL_t} = \frac{\frac{\alpha}{\tilde{k}_t^{1-\alpha}}K_t}{\frac{\alpha}{\tilde{k}_t^{1-\alpha}}K_t + \frac{1-\alpha}{\tilde{k}_t^{1-\alpha}}K_t} = \frac{\alpha}{\alpha+1-\alpha} = \alpha$$

Since we are talking about shares, the relative share of labour is one minus the share of capital, and therefore it is $1-\alpha$. Hence, we know that $r_tK_t = \alpha Y_t$ and $w_tL_t = (1-\alpha)Y_t$.

Part 3 - Equilibrium and dynamics of the aggregate economy

g. Write the equilibrium condition on the labor market. At which rate does the labor force grow in this model?

The labour market is composed by offer and demand of labour. The price is the salary. The demand for labour is given by the firm, which would like to use L_t units of labour to produce. The offer of labour is given by the population of you people, who offer a fixed amount of labour N_t^y (the amount of young people in the population). Therefore, when these two quantities are equal, we are in equilibrium in the labour market $L_t = N_t^y$.

The growth rate of the labour force is computed as usual (don't forget we are in discrete time):

$$\begin{aligned}\frac{L_{t+1}-L_t}{L_t} &= \frac{N_{t+1}^y - N_t^y}{N_t^y}\\ &= \frac{(1+n)N_t^y - N_t^y}{N_t^y}\\ &= n\end{aligned}$$

Or, you can also notice that since in equilibrium $L_t = N_t^y$ then these two quantities have the same growth rate.

h. Write the equilibrium condition on the capital market. Use it to obtain the following:

$$K_{t+1} = (1-\alpha)(1-\gamma)Y_t$$

In the capital market we have people lending capital to firms, which use it for production. Hence, we are in equilibrium when what agents want to lend is the same quantity that firms want to employ for production. The price in this market is the interest rate. The people who save are the young, hence the total amount of savings is $s_t N_t^y$. The equilibrium condition is therefore:

$$\begin{aligned} K_{t+1} &= s_t N_t^y \\ &= (1-\gamma)w_t L_t \qquad \text{By } s_t = (1-\gamma)w_t \text{ and } N_t^y = L_t \\ &= (1-\gamma)(1-\alpha)Y_t \qquad \text{By } w_t L_t = (1-\alpha)Y_t \end{aligned}$$

which is the condition we have in the question.

Part 4 - The balanced growth path

i. Show that $\tilde{k}_t$ evolves through time according to the following law of motion:

$$\tilde{k}_{t+1} = \frac{(1-\alpha)(1-\gamma)}{(1+n)(1+g)}\tilde{k}_t^\alpha$$

(Note: no need to use approximations for growth rates here.) Is there a dilution effect in the model? Explain.

Let's start by expressing $\tilde{k}_{t+1}$. We will see that by elaborating a little bit we can find the requested result.

$$\begin{aligned} \tilde{k}_{t+1} &= \frac{K_{t+1}}{A_{t+1}L_{t+1}} \\ &= \frac{(1-\alpha)(1-\gamma)Y_t}{(1+g)A_t(1+n)L_t} \\ &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\frac{Y_t}{A_t L_t} \\ &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\tilde{y}_t \\ &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\tilde{k}_t^\alpha \end{aligned}$$

You can see that the last step is true as:

$$\tilde{y}_t = \frac{Y_t}{A_t L_t} = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^\alpha = \tilde{k}_t^\alpha$$

Since the coefficient of $\tilde{k}_t$ is less than 1 there is indeed a dilution effect, which means that capital per unit of effective labour decreases. This is, of course, due to the growth of both labour (population) and effectiveness (technology).

j. What is the value of $\tilde{k}_t$ on the balanced growth path?

Remember that on the balanced growth path at any time t capital per capita remains fixed $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^*$. We can directly elaborate on the solution to the previous point to answer this question:

$$\begin{aligned}
\tilde{k}^* &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}(\tilde{k}^*)^\alpha \\
(\tilde{k}^*)^{1-\alpha} &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)} \\
\tilde{k}^* &= \left[\frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\right]^{\frac{1}{1-\alpha}}
\end{aligned}$$

which is $\tilde{k}^*$ on the balanced growth path expressed solely as a function of exogenous variables.

k. What is the value of g_y, the growth rate of income per capita, on the BGP? What about g_Y, the growth rate of aggregate output?

Remember that $y = \frac{Y}{L}$, and by the rules of growth rates its growth rate is the difference between the rate of the numerator and the rate of the denominator. However, it would be nice to express it as a function of growth rates that we know. We can notice that $y = \frac{Y}{AL}A = \tilde{y}A$. Therefore:

$$g_y = g_{\tilde{y}} + g_A = g$$

Remember that on the balanced growth path the growth rate of $\tilde{y}$ must be zero. As for g_Y, we can perform an operation similar to the one above $Y = \frac{Y}{AL}AL = \tilde{y}AL$ and obtain:

$$g_Y = g_{\tilde{y}} + g_A + g_L = g + n$$

l. Does this model respect the Kaldor facts about labor productivity and the interest rate, once the economy has reached its balanced growth path? Justify your answer using previous questions.

The Kaldor fact about productivity states that labour productivity grows at a sustained rate, while the Kaldor fact about the interest rate states that it is stable. The first is indeed verified, as $g_y = g$ which is positive. The second is also verified, as on the BGP $r = \frac{\alpha}{\tilde{k}^{1-\alpha}}$ is stable.

m. Conclusion: how can you compare this OLG growth model to the Solow model with population growth and technological progress seen in class? Discuss both the assumptions (among which the one about savings) and the results. What about depreciation?

This model differs from Solow Swan in that it endogenizes the choice of the savings s, which were given in Solow-Swan. Hence, it is a little bit more realistic in this dimension. Other assumptions, about preferences and the production function are not different (you can check, as an example, that the Inada conditions hold). In fact, the results are the same on the BGP. The only difference is that s is not given but computed inside the model. They both satisfy the two Kaldor facts that we checked above.

n. How does time preferences influence $\tilde{y}$ and c_t^y on the balanced growth path?

Time preferences are captured by γ. As we noticed before, higher γ lead to higher consumption when young rather that old. But this is from an individual perspective, here we have to study on the BGP. For $\tilde{k}$ we can rely on the equation for $\tilde{k}$:

$$\tilde{y}^* = (\tilde{k}^*)^\alpha = \left[\frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\right]^{\frac{\alpha}{1-\alpha}}$$

we can see that higher γ leads to lower $\tilde{y}^*$. This is due to the fact that people save less and income decreases.

As for c_t^y, we have:

$$\begin{aligned} c_t^y &= w_t \gamma \\ &= \gamma(1-\alpha)A_t(\tilde{k}^*)^\alpha \\ &= \gamma(1-\alpha)A_t \left[\frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\right]^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

We can see that γ as the same effect on c_t^y on the balanced growth path, contrary to what intuition suggests. This is because even if agents consume more immediately when they do not save, in the long run the lack of savings will produce less income, and less consumption, even for the young.

🔟

TD 10

1 Review

h. Considering the estimation of the Cagan model, hyperinflation in the Weimar republic was driven only by an excess in money supply.

▼ *Answer*

False: The Cagan model is all about the interplay between money supply and the expectation of inflation. He shows that also expectations mattered in that episode.

i. An efficient way to end hyperinflations is to change the legal currency used in the country.

▼ *Answer*

Maybe: It can be efficient, although not necessary sufficient. It worked in the historical cases of the Weimar Republic (when the Rentenmark replaced the Papiermark), Zimbabwe (in which the Zimbabwean dollar ceased to exist and was replaced by the US dollar, the Euro and the South-African rand) and in Brazil when the real was introduced. However, people also need to trust that this new currency will be stable and the government needs to have a balanced budget which will not necessitate balancing through by issuing more money.

j. In the Cagan model, prices rise whenever real money demand falls.

▼ *Answer*

Maybe: There are different factors which affect money demand, as shown at page 12 of your notes. Everything else fixed, the statement is true.

k. All else being equal, hyperinflation is more likely to feed on itself (in other words, hyperinflation is more likely to be driven by momentum) if expected future inflation depends a lot on the past realisation of inflation.

▼ *Answer*

True: As we will see in the following exercise, for a low level of λ this is indeed true, check $\pi_t^e = \lambda\pi_{t-1}^e + (1-\lambda)(p_t - p_{t-1})$.

l. All else being equal, momentum-driven hyperinflation is more likely when real money demand does not react much to expected future inflation.

▼ *Answer*

False: We will also see this in the exercise. You can see from $m_t^d - p_t = -\alpha\pi_t^e$ that for higher values of α hyperinflation is more likely.

5 Cagan's model

In 1956, P. Cagan published a study of hyper inflation episodes, in which expectations seem to be crucial. This problem is inspired from Cagan 's model, but modifies the way expectations are modelled. We assume that real variables (output, interest rate) are constant, and we study the interactions between the general price level and the money supply, using a money demand equation.

a. The Cagan's money demand equation is specified as follows

$$log\left(\frac{M_t}{P_t}\right) = \alpha_0 + \alpha_1 logY_t + \alpha_2 i_t + u_t$$

where M_t is the monetary aggregate, P_t is the price level, i_t is the nominal interest rate and u_t is the error term reflecting a random shock that affects $log\left(\frac{M_t}{P_t}\right)$, with mean zero. Comment this equation. What are a priori the signs of the coefficients?

The relation between money, output and interest rate might remind you the very stylized IS-LM model. If output Y is higher, this means that people need to more money to buy and sell the product produced. We expect α_1 to be positive. On the contrary, every time someone holds money he is renouncing the possibility of investing it to get a return from a fruitful asset. The higher this return, the lower the money demanded. According to this logic, α_2 should be negative.

b. Given the Fisher equation $i_t = r_t + \pi_t^e$ where r is the real interest rate and $\pi_t^e = p_{t+1}^e - p_t$ is expected level of inflation, and under the assumption (in the very short run) that $y_t = y$ and $r_t = r$, show that the money demand can be written as

$$m_t - p_t = \gamma - \alpha\pi_t^e + u_t \tag{1}$$

where m, p and y are the natural logarithms of M, P and Y and γ, α some constants to define.

The only thing we can do is to start from the expression we have. By substituting the logarithms we should reach the solution (remember that $log\left(\frac{x}{y}\right) = log(x) - log(y)$):

$$\begin{aligned} log\left(\frac{M_t}{P_t}\right) &= \alpha_0 + \alpha_1 logY_t + \alpha_2 i_t + u_t \\ m_t - p_t &= \alpha_0 + \alpha_1 y_t + \alpha_2 i_t + u_t \\ m_t - p_t &= \alpha_0 + \alpha_1 y_t + \alpha_2(r_t + \pi_t^e) + u_t \end{aligned}$$

The left hand-side is fine. The error term u_t is also there, together with π_t^e. We just have to understand which is the coefficient of π_t^e and collect everything else as γ:

$$\begin{aligned} m_t - p_t &= \alpha_0 + \alpha_1 y_t + \alpha_2(r_t + \pi_t^e) + u_t \\ m_t - p_t &= \underbrace{\alpha_0 + \alpha_1 y_t + \alpha_2 r_t}_{\gamma} + \underbrace{\alpha_2}_{-\alpha}\pi_t^e + u_t \\ m_t - p_t &= \gamma - \alpha\pi_t^e + u_t \end{aligned}$$

which is the expression we wanted. Notice that this new equation tells us that a higher level of inflation has a negative impact on $log\left(\frac{M_t}{P_t}\right)$. This is because $\alpha_2 < 0 \Rightarrow -\alpha_2 > 0 \Rightarrow \alpha > 0 \Rightarrow -\alpha < 0$. Hence, when π_t^e is high, $log\left(\frac{M_t}{P_t}\right)$ is low.

c. The following table shows Cagan's observations concerning hyperinflation episodes. The last column presents the minimum level of real balances $\frac{M}{P}$ over the period, as a percentage of the initial level of real balances. It reflects how low money demand dipped during the episode. Comment on this table. Are these observations compatible with the money demand equation?

Country	Period	Average inflation rate (% per month)	Real balances (minimum/initial)
Austria	Oct 1921–Aug 1922	47.1	0.35
Germany	Aug 1922–Nov 1923	322	0.03
Greece	Nov 1943–Nov 1944	365	0.007
Hungary	March 1923–Feb 1924	46	0.39
Hungary	Aug 1945–June 1946	19800	0.0003
Poland	Jan 1923–Jan 1924	81.1	0.34
Russia	Dec 1921–Jan 1924	57	0.27

The table confirms the relation we computed above. Consider the second observation for Hungary, as an example. Inflation is very high and, in fact, the minimum value for real balances is close to 0. The following graph plots the correlation coefficient based on the data of this table. You see that the line has clearly a downward slope.

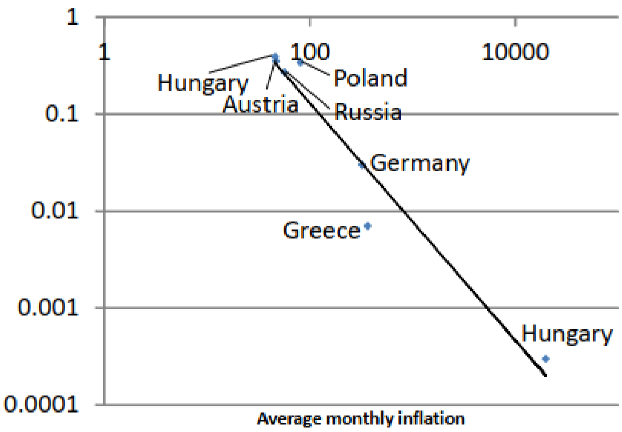


Figure 2: Real balances/initial real balances and inflation during hyperinflation episodes.

d. Why can't we estimate directly equation (1) using available data?

Simply because is very hard to have data about π_t^e, expected inflation. We should ask many people to tell us how much inflation they think there will be in the future (actually, the mere fact of them revealing their expectations might change other people expectations...). Hence, we need to draw some inferences about π_t^e by constructing a theory of how it is determined.

e. Working under the supervision of Friedman, Cagan assumed that expectations were adaptative:

$$\pi_t^e - \pi_{t-1}^e = (1-\lambda)(\pi_{t-1} - \pi_{t-1}^e)$$

Comment this expectation equation.

To interpret it easily we could rewrite it as follows:

$$\pi_t^e = \pi_{t-1} - \lambda\pi_{t-1} - \pi_{t-1}^e + \lambda\pi_{t-1}^e + \pi_{t-1}^e$$
$$\pi_t^e = \lambda\pi_{t-1}^e + (1-\lambda)\pi_{t-1}$$

So expectations about inflation at time t are a combination of realized inflation at time $t-1$ and expectations about it. The variable λ represents the relative weight the agents gives to expectations rather than realised inflation. Notice that if $\pi_{t-1} = \pi_{t-1}^e$ then:

$$\pi_t^e = \pi_{t-1}$$

If the prediction was correct in the past, the agents will not change it.

f. Show that expected inflation π_t^e can be written as a weighted sum of past inflation rates π_{t-i}, assuming $0 < \lambda < 1$. What is the meaning of the λ parameter?

As in one of our previous exercises, we could use a brute force technique here to express present expectations as a function of past expectations. Consider expectations at time t:

$$\begin{aligned}
\pi_t^e &= (1-\lambda)\pi_{t-1} + \lambda\pi_{t-1}^e \\
&= (1-\lambda)\pi_{t-1} + \lambda((1-\lambda)\pi_{t-2} + \lambda\pi_{t-2}^e) \\
&= (1-\lambda)\pi_{t-1} + \lambda(1-\lambda)\pi_{t-2} + \lambda^2\pi_{t-2}^e \\
&= (1-\lambda)\pi_{t-1} + \lambda(1-\lambda)\pi_{t-2} + \lambda^2((1-\lambda)\pi_{t-3} + \lambda\pi_{t-3}^e) \\
&= (1-\lambda)\pi_{t-1} + \lambda(1-\lambda)\pi_{t-2} + \lambda^2(1-\lambda)\pi_{t-3} + \lambda^3\pi_{t-3}^e \\
&\vdots
\end{aligned}$$

You see that any π_{t-i} is multiplied by $(1-\lambda)$ and λ^{i-1}. We may recollect the terms as:

$$\pi_t^e = \sum_{i=1}^{\infty}(1-\lambda)\lambda^{i-1}\pi_{t-i}$$

The last step is to express this as a weighted sum. Of course the weights should be the λ^i , hence it is enough to back out $\frac{1-\lambda}{\lambda}$.

$$\pi_t^e = \frac{1-\lambda}{\lambda}\sum_{i=1}^{\infty}\lambda^i\pi_{t-i}$$

The furthest is π_{t-i} from π_t (high i) the less the agent care about past realisations, as since $\lambda < 1$ we have that $\lambda^i > \lambda^{i+1}$.

g. Using the expectation formula given in e), solve the model to get $m_t - p_t$ as a function of observable variables π_t, $m_t - p_t$, u_t and u_{t-1}.

Now that we have a theory of how expectations are formed, we can use the observable variables that we have to estimate the regression at the beginning of the exercise. Recall that we had:

$$m_t - p_t = \gamma - \alpha\pi_t^e + u_t$$

By plugging in $\pi_t^e = \lambda\pi_{t-1}^e + (1-\lambda)\pi_{t-1}$:

$$\begin{aligned}
m_t - p_t &= \gamma - \alpha\left(\lambda\pi_{t-1}^e + (1-\lambda)\pi_{t-1}\right) + u_t \\
&= \gamma - \alpha\lambda\pi_{t-1}^e - \alpha(1-\lambda)\pi_{t-1} + u_t
\end{aligned}$$

We still have the expectation π_{t-1}^e in the expression, but we can back it out from the previous period regression:

$$m_{t-1} - p_{t-1} = \gamma - \alpha\pi_{t-1}^e + u_{t-1} \Rightarrow \pi_{t-1}^e = -\frac{1}{\alpha}\left(m_{t-1} - p_{t-1} - \gamma - u_{t-1}\right)$$

By substituting again:

$$
\begin{aligned}
m_t - p_t &= \gamma - \alpha\lambda\pi^e_{t-1} - \alpha(1-\lambda)\pi_{t-1} + u_t \\
&= \gamma - \alpha\lambda\left[-\frac{1}{\alpha}\left(m_{t-1} - p_{t-1} - \gamma - u_{t-1}\right)\right] - \alpha(1-\lambda)\pi_{t-1} + u_t \\
&= \gamma + \lambda\left[\left(m_{t-1} - p_{t-1} - \gamma - u_{t-1}\right)\right] - \alpha(1-\lambda)\pi_{t-1} + u_t \\
&= (1-\lambda)\gamma + \lambda\left(m_{t-1} - p_{t-1}\right) - \alpha(1-\lambda)\pi_{t-1} + u_t - \lambda u_{t-1}
\end{aligned}
$$

Which gives us a regression on past observable variables.

h. Assume that money supply is exogenous. Rewrite the model's solution as $p_t = f(p_{t-1}, m_t, m_{t-1}, u_t, u_{t-1})$. We further assume that m_t me_t u_t = 0 (what?). Under which condition is the model stable (i.e. converging to a finite limit).

The equation above is exactly a function of $(p_{t-1}, m_t, m_{t-1}, u_t, u_{t-1})$, so it is enough to explicit p_t:

$$
\begin{aligned}
m_t - p_t &= (1-\lambda)\gamma + \lambda\left(m_{t-1} - p_{t-1}\right) - \alpha(1-\lambda)\pi_{t-1} + \\
m_t - p_t &= (1-\lambda)\gamma + \lambda\left(m_{t-1} - p_{t-1}\right) - \alpha(1-\lambda)(p_t - p_{\cdot} \\
-(1-\lambda)\gamma - \lambda m_{t-1} + m_t + \left(\lambda - \alpha(1-\lambda)\right)p_{t-1} - u_t + \lambda u_{t-1} &= \left(1 - \alpha(1-\lambda)\right)p_t \\
\frac{-(1-\lambda)\gamma - \lambda m_{t-1} + m_t + \left(\lambda - \alpha(1-\lambda)\right)p_{t-1} - u_t + \lambda u_{t-1}}{\left(1 - \alpha(1-\lambda)\right)} &= p_t
\end{aligned}
$$

The series does not explode if the price does not increase continuously. This happens when the coefficient of past price on current price is less than 1, which in our model translates into $\frac{\lambda - \alpha(1-\lambda)}{1 - \alpha(1-\lambda)} < 1$.

i. Cagan estimates the following equation:

$$
p_t = \beta_0 + \beta_1 p_{t-1} + \beta_2 m_t - \beta_3 m_{t-1} + \varepsilon_t
$$

where β_0 is a constant term and ε_t is a residual. The estimates lead to the following results

Country	Period	Coefficients		
		$\hat\beta_1$ $\frac{\hat\lambda - \hat\alpha(1-\hat\lambda)}{1-\hat\alpha(1-\hat\lambda)}$	$\hat\beta_2$ $\frac{1}{1-\hat\alpha(1-\hat\lambda)}$	$\hat\beta_3$ $\frac{\hat\lambda}{1-\hat\alpha(1-\hat\lambda)}$
Austria	Oct 1921–Aug 1922	0.928	0.556	0.516
Germany	Aug 1922–Nov 1923	3.17	-0.092	-0.292
Greece	Nov 1943–Nov 1944	0.611	0.386	0.236
Hungary	March 1923–Feb 1924	0.23	0.13	0.03
Hungary	Aug 1945–June 1946	0.67	0.455	0.305
Poland	Jan 1923–Jan 1924	0.032	0.31	0.01
Russia	Dec 1921–Jan 1924	5.92	-0.07	-0.421

Are there countries in which hyperinflation can occur without any explosion of the money supply? Why?

This table provides the estimates of the coefficient of p_{t-1} on p_t that we computed above. We just realised that if that coefficient is greater than 1 than prices explodes in a finite amount of time. This condition is respected both by Germany and by Russia.

j. The following table shows Cagan 's results from estimating α and λ. Note that there is not a perfect correspondence with $\hat\beta_1$, $\hat\beta_2$, and $\hat\beta_3$ since the model is overidentified.

Country	Period	Param. $\hat{\alpha}$	$\hat{\lambda}$
Austria	Oct 1921–Aug 1922	8.55	0.95
Germany	Aug 1922–Nov 1923	5.46	0.8
Greece	Nov 1943–Nov 1944	4.09	0.85
Hungary	March 1923–Feb 1924	8.7	0.9
Hungary	Aug 1945–June 1946	3.63	0.85
Poland	Jan 1923–Jan 1924	2.3	0.7
Russia	Dec 1921–Jan 1924	3.06	0.65

What do these results imply about inflation expectations? What do they imply about the reaction of money demand to inflation expectations?

Remember that we had $m_t - p_t = \gamma - \alpha\pi_t^e + u_t$. So, in the table we can see that α is quite high. This means that people reaction to inflation is significant, which could in turn lead to a price explosion and thus hyperinflation. However, this effect is countered by the high (sometimes close to 1) λ. Recall that $\pi_t^e = \lambda\pi_{t-1}^e + (1-\lambda)\pi_{t-1}$, so people are strongly anchored to their expectations and will slowly alter their beliefs.