

# 6

## TD 6

### 2 Review (from TD2)

e. In the model of the Easter Island seen in class, there is a threshold amount of resource such that population is growing above that threshold, and decreasing below that threshold.

▼ Answer

**True:** You can easily see it from the phase diagram. We have a line along which  $S$  is fixed, above that line it increases and below it decreases. The expression of that line is  $S = \frac{d-b}{\phi\alpha\beta}$ , therefore if  $S$  is greater than  $\frac{d-b}{\phi\alpha\beta}$  the population is growing, otherwise it is decreasing.

f. In the model of the Easter Island seen in class, the net growth of the resource (i.e. net of harvesting by humans) is decreasing in the numbers of humans present in the ecosystem, for a given value of the amount of the resource.

▼ Answer

**True:** The net growth is given by the replenish of the resource minus the harvest, which is the part affected by the number of humans (population).

$$\dot{S}(t) = \underbrace{rS(t) \left(1 - \frac{S(t)}{K}\right)}_{\text{Natural growth}} - \underbrace{\alpha\beta S(t)L(t)}_{\text{Harvest}}$$

We can clearly see that an increase in  $L(t)$  decreases the net growth. If you want to be more precise its enough to take the derivative with respect to  $L(t)$ :

$$\frac{\delta \dot{S}(t)}{\delta L(t)} = -\alpha \beta S(t) < 0$$

What you get is how much growth is smaller after an infinitesimal change in  $L(t)$ .

## Midterm 2020

**Remark:** Some solutions here are way more elaborate than what was needed to get full points! This is just to make you understand.

## 2 - Production in the Solow-Swan model

Here we have a Solow-Swan model with population growth  $n \in (0,1)$ , no technological growth, where  $s \in (0,1)$  is the saving rate,  $K$  is the stock of capital,  $k = \frac{K}{L}$  is capital per worker,  $F(K,L)$  is the production function with the usual assumptions,  $f(k) = \frac{F(K,L)}{L}$  is production per worker,  $\delta \in (0,1)$  is the rate of

capital depreciation, and finally  $k^*$  denotes  $k$  on the balanced growth path.

In this exercise I put both the general answer and the particular case in which  $f(k) = k^\alpha$  to show you that the solutions make sense and to make you visualise them better.

*Is  $s \frac{\delta f(k)}{\delta k} - \delta - n$  greater than zero, equal to zero or smaller than zero or we cannot say without more information?*

▼ 1. *If  $k \rightarrow 0$ .* In this case  $\frac{\delta f(k)}{\delta k} \rightarrow \infty$  due to the Inada condition, therefore the left side of the equation is way bigger than the right side and therefore the expression is positive.

▼ 2. *If  $k = k^*$ .* A lot of students got this wrong. The standard logic I saw was that since  $k = k^*$  we are on the balanced growth path and therefore  $sf(k^*) = (\delta - n)k^*$ , hence  $sf(k^*)' = (\delta + n)$ . However, this logic is fallacious. If two functions are equal in one point, which in our case is  $k^*$ , then it is not true that their derivative is equal in that point in general. In fact, it is true only if those two functions are equal in every point. This is kind of a technical argument, I understand it is not straightforward, but maybe you will be convinced by the solution. Remember that the derivative of a function tells us how much that function varies after an infinitesimal change in the variable you take the derivative with respect to. Moreover, you know two things. For  $k$  a little bit smaller than  $k^*$ , say  $k_-$  we have that  $sf(k_-) > (\delta + n)k_-$  (you can see it from the standard graph). Also, if  $k$  is a little bit higher than  $k^*$ , call it  $k_+$  it holds that  $sf(k_+) < (\delta + n)k_+$ . This means that in the process of going from  $k_-$  to  $k_+$  the function  $(\delta + n)k$  had a higher increase than  $sf(k)$ , and therefore its derivative  $(\delta + n)$  is higher than the derivative  $sf'(k)$ . Since this holds for all  $k_- < k^* < k_+$  then it also holds for  $k^*$ .

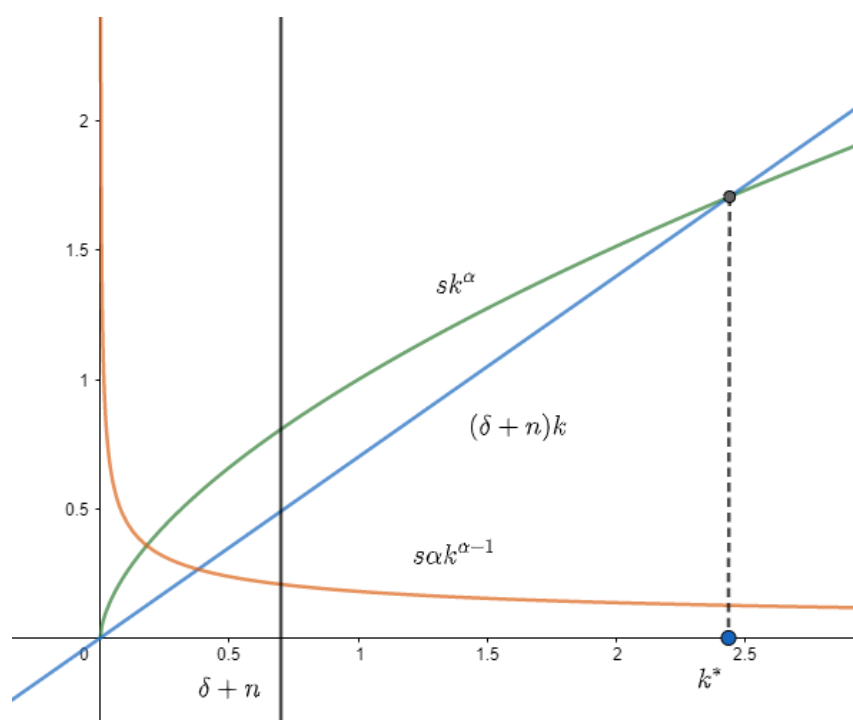


Figure 1: Graph with  $f(k) = k^\alpha$ ,  $s = 0.3$ ,  $\alpha = 0.6$ ,  $\delta = 0.3$ ,  $n = 0.4$ .

▼ 3. *If  $k \rightarrow \infty$ .* This is the converse of the point we had before. Due to the Inada conditions  $\frac{\delta f(k)}{\delta k} \rightarrow 0$  and therefore only the right (negative) side of the equation is left.

▼ 4. *Is  $\frac{\delta f(k)}{\delta k} - \delta - n$  greater than zero, equal to zero or smaller than zero or we cannot say without more information if  $k = k^*$ ? (Note the difference is that here there is no  $s$  in the expression.).* I saw two different kind of mistakes in this question. The first one was to assume that we were in the golden rule and therefore  $\frac{\delta f(k^*)}{\delta k} = \delta - n$ . However, this was not specified by the question. We could have any  $s \in (0,1)$ . The other typical mistake was to say that since  $s \frac{\delta f(k^*)}{\delta k} = \delta - n$  and  $s < 1$  then if you remove it you get  $\frac{\delta f(k^*)}{\delta k} > \delta - n$ . This logic is faulty, as it does not consider that  $\frac{\delta f(k^*)}{\delta k}$  itself depends on  $s$ , and therefore it is different for different values of  $s$ . Hence, we can not determine the relationship between the two quantities.

### 3 - A general purpose technology

Consider the production function with general purpose technology  $Y = F(A, K, L) = AK^\alpha L^{1-\alpha}$  where  $\alpha \in (0,1)$ ,  $K$  is the capital stock and  $L$  is the labor. Assume that  $A_{t+1} = (1+g)A_t$  and  $L_{t+1} = (1+n)L_t$  where  $g$  and  $n$  are exogenous constants. Capital depreciates at rate  $\delta$ . You can use the same approximation as in the TD for the growth of some variable  $X$ :  $g_X = \log\left(\frac{X_{t+1}}{X_t}\right)$ .

This exercise consists in playing with growth rates, the techniques are the same we used in the TDs.

▼ 1. *If the growth rate of capital at time  $t$  is  $g_{K,t}$ , compute the growth rate of  $Y$  at time  $t$  in terms of  $g_K, t$  and the growth rates of technology and labor.* Most if you got this question right. The idea is to exploit the definition given

in the text and do the calculations we did in the TDs. Some of you thought that the  $\alpha$  exponent was also on  $A$ , probably it was just a misreading of the text. Also, some students took derivatives with respect to time, but here time is discrete!

$$\begin{aligned}
 g_{Y,t} &\approx \log \left( \frac{Y_{t+1}}{Y_t} \right) \\
 &= \log \left( \frac{A_{t+1} K_{t+1}^\alpha L_{t+1}^{1-\alpha}}{A_t K_t^\alpha L_t^{1-\alpha}} \right) \\
 &= \log \left( \frac{A_{t+1}}{A_t} \right) + \alpha \log \left( \frac{K_{t+1}}{K_t} \right) + (1-\alpha) \log \left( \frac{L_{t+1}}{L_t} \right) \\
 &= g + \alpha g_{K,t} + (1-\alpha)n
 \end{aligned}$$

▼ 2. *Let's now define general-purpose technology-adjusted labor as  $\bar{L} = A^{\frac{1}{1-\alpha}} L$ . What is the growth rate of  $\bar{L}$ ?* The procedure for answering this question is the same as before, we just need to use the expression of  $\bar{L}$ . Most of you got this correct.

$$\begin{aligned}
 g_{\bar{L}} &= \log \left( \frac{A_{t+1}^{\frac{1}{1-\alpha}} L_{t+1}}{A_t^{\frac{1}{1-\alpha}} L_t} \right) \\
 &= \frac{1}{1-\alpha} \log \left( \frac{A_{t+1}}{A_t} \right) + \log \left( \frac{L_{t+1}}{L_t} \right) \\
 &= \frac{1}{1-\alpha} g + n
 \end{aligned}$$

▼ 3. *What will be the growth of output per worker in the long run?* As always, output per worker is  $\frac{Y}{L}$ , some of you got confused by the  $\bar{L}$  and computed the wrong growth rate. So we have (from now on I omit the logarithm transformation):

$$\frac{Y}{L} = \frac{AK^\alpha L^{1-\alpha}}{L} = A \left( \frac{K}{L} \right)^\alpha$$

$$g_y = g + \alpha(g_{K,t} - n)$$

▼ 4. *What will be the growth rate of real wages in the long run?* Real wage is nominal wage times units of technology adjusted labour over number of workers. Before starting, notice that First, notice that  $g_{\bar{A}} = \frac{g}{1-\alpha}$ . We can now compute the growth rate:

$$w_r = \frac{wLA^{\frac{1}{1-\alpha}}}{L} = wA^{\frac{1}{1-\alpha}}$$

$$g_{w_r} = g_w + g_{\bar{A}} = \frac{g}{1-\alpha}$$

▼ 5. *What will be the growth rate of the real interest rate in the long run?* From the lecture notes you know that  $r$  is stable, which implies that it's growth rate is 0 (not generally constant!).

## 5 - Steady-States

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In this exercise you had to deal with a strange production function. We briefly talked about it in one of the TDs, but if you did not remember you had to think carefully to get the answer correct.

▼ 1. *In the Solow-Swan model with constant population and technology, how many steady-states are there in total if  $\frac{\partial F(K)}{\partial K} = c$  where  $c$  is a constant?* Here it was key to understand that the production function is linear. Consider  $F(K) = cK$ , then you have  $\frac{\partial F(K)}{\partial K} = c$ . We already saw what happens if the production function is linear in TD2. First, there is a steady state in 0. Second, there could be other infinitely many steady states if  $\delta = sc$ , which is in fact

the condition for being in a steady state. The answer to this question should be clear from the graph below.

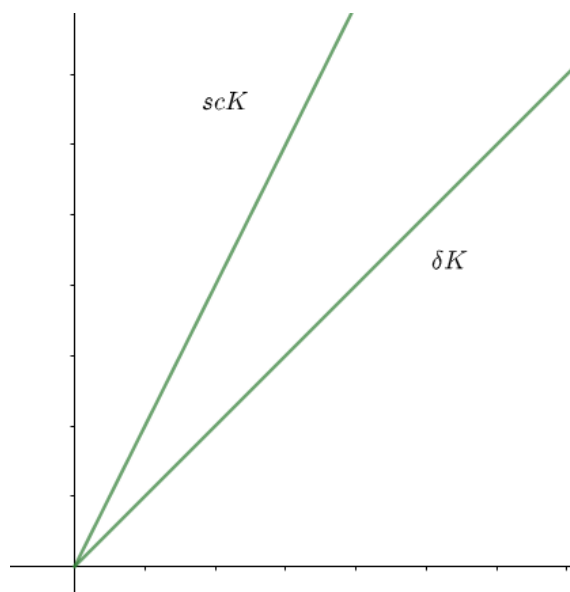


Figure 2: Steady state with linear production function.

▼ 2. (Bonus) *In the Solow-Swan model with constant population and technology, how many stable steady-states are there in total if  $\frac{\delta F(K)}{dK} = c$  where  $c$  is a constant? A steady-state is stable if for small deviation around  $K$ , the economy returns to  $K$  automatically.* This question was a bonus as you did not work with the "stability" concept before. However, is exactly what we did in the previous TD, so you should be a little bit familiar with it by now. Consider the two different cases we studied before. First, it could be that  $sc \neq \delta$  and therefore we only have one steady state at 0. Is it stable? What happens if we perturb it a little bit and move on  $0 + \epsilon$ ? It depends. If  $sc > \delta$  then there is more investment than depreciation, and  $K$  increases more than what is lost due to  $\delta$ . In this case the steady state is not stable, as we do not return to 0. If instead  $sc < \delta$ , then after a small increase capital still depreciates at a higher rate than what is saved. Hence, we return back to 0 and the state is stable. In the case  $sc = \delta$  any point is a steady state and therefore no state is

stable, as if we move a little bit we are already in a new steady state.

## 6 - An Algal Boom

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Consider the dynamics of the fish population with fishing. At time  $t-1$ , the stock of fishes is at its long term equilibrium  $S^* > 0$ . Then, at time  $t$ , an algal bloom kills half of the fish population. The bloom lasts a single period. At  $t+1$ , the algal bloom has ended and the ecosystem is back to where it was before (except that fishes have died of course).

This exercise did not require hard computations, you had to reason about the question and understand more or less intuitively the direction of the answer.

▼ 1. *At  $t+1$ , is the net growth of the stock  $\Delta S_{t+1} = S_{t+2} - S_{t+1}$  higher, equal or lower than the net growth at  $t-2$ , or we don't have enough information to tell. Assume that the number of boats have not changed between  $t-2$  and  $t+1$ .* The question here asks to compare  $\Delta S_{t+1}$  to  $\Delta S_{t-2} = S_{t-1} - S_{t-2}$ . Since before  $t$  we were in steady state, we must have that  $\Delta S_{t-2} = 0$  and  $S_{t-1} = S_{t-2} = S^*$ . Hence, the question becomes: is  $\Delta S_{t+1} > 0$ ? At time  $t$  we have the algal bloom, so the steady state population gets halved and we have that  $S_t = \frac{S^*}{2}$ . At  $t+1$ , even if the bloom is over, we are not at the steady state as  $\frac{S^*}{2} < S^*$ . To reach the steady state again the stock of fish must grow positively. For some time periods  $\tau$ , we will have that  $S_{\tau+1} > S_\tau$ . We reached the conclusion that  $\Delta S_{t+1} > 0$ . Not all of you got this correct, I think it may be due to the confusion with the definition of net growth rate.

▼ 2. *At  $t+1$ , is the natural growth of the stock  $\tau(S_{t+1})$  higher, equal or lower than the natural growth at  $t-2$   $\tau(S_{t-2})$ , or we don't have enough information to tell. Assume*



that the number of boats have not changed between  $t-2$  and  $t+1$ . This question asks to compare the levels of the parabola for different values of  $S_t$ . In  $S_{t-2}$  we were in steady state, but we do not know where! As you can see from the graph, values of  $\tau\left(\frac{S^*}{2}\right)$  could be both higher or lower than  $\tau(S^*)$ .

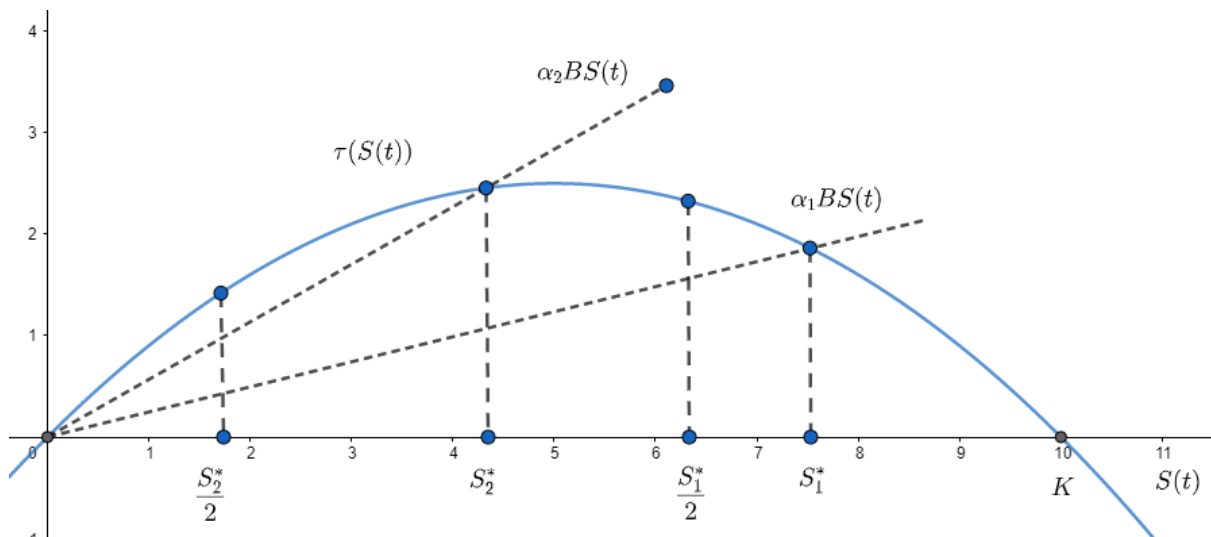


Figure 3: Steady states and growth rate for different fishing intensities.

▼ 3. *If commercial fishing was very intense, could this bloom cause the population to go extinct?* Here you should consider that fishermen's technology is  $\alpha < 1$ , therefore they will always be able to catch a fraction of the fish that is lower than the total amount available. If  $\alpha \rightarrow 1$  there will still be a small fraction  $\epsilon$  of fishes around. You can see from the graph of the growth rate  $\tau$  that this is positive at any value greater than 0, so extinction is not possible in this model (does not mean it is not possible in reality!).

## 5 Equilibrium on Easter island (from TD2)

In this exercise we just have to reason a little bit with the phase diagram to get the answer. I do not want to focus on the calculations which you can find in your notes, therefore I report the fundamental equations here:

$$\dot{S}(t) = \left( r \left( 1 - \frac{S(t)}{K} \right) - \alpha\beta L(t) \right) S(t)$$

$$\dot{L}(t) = (b - d + \phi\alpha\beta S(t)) L(t)$$

To generate the graph we have to find the loci of points in which the two variables are not moving. To do this is enough to set the growth (derivative with respect to time) equal to 0. We obtain the two lines:

$$S^* = K - \frac{K\alpha\beta}{r} L^*$$

$$S^* = \frac{d - b}{\phi\alpha\beta}$$

Here you can find the phase diagrams with these two loci depicted.

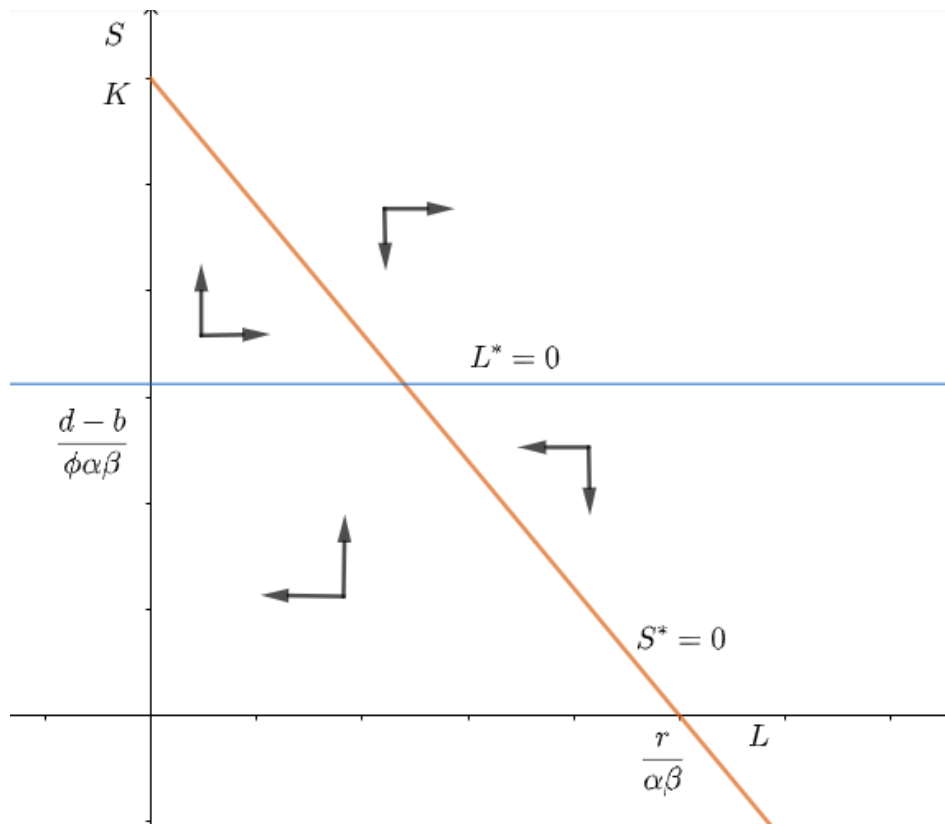


Figure 4: Phase diagram with  $r = 0.04$ ,  $\alpha = 0.00001$ ,  $b - d = 0.1$ ,  $\phi = 4$ ,  $\beta = 0.4$ ,  $K = 12000$ .

In the rest of the solution I indicate with dashed lines the old loci, while continuous lines represents the new ones after the variable change. We assume we are in the nontrivial internal steady state.

**a. What happens if  $r$  goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.**

We can see that  $r$  only appears in one of the two equations, namely  $S^* = K - \frac{K\alpha\beta}{r}L^*$ . Since  $r$  is positively affecting its slope, but not the intercept, we have to rotate it counterclockwise. The other locus is not affected.

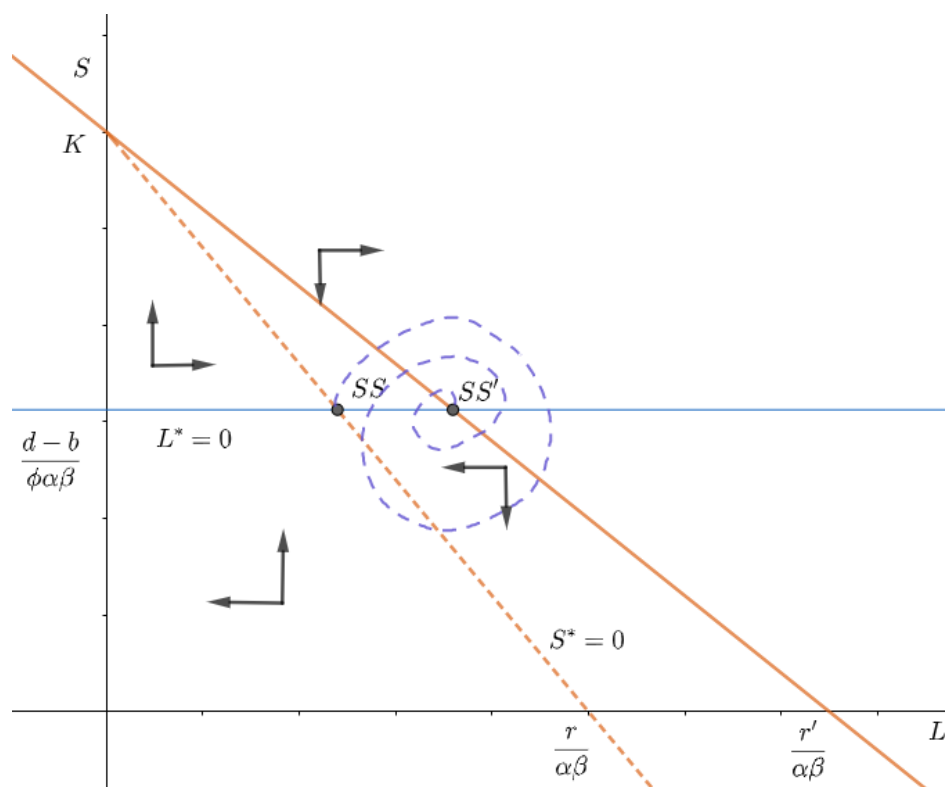


Figure 5: Phase diagram with new  $r' = 0.06$ .

What we see is that in the new steady state  $S^*$  is not affected, but  $L^*$  is higher. This is due to the fact that the increase of availability of resources is completely offset by

the increase in population, which was possible because of the increase of the regeneration rate.

**b. What happens if  $\alpha$  goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.**

The rate  $\alpha$  appears again in the intercept of the locus for  $\dot{S}=0$ . However, in this case the rotation is clockwise as it is negatively affected by an increase in  $\alpha$ . Moreover,  $\alpha$  is also in the intercept for  $\dot{L}=0$ , which goes down due to their negative relationship.

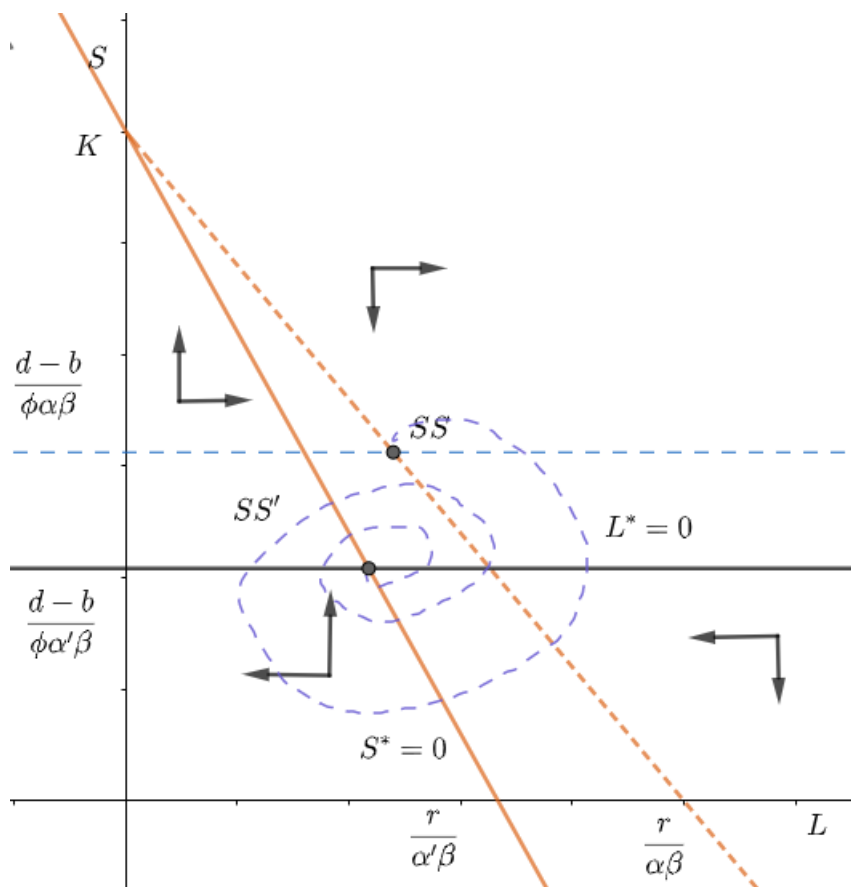


Figure 6: Phase diagram with new  $\alpha' = 0.000015$ .

In the new steady state we have unambiguously lower  $S^*$ , however the impact on  $L^*$  depends on the entity of the increase of  $\alpha$ . More efficient farming means that we need less workers to keep the amount of resources fixed (clockwise

rotation). On the other hand, given the increase in efficiency we need less resources to keep population constant (horizontal line decrease).

c. What happens if  $K$  goes down? Show on the graph the change to each condition and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

The capacity  $K$  only appears in the intercept and slope of  $\dot{S} = 0$ , which then must rotate counterclockwise.

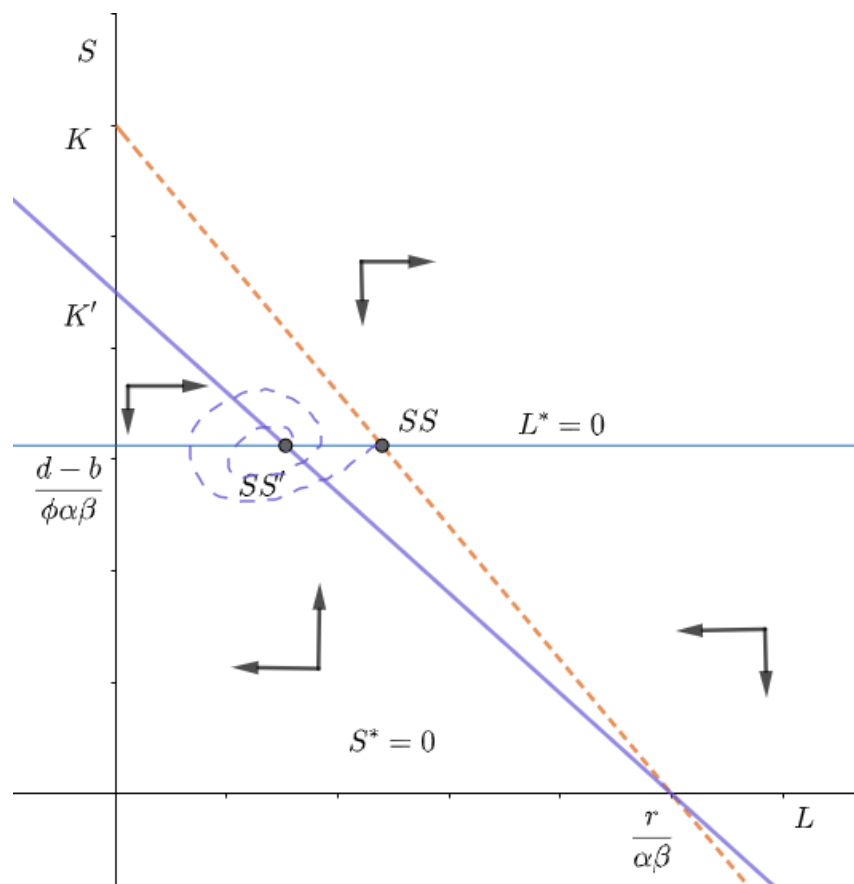


Figure 7: Phase diagram with new  $K' = 9000$ .

This time  $S^*$  is not affected while the impact on  $L^*$  is unambiguous. Due to the decrease in maximum capacity, the stock of resources grows slower. You can see this from the growth rate  $\dot{S}$ . This implies that  $L^*$  is lower than before.