TD 5

1 Review Questions

c. An improvement in extractive technology always increases fish production if fishing is socially optimal.

▼ *Answer*

True: From you notes (page 9) you can see that the expression for the total harvest in the social planner solution is $H_O^* = \frac{rK}{4}\left(1-\left(\frac{c}{p\alpha K}\right)^2\right)$. You can see from the expression that an increase in α will increase total harvest even without taking derivatives.

d. An improvement in extractive technology is always a bad thing from an environmental point of view.

▼ *Answer*

True: We can see it from the equilibrium expression of the stock of natural resources. We have that $S^* = \frac{c}{p\alpha}$. If α increase then the stock of natural resources decreases.

3 The Dynamics of a fish population with threshold

f. If there are B boats catching fishes, their total catches are $H(t) = \alpha BS(t)$. The net growth rate (the law of motion) of the stock is $\dot{S}(t) = N(t) - H(t)$. With B boats in the ocean, what is (are) the steady-state population(s) of fish?

First, notice that $\dot{S}_t$ is just notation for $\frac{\partial S(t)}{\partial t}$ which is the derivative of the stock of fish with respect to time. It is the equivalent of the law of motion of the Solow - Swan growth model, so you should treat it exactly as we did with that model. This observation helps us answering this question. In fact, the steady state population of fish is characterised by setting its growth rate equal to 0, which is the same as saying that $N(t) = H(t) \Leftrightarrow N(t) = \alpha BS(t)$.

$$\begin{aligned}
\dot{S}_t = 0 \Leftrightarrow & \, r(S(t) - T)\left(1 - \frac{S(t)}{K}\right) - \alpha BS(t) = 0 \\
& \Leftrightarrow rS(t) - 2T - \frac{rS(t)^2}{K} - \frac{rTS(t)}{K}\alpha BS(t) = 0 \\
& \Leftrightarrow -\frac{rS(t)^2}{K} + S(t)\left(r + \frac{rT}{K} - \alpha B\right) - rT = 0 \\
& \Leftrightarrow S(t)^2\frac{r}{K} - S(t)\left(r + \frac{rT}{K} - \alpha B\right) + rT = 0 \\
& \Leftrightarrow S(t)^2 - S(t)\left(K + T - \frac{\alpha BK}{r}\right) + TK = 0 \\
& \Leftrightarrow S(t)^2 + S(t)\left(\frac{\alpha BK}{r} - K - T\right) + TK = 0
\end{aligned}$$

We have a second order degree equation of which we have to find the roots by the usual formula. We have to solutions that we label S_U and S_S (you will soon see why).

$$S_U = \frac{K + T - \frac{\alpha BK}{r} - \sqrt{\left(\frac{\alpha BK}{r} - K -\right)^2 - 4TK}}{2}$$

$$S_S = \frac{K + T - \frac{\alpha BK}{r} + \sqrt{\left(\frac{\alpha BK}{r} - K - T\right)^2 - 4TK}}{2}$$

g. Graph the dynamics of the stock with resource extraction and identify the equilibrium population(s) of fish. Show with arrows how population dynamics pushes S to increase or decrease.

Here I put the picture where I added the solutions we computed in the previous point. Notice that here the growth rate is given by the difference $N(t) - H(t)$. Hence, to check the stability of steady states you have to see which one is above the other. Consider S_S in the picture, as an example. If you perturb it towards the right (i.e. $S_S + \epsilon$), you see that $H(t)$ is greater than $N(t)$, hence $\dot{S}(t)$ is negative and we get back to S_S. By performing the same reasoning for any perturbation you can easily see that S_U is unstable while S_S is stable.

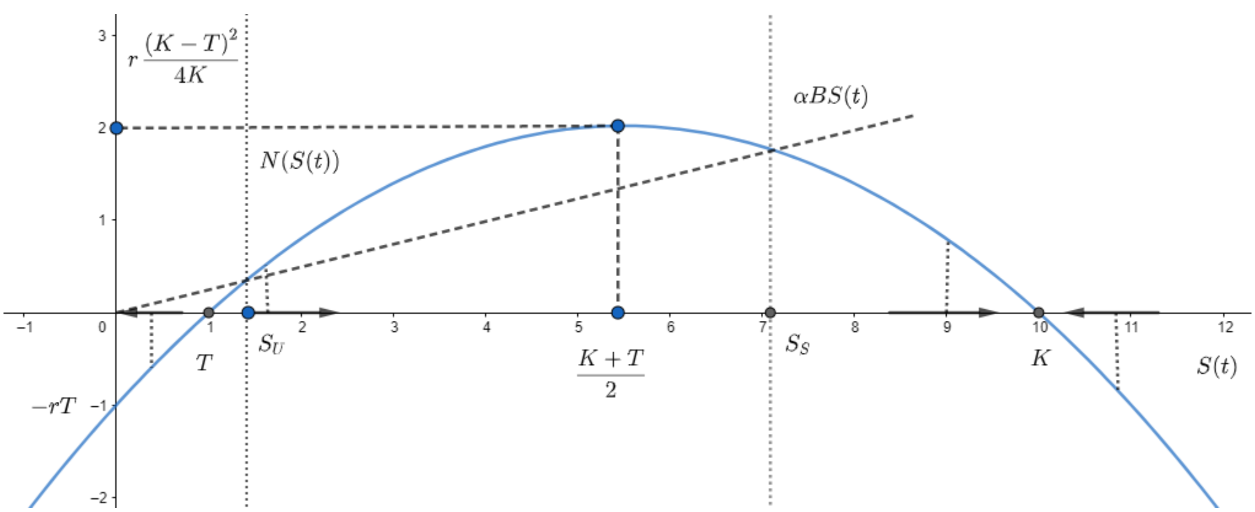

Figure 1: Same graph as before with S_U and S_S. Here I picked $\alpha = \frac{1}{8}$ and $B = 4$.

h. Is there an intensity of fishing (αB) so high that no sustainable fishing is possible? What is it?

To answer this question it is enough to notice that if you increase αB by a lot then the line $\alpha BS(t)$ will not cross $N(S(t))$ anymore, which means in fact that no sustainable fishing is possible. This will happen when there is no solution to the previous second degree equation, that is when the quantity below the square root is negative (imaginary solution). Therefore we just have to check when this condition is satisfied.

$$\left(\frac{\alpha BK}{r} - K - T\right)^2 - 4TK < 0$$

$$\pm\left(\frac{\alpha BK}{r} - K - T\right) < \sqrt{4TK}$$

$$\frac{\alpha BK}{r} > K + T - 2\sqrt{KT} \quad \text{In the minus(-) case}$$

$$\alpha B > \frac{r}{K}\left(T + K - 2\sqrt{KT}\right)$$

$$\alpha B > \frac{r}{K}\left(\sqrt{T} - \sqrt{K}\right)^2$$

$$\alpha B > r\left(\frac{\sqrt{K}}{\sqrt{K}} - \frac{\sqrt{T}}{\sqrt{K}}\right)^2$$

$$\alpha B > r\left(1 - \sqrt{\frac{T}{K}}\right)^2$$

If you perform the same operation in the plus(+) case you will find that $\alpha B < r\left(1 + \sqrt{\frac{T}{K}}\right)^2$, hence we obtain that no sustainable fishing is possible when $r\left(1 - \sqrt{\frac{T}{K}}\right)^2 < \alpha B < r\left(1 + \sqrt{\frac{T}{K}}\right)^2$. Just as a remark, notice that in the numerical example from which I plotted the graph indeed we have $\alpha B < \frac{K}{r}\left(1 - \sqrt{\frac{T}{K}}\right)^2$ and therefore we have the two solutions.

i. The profit from a boat is $\pi(t) = p\alpha S(t) - c$. If there is free entry, fishing boats will enter as long as profits are positive. What is the free market equilibrium value of the stock S_F^* in the steady state.

If boats will continue to enter as long as profits are positive, then they will stop when profits are 0. Therefore, as in class, to find the free market equilibrium value of $S(t)$ we just need to check when this condition is satisfied.

$$\pi(t) = 0 \Leftrightarrow p\alpha S_F^* - c = 0 \Leftrightarrow S_F^* = \frac{c}{p\alpha}$$

However, notice that in class we had $T = 0$, and since $\frac{c}{p\alpha}$ is always weakly greater than 0 we never had any problem. In this case, if $\frac{c}{p\alpha} < T$ the growth is negative and the stock goes to 0.

4 Taxation to obtain optimum resource extraction

This exercise make you compute Pigouvian taxes. These kind of taxes are classical in the economics literature. Their aim is to correct for externalities that affect the market outcome without passing trough the channel of prices. In fact, in the fishing model an increase of boat affect the growth of natural resources in a way that is not transmitted to the market with the price p. Recall that the free market optimal number of boats was $B_F^* = \frac{r}{\alpha}\left(1 - \frac{c}{p\alpha K}\right)$, while from a social planner perspective we should have $B_O^* = \frac{B_F^*}{2}$.

a. Show that the optimal number of boats could be obtained by a tax per boat $t = \frac{p\alpha K - c}{2}$.

There are two ways to answer to this question. The first way, which is the one you will have in the professor's solution, is to ask "which is the value of t such that if boats pay the new cost $c' = c + t$ then $B_O^* = B_{F'}^*$?". This means solving the following equation:

$$B^*_{F'} = \frac{r}{\alpha}\left(1 - \frac{c+t}{p\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B^*_O$$

and realise that $t = \frac{p\alpha K - c}{2}$. Since you have this way already explained in your solution I will show you the second way. I think it is less smart but more algorithmic, in case you do not have the intuition to frame the problem in the way I just exposed.

The second way amounts to perform the same step you did in the class, but the profits are $\pi(t) = \alpha p S(t) - (c+t) = \alpha p S(t) - (c + \frac{p\alpha K - c}{2})$ and realise that the free market equilibrium boats are equal to the social optimum. In optimum we must always have that profits are equal to 0, therefore:

$$\begin{aligned}
\alpha p S^*(B^*) - \left(c + \frac{p\alpha K - c}{2}\right) &= 0 \\
\alpha p K\left(1 - \frac{\alpha B}{r}\right) - \left(c + \frac{p\alpha K - c}{2}\right) &= 0 \\
p\alpha K - \frac{p\alpha K \alpha B}{r} &= c + \frac{p\alpha K - c}{2} \\
1 - \frac{c}{p\alpha K} - \frac{1}{2} + \frac{c}{2p\alpha K} &= \frac{\alpha B}{r} \\
\frac{1}{2} - \frac{c}{2p\alpha K} &= \frac{\alpha B}{r} \\
\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} = B^*_{F'} &= B^*_O
\end{aligned}$$

Which is the result we wanted.

b. Illustrate this tax on the graph of the revenue of the fishing industry.

The change in marginal revenues due to the introduction of the tax change the point in which this line intersect the marginal cost c. You could also interpret it by saying that the new marginal cost is $c+t$ and the optimality condition requires the blue line to intersect with the new marginal cost.

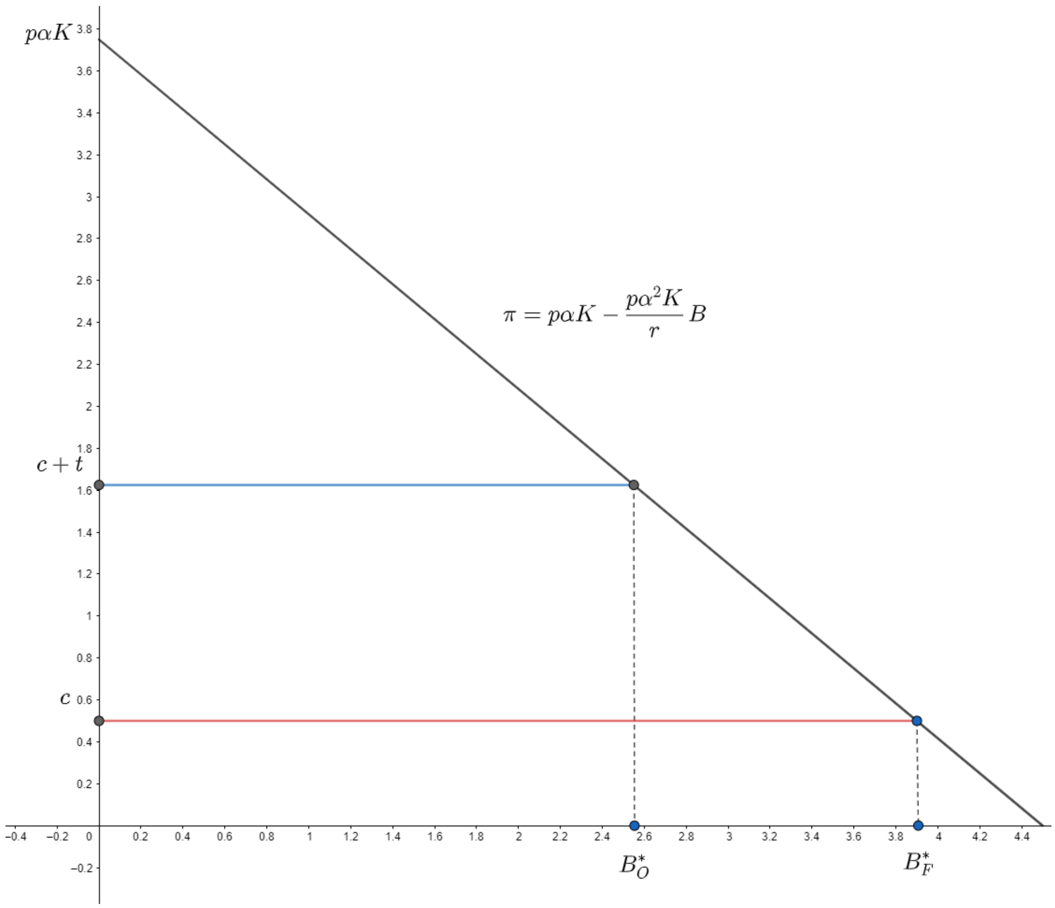

Figure 2: Profits and marginal costs for $p=3$, $c=0.5$ and, as before, $\alpha=\frac{1}{8}$, $K=10$, $r=1$.

c. Show that an ad valorem tax on fish sales of $\tau = \frac{p\alpha K - c}{p\alpha K + c}$ would achieve the optimum as well.

Exactly as before we can solve this problem in two ways. The first one is to compute the τ proportional tax on π that would make $B_{F'}^* = B_0^*$. This is equivalent to solving the following equation:

$$B_{F'}^* = \frac{r}{\alpha}\left(1 - \frac{c}{p(1-\tau)\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B_O^*$$

You will have the detail of this method in the professor's solution.

The second method follows the same state we did in the previous point. We just change the expression for profits and find the optimal number of boats in free markets $B^*_{F'}$. The new profits are $\pi(t) = p(1-\tau)\alpha S(t) - c = p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(t) - c$ and we must set them equal to 0.

$$\begin{aligned}
p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(B^*) - c &= 0 \\
p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha K\left(1 - \frac{\alpha B}{r}\right) - c &= 0 \\
p\left(\frac{2c}{p\alpha K + c}\right)\alpha K\left(1 - \frac{\alpha B}{r}\right) - c &= 0 \\
\frac{p\alpha K 2c}{p\alpha K + c} - \frac{p\alpha K 2c\alpha B}{r(p\alpha K + c)} - c &= 0 \\
\frac{p\alpha K 2c}{p\alpha K + c} - c &= \frac{p\alpha K 2c\alpha B}{r(p\alpha K + c)} \\
1 - \frac{\cancel{c}(p\alpha K + c)}{p\alpha K 2\cancel{c}} &= \frac{\alpha B}{r} \\
1 - \frac{1}{2} - \frac{c}{p\alpha K 2} &= \frac{\alpha B}{r} \\
\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} &= B^*_{F'} = B^*_O
\end{aligned}$$

Which is again the solution we wanted.