TD 6

1 Review (from TD2)

e. In the model of the Easter Island seen in class, there is a threshold amount of resource such that population is growing above that threshold, and decreasing below that threshold.

▼ *Answer*

True: You can easily see it from the phase diagram. We have a line along which S is fixed, above that line it increases and below it decreases. The expression of that line is $S = \frac{d-b}{\phi\alpha\beta}$, therefore if S is greater than $\frac{d-b}{\phi\alpha\beta}$ the population is growing, otherwise it is decreasing.

f. In the model of the Easter Island seen in class, the net growth of the resource (i.e. net of harvesting by humans) is decreasing in the numbers of humans present in the ecosystem, for a given value of the amount of the resource.

▼ *Answer*

True: The net growth is given by the replenish of the resource minus the harvest, which is the part affected by the number of humans (population).

$$\dot{S}(t) = \underbrace{rS(t)\left(1 - \frac{S(t)}{K}\right)}_{\text{Natural growth}} - \underbrace{\alpha\beta S(t)L(t)}_{\text{Harvest}}$$

We can clearly see that an increase in $L(t)$ decreases the net growth. If you want to be more precise its enough to take the derivative with respect to $L(t)$:

$$\frac{\partial \dot{S}(t)}{\partial L(t)} = -\alpha\beta S(t) < 0$$

What you get is how much growth is smaller after an infinitesimal change in $L(t)$.

5 Equilibrium on Easter island (from TD2)

In this exercise we just have to reason a little bit with the phase diagram to get the answer. First, let's derive again the fundamental equations. Here we work in a system in which there is both the growth of a natural resource, as in the fishing model, and population growth. As we just saw in the review question, the net growth of the natural resource is given by its natural growth rate minus the harvest (exactly as in the fishing model). Hence we have:

$$\begin{aligned}\dot{S}(t) &= \underbrace{rS(t)\left(1-\frac{S(t)}{K}\right)}_{\text{Natural growth}} - \underbrace{\alpha\beta S(t)L(t)}_{\text{Harvest}} \\ &= \left(r\left(1-\frac{S(t)}{K}\right) - \alpha\beta L(t)\right) S(t)\end{aligned}$$

As for population growth, we can easily recover it by using the fundamental parameters of the model. We have that people die at a rate d and born at a rate b plus a percentage of per capita harvest $\phi\frac{H(t)}{L(t)}$. Of course, do not forget that the growth itself at time t depends on how many people $L(t)$ we have. Hence:

$$\begin{aligned}\dot{L}(t) &= \left(b-d+\phi\frac{H(t)}{L(t)}\right) L(t) \\ &= \left(b-d+\phi\frac{\alpha\beta S(t)\cancel{L(t)}}{\cancel{L(t)}}\right) L(t) \\ &= \left(b-d+\phi\alpha\beta S(t)\right) L(t)\end{aligned}$$

These are the two **law of motion** of the variables of interest in our model, namely $S(t)$ and $L(t)$. In the previous model with fish extraction we were used to graph the growth rate and see the points which constituted the stable states. Here the problem is a bit harder, as we have two variable with two law of motions, not only one. Graphing the two growth rates separately will not help much in understanding the dynamics of the system. Hence, instead of employing a graph which on one axis has $S(t)$ and on the other one has its growth rate, we use a graph with the two variables of interest, both $S(t)$ and $L(t)$. But what do we represent?

We have a system with two variables and two growth rates, which means that each law of motion will have more than one couples of $S(t)$ and $L(t)$ in which it is zero. Take $\dot{S}(t)$, as an example. In the previous exercise we found the value of $S(t)$ for which it was equal to 0. We will do the same here, but we will not find a value of $S(t)$, but a relation between $S(t)$ and $L(t)$! Let's do it, when is $\dot{S}(t)$ equal to 0?

$$\dot{S}(t) = 0 \Leftrightarrow \left(r\left(1 - \frac{S(t)}{K}\right) - \alpha\beta L(t)\right) S(t) = 0$$

This has two solutions: the first is $S(t) = 0$, for any value of $L(t)$, while the second is $r\left(1 - \frac{S(t)}{K}\right) - \alpha\beta L(t) = 0$ which gives us $S(t) = K - \frac{K\alpha\beta}{r}L(t)$. You see now that $\dot{S}(t)$ is not zero only for some values of $S(t)$, but for couples of values of $S(t)$ and $L(t)$. In all the points of the graph with axis $(L(t), S(t))$ the law of motion of S is equal to 0.

We must now repeat the same exercise with $\dot{L}(t)$.

$$\dot{L}(t) = 0 \Leftrightarrow (b - d + \phi\alpha\beta S(t))\, L(t)$$

We have two solutions here too: the law of motion is null when $L(t) = 0$, for any value of $S(t)$, but also when $b - d + \phi\alpha\beta S(t) = 0$ which gives $S(t) = \frac{d-b}{\phi\alpha\beta}$.

Hence, the equations that give us the set of points $S(t)$ and $L(t)$ in which the two law of motions are zero are given by:

$$S^* = K - \frac{K\alpha\beta}{r}L^* \qquad\qquad and \qquad\qquad S^* = 0$$

$$S^* = \frac{d-b}{\phi\alpha\beta} \qquad and \qquad L^* = 0$$

When our variables respect these conditions there is no movement of S (first equation) or L (second equation). The system completely stops when we are in the steady state, which in this case is the circumstance in which both these equations are satisfied, i.e. neither S nor L are moving. Every time we are not in a point in which these two lines intersect we have movement in the system. If we forget about the 0 solutions we can graph the two lines with the dynamics when we are out of the steady state here:

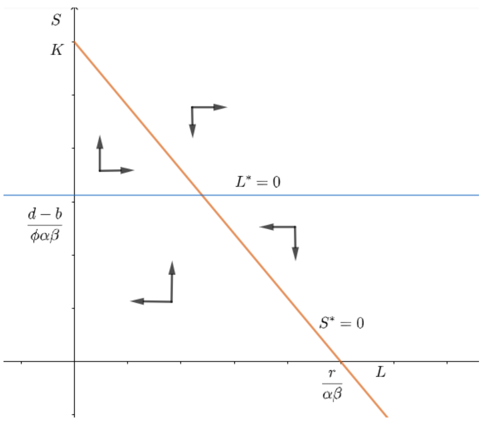

Figure 4: Phase diagram with $r = 0.04$, $\alpha = 10^{-6}$, $b - d = 0.1$, $\phi = 4$, $\beta = 0.4$, $K = 12000$.

In the rest of the solution I indicate with dashed lines the old equations, while continuous lines represents the new ones after the variable changes. We assume we are in the nontrivial internal steady state.

a. What happens if r goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

Let's start by breaking down all the components of the relevant equations.

$$S^* = \underbrace{K}_{intercept} - \underbrace{\frac{K\alpha\beta}{r}}_{slope} L^*$$

$$S^* = \underbrace{\frac{d-b}{\phi\alpha\beta}}_{intercept}$$

We can see that r only appears in one of the two equations, namely $S^* = K - \frac{K\alpha\beta}{r}L^*$. Since r is positively affecting its slope, but not the intercept, we have to rotate it counter-clockwise (as r increases). The other equation is not affected.

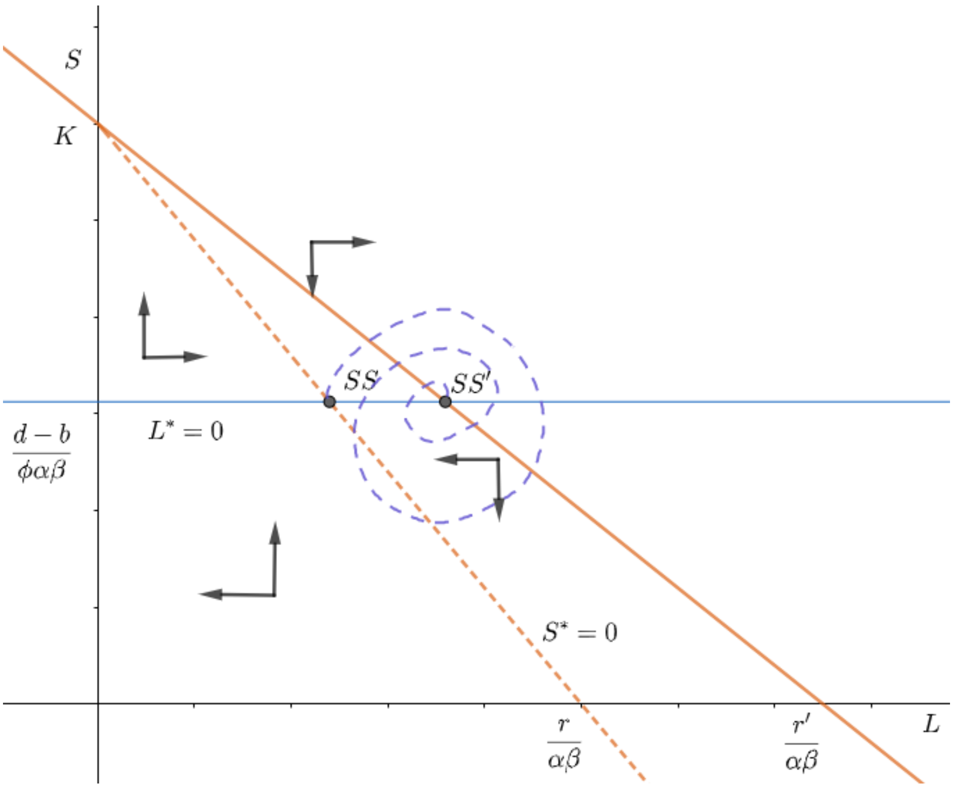

Figure 5: Phase diagram with new $r' = 0.06$.

What we see is that in the new steady state S^* is not affected, but L^* is higher. This is due to the fact that the increase of availability of resources is completely offset by the increase in population, which was possible because of the increase of the regeneration rate.

b. What happens if α goes up? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

The rate α appears again in the intercept of the locus for $\dot{S} = 0$. However, in this case the rotation is clockwise as it is negatively affected by an increase in α. Moreover, α is also in the intercept for $\dot{L} = 0$, which goes down due to their negative relationship.

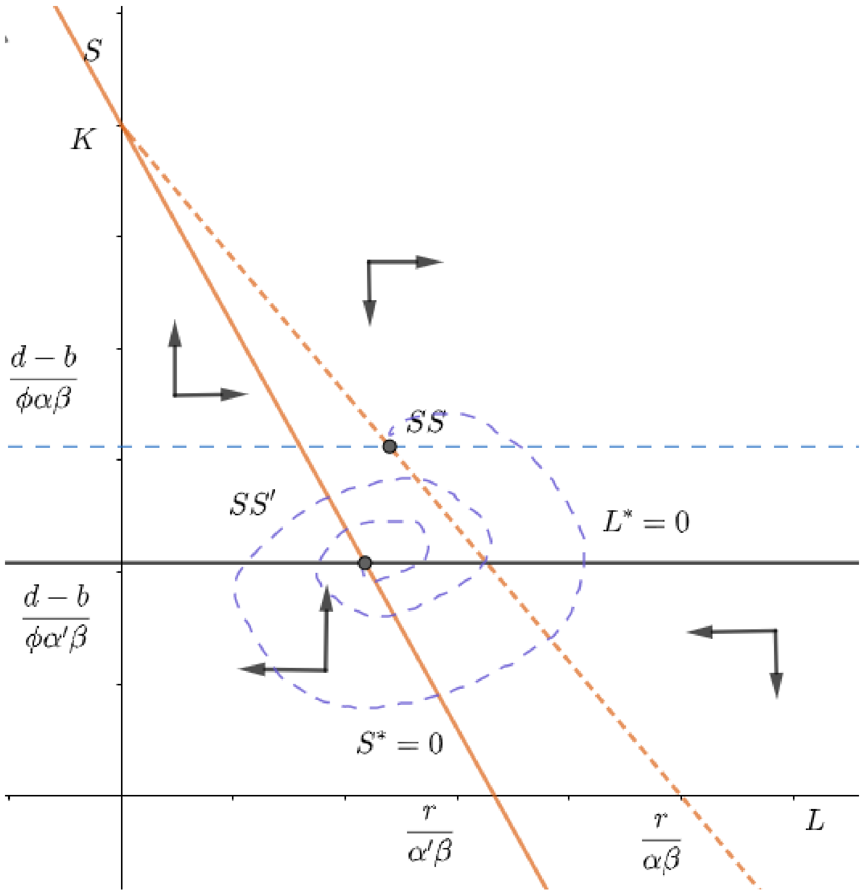

Figure 6: Phase diagram with new $\alpha' = 0.000015$.

In the new steady state we have unambiguously lower S^*, however the impact on L^* depends on the entity of the increase of α. More efficient farming means that we need less workers to keep the amount of resources fixed (clockwise rotation). On the other hand, given the increasy efficiency we need less resources to keep population constant (horizontal line decrease).

c. What happens if K goes down? Show on the graph the change to each conditions and the approximate dynamic transition to the new equilibrium if the convergence (spiral node with cyclical convergence). Interpret in a few words.

The capacity K only appears in the intercept and slope of $\dot{S} = 0$, which then must rotate counterclockwise.

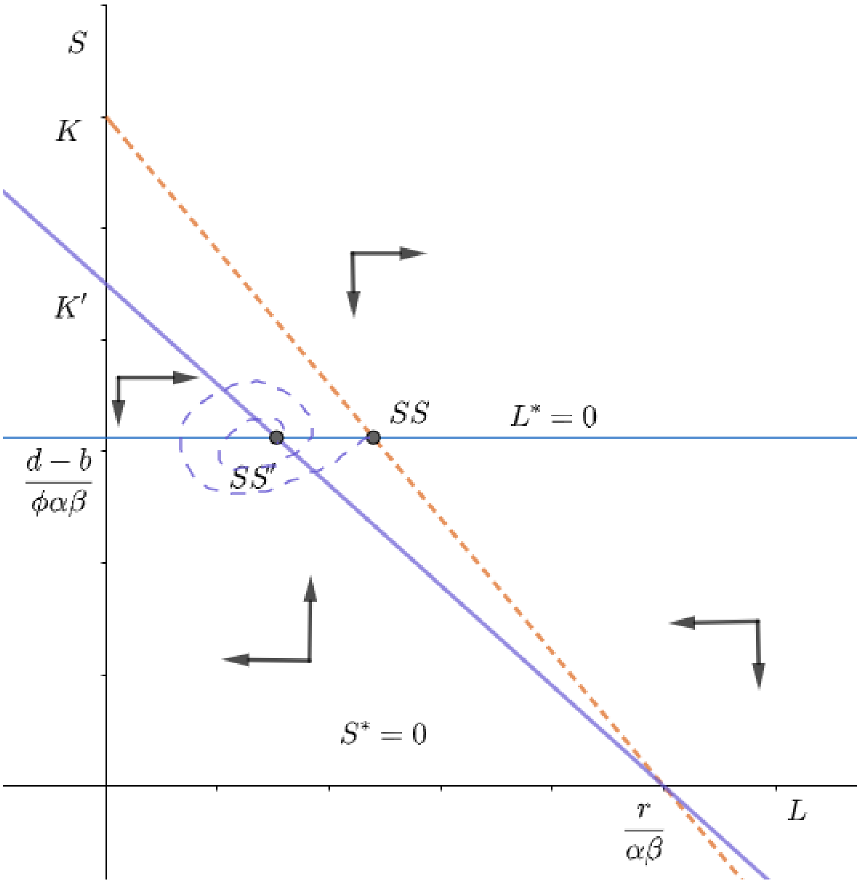

Figure 7: Phase diagram with new $K' = 9000$.

This time S^* is not affected while the impact on L^* is unambiguous. Due to the decrease in maximum capacity, the stock of resources grows slower. You can see this from the growth rate $\dot{S}$. This implies that L^* is lower than before.