9

TD 9

1 Review questions

f. At the OLG equilibrium with money, the real value of money increases at the same rate as the size of each generation, whatever the changes in aggregate money supply.

▼ Answer

False: The real value of money depends on the money supply and the level of prices. Individuals can not control the money supply, but is it changes it will also affect the real value of money. Therefore, the statement is true only if the supply of money does not change.

g. In the long run, nominal variables are disconnected from real ones.

▼ *Answer*

True: This is an empirical question. Apparently it seems that these two variables are connected. You can check the paper "some monetary facts" by (McCandless & Weber, 1995), there are a lot of interesting things about money. They say that "growth rates of the money supply and the general price level are highly correlated".

3 A market for time preferences

Recall some data from the problem. We are dealing with this E consumer with utility function and constraint as follows:

$$U(c_{1E}, c_{2E}) = c_{1E} + \beta c_{2E}$$

$$c_{1E} + \frac{c_{2E}}{1+r} = y_1 + \frac{y_2}{1+r}$$

d. State the elastic household's optimization problem and calculate the optimal choice of c^*_{1E}, c^*_{2E} and s^*_E, individual E's savings, in terms of the model parameters and r. Interpret the result.

The problem is the same as the one we have been solving until now, optimising an objective function with a constraint. We still use the Lagrangian.

$$\max_{c_{1E}, c_{2E}, \lambda} \mathcal{L}(c_{1E}, c_{2E}, \lambda) = \max_{c_{1E}, c_{2E}, \lambda} c_{1E} + \beta c_{2E} - \lambda \left[c_{1E} + \frac{c_{2E}}{1+r} - y_1 - \frac{y_2}{1+r} \right]$$

The first order conditions are:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial c_{1E}} = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{2E}} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} 1 - \lambda = 0 \\ \beta - \frac{\lambda}{1+r} = 0 \\ c_{1E} + \frac{c_{2E}}{1+r} - y_1 - \frac{y_2}{1+r} = 0 \end{cases} \Rightarrow \begin{cases} 1 = \lambda & (1) \\ \beta = \frac{\lambda}{1+r} & (2) \\ c_{1E} + \frac{c_{2E}}{1+r} = y_1 + \frac{y_2}{1+r} & (3) \end{cases}$$

From (1) and (2) we get that $\beta(1+r) = 1$. This means that the fact that a consumption bundle is the optimal one is only related to exogenous parameters. What does this mean? This utility function is linear! You should remember that with linear utility function we could be in the two corner solutions or any bundle of consumption that satisfy the budget constraint is fine. This depends on the ratio of prices as compared to the marginal rate of substitution, which in this case is $\frac{1}{\beta}$, but what are prices here? We have that $\frac{1}{1+r}$ is the "price" of consuming tomorrow (not consuming today) and the ratio of prices is $(1+r)$.

Hence, we have that when $\frac{1}{\beta} > 1 + r \Rightarrow \beta(1+r) < 1$ we have a corner solution and consume only $c_{1E} = y_1 + \frac{y_2}{1+r}$. If $\frac{1}{\beta} < 1 + r \Rightarrow \beta(1+r) > 1$ we have the opposite corner solution and $c_{2E} = y_1 + \frac{y_2}{1+r}$. When $\frac{1}{\beta} = 1 + r \Rightarrow \beta(1+r) = 1$ any value will do the job.

e. Why wouldn’t a straight application of the Euler Equation work for type-E individuals?

Technically because the utility function is not concave, which is an important condition in an optimisation problem. Economically, it is nice to see it from the standard graph.

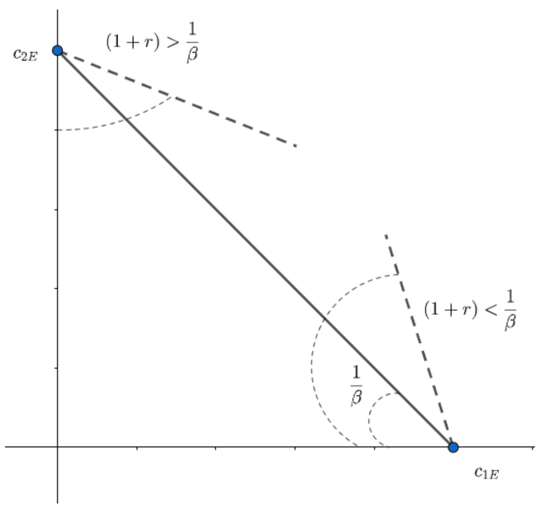

Figure 1: Corner solutions with linear utility.

Whether we are in a corner solution or not is related to the marginal rate of substitution and its magnitude with respect to the ratio of prices.

f. State the market-clearing condition in the bond market. Show that this market will only clear if $\beta(1+r) \leq 1$. Do not find the solution for r though (or if you do, I'm impressed!).

What we study here is a general equilibrium problem in which the two agents are I and E. They are demanding and supplying credit. When does the market of credit clears? Since we have only two individual here, the market clears when what someone is saving is the same as what the other one wants as credit. Recall that I only gets resources in period 1 y_I, while agent E gets y_1 and y_2. This means that agent I can not borrow (how would he repay his debt?), he can also save. Therefore, in the market clearing we must have that his savings are weakly greater than 0, $s_I \geq 0$. In turn, this implies that agent E savings must be weakly lower than zero (he can not give credit) $s_E \leq 0$. First, we exclude the corner solution in which $s_I = 0$. From point **b.** we had

$$
\begin{aligned}
s_I^* &= y_I - \frac{y_I}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]} \\
&= y_I \left[1 - \frac{1}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}\right] \\
&= y_I \left[\frac{\cancel{1}+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}} - \cancel{1}}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}\right] \\
&= y_I \frac{\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}}{1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}}
\end{aligned}
$$

Which is zero only if $\beta = 0$ or $r = -1$. However, this would mean that $\beta(1+r) = 0 < 1$, which means that agent E would like to borrow, so no market clearing in this case. Hence, we must have $s_I > 0$ and $s_E < 0$ with $s_I = -s_E$. The exact amount is given by the exogenous variables β and the equilibrium price r. Tell me if you are strange enough to be willing to find the equilibrium interest rate! Hint: think about an Edgeworth Box in which the goods are c_1 and c_2 and the prices are as exposed in the previous point.

g. From now on, assume for the sake of simplicity that $\alpha = 1$ (You'll admit that the per-period utility function is then $u(c) = ln(c)$). Also assume that $\beta(1+r) = 1$. Solve for the equilibrium $(c_{1I}^*, c_{2I}^*, c_{1E}^*, c_{2E}^*)$. Which parameters allow this equilibrium to exist?

We can use the same reasoning as we did in the previous exercise to "prove" that $\alpha = 1$ implies a logarithmic function. To find the equilibrium values we look at the results of the previous points for consumption. For individual I we had:

$$
c_{1I} = \frac{y_I}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}
$$

$$c_{2I} = \frac{y_I \left[\beta(1+r)\right]^{\frac{1}{\alpha}}}{\left[1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

Substituting $\beta(1+r) = 1$ and $\alpha = 1$ we get:

$$c_{1I} = \frac{y_I}{1+\beta}$$

$$c_{1I} = \frac{y_I}{1+\beta}$$

The individual chooses to perfectly smooth consumption, due to the values of β and r. Type E individual is exactly indifferent between any consumption bundle, as we saw in point **.d**. Therefore, to find his value of consumption which clear the market we must check when what he consumes in each period plus what individual I consume in each period satisfies the budget constraint (market clearing conditions). Hence, we must have:

$$c_{1I} + c_{1E} = \frac{y_I}{1+\beta} + c_{1E} = y_1 + y_I$$
$$c_{1E} = y_1 + y_I - \frac{y_I}{1+\beta}$$

And the same at period 2:

$$c_{2I} + c_{2E} = \frac{y_I}{1+r} + c_{2E} = y_2$$
$$c_{2E} = y_2 - \frac{y_I}{1+r}$$

Since we have a minus sign, we must also specify that this solution holds when the values of parameters y_2, y_I and r are such that c_{2E} is positive, which answer the last bit of the question.

h. Assume again that $\alpha = 1$ (You'll admit that the per-period utility function is then $u(c) = ln(c)$). Now assume that $\beta(1+r) < 1$. Solve for the equilibrium

$(c^*_{1I}, c^*_{2I}, c^*_{1E}, c^*_{2E})$. **Use these solutions to compute** $\hat{U}(c^*_{1I}, c^*_{2I})$.

Here we don't have a specific value for $\beta(1+r)$, therefore we must rely on the market clearing condition which we found at point **f.** $s_I = -s_E$. Given that $\beta(1+r) < 1$ individual E only consumes in the first period $c_{1E} = y_1 + \frac{y_2}{1+r}$ and $s_E = y_1 - c_{1E} = -\frac{y_2}{1+r}$. By working out the condition we get:

$$y_I \frac{\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}}{1+\beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}} = -\left(-\frac{y_2}{1+r}\right)$$
$$y_I \frac{\beta}{1+\beta} = \frac{y_2}{1+r}$$
$$r = \frac{y_2}{y_I}\frac{1+\beta}{\beta} - 1$$

To see what this implies, we substitute this expression in the condition $\beta(1+r) < 1$:

$$\cancel{\beta}(\cancel{1} + \frac{y_2}{y_I}\frac{1+\beta}{\cancel{\beta}} - \cancel{1}) < 1$$
$$y_2(1+\beta) < 1$$

which therefore must hold in equilibrium. Since we have the equilibrium r it is enough to substitute it into the consumption decisions. We already now that $c_{2E} = 0$ (since $\beta(1+r) < 1$). Moreover:

$$c_{1E} = y_1 + \frac{y_2}{1+r}$$
$$c_{1E} = y_1 + \frac{y_2}{\cancel{1} + \frac{y_2}{y_I}\frac{1+\beta}{\beta} - \cancel{1}}$$

For individual I:

$$c_{1I} = \frac{y_I}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$
$$c_{1I} = \frac{y_I}{1+\beta}$$

$$c_{2I} = \frac{y_I\left[\beta(1+r)\right]^{\frac{1}{\alpha}}}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$
$$c_{2I} = \frac{y_I\beta(1+r)}{1+\beta}$$
$$c_{2I} = \frac{\cancel{y_I\beta}(\cancel{1} + \frac{y_2}{\cancel{y_I}}\frac{\cancel{1+\beta}}{\cancel{\beta}} - \cancel{1})}{\cancel{1+\beta}}$$
$$c_{2I} = y_2$$

To compute $\hat{U}(c^*_{1I}, c^*_{2I})$, just substitute the equilibrium values (remember that we had the logarythm):

$$\hat{U}(c^*_{1I}, c^*_{2I}) = ln\,\frac{y_I}{1+\beta} + \beta ln\, y_2$$

4 An OLG growth model

In this problem, we shall consider an overlapping-generation (OLG) model with production, capital accumulation, technological progress and population growth. The aim of the problem is to check that an economy represented by this model attains a balanced growth path (BGP) in the long run, and verifies some Kaldor facts on the BGP.

Time is considered discrete and denoted by t. In any period, a closed economy is populated by 2 generations: the young, born in t, and the old, born in $(t-1)$. In what follows, all variables relative to an individual will be denoted with a y superscript if the variable concerns a young individual, and with an o superscript for an old individual; variables will also have a time index that represents the current period (and not the generation as in the lecture).

The total population in time t is

$$N_t = N_t^y + N_t^o$$

with $N_t^y = (1+n)N_t^o = (1+n)N_{t-1}^y$, and $n > 0$ is a constant.

Individuals from the generation born in t seek to maximise their lifetime (or intertemporal) utility, defined as follows:

$$U(c_t^y, c_{t+1}^o) = \gamma ln\ c_t^y + (1-\gamma) ln\ c_{t+1}^o$$

with c denoting individual consumption of a final good of price 1.

It is further assumed that individuals obtain some labor income w_t by working when young, and can save an amount s_t of this income to accumulate capital. In the next period, individuals (who are now old) are paid a real return r_{t+1} on this capital: the capital income which is generated is then fully consumed. Assume full depreciation of capital after one period of production, so that the real interest rate r is also the return on capital net of depreciation. To avoid ambiguities, old people do not work.

The final good (which can be consumed or saved to accumulate capital) is produced according to the following aggregate production function:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

where K_t denotes aggregate capital available for production at the beginning of t, L_t is labor and A_t represents labor-augmenting technology. Assume $0 < \alpha < 1$.

You are given the following laws of motion for K and A:

$$K_{t+1} = I_t$$

$$\Delta A_t = g A_t$$

where $g > 0$ is constant, and I_t denotes aggregate investment.

Part 1 - Individual choices of consumption and savings

a. Prove that the intertemporal budget constraint of this individual is:

$$c_t^y + \frac{c_{t+1}^o}{r_{t+1}} = w_t$$

In this exercise we just have heavier notation, but the steps to get the budget constraints are the same as before. When an individual is young, he gets w_t and he must choose how to divide it between consumption and savings:

$$c_t^y + s_t \leq w_t$$

When old individuals do not have any income, they only have the return of their savings. Notice, however, that here only the return $r_{t+1}s_t$ is available, not the amount saved s_t plus the return. This is due to the full depreciation of capital. The budget constraint for old individuals is therefore:

$$c_{t+1}^o \leq s_t(1 + r_{t+1}) - s_t = r_{t+1}s_t$$

As before, from preferences we understand that agents do not want to waste money, they will use all they have. These constraint will then hold with equality. As usual, we must explicit savings from the first constraint and substitute into the second to find the intertemporal budget constraint.

$$s_t = w_t - c_t^y$$

$$c_{t+1}^o = r_{t+1}(w_t - c_t^y)$$

$$c_t^y + \frac{c_{t+1}^o}{r_{t+1}} = w_t$$

which is the expression given by the question.

b. Show that when the individual chooses optimally consumption and savings, one obtains the following Euler equation:

$$\frac{c_{t+1}^o}{c_y^y} = \frac{1-\gamma}{\gamma} r_{t+1}$$

To find the optimal choice we rely on the Lagrangian procedure. Remember, the Lagrangian is the sum of the objective function (utility) minus λ times the constraint with everything on one side:

$$c_t^y + \frac{c_{t+1}^o}{r_{t+1}} - w_t = 0$$

The Lagrangian is:

$$\max_{c_t^y, c_{t+1}^o, \lambda} \mathcal{L}(c_t^y, c_{t+1}^o, \lambda) = \max_{c_t^y, c_{t+1}^o, \lambda} \gamma ln\, c_t^y + (1-\gamma) ln\, c_{t+1}^o - \lambda \left[c_t^y + \frac{c_{t+1}^o}{r_{t+1}} - w_t \right]$$

Exactly as before, we take derivatives for the first order conditions:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial c_t^y} = 0 \\ \frac{\partial \mathcal{L}}{\partial c_{t+1}^o} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\gamma}{c_t^y} - \lambda = 0 \\ \frac{1-\gamma}{c_{t+1}^o} - \frac{\lambda}{r_{t+1}} = 0 \\ c_t^y + \frac{c_{t+1}^o}{r_{t+1}} - w_t = 0 \end{cases} \Rightarrow \begin{cases} c_t^y = \frac{\gamma}{\lambda} & (1) \\ c_{t+1}^o = \frac{r_{t+1}(1-\gamma)}{\lambda} & (2) \\ c_t^y + \frac{c_{t+1}^o}{r_{t+1}} = w_t & (3) \end{cases}$$

To find the Euler equation dividing condition (2) by condition (1).

$$\frac{c_{t+1}^o}{c_t^y} = \frac{1-\gamma}{\gamma} r_{t+1}$$

c. Compute c_t^y and c_{t+1}^o, for prices w_t and r_{t+1} given.

To find the optimal amounts of consumption, we must use the relation between c_t^y and c_{t+1}^o specified by the Euler equation and substitute it into condition (3), the budget constraint. From Euler:

$$c_t^y = c_{t+1}^o \frac{\gamma}{1-\gamma} \frac{1}{r_{t+1}}$$

Plugging this into (3):

$$\begin{aligned} c_{t+1}^o \frac{\gamma}{1-\gamma} \frac{1}{r_t} + \frac{c_{t+1}^o}{r_{t+1}} &= w_t \\ c_{t+1} \frac{1}{r_{t+1}} \left(\frac{\gamma}{1-\gamma} + 1 \right) &= w_t \\ c_{t+1} \frac{1}{r_{t+1}} \left(\frac{1}{1-\gamma} \right) &= w_t \\ c_{t+1}^o &= w_t r_{t+1} (1-\gamma) \end{aligned}$$

From Euler we can recover c_t^y:

$$\begin{aligned} c_t^y &= c_{t+1}^o \frac{\gamma}{1-\gamma} \frac{1}{r_{t+1}} \\ &= w_t \cancel{r_{t+1}} \cancel{(1-\gamma)} \frac{\gamma}{\cancel{1-\gamma}} \frac{1}{\cancel{r_{t+1}}} \\ c_t^y &= w_t \gamma \end{aligned}$$

Notice that c_t^y is a fraction γ of income, while c_t^o is what remains from that fraction $(1-\gamma)$ multiplied by the interest rate! This is useful to answer the next question.

d. From the previous question, show that individual savings are $s_t = (1-\gamma) w_t$. What does a high value of mean γ in terms of individual preferences? Explain the link with savings.

To compute savings we use the expression we found before (or, if you forget, just think that savings are what you get minus what you consume).

$$
\begin{aligned}
s_t &= w_t - c_t^y \\
&= w_t - \gamma w_t \\
&= w_t(1-\gamma)
\end{aligned}
$$

Notice, in fact, that indeed what you consume when you are old are your savings times the interest rate! The parameter γ identifies your relative preference for present and future consumption. Higher γ means you are more impatient and you care more about consumption today then tomorrow. The contrary holds for lower values of γ.

Part 2 - Production, factor prices and factor shares

Let $\tilde{k}_t = \frac{K_t}{A_t L_t}$. Assume that production is managed by a firm operating in perfectly competitive markets. Note: contrary to class, define the wage bill paid by the firm by wL (not wAL).

e. Prove that $w_t = (1-\alpha)A_t\tilde{k}_t^\alpha$ and $r_t = \frac{\alpha}{\tilde{k}_t^{1-\alpha}}$.

To answer this question we must realise that wages and interest rates are costs for the firms. Therefore, they will be determined by the optimal choice about how much work and capital they want to use to produce. This is why they are determined by the optimal choice of the firm. In particular, firms maximise profits, which is production minus costs.

$$
\Pi_t(K_t, L_t) = K_t^\alpha (A_t L_t)^{1-\alpha} - w_t L_t - r_t K_t
$$

Notice that firms pay r_t as they employ capital when they receive it (contrary to the households that have their returns the time after they save). To maximise profits we must take first order conditions with respect to the production factors:

$$
\begin{cases} \frac{\partial \Pi_t}{\partial K_t} = 0 \\ \frac{\partial \Pi_t}{\partial L_t} = 0 \end{cases} \Rightarrow \begin{cases} \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} - r_t = 0 \\ (1-\alpha)A_t K_t^\alpha (A_t L_t)^{-\alpha} - w_t = 0 \end{cases} \Rightarrow \begin{cases} \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} = r_t & (1) \\ (1-\alpha)A_t K_t^\alpha (A_t L_t)^{-\alpha} = w_t & (2) \end{cases}
$$

The last step is to find these costs as a function of $\tilde{k}$ as expressed in the text. By rewriting condition (1):

$$
\begin{aligned}
\alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} &= r_t \\
\alpha \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} &= r_t \\
\frac{\alpha}{\tilde{k}_t^{1-\alpha}} &= r_t
\end{aligned}
$$

Instead, by rewriting condition (2):

$$
\begin{aligned}
(1-\alpha)A_tK_t^{\alpha}(A_tL_t)^{-\alpha} &= w_t \\
(1-\alpha)A_t\left(\frac{K_t}{A_tL_t}\right)^{\alpha} &= w_t \\
(1-\alpha)A_t\tilde{k}_t^{\alpha} &= w_t
\end{aligned}
$$

which are the two expressions given by the text.

f. Factor shares are defined as the shares of each factor of production in total income. Show that the factor share of capital is α, and the factor share of labor is $(1-\alpha)$.

This question asks the relative shares of employed capital and labour by firms. For the purpose of this calculation, we should first notice that:

$$
w_tL_t = (1-\alpha)A_t\tilde{k}_t^{\alpha}L_t = (1-\alpha)\frac{A_tL_t}{K_t}\tilde{k}_t^{\alpha}K_t = (1-\alpha)\tilde{k}_t^{-1}\tilde{k}_t^{\alpha}K_t = (1-\alpha)\tilde{k}_t^{\alpha-1}K_t = \frac{1-\alpha}{\tilde{k}_t^{1-\alpha}}K_t
$$

Now, the quantity we are looking for (for capital), is (you can check that $Y_t = r_tK_t + w_tL_t$):

$$
\frac{r_tK_t}{r_tK_t+w_tL_t} = \frac{\frac{\alpha}{\tilde{k}_t^{1-\alpha}}K_t}{\frac{\alpha}{\tilde{k}_t^{1-\alpha}}K_t + \frac{1-\alpha}{\tilde{k}_t^{1-\alpha}}K_t} = \frac{\alpha}{\alpha+1-\alpha} = \alpha
$$

Since we are talking about shares, the relative share of labour is one minus the share of capital, and therefore it is $1-\alpha$. Hence, we know that $r_tK_t = \alpha Y_t$ and $w_tL_t = (1-\alpha)Y_t$.

Part 3 - Equilibrium and dynamics of the aggregate economy

g. Write the equilibrium condition on the labor market. At which rate does the labor force grow in this model?

The labour market is composed by offer and demand of labour. The price is the salary. The demand for labour is given by the firm, which would like to use L_t units of labour to produce. The offer of labour is given by the population of you people, who offer a fixed amount of labour N_t^y (the amount of young people in the population). Therefore, when these two quantities are equal, we are in equilibrium in the labour market $L_t = N_t^y$.

The growth rate of the labour force is computed as usual (don't forget we are in discrete time):

$$
\begin{aligned}
\frac{L_{t+1}-L_t}{L_t} &= \frac{N_{t+1}^y - N_t^y}{N_t^y} \\
&= \frac{(1+n)N_t^y - N_t^y}{N_t^y} \\
&= n
\end{aligned}
$$

Or, you can also notice that since in equilibrium $L_t = N_t^y$ then these two quantities have the same growth rate.

h. Write the equilibrium condition on the capital market. Use it to obtain the following:

$$K_{t+1} = (1-\alpha)(1-\gamma)Y_t$$

In the capital market we have people lending capital to firms, which use it for production. Hence, we are in equilibrium when what agents want to lend is the same quantity that firms want to employ for production. The price in this market is the interest rate. The people who save are the young, hence the total amount of savings is $s_t N_t^y$. The equilibrium condition is therefore:

$$\begin{aligned} K_{t+1} &= s_t N_t^y \\ &= (1-\gamma)w_t L_t \qquad \text{By } s_t = (1-\gamma)w_t \text{ and } N_t^y = L_t \\ &= (1-\gamma)(1-\alpha)Y_t \qquad \text{By } w_t L_t = (1-\alpha)Y_t \end{aligned}$$

which is the condition we have in the question.

Part 4 - The balanced growth path

i. Show that $\tilde{k}_t$ evolves through time according to the following law of motion:

$$\tilde{k}_{t+1} = \frac{(1-\alpha)(1-\gamma)}{(1+n)(1+g)}\tilde{k}_t^\alpha$$

(Note: no need to use approximations for growth rates here.) Is there a dilution effect in the model? Explain.

Let's start by expressing $\tilde{k}_{t+1}$. We will see that by elaborating a little bit we can find the requested result.

$$\begin{aligned} \tilde{k}_{t+1} &= \frac{K_{t+1}}{A_{t+1}L_{t+1}} \\ &= \frac{(1-\alpha)(1-\gamma)Y_t}{(1+g)A_t(1+n)L_t} \\ &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\frac{Y_t}{A_t L_t} \\ &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\tilde{y}_t \\ &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\tilde{k}_t^\alpha \end{aligned}$$

You can see that the last step is true as:

$$\tilde{y}_t = \frac{Y_t}{A_t L_t} = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^\alpha = \tilde{k}_t^\alpha$$

Since the coefficient of $\tilde{k}_t$ is less than 1 there is indeed a dilution effect, which means that capital per unit of effective labour decreases. This is, of course, due to the growth of both labour (population) and effectiveness (technology).

j. What is the value of $\tilde{k}_t$ on the balanced growth path?

Remember that on the balanced growth path at any time t capital per capita remains fixed $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^*$. We can directly elaborate on the solution to the previous point to answer this question:

$$\begin{aligned}
\tilde{k}^* &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}(\tilde{k}^*)^\alpha \\
(\tilde{k}^*)^{1-\alpha} &= \frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)} \\
\tilde{k}^* &= \left[\frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\right]^{\frac{1}{1-\alpha}}
\end{aligned}$$

which is $\tilde{k}^*$ on the balanced growth path expressed solely as a function of exogenous variables.

k. What is the value of g_y, the growth rate of income per capita, on the BGP? What about g_Y, the growth rate of aggregate output?

Remember that $y = \frac{Y}{L}$, and by the rules of growth rates its growth rate is the difference between the rate of the numerator and the rate of the denominator. However, it would be nice to express it as a function of growth rates that we know. We can notice that $y = \frac{Y}{AL}A = \tilde{y}A$. Therefore:

$$g_y = g_{\tilde{y}} + g_A = g$$

Remember that on the balanced growth path the growth rate of $\tilde{y}$ must be zero. As for g_Y, we can perform an operation similar to the one above $Y = \frac{Y}{AL}AL = \tilde{y}AL$ and obtain:

$$g_Y = g_{\tilde{y}} + g_A + g_L = g + n$$

l. Does this model respect the Kaldor facts about labor productivity and the interest rate, once the economy has reached its balanced growth path? Justify your answer using previous questions.

The Kaldor fact about productivity states that labour productivity grows at a sustained rate, while the Kaldor fact about the interest rate states that it is stable. The first is indeed verified, as $g_y = g$ which is positive. The second is also verified, as on the BGP $r = \frac{\alpha}{\tilde{k}^{1-\alpha}}$ is stable.

m. Conclusion: how can you compare this OLG growth model to the Solow model with population growth and technological progress seen in class? Discuss both the assumptions (among which the one about savings) and the results. What about depreciation?

This model differs from Solow Swan in that it endogenizes the choice of the savings s, which were given in Solow-Swan. Hence, it is a little bit more realistic in this dimension. Other assumptions, about preferences and the production function are not different (you can check, as an example, that the Inada conditions hold). In fact, the results are the same on the BGP. The only difference is that s is not given but computed inside the model. They both satisfy the two Kaldor facts that we checked above.

n. How does time preferences influence $\tilde{y}$ and c_t^y on the balanced growth path?

Time preferences are captured by γ. As we noticed before, higher γ lead to higher consumption when young rather that old. But this is from an individual perspective, here we have to study on the BGP. For $\tilde{k}$ we can rely on the equation for $\tilde{k}$:

$$\tilde{y}^* = (\tilde{k}^*)^\alpha = \left[\frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\right]^{\frac{\alpha}{1-\alpha}}$$

we can see that higher γ leads to lower $\tilde{y}^*$. This is due to the fact that people save less and income decreases.

As for c_t^y, we have:

$$\begin{aligned} c_t^y &= w_t\gamma \\ &= \gamma(1-\alpha)A_t(\tilde{k}^*)^\alpha \\ &= \gamma(1-\alpha)A_t\left[\frac{(1-\alpha)(1-\gamma)}{(1+g)(1+n)}\right]^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

We can see that γ as the same effect on c_t^y on the balanced growth path, contrary to what intuition suggests. This is because even if agents consume more immediately when they do not save, in the long run the lack of savings will produce less income, and less consumption, even for the young.