TD 2

1 Review Questions

e. At the steady state, investments are equal to what is lost to depreciation, population growth and technological progress.

▼ Answer

TRUE: At the steady state we must have that capital per unit of effective labour must not grow. From the results of the class we have

$$\Delta\tilde{k} = sf\left(\tilde{k}\right) - (\delta + n + g)\tilde{k}$$

Remember that here $\tilde{k} = \frac{K}{AL}$. By setting $\Delta\tilde{k} = 0$ we get:

$$sf\left(\tilde{k}^*\right) = (\delta + n + g)\tilde{k}^*$$

Which exactly means that investments $\tilde{i}^*$ are equal to the loss in capital due to depreciation, population growth and technological progress.

f. The golden rule of savings states that in steady state, capital should be barely productive enough to compensate for depreciation, population growth and technological progress.

▼ Answer

TRUE: The golden rule of savings tells us what is the optimal $\tilde{k}$ (or s, in the previous TD we found a relation between these two problems) to maximise steady state consumption $\tilde{c}^*$. To work it out we need to recall how to express consumption. We consume what we produce minus what we invest, therefore $\tilde{c}^* = (1-s)\tilde{y}^* = (1-s)f\left(\tilde{k}^*\right)$. By the result in the previous question we can rewrite this expression:

$$
\begin{aligned}
\tilde{c}^* &= (1-s) f\left(\tilde{k}^*\right) \\
&= f\left(\tilde{k}^*\right) - s f\left(\tilde{k}^*\right) \\
&= f\left(\tilde{k}^*\right) - (\delta + n + g)\tilde{k}^*
\end{aligned}
$$

What is the capital that maximises $\tilde{c}^*$? We have to solve a standard optimisation problem.

$$
\underset{\tilde{k}^*}{max} \quad \tilde{c}^* \Rightarrow \underset{\tilde{k}^*}{max} \quad f\left(\tilde{k}^*\right) - (\delta + n + g)\tilde{k}^*
$$

$$
\frac{\partial \tilde{c}^*}{\partial \tilde{k}^*} = 0 \Rightarrow f'\left(\tilde{k}^*\right) - (\delta + n + g) = 0 \Rightarrow f'\left(\tilde{k}^*\right) = \delta + n + g
$$

Which exactly tells us that the marginal productivity of capital must exactly offset the loss due to depreciation, population growth and technological progress (remember that $f'\left(\tilde{k}^*\right)$ tells us how much production increases after an infinitesimal change in $\tilde{k}^*$, namely the marginal product).

3 Exercise - Convergence Towards the steady state

The aim of this exercise is to understand the role of assumptions in the Solow - Swan model. It may be tempting to read assumptions once and then forget about them, but they are of crucial importance in these and in all the other models in economics (science and reasoning in general).

Our production function for the first point is $F(K_t, L_t) = (K_t L_t)^{\frac{1}{2}} = \sqrt{K_t L_t}$. The saving rate is s, population growth rate is $\frac{\Delta L_t}{L_t} = n$ and capital depreciates at rate δ.

a. Represent graphically in the (k, y) plane the dynamics of the Solow model with population growth and no technological change.

The first step is to transform all the variables in per capita quantities. We perform this step because we are interested in the steady state of capital per worker, and not in its absolute value.

1. $k_t = \frac{K_t}{L_t}$
2. $y_t = \frac{Y_t}{L_t}$
3. $c_t = \frac{C_t}{L_t}$
4. $i_t = \frac{I_T}{L_t}$

As for the production function, we perform the same changes by dividing with L_t and exploiting constant returns to scale (*CRS*).

$$\frac{1}{L_t}F(K_t, L_t) = \left(\frac{K_t}{L_t}, \frac{\cancel{L_t}}{\cancel{L_t}}\right)^{\frac{1}{2}}$$
$$= \left(\frac{K_t}{L_t}\right)^{\frac{1}{2}}$$
$$f(k_t) = (k_t)^{\frac{1}{2}}$$

To check for the dynamics of the model we need the *law of motion of capital,* as capital is the principal state (endogenous) variable which determines what happens in the economy as time changes. The law of motion tells us how capital evolves in time. We ask ourselves the question: "if at time t I have capital k_t (per capita), how much capital k_{t+1} do I have in the next period? To answer this question we need to know what is the relation between these two variables.

On the one hand, we have the saved resources that we can use in the next period $sy = sf(k_t)$, on the other hand, the capital we have depreciates at rate δ, and therefore we lose δk. Moreover, in this formulation of the model we also have population growth. An increase in population decreases the amount of capital per capita. These are the two important factor that affects the dynamics of capital. To find the law of motion we rely on the rules of growth rates of products and ratios, in particular:

$$\frac{\Delta k_t}{k_t} = \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} \qquad \text{Check page 17 of the lecture notes to verify that this is true}$$

We already know that $\frac{\Delta L_t}{L_t} = n$. Moreover, we know from the standard Solow model that $\Delta K_t = sF(K_t, L_t) - \delta K_t$. Therefore,

$$\frac{k_{t+1}-k_t}{k_t} = \frac{\Delta k_t}{k_t} = \frac{sF(K_t,L_t)-\delta K_t}{K_t} - n$$
$$\frac{\Delta k_t}{k_t} = \frac{\frac{1}{L_t}\left(sF(K_t,L_t)-\delta K_t\right)}{\frac{1}{L_t}K_t} - n$$
$$\frac{\Delta k_t}{k_t} = \frac{sf(k_t)-\delta k_t}{k_t} - n$$

By multiplying on the right and on the left by k_t we get the final expression:

$$\Delta k_t = \underbrace{sf(k_t)}_{\text{Saved resources}} - \underbrace{\delta k_t}_{\text{Depreciation loss}} - \underbrace{nk_t}_{\text{Population growth loss}}$$

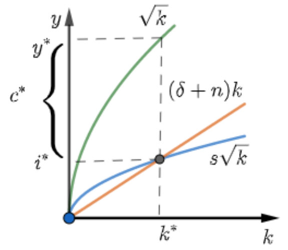
$$\Delta k_t = sf(k_t) - (\delta+n)k_t$$

In the steady state variables do not change over time, therefore the capital per capita will be stable, which means that $\Delta k_t = 0$.

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) = (\delta+n)k_t^*$$

Investment exactly offsets the loss due to depreciation and population growth.

In our case $f(k_t) = (k_t)^{\frac{1}{2}} = \sqrt{k}$, therefore the three elements of our graph are:



Graph1: Solow Model with Population Growth

1. $\sqrt{k}$
2. $(\delta+n)k_t$
3. $s\sqrt{k}$

As always we have that $y^* = f(k^*)$, $i^* = sf(k^*)$ and $c^* = f(k^*) - sf(k^*) = y^* - i^*$, as shown in the graph.

b. Same question for the AK production function $F(K_t,L_t) = AK_t$, assuming $sA > \delta + n$. Does k^* exist in this case?

For this point I offer a different path to reach the solution with respect to the one you will receive from the professor. I think this way is more in line with the standard method, but you choose which you prefer.

To answer this question we proceed as we did in the previous point, but of course, we have to take into account the different production function and the technological change. First, let's express F as a function of per capita capital. We perform the same computation as before.

$$\frac{1}{L_t}F(K_t, L_t) = \left(A\frac{K_t}{L_t}\right)$$
$$f(k_t) = Ak_t$$

Notice that, contrary to what we do in the lecture notes, the exercise asks to draw the graph in the space (k, y) and not $\left(\tilde{k}, \tilde{y}\right)$, that's why we do not divide by units of effective labour AL.

<u>*Question*</u>: What if we had $\tilde{k} = \frac{K_t}{AL}$? Due growth rates rules:

$$\begin{aligned}\frac{\Delta\tilde{k}}{\tilde{k}} &= \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t} - \frac{\Delta A}{A} \\ &= \frac{sF(K_t, L_t) - \delta K_t}{K_t} - n - g \\ \Delta\tilde{k} &= sf\left(\tilde{k}\right) - (\delta + n + g)\tilde{k}\end{aligned}$$

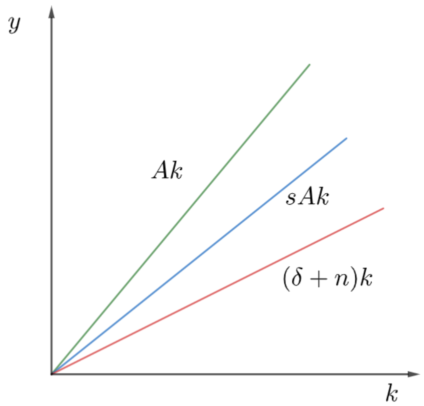
However, we are still in the (k, y) space, so we are interested in the law of motion of capital per capita k, which is the same we had in the previous point (except for the different production function). We have the following:

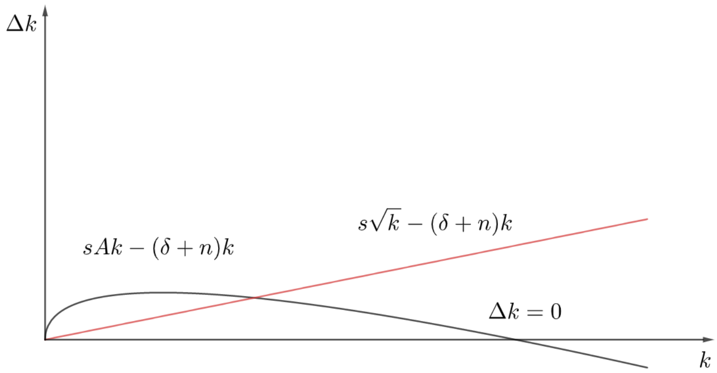
$$\begin{aligned}\Delta k_t &= sf(k_t) - (\delta + n)k_t \\ &= sAk_t - (\delta + n)k_t\end{aligned}$$

So, does k^* exists in this case? The steady state condition is always the same:

$$\Delta k_t = 0 \Leftrightarrow sf(k_t^*) = (\delta + n)k_t^* \Leftrightarrow sA\cancel{k_t^*} = (\delta + n)\cancel{k_t^*} \Leftrightarrow sA = (\delta + n)$$

However, in this model, this can never be true as we have $sA > \delta + n$! So the answer is no, there exists no steady state capital k^* (notice however that $k^* = 0$ is a solution, in fact, the condition $sAk_t^* = (\delta + n)k_t^*$ is respected in this case). The reason is also apparent from the graph. Inspired by a question of one of your classmates (and the mistake in the text) I also plotted a graph of the two economies we just studied in the $(k, \Delta k)$ space.





Graph 2 and 3: Solow model with technological change and linear production function. _Question_: Can you guess what point the intersection between the curve and the x-axis is?

The problem here is that the saving rate and the technological change outset the decrease of capital per capita due to depreciation and population growth. From a mathematical point of view, this is due to the fact that the coefficient of the (linear) savings function sA is greater than the coefficient of the depreciation $\delta + n$. Therefore, the increase in capital will always be greater than its loss. Its growth will never stop, it will continue to increase in every time t. This example shows why assumptions are a fundamental ingredient of the model and not something we use just for convenience and that we can forget by putting them below the carpet. We will discuss this in the following point.

Question: Do you think the results would have been different if we checked capital per units of effective labour $\left(\tilde{k} = \frac{K_t}{AL_t}\right)$?

c. Make a list of the properties that the AK function does not satisfy with regards to the Solow model. Which one explains the previous result?

In the Solow model, we have three assumptions on the production function and three extra assumptions that are denominated *Inada Conditions*.

1. $F(K) > 0$ if $K > 0$. This condition just ensures that our production is positive if we use a positive amount of capital;
2. $\frac{\partial F(K)}{\partial K} > 0$. This condition tells us that we have a *positive marginal product*, that is, increasing capital always increases production;
3. $\frac{\partial^2 F(K)}{\partial K^2} < 0$. This condition is a crucial one in this exercise. It ensures that the marginal product is decreasing in K. This means that the n^{th+1} unit of capital K will increase the production less than the n^{th} one. We will show that this does not hold in the previous point.

Then we have the *Inada Conditions.*

1. $F(0) = 0$. This only imposes that you can not have a positive production by employing zero capital;
2. $\lim_{K \to 0} \frac{\partial F(K)}{\partial K} = \infty$. This assumption just tells us that a little bit of capital is infinitely productive, as we go from 0 production to positive production;
3. $\lim_{K \to \infty} \frac{\partial F(K)}{\partial K} = 0$. This assumption is also broken in this exercise. It ensures that employing an infinite amount of capital is not convenient, as the marginal product will eventually reach 0 so that using capital will not be productive at all and therefore will be wasted.

Let's check that $F(K_t, L_t) = AK_t$ does not satisfy assumption 3 and Inada condition 3. We have

$$\frac{\partial F(K)}{\partial K} = A > 0$$

$$\frac{\partial^2 F(K)}{\partial K^2} = 0 \geq 0$$

$$\lim_{K \to \infty} \frac{\partial F(K)}{\partial K} = A \neq 0$$

Therefore, as we noticed before, capital is too productive and its increase due to production always offset its loss due to depreciation and population growth.

Question: Check that the production function in the first part of the exercise indeed satisfies all these assumptions.