7️⃣

TD 7

1 Review

a. Before the demographic transition, increases in income per capita always caused an increase in the growth rate of population.

▼ *Answer*

True: As you found in your lecture, in the pre industrialised world technological progress and land expansion caused temporary increase in income per capita which in turn increased the growth rate of population. As an example see the graph at page 3.

b. In the contemporary world, an increase in income per capita is associated to a decrease in the growth rate of population.

▼ *Answer*

True: The graph at page 12 of your lecture notes is self explanatory. Higher average income correlates with lower births per woman. An explanation taken from your lecture notes could be the effects of urbanization.

c. Decreases in the various measures of fertility came after decreases in mortality.

▼ *Answer*

True: You can see from the graph at page 11 of your lecture notes that the fertility rate has decreased from 1950. For sure the mortality rate had significantly declined when compared to pre 1900 times. This statement is also consistent with the fact that fertility rates are lower in developed countries where the mortality rate is lower.

d. The demographic transition is now over for most of the world population.

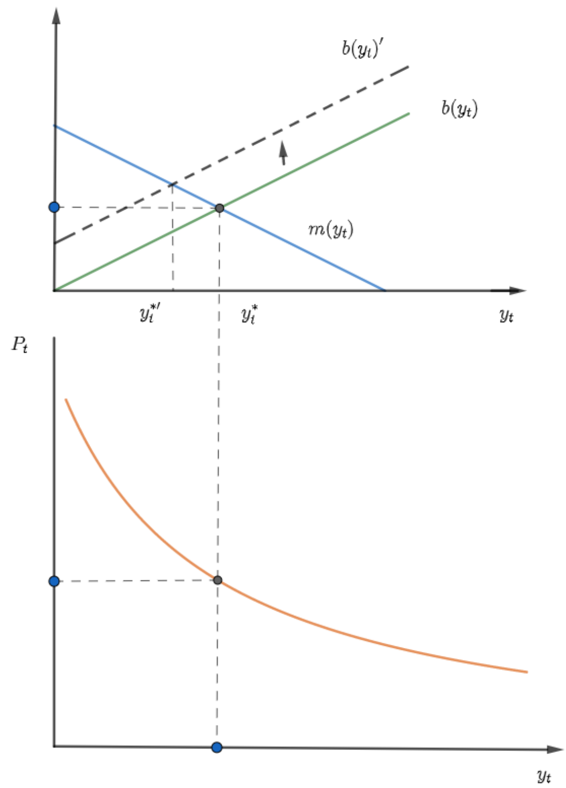
▼ *Answer*

True: Quoting from page 11 of your notes "*as of 2017, more than 80% of the world fertility was already at or below replacement rate*".

e. In the model of the Malthusian regime seen in class, an exogenous increase in the birth rate translates into a lower level of steady-state income per capita.

▼ *Answer*

Maybe: If this shock implies that $b(y_t)$ is higher, keeping other things fixed, then the statement is true, you can see it from the graph presented here.



2 The Malthusian Regime

This exercise studies a particular example of the model seen in class, with specifications for the primitives of the model. I report them here.

The birth rate is given by:

$$b(y_t) = \alpha_b + \beta_b y_t \tag{1}$$

The mortality rate is:

$$m(y_t) = \alpha_m - \beta_m y_t \tag{2}$$

The production function is:

$$Y(P_t) = \alpha_y + \beta_y P_t \tag{3}$$

Y is the total production (or income), P is the population, y is income per capita. It is assumed that all the population works and that there is no immigration nor emigration. All coefficients of the model are positive. It is further assumed that $\alpha_m > \alpha_b$ and $\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m) > 0$.

a. Discuss equations (1) and (3): how do they relate to the model seen in the lecture?

In your lectures you saw that the birth rate depends positively on income per capita, which translates into $b(y_t)$ with $\frac{\partial b}{\partial y_t} > 0$. This condition is indeed respected in the specific function we have in this exercise, as $\frac{\partial b}{\partial y_t} = \beta_b > 0$.

As for the technology, the condition you had in the lecture were positive marginal product $\frac{\partial Y}{\partial P_t} > 0$ and decreasing returns $\frac{\partial^2 Y}{\partial P_t^2} < 0$. It is easy to check that the first holds while the second does not. In fact, $\frac{\partial Y}{\partial P_t} = \beta_y > 0$ and $\frac{\partial^2 Y}{\partial P_t^2} = 0 \nless 0$. This production function is similar to what you saw in TD2, it is affine! Since it is a line it does not have decreasing returns.

b. Compute the marginal and average productivity of labor. Comment.

To compute average and marginal productivity, we first need productivity. As usual we divide by the population P_t.

$$
\begin{aligned}
y(P_t) &= \frac{Y(P_t)}{P_t} \\
&= \frac{\alpha_y + \beta_y P_t}{P_t} \\
&= \frac{\alpha_y}{P_t} + \beta_y
\end{aligned}
$$

Marginal productivity indicates how much more productive we are by increasing P_t by an infinitesimal amount. By taking derivatives we get:

$$
\frac{\partial y(P_t)}{\partial P_t} = -\frac{\alpha_y}{P_t^2} < 0
$$

Hence, by adding labour we become less productive. As for average productivity we just have to divide by the number of workers

$$
\frac{y(P_t)}{P_t} = \frac{\alpha_y}{P_t^2} + \frac{\beta_y}{P_t}
$$

Nothing special here, as we increase the number of workers the productivity per worker decreases.

c. Compute the steady-state level of total income, per capita income and population. Show graphically how those steady-state values are determined.

The law of motion of population in this model is $\dot{P} = [(b(y_t) - m(y_t)] P$. As usual, we are in steady state when the law of motion is equal to 0, i.e. population is not growing $\dot{P} = 0$. This when the mortality rate is equal to the birth rate (actually, also when $P = 0$, but we are not interested in this case). We have to set $b(y_t) = m(y_t)$ to get:

$$
\begin{aligned}
\alpha_b + \beta_b y^* &= \alpha_m - \beta_m y^* \\
y^*(\beta_b + \beta_m) &= \alpha_m - \alpha_b \\
y^* = \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m}
\end{aligned}
$$

which is the steady state level of per capita income. We still need to compute the steady-state level of total income and population. Total income is itself a function of the population, so we have to find the steady state of this one first. We can rely on y^* which we just found.

$$
\begin{aligned}
y^* &= \frac{\alpha_y}{P^*} + \beta_y \\
\frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} &= \frac{\alpha_y}{P^*} + \beta_y \\
\frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} - \beta_y &= \frac{\alpha_y}{P^*} \\
P^* \left(\frac{\alpha_m - \alpha_b}{\beta_m + \beta_b} - \beta_y \right) &= \alpha_y \\
P^* \left(\frac{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}{\beta_m + \beta_b} \right) &= \alpha_y \\
P^* &= \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}
\end{aligned}
$$

Now that we have y^* and P^* we are ready to compute the steady state level of total income Y^*. Its expression is given in the text:

$$
\begin{aligned}
Y^* = Y(P^*) &= \alpha_y + \beta_y P^* \\
&= \alpha_y + \beta_y \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
&= \frac{\alpha_y \alpha_m - \alpha_y \alpha_b - \cancel{\alpha_y \beta_y \beta_b} - \cancel{\alpha_y \beta_y \beta_m} + \cancel{\alpha_y \beta_y \beta_b} + \cancel{\alpha_y \beta_y \beta_m}}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} \\
Y^* &= \frac{\alpha_y(\alpha_m - \alpha_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)}
\end{aligned}
$$

In the following figure I represented the equilibrium for specific values of the paramenters.

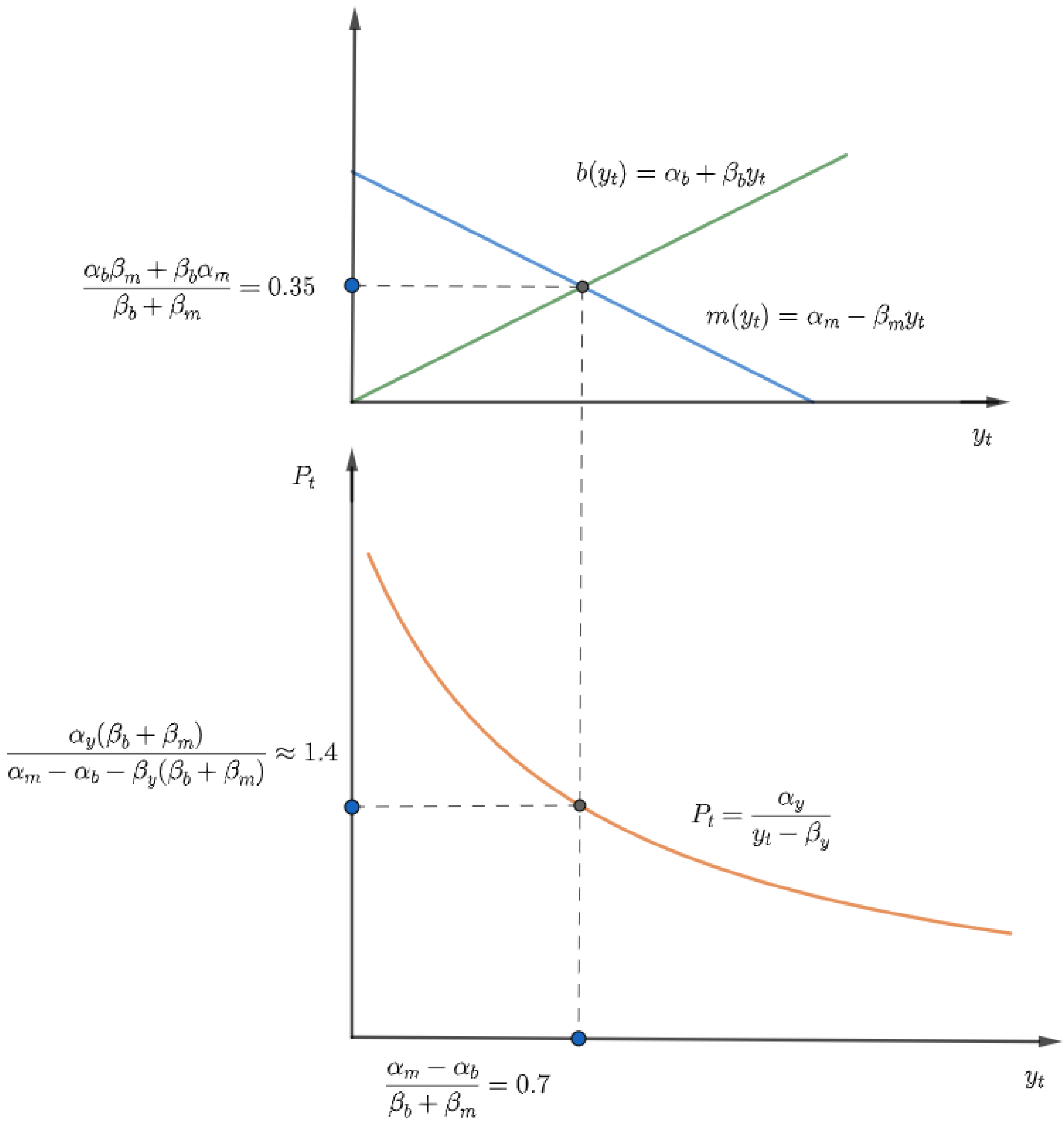


Figure 1: Steady state for $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = 0.7$. Here $Y^* = \alpha_y + \beta_y P^* = 1$.

d. For the rest of the exercise, we assume $\alpha_b = \beta_y = 0$, $\beta_b = \beta_m = 0.5$, $\alpha_y = 1$ and $\alpha_m = \alpha > 0$. What are the steady-state levels for this configuration of parameters?

I more or less computed them in the graph, but I assumed a specific value for α. We have again that $Y^* = \alpha_y + \beta_y P^* = 1 + 0P^* = 1$. As for y^* and P^*:

$$y^* = \frac{\alpha_m - \alpha_b}{\beta_b + \beta_m} = \frac{\alpha}{1} = \alpha$$

$$P^* = \frac{\alpha_y(\beta_m + \beta_b)}{\alpha_m - \alpha_b - \beta_y(\beta_b + \beta_m)} = \frac{1}{\alpha}$$

e. Show that the model dynamics can be summarized by a first-order difference equation in P_t (of the type $P_{t+1} = f(P_t)$, with f some function that you need to find; you can also look for an equation of the type $\Delta P_t = g(P_t)$ with g some function to find, if it is easier for you to do so).

This question is a very involved way of asking: what are the time dynamics of P_t? You know from your lecture notes that $\dot{P} = [b(y_t) - m(y_t)]\,P_t$. However, we are in discrete time here, as the question asks for a difference equation (not differential), therefore in this case $\dot{P}$ is substituted by $P_{t+1} - P_t$. We just have to work out the expression above and plug values for the parameters.

$$\begin{aligned}
P_{t+1} - P_t &= [b(y_t) - m(y_t)]\,P_t \\
&= [\cancel{\alpha_b} + \beta_b y_t - \alpha_m + \beta_m y_t]\,P_t && \text{since } \alpha_b = 0 \\
&= [(\beta_b + \beta_m) y_t - \alpha]\,P_t && \text{since } \alpha_m = \alpha \\
&= [y_t - \alpha]\,P_t && \text{since } \beta_b + \beta_m = 1 \\
&= \left[\frac{\alpha_y}{P_t} + b_y - \alpha\right] P_t && \text{substituting } y_t(P_t) \\
&= \left[\frac{1}{P_t} - \alpha\right] P_t && \text{since } \alpha_y = 1 \text{ and } \beta_y = 0 \\
P_{t+1} - P_t &= 1 - \alpha P_t \\
P_{t+1} &= P_t(1 - \alpha) + 1
\end{aligned}$$

f. Study the convergence of population to its steady state starting from an initial value of population P_0 close to 0 for the following values of α: $(i)\ 0 < \alpha < 1$, $(ii)\ \alpha = 1$, $(iii)\ 1 < \alpha < 2$.

This question basically asks you to study the dynamics of population for different values of α. It is more or less about plugging numbers. Let's start from $t = 1$ and see what the dynamics look like. Since $1 - \alpha$ is a bit uncomfortable I substitute it with $\gamma = 1 - \alpha$.

$$\begin{aligned}
P_1 &= \gamma P_0 + 1 \\
P_2 &= \gamma P_1 + 1 \\
&= (\gamma P_0 + 1)\gamma + 1 \\
&= \gamma^2 P_0 + \gamma + 1 \\
P_3 &= \gamma P_2 + 1 \\
&= (\gamma^2 P_0 + \gamma + 1)\gamma + 1 \\
&= \gamma^3 P_0 + \gamma^2 + \gamma + 1
\end{aligned}$$

You see the pattern. By thinking a little bit we realise that we can express P_t in the following way:

$$P_t(\gamma) = \gamma^t P_0 + \sum_{s=0}^{t-1} \gamma^s$$

For $t \to \infty$, by the rules of power series, we have:

$$\begin{aligned}
P_\infty(\gamma) &= \gamma^\infty P_0 + \sum_{s=0}^{\infty} \gamma^s \\
&= \gamma^\infty P_0 + \frac{1}{1-\gamma}
\end{aligned}$$

We are ready to evaluate the convergence. The following table gives a relationship between $1 - \alpha$ and γ.

	α	γ
(i)	$0 < \alpha < 1$	$0 < \gamma < 1$
(ii)	1	0
(iii)	$1 < \alpha < 2$	$-1 < \gamma < 0$

Case (ii) is the easiest. If $\gamma = 0$ then $P_t = 1$ for any t. Population is fixed since the beginning, so in some sense we already converged from the start.

In case (i) we have $0 < \gamma < 1$. If we have no clue we can take one number and see what happens. Let's try $\gamma = 0.5$. We have the following series (assuming P_0 is close to 0):

$$\begin{aligned}
P_1 &= 0.5P_0 + 1 \approx 1 \\
P_2 &= 0.5(1) + 1 = 1.5 \\
P_3 &= 0.5(1.5) + 1 = 1.75 \\
P_4 &= 0.5(1.75) + 1 = 1.875 \\
P_5 &= 1.9375 \\
P_6 &= 1.96875 \\
&\vdots
\end{aligned}$$

In the following picture you can see the series grahpically:

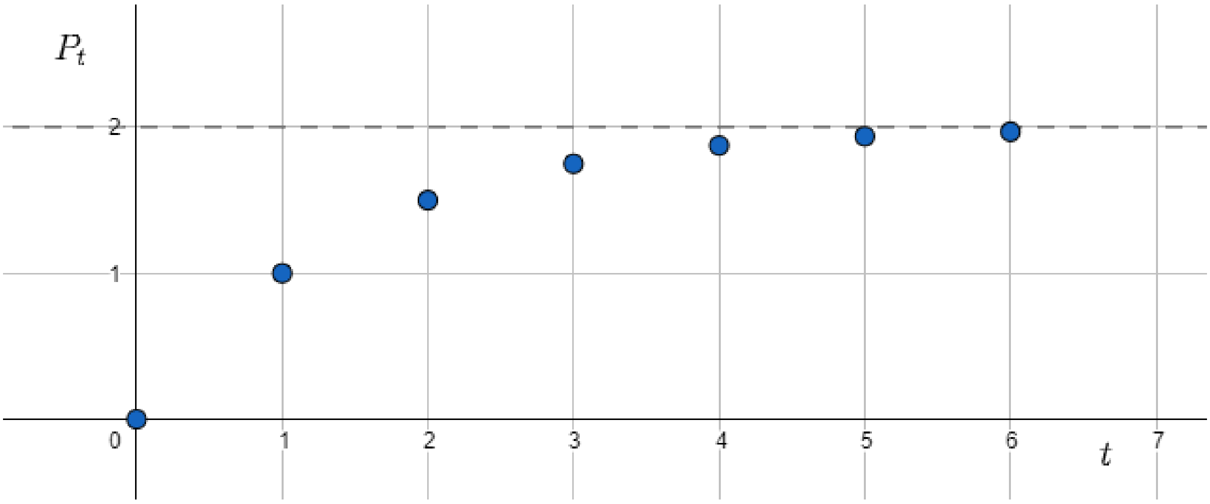


Figure 2: Series with $\gamma = 0.5$.

You can see where we are going. We can immediately compute P_∞ from the expression above (remember that for any $-1 < \gamma < 1$ we have that $\gamma^\infty = 0$):

$$P_\infty(0.5) = 0P_0 + \frac{1}{1-0.5} = \frac{1}{0.5} = 2$$

What we conclude is that for a value of α between 0 and 1 population grows, slower at each step, and eventually reaches a steady state level (different for different values of γ).

As for case (iii), with $-1 < \gamma < 0$, we use the same strategy, namely plugging numbers for the specific value $\gamma = -0.5$. The series looks like this:

$$\begin{aligned}
P_1 &= -0.5P_0 + 1 \approx 1 \\
P_2 &= -0.5(1) + 1 = 0.5 \\
P_3 &= -0.5(0.5) + 1 = 0.75 \\
P_4 &= -0.5(0.75) + 1 = 0.625 \\
P_5 &= 0.6875 \\
P_6 &= 0.65625 \\
P_7 &= 0.671875 \\
&\vdots
\end{aligned}$$

As before, I plotted the series:

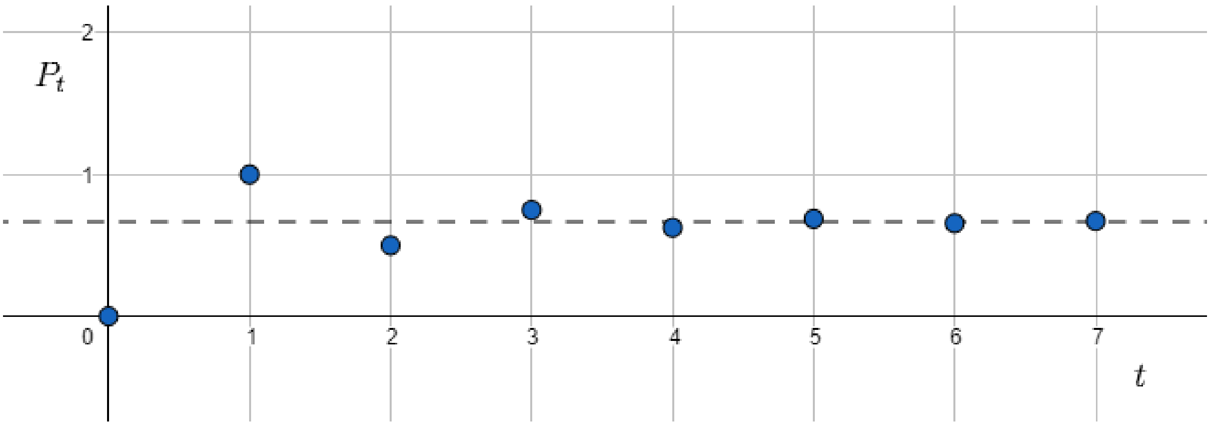


Figure 3: Series with $\gamma = -0.5$.

As you can see the series gere goes up and down, it is not monotonic in its growth, contrary to the previous case. However, we can see where it converges to:

$$P_\infty(-0.5) = 0P_0 + \frac{1}{1-(-0.5)} = \frac{1}{1.5} = 0.\bar{6}$$

Interestingly, notice that the term P_0 has in both case no role in determining the convergence, only shaped by γ.