TD 5

1 Review Questions

c. An improvement in extractive technology always increases fish production if fishing is socially optimal.

▼ *Answer*

True: From you notes (page 9) you can see that the expression for the total harvest in the social planner solution is $H_O^* = \frac{rK}{4}\left(1 - \left(\frac{c}{p\alpha K}\right)^2\right)$. You can see from the expression that an increase in α will increase total harvest even without taking derivatives.

d. An improvement in extractive technology is always a bad thing from an environmental point of view.

▼ *Answer*

True: We can see it from the equilibrium expression of the stock of natural resources. We have that $S^* = \frac{c}{p\alpha}$. If α increase then the stock of natural resources decreases.

3 The Dynamics of a fish population with threshold

One of the problems that the fishing model has is that the only circumstance in which there is an extinction of fishes (or natural resources in general) is when the starting stock is equal to 0. Of course, this is counterfactual with reality as we go from a state in which there is a positive amount of resources to a state in which they are extinct. The aim of this exercise is to augment to model by assuming that when the stock of fishes goes below a threshold T then it is destined to converge to 0. I think it is an interesting exercise, it helps interpreting some real world facts.

a. Find the values of $S(t) > 0$ for which the fish stock does not grow naturally.

As always, we first need to understan what the question is asking. When does the fish stock grows? When its growth rate is different then 0. If the growth rate is equal to zero then the stock will not grow. The question is asking ask to determine for which values of $S(t)$ the growth rate is equal to 0. The growth rate is:

$$N(S(t)) = r(S(t) - T)\left(1 - \frac{S(t)}{K}\right)$$

Since $r > 0$ and the expression is a product, we have that $N(S(t)) = 0$ when one of the two element of the product is zero.

$$N(S(t)) = 0 \Leftrightarrow \begin{cases} S(t) - T = 0 \\ 1 - \frac{S(t)}{K} = 0 \end{cases} \Leftrightarrow \begin{cases} S(t) = T \\ S(t) = K \end{cases}$$

Hence, fishes will not grow when their stock is exactly equal to their maximum capacity K and when the stock is equal to the minimum treshold T. Notice that this is clear also from the plot of the growth rate.

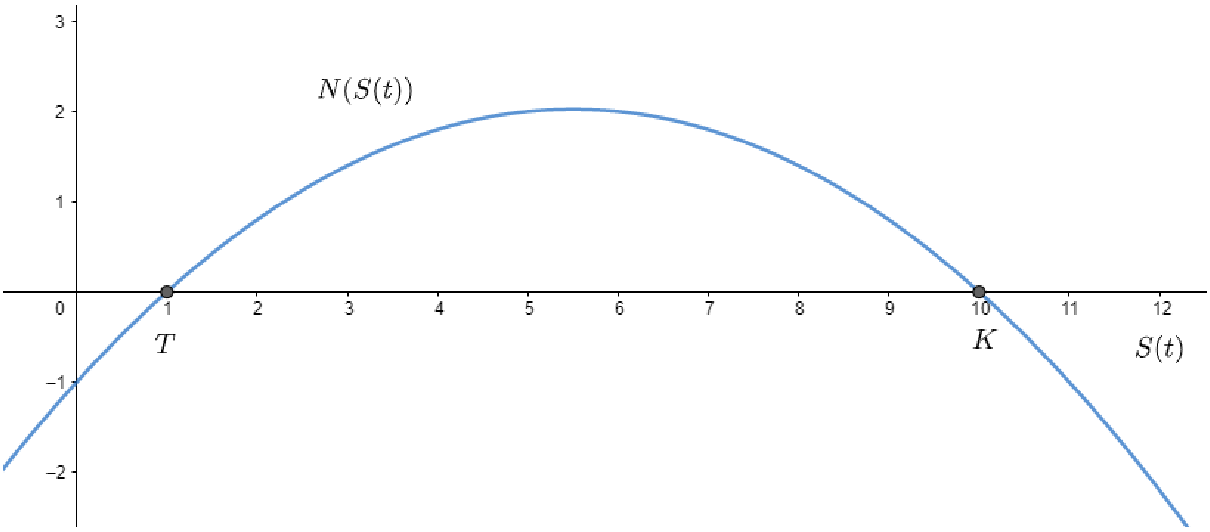

Figure 1: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$.

b. Of these values, which are stable, which are not?

First of all, what does stable mean in this context? When speaking about steady states (growth rates equal to 0), we say that a steady state is stable if a small perturbation of the system from a steady steady returns to the previous point autonomously. In this case, a steady state is stable if by slightly increasing or decreasing the stock of fishes from $S(t) = T$ or $S(t) = K$ we then return to the previous steady state or the system evolves in a different direction.

Checking for stability seems a daunting task, but if you have the graph it becomes easier. Here I will show you to check for stability with both a graphical and a mathematical technique. Let's start from the graphical one. Consider the steady state $S(t) = T$, what happens if we move slightly on the right (e.g. $S(t) = T + \epsilon$) ? You see that $N(S(T) + \epsilon)$ is positive. Hence, the stock will continue to grow and will become significantly different with respect to $S(T)$. In the same way, if we perturb the stock in the other direction ($S(T) - \epsilon$) we can see that $N(S(T) - \epsilon)$ is negative, the stock will become lower and lower. This steady state is not stable. If we perturb it slightly it will not return to its original point. Now consider the second steady state, when $S(t) = K$. If we perturb it by moving slightly on the right (

$S(K)+\epsilon)$ we see that $N(S(K)+\epsilon))$ is negative, therefore the stock of fish will decrease until it returns to its stable value $S(t)=K$. The same happens when you perturb the stock in the other direction, we have that $N(S(K)-\epsilon))$ is positive, which will make the stock increase until it is stable in $S(t)=K$. We concluded that $S(t)=T$ is not stable while $S(t)=K$ is stable. The graph below captures this reasoning pattern.

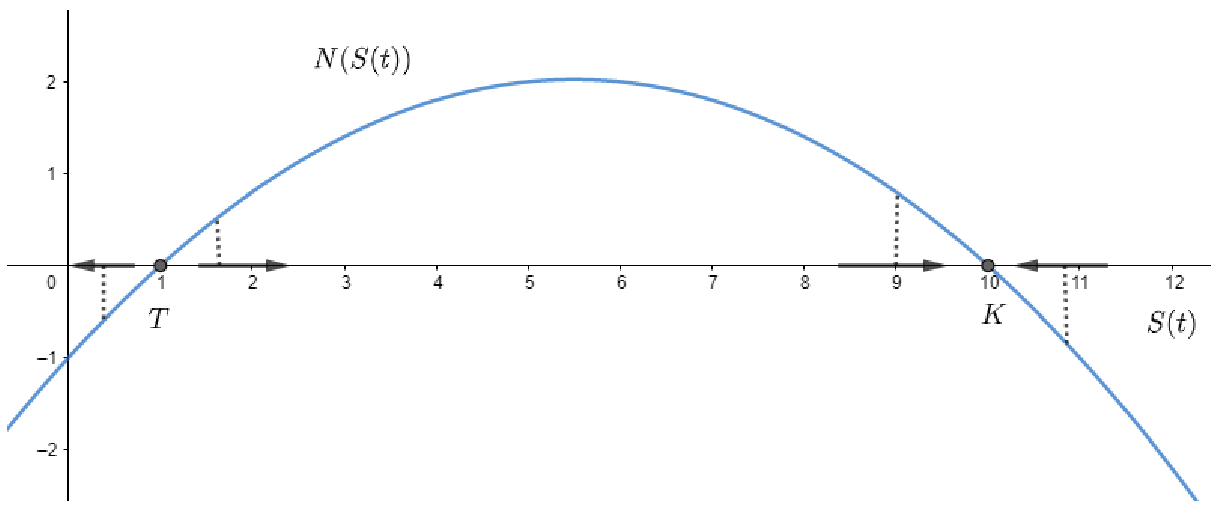

Figure 2: Stability of steady states.

If you don't like this graphical reasoning, there is also the math way. This perspective amounts to taking the derivative of $N(S(t))$ with respect to $S(t)$, which measures the change in growth by a change in stock of fish. By evaluating the derivative in the two steady states we can check its sign. If the sign of the derivative is positive, this means that a positive variation will be magnified even more, and therefore the steady state is not stable. If the sign is negative, it means that after a positive variation the stock will decrease and return to its original value. This would mean that the steady state is stable. Let's start taking the derivative. It might look a little bit scary to the the derivative with respect to $S(t)$, but you just have to consider the entire expression as a single variable and derive it with the rules you know (you can look at TD1 for an explanation on how to derive products).

$$
\begin{aligned}
\frac{\partial N(S(t))}{\partial S(t)} &= r\left[\left(1-\frac{S(t)}{K}\right)+(S(t)-T)\left(-\frac{1}{K}\right)\right] \\
&= r\left[1-\frac{S(t)}{K}-\frac{S(t)}{K}+\frac{T}{K}\right] \\
N'(S(t)) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right]
\end{aligned}
$$

We can now evaluate the derivative in the two points of interest. Recall that $r > 0$.

$$
\begin{aligned}
N'(T) &= r\left[1-\frac{2S(t)}{K}+\frac{T}{K}\right] \\
&= r\left[1-\frac{2T}{K}+\frac{T}{K}\right] \\
&= r\underbrace{\left[1-\underbrace{\frac{T}{K}}_{<1}\right]}_{>0} > 0
\end{aligned}
$$

This result confirms our graphical analysis. Since the derivative at T is greater than 0, this means that a positive perturbation of $S(t)$ at T will increase the stock even more, and therefore will push the system far from the original state. On the contrary, for K we have:

$$
\begin{aligned}
N'(K) &= r\left[1-\frac{2K}{K}+\frac{T}{K}\right] \\
&= r\left[1-2+\frac{T}{K}\right] \\
&= r\underbrace{\left[\underbrace{\frac{T}{K}}_{<1}-1\right]}_{<0} < 0
\end{aligned}
$$

Which again goes in the same direction as the graphical intuition. If we positively perturb the steady state at K, the stock of fish will decrease until we reach the previous state again.

c. What is the natural growth of the fish population at t if $S(t) = 0$? Is it also an equilibrium?

To answer this question we just need to evaluate the growth rate in the point $S(t) = 0$.

$$N(0) = r(0 - T)(1 - 0) = -rT$$

We should have expected this result, as we know that T is a threshold for the fish to grow and $S(t) = 0 < T$. Since the computed growth rate is negative and since the stock can not go lower than 0, we conclude that $S(t) = 0$ is also a steady state. Notice that $S(t)$ is the point in which the growth rate crosses the y axis, as shown in the picture below.

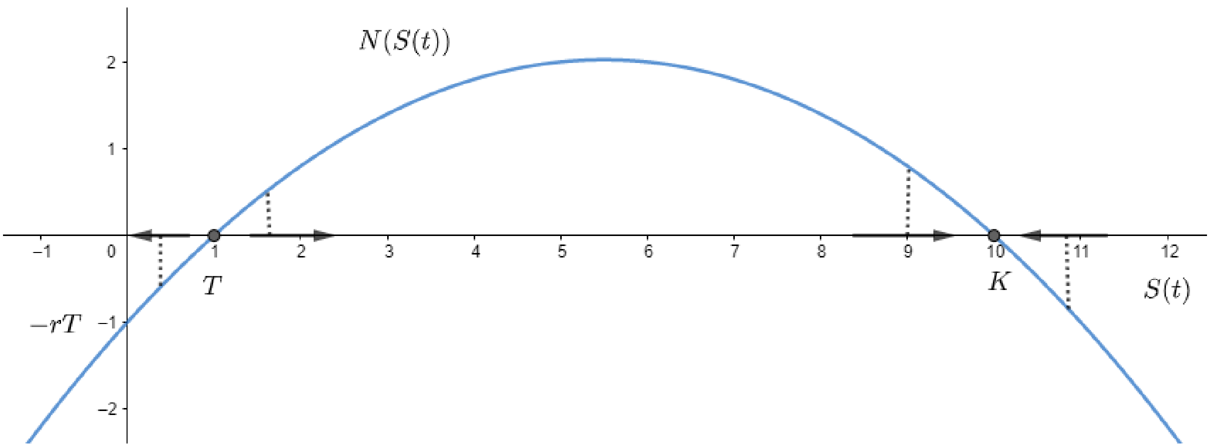

Figure 3: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Notice that $-rT = -1(1) = -1$, where the growth rate is negative and the stock is 0.

d. What is the maximum number of fish that can be caught per unit of time such that the fish population is constant? This

is also called the maximum sustained yield. What is the fish stock $S(t)$ at this value?

To answer this question we must ask when the growth rate of fishes is the highest. This would allow us to capture the maximum number of fishes every time t and then obtain for $t + \epsilon$ the greatest amount of growth so that we can always maximise our catches. Hence, we must maximise the growth rate of fishes with respect to the stock. We already have the derivative. To check for the maximum we need to find the value of $S(t)$ for which the derivative is equal to 0.

$$\begin{aligned}\frac{\partial N(S(t))}{\partial S(t)} = 0 &\Leftrightarrow r\left[\left(1 - \frac{S(t)}{K}\right) + (S(t) - T)\left(-\frac{1}{K}\right)\right] = 0 \\ &\Leftrightarrow \cancel{r}\left[1 - \frac{2S(t)}{K} + \frac{T}{K}\right] = 0 \\ &\Leftrightarrow K - 2S(t) + T = 0 \\ &\Leftrightarrow S(t) = \frac{K+T}{2}\end{aligned}$$

Now that we have the stock of fishes that maximises growth we can ask by how much fishes grow for this value of the stock. Of course, to answer this question we just need to plug the value we just found in the growth rate.

$$\begin{aligned}N\left(\frac{K+T}{2}\right) &= r\left(\frac{K+T}{2} - T\right)\left(1 - \frac{K+T}{2K}\right) \\ &= r\left(\frac{K+T-2T}{2}\right)\left(\frac{2K-K-T}{2K}\right) \\ &= r\left(\frac{K-T}{2}\right)\left(\frac{K-T}{2K}\right) \\ &= r\frac{(K-T)^2}{4K}\end{aligned}$$

This expression tells us by how much the fish grows at the optimal stock.

e. Graph the fish growth function $S(t)$. Place all the elements previously computed on the graph.

We already did a big part of the graph, the one below has also the answers to the last question.

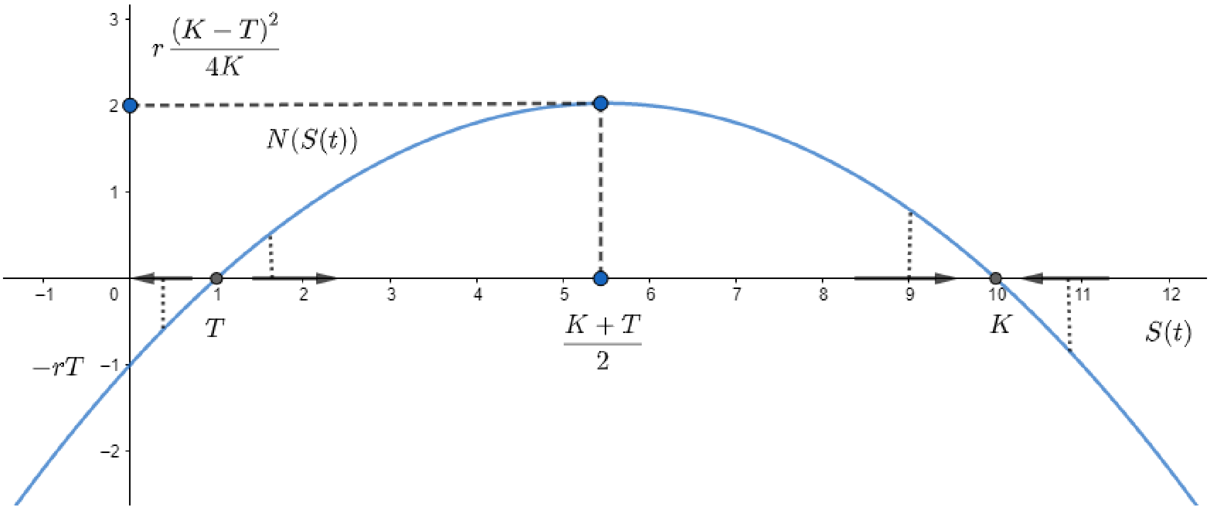

Figure 4: Graph of the growth rate of fishes for $T = 1$, $K = 10$ and $r = 1$. Here $S^*(t) = \frac{10+1}{2} = 5.5$ and $N(S^*(t)) = \frac{9^2}{40} \approx 2$.

f. If there are B boats catching fishes, their total catches are $H(t) = \alpha BS(t)$. The net growth rate (the law of motion) of the stock is $\dot{S}(t) = N(t) - H(t)$. With B boats in the ocean, what is (are) the steady-state population(s) of fish?

First, notice that $\dot{S}_t$ is just notation for $\frac{\partial S(t)}{\partial t}$ which is the derivative of the stock of fish with respect to time. It is the equivalent of the law of motion of the Solow - Swan growth model, so you should treat it exactly as we did with that model. This observation helps us answering this question. In fact, the steady state population of fish is characterised by setting its growth rate equal to 0, which is the same as saying that $N(t) = H(t) \Leftrightarrow N(t) = \alpha BS(t)$.

$$
\begin{aligned}
\dot{S}_t = 0 &\Leftrightarrow r(S(t)-T)\left(1-\frac{S(t)}{K}\right) - \alpha BS(t) = 0 \\
&\Leftrightarrow rS(t) - 2T - \frac{rS(t)^2}{K} - \frac{rTS(t)}{K}\alpha BS(t) = 0 \\
&\Leftrightarrow -\frac{rS(t)^2}{K} + S(t)\left(r + \frac{rT}{K} - \alpha B\right) - rT = 0 \\
&\Leftrightarrow S(t)^2\frac{r}{K} - S(t)\left(r + \frac{rT}{K} - \alpha B\right) + rT = 0 \\
&\Leftrightarrow S(t)^2 - S(t)\left(K + T - \frac{\alpha BK}{r}\right) + TK = 0
\end{aligned}
$$

We have a second order degree equation of which we have to find the roots by the usual formula. We have to solutions that we label S_U and S_S (you will soon see why).

$$
S_U = \frac{K + T - \frac{\alpha BK}{r} - \sqrt{\left(K + T - \frac{\alpha BK}{r}\right)^2 - 4TK}}{2}
$$

$$
S_S = \frac{K + T - \frac{\alpha BK}{r} + \sqrt{\left(K + T - \frac{\alpha BK}{r}\right)^2 - 4TK}}{2}
$$

g. Graph the dynamics of the stock with resource extraction and identify the equilibrium population(s) of fish. Show with arrows how population dynamics pushes S to increase or decrease.

Here I put the picture where I added the solutions we computed in the previous point. By performing the same reasoning as before, you can easily see that S_U is unstable while S_S is stable.

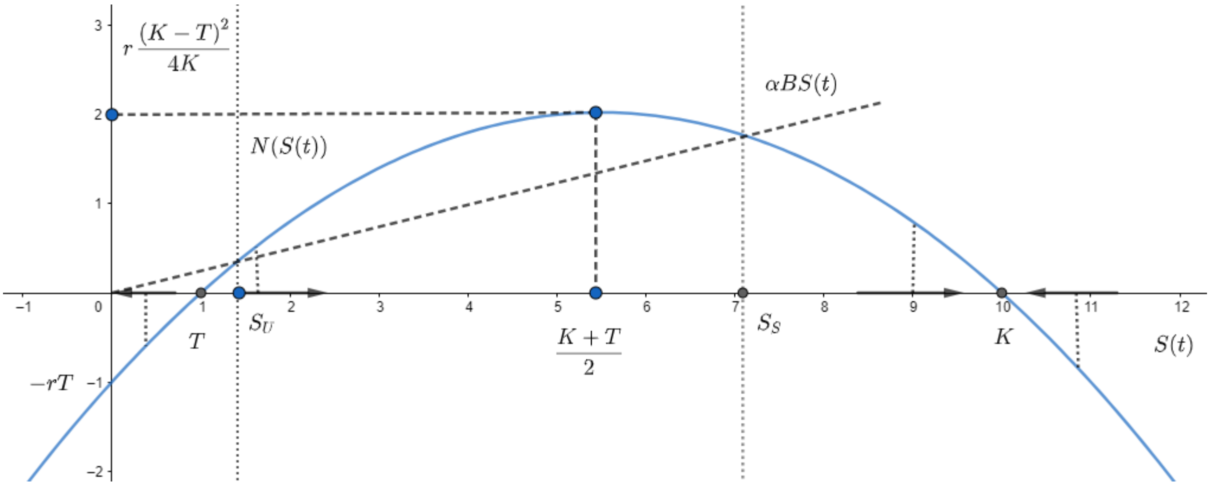

Figure 5: Same graph as before with S_U and S_S. Here I picked $\alpha = \frac{1}{8}$ and $B = 4$.

h. Is there an intensity of fishing (αB) so high that no sustainable fishing is possible? What is it?

To answer this question it is enough to notice that if you increase αB by a lot then the line $\alpha BS(t)$ will not cross $N(S(t))$ anymore, which means in fact that no sustainable fishing is possible. This will happen when there is no solution to the previous second degree equation, that is when the quantity below the square root is negative (imaginary solution). Therefore we just have to check when this condition is satisfied.

$$\left(K + T - \frac{\alpha BK}{r}\right)^2 - 4TK < 0$$

$$\frac{\alpha BK}{r} > K + T - 2\sqrt{KT}$$

$$\alpha B > \frac{K}{r}\left(T + K - 2\sqrt{KT}\right)$$

$$\alpha B > \frac{K}{r}\left(\sqrt{T} - \sqrt{K}\right)^2$$

$$\alpha B > \frac{K}{r}\left(\sqrt{T} - \sqrt{K}\right)^2$$

$$\alpha B > \frac{K}{r}\left(1 - \sqrt{\frac{T}{K}}\right)^2$$

Just as a remark, notice that in the numerical example from which I plotted the graph indeed we have $\alpha B < \frac{K}{r}\left(1-\sqrt{\frac{T}{K}}\right)^2$ and therefore we have the two solutions.

i. The profit from a boat is $\pi(t) = p\alpha S(t) - c$. If there is free entry, fishing boats will enter as long as profits are positive. What is the free market equilibrium value of the stock S_F^* in the steady state.

If boats will continue to enter as long as profits are positive, then they will stop when profits are 0. Therefore, as in class, to find the free market equilibrium value of $S(t)$ we just need to check when this condition is satisfied.

$$\pi(t) = 0 \Leftrightarrow p\alpha S_F^* - c = 0 \Leftrightarrow S_F^* = \frac{c}{p\alpha}$$

However, notice that in class we had $T = 0$, and since $\frac{c}{p\alpha}$ is always weakly greater than 0 we never had any problem. In this case, if $\frac{c}{p\alpha} < T$ the growth is negative and the stock goes to 0.

4 Taxation to obtain optimum resource extraction

This exercise make you compute Pigouvian taxes. These kind of taxes are classical in the economics literature. Their aim is to correct for externalities that affect the market outcome without passing trough the channel of prices. In fact, in the fishing model an increase of boat affect the growth of natural resources in a way that is not transmitted to the market with the price p. Recall that the free market optimal number of boats was $B_F^* = \frac{r}{\alpha}\left(1-\frac{c}{p\alpha K}\right)$, while from a social planner perspective we should have $B_O^* = \frac{B_F^*}{2}$.

a. Show that the optimal number of boats could be obtained by a tax per boat $t = \frac{p\alpha K - c}{2}$.

There are two ways to answer to this question. The first way, which is the one you will have in the professor's solution, is to ask "which is the value of t such that if boats pay the new cost $c' = c + t$ then $B^*_O = B^*_{F'}$?". This means solving the following equation:

$$B^*_{F'} = \frac{r}{\alpha}\left(1 - \frac{c+t}{p\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B^*_O$$

and realise that $t = \frac{p\alpha K - c}{2}$. Since you have this way already explained in your solution I will show you the second way. I think it is less smart but more algorithmic, in case you do not have the intuition to frame the problem in the way I just exposed.

The second way amounts to perform the same step you did in the class, but the profits are $\pi(t) = \alpha p S(t) - (c + t) = \alpha p S(t) - (c + \frac{p\alpha K - c}{2})$ and realise that the free market equilibrium boats are equal to the social optimum. In optimum we must always have that profits are equal to 0, therefore:

$$\begin{aligned}
\alpha p S^*(B^*) - \left(c + \frac{p\alpha K - c}{2}\right) &= 0 \\
\alpha p K\left(1 - \frac{\alpha B}{r}\right) - \left(c + \frac{p\alpha K - c}{2}\right) &= 0 \\
p\alpha K - \frac{p\alpha K \alpha B}{r} &= c + \frac{p\alpha K - c}{2} \\
1 - \frac{c}{p\alpha K} - \frac{1}{2} + \frac{c}{2p\alpha K} &= \frac{\alpha B}{r} \\
\frac{1}{2} - \frac{c}{2p\alpha K} &= \frac{\alpha B}{r} \\
\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} &= B^*_{F'} = B^*_O
\end{aligned}$$

Which is the result we wanted.

b. Illustrate this tax on the graph of the revenue of the fishing industry.

The change in marginal revenues due to the introduction of the tax change the point in which this line intersect the marginal cost c. You could also interpret it by saying that the new marginal cost is $c + t$ and the optimality condition requires the blue line to intersect with the new marginal cost.

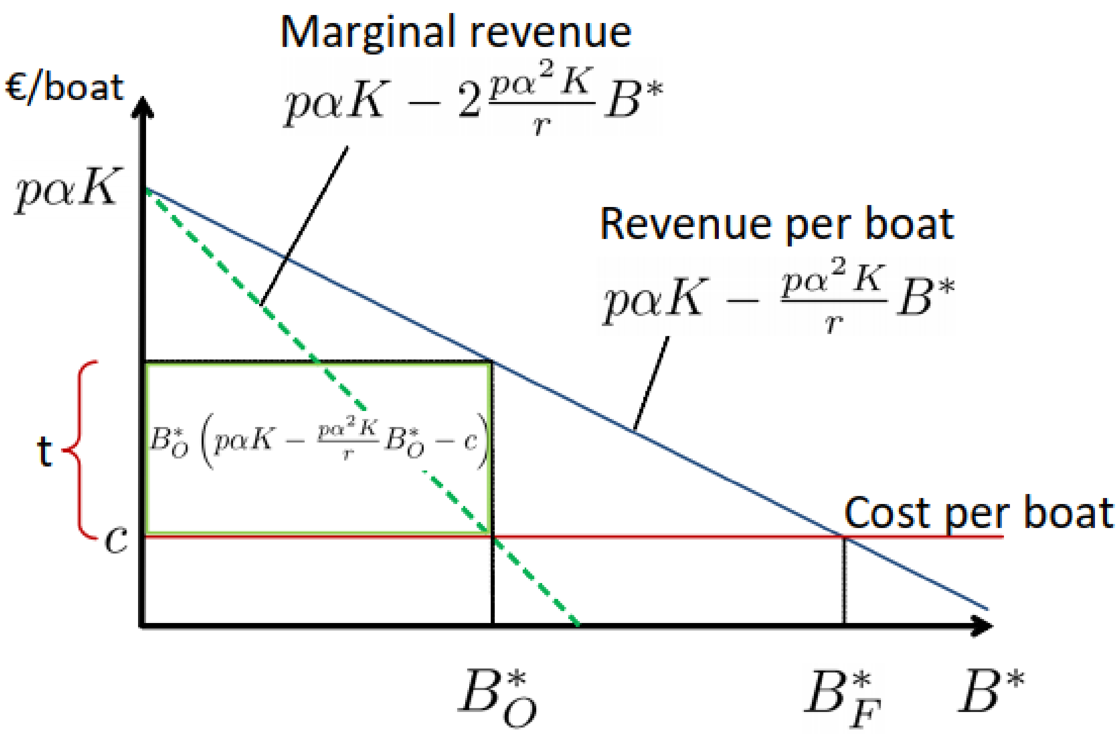

Figure 6: Representation of the fishing industry with a lump-sum tax.

c. Show that an ad valorem tax on fish sales of $\tau = \frac{p\alpha K - c}{p\alpha K + c}$ would achieve the optimum as well.

Exactly as before we can solve this problem in two ways. The first one is to compute the τ proportional tax on p that would make $B_{F'}^* = B_0^*$. This is equivalent to solving the following equation:

$$B_{F'}^* = \frac{r}{\alpha}\left(1 - \frac{c}{p(1-\tau)\alpha K}\right) = \frac{r}{2\alpha}\left(1 - \frac{c}{p\alpha K}\right) = B_O^*$$

You will have the detail of this method in the professor's solution.

The second method follows the same state we did in the previous point. We just change the expression for profits and find the optimal number of boats in free markets $B^*_{F'}$. The new profits are $\pi(t) = p(1-\tau)\alpha S(t) - c =$ $p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(t) - c$ and we must set them equal to 0.

$$\begin{aligned}
p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha S(B^*) - c = 0 \\
p\left(1 - \frac{p\alpha K - c}{p\alpha K + c}\right)\alpha K\left(1 - \frac{\alpha B}{r}\right) - c = 0 \\
p\left(\frac{2c}{p\alpha K + c}\right)\alpha K\left(1 - \frac{\alpha B}{r}\right) - c = 0 \\
\frac{p\alpha K 2c}{p\alpha K + c} - \frac{p\alpha K 2c\alpha B}{r(p\alpha K + c)} - c = 0 \\
\frac{p\alpha K 2c}{p\alpha K + c} - c = \frac{p\alpha K 2c\alpha B}{r(p\alpha K + c)} \\
1 - \frac{\cancel{c}(p\alpha K + c)}{p\alpha K 2\cancel{c}} = \frac{\alpha B}{r} \\
1 - \frac{1}{2} - \frac{c}{p\alpha K 2} = \frac{\alpha B}{r} \\
\frac{1}{2}\left(1 - \frac{c}{p\alpha K}\right)\frac{r}{\alpha} = B^*_{F'} = B^*_O
\end{aligned}$$

Which is again the solution we wanted.