TD 1

1 Review Questions

a. Inequality of income across countries was already very wide at the end of the medieval period.

▼ *Answer*

False: It started widening after the industrial revolution. This is due to the fact that before the revolution growth was almost null around the world. The production was very close to subsistence level and there were no investments. The technological progress of the revolution was the catalyst of growth which developed differently and caused inequalities among different countries.

2 The Golden Rule for Savings

The aim of this problem is to get you used with the basic calculations of the Solow model. This exercise is a particular case in which we have a Cobb-Douglas production function, population growth and no technology. Therefore we have $n > 0$ and $g = 0$ and the production function $f(k) = k^\alpha$ with $0 < \alpha < 1$.

Question: Can you get what is the original production function $F(K_t, L_t)$?

a. Express the steady-state level of consumption c^* as a function of k^* and the exogenous parameters n, δ and α (but not s).

First, we have to recall how we express consumption in this model. What we consume is what we produce minus what we invest, therefore $c = y - sy$. However, production is a function of capital $y = f(k)$. We are also reminded that $sf(k^*) = (\delta + n)k^*$. This is key to express c as a function of the exogenous variables and k, but not s:

$$
\begin{aligned}
c^* &= (1-s)y^* \\
&= (1-s)f(k^*) \\
&= f(k^*) - (\delta + n)k^* \quad \text{By substituting the steady state condition} \\
&= (k^*)^\alpha - (\delta + n)k^*
\end{aligned}
$$

Here we have consumption as a function of k^*, n, δ and α.

b. Use the result to the previous question to find the optimum level of k^* from the point of view of consumption.

This question is asking us to maximise consumption with respect to capital. In fact, we want to answer the question: what is the level of k^* that gives me the maximum c^*? The first order condition of this problem amounts to set the first derivative to be equal to 0, as we do in the following:

$$\frac{\partial c^*}{\partial k^*} = 0 \Rightarrow \alpha(k^*)^{\alpha-1} - (\delta + n) = 0 \Rightarrow k^* = \left(\frac{\delta + n}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{1}$$

Remember that you can express this quantity in a different way by playing with the exponent!

Question: what about the second order condition?

c. Express the steady-state level of consumption c^* as a function of the exogenous parameters only (n, δ, α, also including s).

We have to perform the same operation as before, but without including k^* in the expression for consumption. Therefore, we must first find k^* as a function of the exogenous variables only, to be able to substitute it in the expression for consumption. We start from the fact that $sf(k^*) = (\delta + n)k^*$:

$$\begin{aligned} sf(k^*) &= (\delta+n)k^* \\ \frac{1}{k^*}(k^*)^\alpha &= \frac{\delta+n}{s} \\ (k^*)^{-1}(k^*)^\alpha &= \frac{\delta+n}{s} \\ (k^*)^{\alpha-1} &= \frac{\delta+n}{s} \\ (k^*)^{\frac{\alpha-1}{\alpha-1}} &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \\ k^* &= \left(\frac{\delta+n}{s}\right)^{\frac{1}{\alpha-1}} \end{aligned} \tag{2}$$

We can now substitute k^* in the expression for c^*, as it is expressed only as a function of exogenous variables:

$$\begin{aligned} c^* &= (1-s)f(k^*) \\ &= (1-s)\left(\frac{\delta+n}{s}\right)^{\frac{\alpha}{\alpha-1}} \end{aligned}$$

Here we have c^* which only depends on s, n, δ and α.

d. Use the result to the previous question to find the optimum level of the savings rate s from the point of view of consumption.

We have to perform the exact same operation as before, but instead of maximising for k^* we do it for s, but first we rewrite c^* to be able to take the derivative easily. We know in fact that $\left(\frac{\delta+n}{s}\right)^{-a} = \left(\frac{s}{\delta+n}\right)^a$. In our case the new fraction is therefore $\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$. We also isolate the variable with respect to which we must take the derivative (only to make the computations easier, there is no need to do this if you are comfortable with derivatives):

$$c^* = (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} = (1-s)(s)^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

Now we are ready to solve the maximisation problem (here I provided the calculations in a very detailed way, if you are comfortable with calculus you do not need to write everything as I do here). First, I recall the rules for deriving a product. In general we have the following:

$$\frac{\partial f(x)g(x)}{\partial x} = f'(x)g(x) + f(x)g'(x)$$

In our case the two functions are $f(s) = (1-s)$ and $g(s) = s^{\frac{\alpha}{1-\alpha}}$. Moreover, everything is multiplied by the constant $\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$ which does not affect the derivative. First, we compute the derivative for each of the function that compose the product:

$$g'(s) = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}-1} = \frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}$$

$$f'(s) = -1$$

Then, by applying the general rule:

$$\begin{aligned}\frac{\partial[f(s)g(s)]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}}{\partial s} &= \frac{\partial[f(s)g(s)]}{\partial s}\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \\ &= \left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

By setting the derivative equal to zero, we can get rid of the constant (equivalent to divide each side of the equality by the constant itself).

$$\left[-1s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}}\frac{1}{s}\right]\cancel{\left(\frac{1}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}} = 0$$

We are therefore left with the following:

$$
\begin{aligned}
\frac{\partial c^*}{\partial s} = 0 \Rightarrow & \left[-1 s^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha} s^{\frac{\alpha}{1-\alpha}} \frac{1}{s}\right] = 0 \\
\Leftrightarrow & \left[-1 \cancel{s^{\frac{\alpha}{1-\alpha}}} + (1-s)\frac{\alpha}{1-\alpha} \cancel{s^{\frac{\alpha}{1-\alpha}}} \frac{1}{s}\right] = 0 \\
\Leftrightarrow & \left[-1 + (1-s)\frac{\alpha}{1-\alpha} \frac{1}{s}\right] = 0 \\
\Leftrightarrow & (1-s)\frac{\alpha}{1-\alpha}\frac{1}{s} = 1 \\
\Leftrightarrow & (1-s)\alpha = (1-\alpha)s \\
\Leftrightarrow & \alpha = s
\end{aligned}
$$

Therefore, the s that maximise consumption is exactly α, the exponent of the production function. This is more or less intuitive, the more your function is relatively productive, as captured by α, the more you save.

Question: Again, what about the second order conditions for this problem?

e. Comment the results obtained to questions 2 and 4: how are they related?

Consider the expressions (1) and (2) that we computed in points **b.** and **c.**

$$k_1^* = \left(\frac{\delta + n}{\alpha}\right)^{\frac{1}{\alpha - 1}} \tag{1}$$

$$k_2^* = \left(\frac{\delta + n}{s}\right)^{\frac{1}{\alpha - 1}} \tag{2}$$

Next, consider the result $\alpha = s$ obtained in point **d.**

$$\alpha = s \tag{3}$$

By combining (1) or (2) with (3) (substituting s or α in one of the two expressions) you find that $k_1^* = k_2^*$! In fact, if

$s \neq \alpha$ then we would have that two different expressions maximise consumption, which is not possible if we only have one maximum in steady state, as in this case.