TD 8

1 Review

a. In the OLG model of the course, the market for financial contracts without money doesn't exist because the interest rate asked for by the Young is too high.

▼ *Answer*

False: There is now way young can be convinced to transfer the consumption good to the old, as they do not have any guarantee that when they will be old the newborn will do the same. Money allows to overcome the perishability of the good and permits to contract both with the previous and the new generation.

b. When individuals optimally choose their consumption through time, any change in income at any date (or combination of changes in several periods) modifies the consumption choices.

▼ *Answer*

False: As we will soon see what matters in the intertemporal budget constraint is the net present value of resources, not particular levels of income at specific dates. If the net present value of resources does not change (i.e. the income shocks balance out), consumption choices do not change.

c. In the interior solution, if an individual receives more income when old than when young, he or she will always want to borrow when young.

▼ *Answer*

False: As we will see in the following exercise, this depends on preferences for consumption, time preferences as captured by the β, the amount of the net present value of income and the interest rate r. The statement is true when the agent likes "*consumption smoothing*", he likes to consume the same amount in every time period.

d. When there is inflation, the real value of money decreases through time.

▼ *Answer*

True: Simply, the real value of money is its nominal value taking into account inflation. If inflation goes from zero to a positive value then money is nominally less valuable. The real value of money at time t is $\frac{1}{P_t}$. If there is inflation the level of prices increases.

2 Intertemporal choices

This exercise is for you to get acquainted with the basic intertemporal consumption model. We have an individual living two periods $t=1$ and $t=2$, he receives an amount of the consumption good ω_t in each period t, and can save an amount of good s of the consumption good in period 1. The real interest rate received on savings in period 2 is denoted r. The utility of the individual over his or her consumption c_1 and c_2 at both periods is defined as follows:

$$U(c_1,c_2)=u(c_1)+\beta u(c_2)=\frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}+\beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}$$

with discount rate $0\leq\beta$ and intertemporal elasticity of consu $\sigma\in(0,1)\cup(\infty)$.

a. Write the budget constraint of the individual in both periods.

Intuitively, in time $t=1$ you can not consume more than your income ω_1, while in time $t=2$ you can not consume more than ω_2 plus the amount of savings from period 1. At time 1 you have to decide how to divide your income in consumption and savings, we have:

$$c_1 + s \leq \omega_1$$

At time 2 the individual does not have the problem of saving, he can just consume. What he will get is $\omega_2 + s(1+r)$, therefore we have:

$$c_2 \leq \omega_2 + (1+r)s$$

b. Write the intertemporal budget constraint of the individual.

The intertemporal budget constraint is the constraint that a forward looking (who considers the future) individual faces when he chooses how to consume and save at time $t = 1$. The first step is to recognise that the per period constraints hold with equality. Why? Because there is no reason individuals should waste resources by not consuming or savings (technically, this is because preferences satisfy "*local non-satiation*"). Hence, we have that:

$$c_1 + s = \omega_1$$

$$c_2 = \omega_2 + (1+r)s$$

The two contraints are linked by s. From period 1 we get $s = \omega_1 - c_1$, by substituting s into the budget constraint for period 2 we get:

$$\begin{aligned} c_2 &= \omega_2 + (\omega_1 - c_1)(1+r) \\ \frac{c_2}{1+r} &= \frac{\omega_2}{1+r} + \omega_1 - c_1 \\ c_1 + \frac{c_2}{1+r} &= \underbrace{\omega_1 + \frac{\omega_2}{1+r}}_{\text{Net present value of resources}} \end{aligned}$$

which is in fact the intertemporal budget constraint.

c. Solve the intertemporal consumption problem, assuming an interior solution with $c_1 > 0$, $c_2 > 0$. Write the Euler equation,

that describes the evolution of consumption through time as a function of β and r at optimality. When does consumption rise through time? When does it fall through time? Interpret the role of r and β. What happens when $\sigma \to 0$? What happens when $\sigma \to \infty$?

Let's start from the first part of the question. The intertemporal consumption problem constitutes, from a technical point of view, a constrained maximisation. In fact, the agents would like to solve the following program:

$$\underset{c_1,c_2}{max}\ U(c_1,c_2) = \underset{c_1,c_2}{max}\ \frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} + \beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}$$

The solution to this problem alone would be to consume an infinite amount of c_1 and c_2, of course. However, agents can not do that, as they are constrained by their resources. These are given by the equation we computed before, namely the net present value of income. The problem therefore becomes to maximise the quantity above subject to the constraint. In particular, since the agent needs to choose c_1 and c_2 given β, r, σ, ω_1 and ω_2 the problem he is solving allows him to distribute consumption in time to maximise his utility.

$$\underset{c_1,c_2}{max}\ \frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} + \beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}$$

$$\text{subject to } \ c_1 + \frac{c_2}{1+r} = \omega_1 + \frac{\omega_2}{1+r}$$

Luckily enough there is a mathematical technique to solve constrained maximisation problems, the Lagrangian technique. It amounts to construct a new objective function from the old one and the constraint. First, let's express the constraint in a slightly different way, as we want everything to be on one side of the equal sign:

$$\text{subject to } \ c_1 + \frac{c_2}{1+r} - \omega_1 - \frac{\omega_2}{1+r} = 0$$

Now, to construct the Lagrangian we just have to place the multiplier λ in front of the left hand side of the constraint expressed in this way, with a minus sign in front, and add the objective function:

$$\max_{c_1,c_2,\lambda} \mathcal{L}(c_1,c_2,\lambda) = \max_{c_1,c_2,\lambda} \frac{c_1^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} + \beta\frac{c_2^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} - \lambda\left[c_1 + \frac{c_2}{1+r} - \omega_1 - \frac{\omega_2}{1+r}\right]$$

Solving this problem is equivalent to solving the previous problem. We take derivatives with respect to the three variables of interest:

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial c_1} = 0 \\ \frac{\partial\mathcal{L}}{\partial c_2} = 0 \\ \frac{\partial\mathcal{L}}{\partial\lambda} = 0 \end{cases} \Rightarrow \begin{cases} \cancel{\frac{\sigma-1}{\sigma}} \frac{c_1^{\frac{\sigma-1}{\sigma}-1}}{\cancel{\frac{\sigma-1}{\sigma}}} - \lambda = 0 \\ \beta\cancel{\frac{\sigma-1}{\sigma}} \frac{c_2^{\frac{\sigma-1}{\sigma}-1}}{\cancel{\frac{\sigma-1}{\sigma}}} + \frac{\lambda}{1+r} = 0 \\ c_1 + \frac{c_2}{1+r} - \omega_1 - \frac{\omega_2}{1+r} = 0 \end{cases} \Rightarrow \begin{cases} c_1^{\frac{\sigma-1}{\sigma}-1} = \lambda & (1) \\ \beta c_2^{\frac{\sigma-1}{\sigma}-1} = \frac{\lambda}{1+r} & (2) \\ c_1 + \frac{c_2}{1+r} = \omega_1 - \frac{\omega_2}{1+r} & (3) \end{cases}$$

As you can see, condition (3) is the constraint itself, this will happen with every Lagrangian you solve. The question asks to find the Euler equation. It is enough to divide condition (2) by condition (1) and rearrange terms. First, notice that $\frac{\sigma-1}{\sigma} - 1 = \frac{\sigma-1-\sigma}{\sigma} = -\frac{1}{\sigma}$ We have the following:

$$\begin{aligned} \frac{\beta c_2^{\frac{\sigma-1}{\sigma}-1}}{c_1^{\frac{\sigma-1}{\sigma}-1}} = \frac{\frac{\cancel{\lambda}}{1+r}}{\cancel{\lambda}} = \beta\left(\frac{c_2}{c_1}\right)^{\frac{\sigma-1}{\sigma}-1} &= \frac{1}{1+r} \\ \left(\frac{c_2}{c_1}\right)^{-\frac{1}{\sigma}} &= \frac{1}{\beta(1+r)} \\ \left(\frac{c_2}{c_1}\right)^{-\frac{1\cdot(-\cancel{\sigma})}{\cancel{\sigma}}} &= \left[\frac{1}{\beta(1+r)}\right]^{-\sigma} \\ \frac{c_2}{c_1} &= [\beta(1+r)]^{\sigma} \end{aligned}$$

The ratio of c_2 over c_1 indicates the evolution of consumption trough time, the higher the ratio the higher future consumption will be relative to present consumption. In particular, we can check if consumption rises trough time be looking at the value of the ratio. In fact, if $c_2 > c_1$ then $\frac{c_2}{c_1} > 1 \Rightarrow [\beta(1+r)]^{\sigma} > 1$. This will happen more likely if the discounting factor β is high (the agent care more about the future), if the interest rate r is high (savings pay more) and σ is high (elasticity between consumption today and tomorrow). Of course, if $\frac{c_2}{c_1} < 1 \Rightarrow [\beta(1+r)]^{\sigma} < 1$, then consumption decreases over time.

To understand what happens for limit values of σ we can look at the marginal rate of substitution between consumption today and tomorrow:

$$\frac{\partial U(c_1, c_2)/\partial c_1}{\partial U(c_1, c_2)/\partial c_2} = \left(\frac{c_2}{c_1}\right)^{-\frac{1}{\sigma}} = \left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}}$$

we have that

$$\lim_{\sigma \to 0} \left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}} = +\infty$$

You may recall that this is the marginal rate of substitution of the Leontief utility function (perfect complements), therefore in this limit case we have that $U(c_1, c_2) = min\{c_1, c_2\}$. There is zero elasticity between consumption today and tomorrow. On the contrary, in the opposite limit case:

$$\lim_{\sigma \to \infty} \left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}} = 1$$

Which is the marginal rate of substitution of a linear utility function (perfect substitute). We have that $U(c_1, c_2) = c_1 + c_2$. The agent has very high elasticity between consumption today and tomorrow, he does not care how he distributes consumption over time.

d. Rewrite the Euler equation when $\sigma = 1$. What is the value of consumption when $\beta = 1$ and $r = 0$? Under the same parameter values, does the individual save ($s > 0$) or borrow ($s < 0$) in the first period? Comment.

When $\sigma = 1$ the Euler equation becomes

$$\frac{c_2}{c_1} = [\beta(1+r)]$$

We have to evaluate it in $\beta = 1$ and $r = 0$. This is a case in which the individual cares about future consumption tomorrow as much as he cares about consumption today ($\beta = 1$), and savings have no returns ($r = 0$). Can you guess what the consumption pattern will be?

$$\frac{c_2}{c_1} = 1 \Rightarrow c_2 = c_1$$

The agent wants to smooth consumption as more as he can (see question **c.** in the Review!). We can now evaluate the full profile of consumption (as a function of exogenous variables). From condition (3) in te Lagrangian (the budget constraint):

$$\begin{aligned} c_1 + \frac{c_2}{1+r} &= \omega_1 - \frac{\omega_2}{1+r} \\ c_1 + c_1 &= \omega_1 + \omega_2 \\ 2c_1 &= \omega_1 + \omega_2 \\ c_1 = c_2 &= \frac{\omega_1 + \omega_2}{2} \end{aligned}$$

Hence, when does the individual saves? If he has $\omega_1 > \omega_2$ he will save to equalise consumption in the two time periods. The opposite holds when $\omega_1 < \omega_2$, he is forced to borrow. We can see it from the expression for the savings:

$$\begin{aligned} s = \omega_1 - c_1 = \omega_1 - \frac{\omega_1 + \omega_2}{2} \\ = \frac{2\omega_1 - \omega_1 - \omega_2}{2} \\ = \frac{\omega_1 - \omega_2}{2} \end{aligned}$$

When is $s > 0$? Indeed, when $\omega_1 > \omega_2$.

3 A market for time preferences

In this problem, we explore how markets allow people with different attitudes towards intertemporal consumption to use financial markets to their advantage.

Suppose that an economy is made up of two types of households. There are $N_E = 1$ (E stands for elastic) households of the first type. These households have infinitely elastic preferences:

$$U(c_{1E}, c_{2E}) = c_{1E} + \beta c_{2E}$$

There are $N_I = 1$ (I is inelastic) households of the second type. These households have the following utility function (with $0 < \alpha < 1$):

$$\hat{U}(c_{1I}, c_{2I}) = \frac{c_{1I}^{1-\alpha}}{1-\alpha} + \beta \frac{c_{2I}^{1-\alpha}}{1-\alpha}$$

Both types of households have the same discount factor β. Finally, assume that elastic households have real incomes y_1 in period 1 and y_2 in period 2, while the inelastic households only earn a revenue y_I in the first period.

a. Write the intertemporal budget constraint for the type-I consumer.

We perform the same step as before, except for the fact that individual I has no income in period 2, he can consume only savings. We get the two periods budget constraints:

$$c_{1I} + s_I \leq y_I$$

$$c_{2I} \leq (1+r)s_I$$

As usual, we recognise that these two will hold with equality, as the individual does not want to waste resources. By explicating s in the first constraint and substituting it in the second we get the intertemporal budget constraint.

$$s_I = y_I - c_{1I}$$

$$c_{2I} = (1+r)(y_I - c_{1I})$$
$$\frac{c_{2I}}{1+r} = y_I - c_{1I}$$
$$c_{1I} + \frac{c_{2I}}{1+r} = y_I$$

As you can see there is no present value of income in period 2.

b. State the inelastic household's optimization problem and calculate the optimal choices c_{1I}^*, c_{2I}^* and s_I^*, individual I's savings, in terms of the model parameters and r.

The problem is very similar to the one we solved in the previous exercise. In particular, the methodology is identical and relies on the maximisation of the Lagrangian. Also the computational steps are really close with the ones of the exercise above.

$$\max_{c_{1I},c_{2I},\lambda} \ \mathcal{L}(c_{1I},c_{2I},\lambda) = \max_{c_{1I},c_{2I},\lambda} \ \frac{c_{1I}^{1-\alpha}}{1-\alpha} + \beta\frac{c_{2I}^{1-\alpha}}{1-\alpha} - \lambda\left[c_{1I} + \frac{c_{2I}}{1+r} - y_I\right]$$

We take derivatives:

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial c_{1I}} = 0 \\ \frac{\partial\mathcal{L}}{\partial c_{2I}} = 0 \\ \frac{\partial\mathcal{L}}{\partial\lambda} = 0 \end{cases} \Rightarrow \begin{cases} \cancel{1-\alpha}\frac{c_{1I}^{1-\alpha-1}}{\cancel{1-\alpha}} - \lambda = 0 \\ \beta\frac{c_{2I}^{1-\alpha-1}}{\cancel{1-\alpha}} - \frac{\lambda}{1+r} = 0 \\ c_{1I} + \frac{c_{2I}}{1+r} - y_I = 0 \end{cases} \Rightarrow \begin{cases} c_{1I}^{-\alpha} = \lambda & (1) \\ \beta c_{2I}^{-\alpha} = \frac{\lambda}{1+r} & (2) \\ c_{1I} + \frac{c_{2I}}{1+r} = y_I & (3) \end{cases}$$

As before, condition (3) is the budget constraint. From condition (1) and (2) we can compute the Euler equation:

$$\begin{aligned} \frac{\beta c_{2I}^{-\alpha}}{c_{1I}^{-\alpha}} = \frac{\frac{\cancel{\lambda}}{1+r}}{\cancel{\lambda}} = \beta\left(\frac{c_2}{c_1}\right)^{-\alpha} &= \frac{1}{1+r} \\ \left(\frac{c_2}{c_1}\right)^{\cancel{-\alpha}\cdot\frac{1}{\cancel{-\alpha}}} &= \left[\frac{1}{\beta(1+r)}\right]^{\frac{1}{-\alpha}} \\ \frac{c_2}{c_1} &= [\beta(1+r)]^{\frac{1}{\alpha}} \\ c_{2I} &= [\beta(1+r)]^{\frac{1}{\alpha}}\, c_{1I} \end{aligned}$$

From the Euler equation we can always express c_2 as a function of c_1 (or the contrary), so that thanks to condition (3) (the budget constraint) we can always compute the optimal value of the two.

$$c_{1I} + \frac{[\beta(1+r)]^{\frac{1}{\alpha}} c_{1I}}{1+r} = y_I$$

$$c_{1I}\left[1 + \frac{[\beta(1+r)]^{\frac{1}{\alpha}}}{1+r}\right] = y_I$$

$$c_{1I}\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right] = y_I$$

$$c_{1I} = \frac{y_I}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

From the Euler equation we can get the expression for c_{2I}:

$$c_{2I} = \frac{y_I\,[\beta(1+r)]^{\frac{1}{\alpha}}}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

Finally, we get expression of optimal savings from $s_I^* = y_I - c_{1I}$:

$$s_I^* = y_I - \frac{y_I}{\left[1 + \beta^{\frac{1}{\alpha}}(1+r)^{\frac{1-\alpha}{\alpha}}\right]}$$

c. Write the intertemporal budget constraint for the type-E consumer.

The steps are the same as usual. I report without commenting, you can see the previous points.

$$c_{1E} + s_E \leq y_1$$

$$c_{2E} \leq y_2 + (1+r)s$$

$$s_E = y_1 - c_{1E}$$

$$c_{2E} = y_2 + (1+r)(y_1 - c_{1E})$$

$$c_{1E} + \frac{c_{2E}}{1+r} = y_1 + \frac{y_2}{1+r}$$

Here, we have the income in period 2, y_2.