Microeconomics 1 Lecture Notes

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Contents

Preamble

These are notes for the first part of the PhD microeconomics sequence. They cover choice under uncertainty and general equilibrium theory. The write-up is not yet fully polished, and I will keep updating it. If you spot mistakes or typos, please let me know. I have aimed for a conversational style rather than the more formal tone of, say, Mas-Colell et al. (1995). I assumed no knowledge of the topics, but, as usual, mathematical sophistication helps (I hope you develope it during the class!). Each lecture outlines what we discuss in class, followed by exercises and suggestions for further reading. Each time a result is proved, I attempted to give the easier proof possible. This means that explanations and proofs are a bit longer than necessary, but more accessible.

Before starting, you may enjoy some non-technical background that frames the topics we'll study: Kreps (1988, ch. 1), Debreu (1959, pp. ix–xi), Myerson (1997, pp. 1–7), and Gilboa (2009, chs. 1–2).

You will occasionally see smaller text like this. These remarks are not essential for following the main exposition, but they add context or point to related ideas. Feel free to skip them on a first pass.

These notes draw on several sources. The main reference is Mas-Colell et al. (1995), but both here and in the text you can find references for alternatives or complements. Below is a short reading list. If you would like more pointers on any topic, just ask—I am happy to discuss questions and related material, just drop me an email.

Have fun!

Choice under uncertainty.

- **Mas-Colell et al. (1995), ch. 6.**
- Kreps (1988), chs. 4-6.
- Fishburn (1970), ch. 8.
- Kreps (2013), chs. 5-6.
- Gilboa (2009).

General equilibrium theory.

- **Mas-Colell et al. (1995) chs. 15–16.**
- Thomson (2011), sec 4.3 (no-envy).
- Kreps (2013), chs. 14–15.
- Debreu (1959).
- McKenzie (2005).

References

Debreu, G. (1959). *Theory of value: An axiomatic analysis of economic equilibrium* (Vol. 17). Yale University Press. 1

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Lecture 1

Introduction to uncertainty

1.1 How to model uncertainty

The outcomes of our decisions are often uncertain, so we want a choice theory that takes uncertainty into account. Let's start by thinking about how to represent uncertainty. Suppose you make a bet with a friend: if a fair coin toss results in heads, you get 10 euros; otherwise you pay 10 euros to your friend. There are two outcomes, 10 and -10, and since the coin is fair, each occurs with probability $1/2$. What are the main ingredients of this example?

First, we started from a set of outcomes, in this case the monetary transfers 10 and -10. Second, we specified the probability of each outcome occurring, $1/2$ for both. We call such an object—a set of outcomes, each associated with a probability—a **lottery**. Denote the set of outcomes by X. Generic elements of X will be denoted x, y, z or sometimes $x_1, x_2, \ldots$. For simplicity, assume that X is finite. Outcomes alone are not enough to describe a lottery: we need probability distributions over outcomes, as in the $1/2$–$1/2$ distribution of the fair coin above. The set of lotteries over X is denoted by $\Delta(X)$.[1] Each element of $\Delta(X)$ is a function $p : X \to [0, 1]$ such that $\sum_{x \in X} p(x) = 1$; it maps each outcome x to a number $p(x) \in [0, 1]$, representing the probability that x occurs.[2] We can write a lottery as a vector, for example $p = (p(x), p(y), p(z))$ if $X = \{x, y, z\}$.

Example 1.1. In the example above, the set of outcomes is $\{10, -10\}$ and the lottery $p \in \Delta(\{10, -10\})$ induced by the fair coin toss satisfies $p(10) = p(-10) = 1/2$. ■

We can depict lotteries using a tree diagram, as in Figure 1.1.

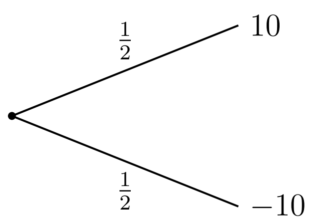


Figure 1.1: Lottery from Example 1.1.

Remark 1.1. *Notice that, in this setup, we are missing something: whether the coin lands on heads or tails is irrelevant; only the probabilities of outcomes matter, not the*

[1] Why the notation Δ? You will realise soon.

[2] Why do we write a sum $\sum_{x \in X} p(x) = 1$ and not an integral?

events that induced them. This is a limitation of this model, which we will address when we introduce a state-space representation of uncertainty.

The set of lotteries $\Delta(X)$ has *structure*: we can combine its elements in a meaningful way. For example, consider a lottery r that yields the lottery p with probability α and q with probability $1-\alpha$, where $\alpha \in [0,1]$. This is a *compound lottery.* It is still an element of $\Delta(X)$, and we write $r = \alpha p + (1-\alpha)q$. For instance, if $p(10) = 1/2$ and $q(10) = 1/4$, the associated compound lottery is represented on the left of Figure 1.2. We can compute the probability that outcome 10 occurs in this compound lottery: $\alpha \times 1/2 + (1-\alpha) \times 1/4 = \frac{1+\alpha}{4}$. By computing the probability of each outcome in a compound lottery, we *reduce* it to a simple lottery, as represented on the right of Figure 1.2.

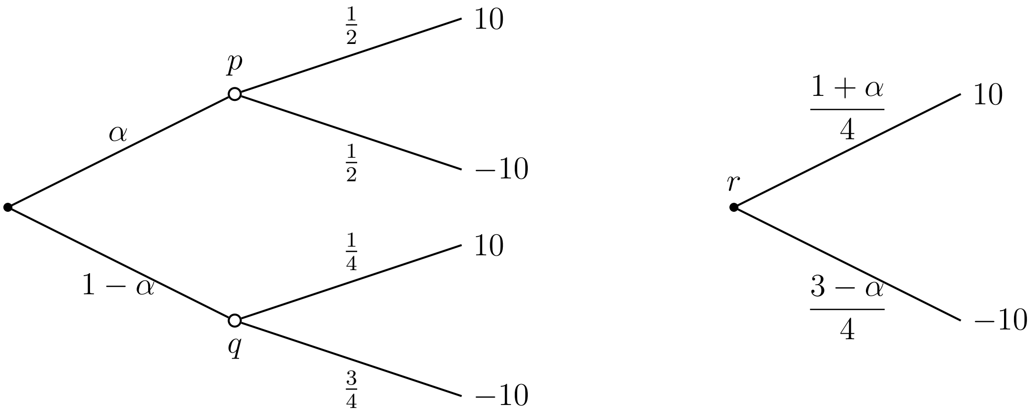


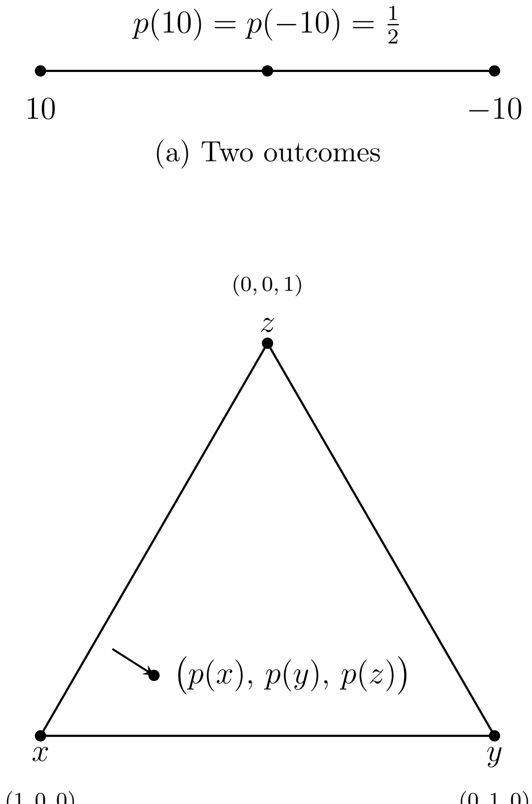
Figure 1.2: Compound lottery (left) and its reduced form (right).

We assume *reduction of compound lotteries*: individuals are indifferent between any compound lottery and its reduced form, i.e., any two lotteries that induce the same probabilities over outcomes are treated as the same.

Can you think of reasons why someone might *not* be indifferent between a compound lottery and its reduced form? Violations of reduction generate interesting phenomena studied in behavioural economics. See, e.g., Segal (1990) and Dillenberger & Raymond (2020).

This lottery *mixing* operation is not possible with an unstructured set of outcomes. As an illustration, suppose the set of outcomes comprises fruits. We can have an apple or a banana, but there is no fruit that is a mixture of an apple and a banana. Imposing structure on the set of elements to be ranked is a key move of microeconomic theory. In fact, we will later assume that the set of outcomes is $\mathbb{R}$, the set of real numbers representing monetary outcomes, which lets us say more than with a generic set of outcomes.

There is another way to represent lotteries graphically. Consider once again the coin toss: obtaining 10 euros with probability $1/2$ and -10 euros with probability $1/2$. We can represent this lottery as the midpoint of the line segment whose endpoints correspond to the degenerate lotteries yielding 10 and -10 with probability 1; see panel (a) of Figure 1.3. More generally, with n outcomes we can represent a lottery as a point in an $(n-1)$-dimensional simplex. For example, with three outcomes we can represent lotteries as points in an equilateral triangle, as in panel (b).[3] The vertices of the triangle correspond to degenerate lotteries that yield one outcome with probability 1. Any other point in the triangle corresponds to a lottery that yields each of the three outcomes with some probability. Roughly, the farther a point is from a vertex, the lower the probability of the corresponding outcome. For example, the lottery p in panel (b) yields outcome x with relatively high probability and outcomes y and z with relatively low probabilities.



(a) Two outcomes

(b) Three outcomes

Figure 1.3: Lotteries as points in simplexes: (a) a two–outcome lottery lies on a line segment; (b) with three outcomes, lotteries lie in an equilateral triangle.

For a finite outcome set X, the probability simplex over X is

$$\Delta(X) = \left\{ p : X \to [0,1] \mid \sum_{x \in X} p(x) = 1 \right\}$$

or equivalently

$$\left\{ (p(x_1), \ldots, p(x_n)) \in \mathbb{R}^n \mid p(x_i) \geq 0, \sum_i p(x_i) = 1 \right\}.$$

[3] That's why the Δ notation!

This is an $n-1$-dimensional simplex whose vertices are the degenerate lotteries (unit vectors), e.g. $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$.

1.2 Preferences over lotteries

Our aim is to understand how individuals choose between lotteries, whether they like or dislike risk, and how to compare different individuals' attitudes towards risk. We therefore need to make statements such as "an individual weakly prefers lottery p to lottery q". To do so, introduce a binary relation $\succsim$ over $\Delta(X)$, so that $p \succsim q$ reads "the individual weakly prefers lottery p to lottery q".[4] Compared to choice under certainty, we are now comparing lotteries—i.e., probability distributions over outcomes—rather than outcomes themselves.

Technically, $\succsim$ is a subset of $\Delta(X) \times \Delta(X)$, i.e., a set of ordered pairs of lotteries. For example, if $p, q, r \in \Delta(X)$, the statements p *is (weakly) preferred to* q imply $(p, q) \in \succsim$.

Recall that we can define strict preference and indifference in terms of weak preference: $p \succ q$, which reads "p is strictly preferred to q" if and only if $p \succsim q$ but not $q \succsim p$; and $p \sim q$, which reads "p is indifferent to q" if and only if both $p \succsim q$ and $q \succsim p$.

In what follows, we consider what assumptions preferences over lotteries might satisfy, and what these assumptions imply for behaviour.

Things to read. See Kreps (1988, pp. 31–33) for a brief intuitive introduction to the lottery model in this chapter. If you want to read a similar treatment from a textbook read Mas-Colell et al. (1995, pp. 168–170).

1.3 Exercises

Exercise 1.1. Can we still represent the set of lotteries and compound lotteries on the simplex if there is no indifference between a compound lottery and its reduced form? Why or why not?

Exercise 1.2. Assume there are three outcomes x, y, z. Draw in the simplex the set of lotteries that yield each outcome with the same probability and the lottery that yields x with certainty. Now draw the set of all mixtures of these two lotteries. Assume that the individual is indifferent between the lottery yielding each outcome with the same

[4] Are you curious why we use the symbol $\succsim$ for preferences and not $\geq$? An historian of economic theory, Ivan Boldyrev, told me it is from Herstein & Milnor (1953), who used the symbol in their famous paper where they provided an axiomatic characterisation of expected utility, which we will do soon.

probability, the lottery yielding x with certainty, and any mixture of the two. Which part of the simplex does this indifference "curve" correspond to? Is it really a curve?

Exercise 1.3. Assuming three outcomes x, y, z, draw in the simplex the set of lotteries that yield outcome x with probability at least $1/2$.

Exercise 1.4. In the main text we assumed that individuals are indifferent between a compound lottery and its reduced form. State this indifference formally as a condition on the preference relation $\succsim$, using the notation introduced.

References

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Lecture 2

Expected utility theory

2.1 Assumptions on preferences

We now impose properties on preferences over lotteries. But first, a brief methodological aside on what we are doing. Before discussing properties of $\succsim$, we should explicit what the interpretation of $\succsim$ is. Different methodological stances are possible. Is $\succsim$ tracking what an individual has in mind? What he would say if asked? How he chose in the past?

Under *revealed preference theory*, we interpret $\succsim$ as a description of how an individual chooses. Therefore, there is no psychological content to $\succsim$. Revealed preference theory has been the standard methodological stance in economics for a long time. But why? Wouldn't it be better to develop a theory that exploits psychological insights? Revealed preference theory is exclusively a methodological stance, not a psychological (or, for what matters, a moral) one. The assumption is not that choices are not driven by psychological motives, but that we abstract from these motives and attempt to find patterns in choices directly. There is a strong advantage in doing so: psychological motives are hard to observe, while choices can be observed easily. The implication is that a choice theory based on revealed preferences is more easily testable: if we observe choices that violate the assumptions of the theory, we can reject the it. Therefore, revealed preference theory is **not** a stance on how individuals make choices or what matters for choices, it is just silent about these issues. This is often misunderstood: there is a plethora of papers claiming that economics views individuals as cold robots.[1]

Such critics mostly come from behavioural economics, which is a field that attempts to incorporate psychological insights into economic models. Is therefore impossible to do behavioural economics under the revealed preference approach? Not at all. Good behavioural theories do what the name of the field suggests: characterise the behavioural content of the theory, so that, as economist, we know how different individuals behave. Two theories with different psychological content that are observationally equivalent, i.e., they make the same predictions about choices, are not equally useful for economists.[2]

An interesting case study is Masatlioglu & Raymond (2016), where the authors show that the famous model by Köszegi & Rabin (2007) is behaviourally equivalent to the intersection of

[1] If you are interested, you can read Thoma (2021) for a discussion of the current status of revealed preference theory.

[2] There is a huge debate on this topic. Among many, I suggest you to read Gul & Pesendorfer (2011) and the response by Camerer (2008). A more recent piece is Spiegler (2019).

rank-dependent utility and quadratic utility, two older models. Another example that is quite relevant today is Eliaz & Spiegler (2006).

In what follows, you can have in mind the interpretation of $\succsim$ that you prefer, but remember that it is important to be clear about it.

Before discussing the properties of preferences over lotteries, let's consider a reasonable functional form for preferences. A natural candidate is the following: the utility of a lottery p is given by

$$\sum_{x \in X} p(x)u(x) \tag{2.1}$$

for some function $u\colon X \to \mathbb{R}$, assigning to each outcome x a number representing its utility $u(x)$. The idea is simple, the outcome x realises with probability p, and when x realises, the individual gets utility $u(x)$. The functional form in Equation (2.1) is called **expected utility**. This is because it is the expectation, computed with the probability p, of the utility the individual gets. Before turning to the properties of preferences that will lead us to this functional form, let's make some observations.

Having expected utility preferences over lotteries implies that indifference curves on the simplex are straight lines. That is, say that $p \sim q$. Then, for any $\alpha \in (0, 1)$ it holds that $\alpha p + (1 - \alpha)q \sim p$, as illustrate in Figure 2.1.

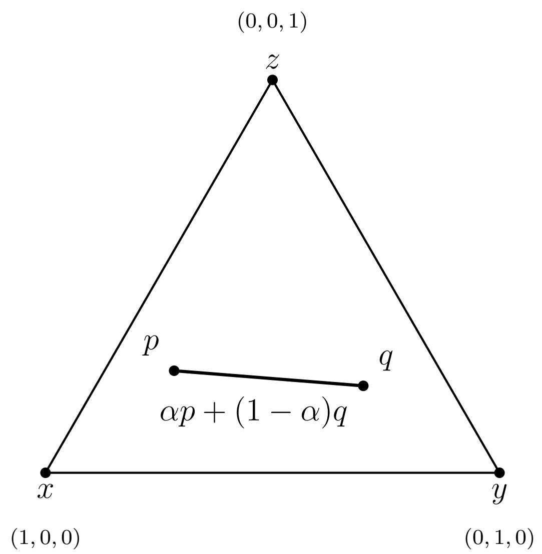

Figure 2.1: If $p \sim q$, then any mixture of p and q is also indifferent to p and q.

Let's show this formally. Assume that $p \sim q$. Then, by definition of expected utility, we have

$$\sum_{x \in X} p(x)u(x) = \sum_{x \in X} q(x)u(x).$$

By applying expected utility again, for any $\alpha \in (0,1)$, the utility of the lottery $\alpha p + (1-\alpha)q$ is given by

$$\begin{aligned}\sum_{x\in X}\big(\alpha p(x) + (1-\alpha)q(x)\big)u(x) &= \sum_{x\in X}\alpha p(x)u(x) + \sum_{x\in X}(1-\alpha)q(x)u(x)\\ &= \alpha\sum_{x\in X}p(x)u(x) + (1-\alpha)\sum_{x\in X}q(x)u(x)\\ &= \alpha\sum_{x\in X}q(x)u(x) + (1-\alpha)\sum_{x\in X}q(x)u(x)\\ &= \sum_{x\in X}q(x)u(x).\end{aligned}$$

Indifference curves are also parallel, you are asked to show this in Exercise 2.2.

Let's now turn to the properties of $\succsim$ we will consider. First, we assume that preferences are a **weak order**.

Axiom 2.1. ***(Weak order)*** *Preferences $\succsim$ are complete and transitive.*

Recall that preferences are **complete** if for any two lotteries p, q, either $p \succsim q$ or $q \succsim p$, or both. They are transitive if for any three lotteries p, q, r, if $p \succsim q$ and $q \succsim r$, then $p \succsim r$.

Axiom 2.2. ***(Continuity)*** *For any three lotteries p, q, r, if $p \succ q \succ r$ then there exist $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$.*

Axiom 2.3. ***(Independence)*** *For any three lotteries p, q, r and for any $\alpha \in (0,1)$, we have $p \succsim q$ if and only if $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$.*

Lemma 2.1. *Let $\succsim$ satisfy Weak order, Continuity, and Independence, then there exist two lotteries $\overline{p}$ and $\underline{p}$ such that $\overline{p} \succsim p \succsim \underline{p}$ for all p.*

Proof. The proof is in two steps.

Step 1. By Weak order, the restriction of $\succsim$ to the set of lotteries giving positive probability to only one outcome $\{\delta_x : x \in X\}$ is a complete and transitive order on a *finite* set. Hence there exist x^*, x_* such that

$$\delta_{x^*} \succsim \delta_x \succsim \delta_{x_*} \qquad \text{for all } x.$$

Fix $\overline{p} := \delta_{x^*}$ and $\underline{p} := \delta_{x_*}$.

Step 2. For $p \in \Delta(X)$ write $supp(p) = \{x \in Z : p(x) > 0\}$ and let $|supp(p)|$ be its size. We prove by induction on $k := |supp(p)|$ that

$$\overline{p} \succsim p \succsim \underline{p}.$$

Base case $k = 1$. If $supp(p) = \{x\}$, then $p = \delta_x$ and the claim follows from Step 1.

Inductive step. Assume the statement holds for all lotteries with support size $\leq k-1$. Let p have support size $k \geq 2$. Pick any $x \in supp(p)$ and write

$$p \; = \; \lambda \, \delta_x + (1-\lambda) \, q, \qquad \lambda := p(x) \in (0,1),$$

where q is the renormalized remainder (so $|supp(q)| \leq k-1$).

By the inductive hypothesis, $\overline{p} \succsim q$; by Step 1, $\overline{p} \succsim \delta_x$. Using Independence twice and transitivity,

$$\overline{p} \; \succsim \; \lambda \, \overline{p} + (1-\lambda) \, q \; \succsim \; \lambda \, \delta_x + (1-\lambda) \, q \; = \; p.$$

(The first relation comes from $\overline{p} \succsim q$ with $t = 1-\lambda$ and $Z = \overline{p}$; the second from $\overline{p} \succsim \delta_x$ with $t = \lambda$ and $Z = q$.)

A symmetric argument gives $p \succsim \underline{p}$: by the inductive hypothesis $q \succsim \underline{p}$ and by Step 1 $\delta_x \succsim \underline{p}$; then, by Independence and transitivity,

$$p = \lambda \, \delta_x + (1-\lambda) \, q \; \succsim \; \lambda \, \underline{p} + (1-\lambda) \, q \; \succsim \; \underline{p}.$$

Thus $\overline{p} \succsim p \succsim \underline{p}$ for all lotteries with support size k, closing the induction.

Therefore, the fixed Diracs $\overline{p} = \delta_{x^*}$ and $\underline{p} = \delta_{x_*}$ bound every lottery $p \in \Delta(Z)$, as claimed. $\square$

2.2 Expected utility representation

Theorem 2.1. *Preferences over lotteries $\succsim$ satisfy Weak order, Continuity, Independence if and only if then there exists a utility function $u \colon X \to \mathbb{R}$ such that:*

$$p \succsim q \text{ if and only if } \sum_{x \in X} p(x) u(x) \geq \sum_{x \in X} q(x) u(x). \tag{2.2}$$

*We say that u **represents** $\succsim$.*

The proof here essentially follows Mas-Colell et al. (1995, pp. 176–178), but it is complemented by intuition and figures.

Proof. We proceed by steps.

Step 1. If $p \succsim q$ then $p \succsim \alpha p + (1-\alpha) q \succsim q$ for any $\alpha \in (0,1)$.

The intuition behind this step is simple: if p is better than q, than any mixture between the two is worse than p and better than q. Figure 2.2 illustrates the idea.

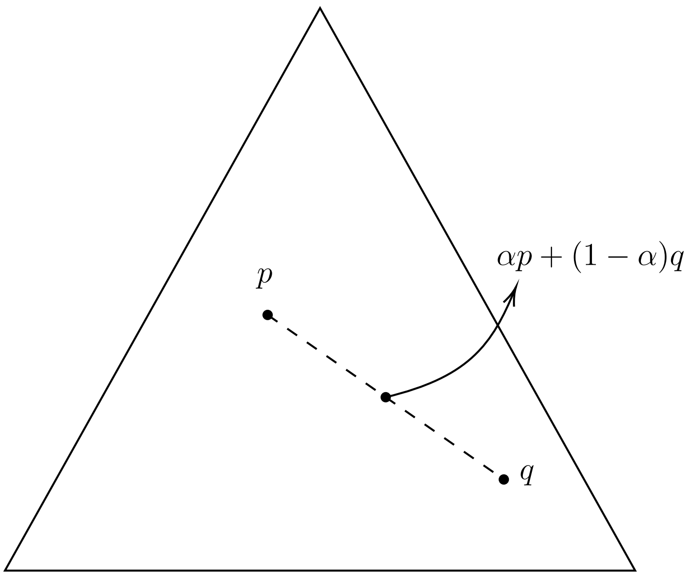

Figure 2.2: Step 1.

This follows from Independence.

$$p \succsim q \implies (1-\alpha)p + \alpha p \succsim (1-\alpha)q + \alpha q \implies p \succsim \alpha p + (1-\alpha)q \succsim q. \tag{2.3}$$

$$p \succsim q \implies \alpha p + (1-\alpha)q \succsim \alpha q + (1-\alpha)q \implies \alpha p + (1-\alpha)q \succsim q. \tag{2.4}$$

The conclusion follows from Equations (2.3) and (2.4).

Step 2. $\beta > \alpha$ if and only if $\beta\overline{p} + (1-\beta)\underline{p} \succ \alpha\overline{p} + (1-\alpha)\underline{p}$.

The idea for this step is the following. From **Step 1**, we know that a mixture between p and q, where $p \succsim q$ is better than q and worse than p. Now, we know that $\overline{p} \succ \alpha\overline{p} + (1-\alpha)\underline{p}$, since $\overline{p}$ is the best lottery available. We want to show that $\beta\overline{p} + (1-\beta)\underline{p}$ is a mixture between $\overline{p}$ and $\alpha\overline{p} + (1-\alpha)\underline{p}$, and therefore, by **Step 1**, better than $\alpha\overline{p} + (1-\alpha)\underline{p}$. The idea is illustrated in Figure 2.3.

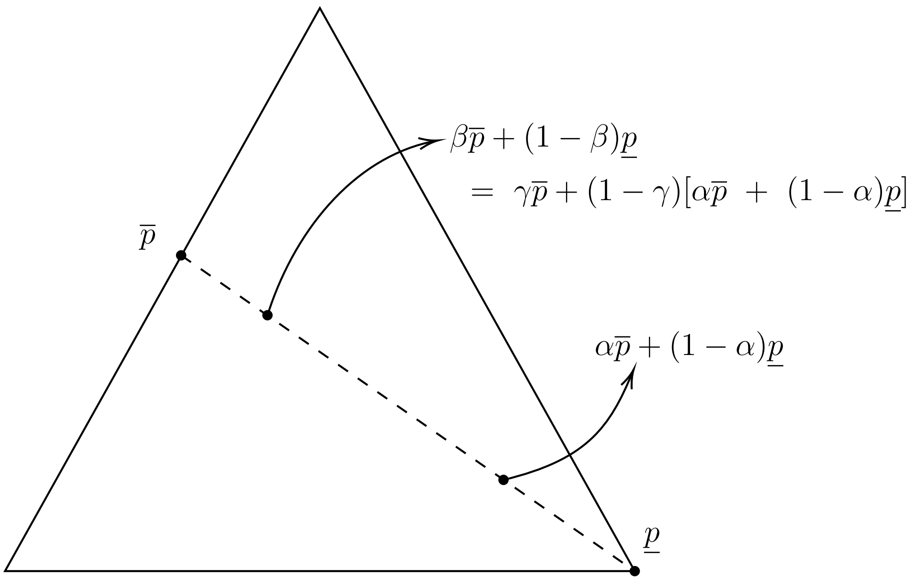

Figure 2.3: Step 2.

We want to write $\beta\overline{p} + (1-\beta)\underline{p}$ as a mixture between $\overline{p}$ and $\alpha\overline{p} + (1-\alpha)\underline{p}$. That is, we want to find $\gamma \in (0,1)$ such that

$$\beta\overline{p} + (1-\beta)\underline{p} = \gamma\overline{p} + (1-\gamma)[\alpha\overline{p} + (1-\alpha)\underline{p}].$$

With some algebra we get that $\gamma = \frac{\beta-\alpha}{1-\alpha}$. By step 1 we know that $\overline{p} \succ \alpha\overline{p} + (1-\alpha)\underline{p}$, therefore, $\gamma\overline{p} + (1-\gamma)[\alpha\overline{p} + (1-\alpha)\underline{p}] \succ \alpha\overline{p} + (1-\alpha)\underline{p}$. Since $\beta\overline{p} + (1-\beta)\underline{p} = \gamma\overline{p} + (1-\gamma)[\alpha\overline{p} + (1-\alpha)\underline{p}]$, the conclusion follows.

Until now we proved that if $\beta > \alpha$ then $\beta\overline{p} + (1-\beta)\underline{p} \succ \alpha\overline{p} + (1-\alpha)\underline{p}$. But the statement says "if and only if", so we have to prove that if $\alpha \geq \beta$ then it is not the case that $\beta\overline{p} + (1-\beta)\underline{p} \succ \alpha\overline{p} + (1-\alpha)\underline{p}$. When $\beta = \alpha$ the two are the same lotteries and therefore are indifferent. The relevant case is when $\alpha > \beta$. By the argument above, we know that $\alpha\overline{p} + (1-\alpha)\underline{p} \succ \beta\overline{p} + (1-\beta)\underline{p}$, and that's all.

Step 3. [3] For any p, there exists a unique $\alpha_p \in [0,1]$ such that $p \sim \alpha_p\overline{p} + (1-\alpha_p)\underline{p}$.

We can derive this step as an implication of previous steps and Continuity. Unfortunately, this step requires a bit of algebra. But you can get some intuition from Figure 2.4.

[3] In this step we make use of proof by contradiction. Before diving in, you should make sure your are familiar with the logic of such proofs.

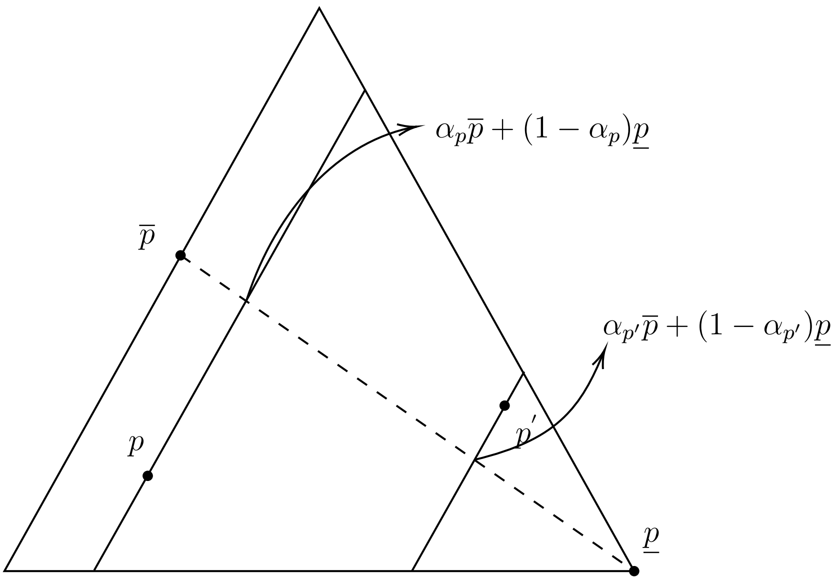

Figure 2.4: Step 3.

First, notice that if α_p exists, it must be unique. Suppose there are two such numbers α_p and α'_p with $\alpha_p > \alpha'_p$, then by Step 2, $\alpha_p\overline{p} + (1-\alpha_p)\underline{p} \succ \alpha'_p\overline{p} + (1-\alpha'_p)\underline{p}$, contradicting indifference to p.

Now we have to show that such α_p exists. If $\overline{p} \sim p$ then $\alpha_p = 1$ works, and if $\underline{p} \sim p$ then $\alpha_p = 0$ works. So we have to look at the interesting case $\overline{p} \succ p \succ \underline{p}$.

Define

$$\alpha_p \ = \ \sup\bigl\{\alpha \in [0,1] \ : \ p \succsim \alpha\overline{p} + (1-\alpha)\underline{p}\bigr\}. \tag{2.5}$$

Since $\alpha = 0$ is in this set, we are sure that the supremum is not over an empty set.

We now have to do a few algebraic arguments.

$$\text{If } 1 \geq \alpha > \alpha_p \ \text{ then } \ \alpha\overline{p} + (1-\alpha)\underline{p} \succ p. \tag{2.6}$$

Indeed, if $p \succsim \alpha\overline{p} + (1-\alpha)\underline{p}$ held for such α, then α_p would not satisfy Equation (2.5). Moreover:

$$\text{If } 0 \leq \alpha < \alpha_p \ \text{ then } \ p \succ \alpha\overline{p} + (1-\alpha)\underline{p}. \tag{2.7}$$

The reason is as follows. By the definition of α_p, there exists α' such that $\alpha < \alpha' \leq \alpha_p$ and $p \succsim \alpha'\overline{p} + (1-\alpha')\underline{p}$. Since $\alpha < \alpha'$, **Step 2** implies that $p \succ \alpha'\overline{p} + (1-\alpha')\underline{p} \succ \alpha\overline{p} + (1-\alpha)\underline{p}$.

Now,there are three possibilities to consider: $\alpha_p\overline{p} + (1-\alpha_p)\underline{p} \succ p$, $p \succ \alpha_p\overline{p} + (1-\alpha_p)\underline{p}$, or they are indifferent. If $\alpha_p\overline{p} + (1-\alpha_p)\underline{p} \succ p$, by Continuity there exists $\beta \in (0,1)$ such that $\beta\bigl(\alpha_p\overline{p} + (1-\alpha_p)\underline{p}\bigr) + (1-\beta)\underline{p} \succ p$. But notice that

$$\begin{aligned}\beta\big(\alpha_p\overline{p}+(1-\alpha_p)\underline{p}\big)+(1-\beta)\underline{p} &= \beta\alpha_p\,\overline{p}+\beta(1-\alpha_p)\,\underline{p}+(1-\beta)\,\underline{p}\\ &= \beta\alpha_p\,\overline{p}+\big[\beta(1-\alpha_p)+(1-\beta)\big]\underline{p}\\ &= \beta\alpha_p\,\overline{p}+\big(1-\beta\alpha_p\big)\underline{p}\ \succ\ p.\end{aligned}$$

Since $\beta\alpha_p < \alpha_p$, by Equation (2.7) we should have $p \succ \beta\alpha_p\,\overline{p}+(1-\beta\alpha_p)\underline{p}$, which leads to contradiction.

If instead $p \succ \alpha_p\overline{p}+(1-\alpha_p)\underline{p}$, by Continuity there exists $\beta \in (0,1)$ such that

$$\begin{aligned} p &\succ\ \beta\big(\alpha_p\overline{p}+(1-\alpha_p)\underline{p}\big)+(1-\beta)\,\overline{p}\\ &=\ \big[\beta\alpha_p+(1-\beta)\big]\overline{p}+\beta(1-\alpha_p)\,\underline{p}\\ &=\ \big(1-\beta(1-\alpha_p)\big)\overline{p}+\beta(1-\alpha_p)\,\underline{p}.\end{aligned}$$

Since $1-\beta(1-\alpha_p) > \alpha_p$, by Equation (2.6) we should have $\big(1-\beta(1-\alpha_p)\big)\overline{p}+\beta(1-\alpha_p)\underline{p}\ \succ\ p$, a contradiction.

Step 4. The utility function $U : \Delta(X) \to \mathbb{R}$, assigning to each lottery a number representing its utility defined as $U(p) = \alpha_p$ represents preferences $\succsim$.

Take two lotteries p and p'. By **Step 3**, there exist unique α_p and α'_p such that

$$p \sim \alpha_p\overline{p}+(1-\alpha_p)\underline{p}, \quad p' \sim \alpha_{p'}\overline{p}+(1-\alpha_{p'})\underline{p},$$

and therefore

$$p \succsim p' \text{ if and only if } \alpha_p\overline{p}+(1-\alpha_p)\underline{p} \succsim \alpha_{p'}\overline{p}+(1-\alpha_{p'})\underline{p}.$$

By **Step 2**,

$$\alpha_p\overline{p}+(1-\alpha_p)\underline{p} \succsim \alpha_{p'}\overline{p}+(1-\alpha_{p'})\underline{p} \text{ if and only if } \alpha_p \geq \alpha_{p'},$$

where the last condition holds if and only if $U(p) \geq U(p')$, which proves the claim.

Step 5. The function U has the expected utility form.

By the previous steps, we know that for any lottery p there is a unique number $U(p) \in [0,1]$ such that

$$p \sim U(p)\,\overline{p}+\big(1-U(p)\big)\underline{p}, \qquad p' \sim U(p')\,\overline{p}+\big(1-U(p')\big)\underline{p}.$$

Apply Independence to get:

$$\begin{aligned}\beta p+(1-\beta)p' &\sim \beta\big[U(p)\overline{p}+\big(1-U(p)\big)\underline{p}\big]+(1-\beta)p'\\ &\sim \beta\big[U(p)\overline{p}+\big(1-U(p)\big)\underline{p}\big]+(1-\beta)\big[U(p')\overline{p}+\big(1-U(p')\big)\underline{p}\big]\\ &= \big[\beta U(p)+(1-\beta)U(p')\big]\overline{p}+\Big(1-\big[\beta U(p)+(1-\beta)U(p')\big]\Big)\underline{p}.\end{aligned}$$

Let $\gamma := \beta U(p) + (1-\beta)U(p')$. By **Step 4**, for the lottery $\beta p + (1-\beta)p'$ there is a *unique* number γ such that $\beta p + (1-\beta)p' \sim \gamma \bar{p} + (1-\gamma)\underline{p}$. Therefore

$$U(\beta p + (1-\beta)p') = \beta U(p) + (1-\beta)U(p').$$

$\square$

Corollary 2.1. *If u represents $\succsim$, then a function $u' \colon X \to \mathbb{R}$ represents $\succsim$ if and only if there exist real numbers $a > 0$ and b such that $u' = au + b$.*

Things to read. There is already quite a lot to read that I mentioned in the main text. If you want a textbook source of the content of this lecture, check Mas-Colell et al. (1995, pp. 170–178).

2.3 Exercises

Exercise 2.1. Prove the direction of Theorem 2.1 that we did not prove. Show that if u represents $\succsim$, then $\succsim$ satisfies Weak order, Continuity, and Independence. (It is not difficult, I promise!)

Exercise 2.2. Show that, if preferences are represented by an expected utility function, then indifference curves in the triangle are parallel lines.

Solution to Exercise 2.2. Let $\{x, y, z\}$ be the outcomes and let

$$U(p) = \sum_{w \in \{x,y,z\}} u(w)\, p(w)$$

be the expected utility of lottery p. Since $p(z) = 1 - p(x) - p(y)$, we can write

$$U(p) \;=\; \big(u(x) - u(z)\big)\, p(x) \;+\; \big(u(y) - u(z)\big)\, p(y) \;+\; u(z).$$

Fix a utility level c. The indifference set $\{p : U(p) = c\}$ satisfies

$$\big(u(x) - u(z)\big)\, p(x) \;+\; \big(u(y) - u(z)\big)\, p(y) \;=\; c - u(z).$$

Solving for $p(y)$ as a function of $p(x)$,

$$p(y) \;=\; \frac{c - u(z)}{u(y) - u(z)} \;-\; \frac{u(x) - u(z)}{u(y) - u(z)}\, p(x).$$

The coefficient of $p(x)$ is

$$-\frac{u(x) - u(z)}{u(y) - u(z)},$$

which does not depend on c. Changing c only changes the intercept $\frac{c-u(z)}{u(y)-u(z)}$. Therefore, all indifference lines, the portions of these lines that lie inside the simplex, have the same slope and are parallel. $\square$

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Lecture 3

Money lotteries

3.1 Structuring the set of outcomes

In the previous section, we studied preferences with the expected utility form over lotteries on a *finite* outcome set X. We now study a setting where the outcome set is the set of real numbers $\mathbb{R}$, representing monetary outcomes. This setting is particularly important in economics and finance, as it allows us to model decisions involving money, such as investments, insurance, and consumption choices.

You may wonder wether a form of Theorem 2.1 can be extended to such a setting. The answer is yes, if you are interested check Kreps (1988, pp. 59-78) or Fishburn (1970, ch. 10).

Since the outcome set is now infinite, we need to be careful about how we define lotteries. We introduce cumulative distribution functions (CDFs) to represent lotteries over monetary outcomes. A CDF $F : \mathbb{R} \to [0, 1]$ maps each monetary outcome x to the probability that the outcome is less than or equal to x. It satisfies the following properties:

- F is non-decreasing: if $x \le y$, then $F(x) \le F(y)$.
- $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$.
- F is right-continuous, i.e. for every $x \in \mathbb{R}$, $\lim_{y\downarrow x} F(y) = F(x)$.[1]

Example 3.1. Consider a lottery that pays 1 dollar with probability $\frac{1}{4}$, 4 dollars with probability $\frac{1}{2}$, and 6 dollars with probability $\frac{1}{4}$. The corresponding CDF F is given by:

$$F(x) = \begin{cases} 0 & \text{if } x < 1, \\ \frac{1}{4} & \text{if } 1 \le x < 4, \\ \frac{3}{4} & \text{if } 4 \le x < 6, \\ 1 & \text{if } x \ge 6, \end{cases}$$

it is represented in Figure 3.1.

[1]The symbol $y \downarrow x$ means that y approaches x from above.

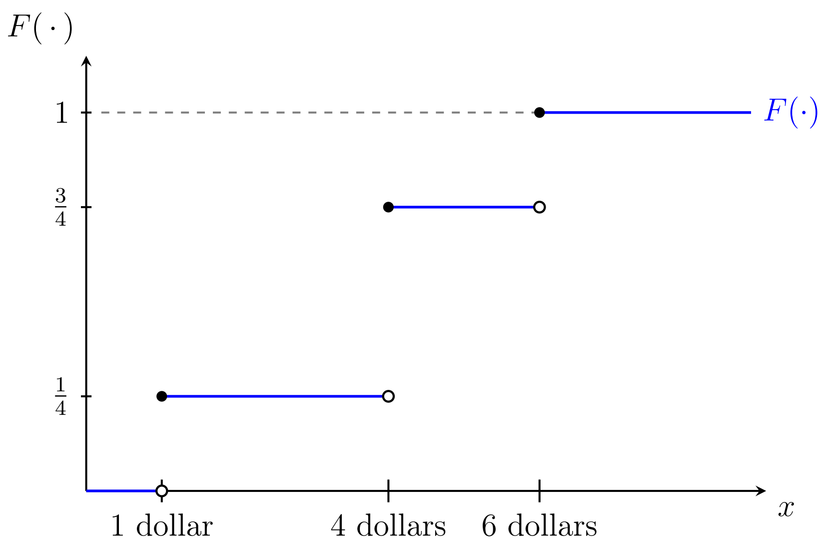

Figure 3.1: Cumulative distribution function (CDF) representing a lottery over monetary outcomes.

■

Notice that mixtures of CDFs are also CDFs, so we can employ the same mixture operation defined in Section 1.1. In particular, given two CDFs F and G, and $\alpha \in [0, 1]$, the mixture $H = \alpha F + (1-\alpha)G$ is also a CDF, where $H(x) = \alpha F(x) + (1-\alpha)G(x)$ for all $x \in \mathbb{R}$.

We now define preferences $\succsim$ over the set of CDFs over non-negative amounts of money that have the expected utility form. The idea is the same, we weight the utility of each monetary outcome by its probability, and sum these weighted utilities to get the expected utility of the lottery. Formally, a preference relation $\succsim$ over the set of CDFs has the expected utility form if there exists a utility function $u : \mathbb{R} \to \mathbb{R}$ such that for any two CDFs F and G:

$$F \succsim G \quad \text{if and only if} \quad \int u(x)dF(x) \geq \int u(x)dG(x).$$

Before we had a utility function over outcomes $u : X \to$, but now the set of outcomes is $\mathbb{R}$, that is why the domain is different. Such distinction allows us to introduce properties of the function u that are specific to monetary outcomes. From now on, we assume the following two.

Definition 3.1. *The utility function u is **increasing** if for any x, y such that $x > y$, we have $u(x) > u(y)$.*

Definition 3.1 captures the idea that more money is always preferred to less money. When the outcome set was X, we could not state this property, as $x > y$ meant nothing.[2]

[2]As an example, if x is an apple and y is a banana, what does $x > y$ means?

Definition 3.2. *The utility function u is* ***continuous*** *if for any x and any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all y with $|x - y| < \delta$, we have $|u(x) - u(y)| < \varepsilon$.*

Definition 3.2 ensures that small changes in monetary outcomes lead to small changes in utility. This property could not be stated with a generic outcome set, as $x - y$ had no meaning.

Definition 3.2 is continuity in *money*. What about continuity in probability?

3.2 Risk aversion

We now have the tools to define and discuss the concept of risk aversion. Defining this concept allows us to answer the question: how much does an individual dislike risk? As we will see, the answer to this question has important implications for economic behavior, such as investment decisions and insurance choices.

The definition of risk aversion is quite intuitive. Consider an individual offered the following opportunity: they can either receive 5 euros, or a lottery that pays 0 euros with probability 0.5 and 10 euros with probability 0.5. Both options have the same expected monetary value of 5 euros. Intuitively, if the individual prefers to receive the certain amount of 5 euros over the lottery, he dislikes risk, prefers getting the mean outcome for sure rather than facing uncertainty. Instead, if the individual prefers the lottery, he likes risk, as he is willing to face uncertainty for the chance of getting a lower payoff.

For each lottery, we define an individual as risk averse if he prefers the certain amount equal to the expected value of the lottery over the lottery itself, as in the example above. For each CDF F, the expected value of the lottery is given by:

$$\int x dF(x). \tag{3.1}$$

An individual evaluates money using the utility function u. Therefore, the certain amount equal to the expected value of the lottery provides utility

$$u\left(\int x dF(x)\right). \tag{3.2}$$

On the other hand, the lottery itself provides expected utility

$$\int u(x) dF(x). \tag{3.3}$$

We say an individual is risk averse if his utility function u is such that, for each CDF F, Equation (3.2) is greater than or equal to Equation (3.3).

Definition 3.3. *An individual with expected utility preferences and utility function u is* ***risk averse*** *if for each CDF F*

$$u\left(\int x dF(x)\right) \geq \int u(x) dF(x). \tag{3.4}$$

You should notice that, if u is not increasing (Definition 3.1), such definition of risk aversion does not make sense.

Equation (3.4) is Jensen's inequality, and it defines concavity of the utility function u. Therefore, the intuitive notion of risk aversion we discussed is technically equivalent to concavity of u, as illustrated in Figure 3.2. Recall that concavity of u, if it is twice differentiable, means that its second derivative is non-positive, i.e. $u''(x) \leq 0$ for all x.

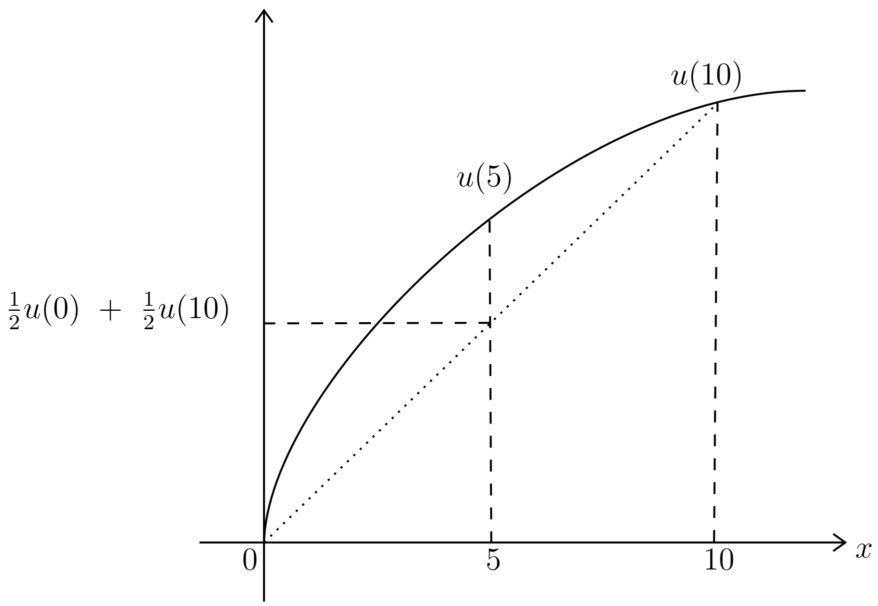

Figure 3.2: Example of a u exhibiting risk aversion.

Equivalently, an individual is risk loving if the inequality in Definition 3.3 is reversed, and risk neutral if the individual is indifferent between the certain amount and the lottery, i.e. if the inequality holds with equality.

There are other ways of defining risk aversion starting from different thought experiments that are equivalent to Definition 3.3. This is good news, it means the definition makes sense! If you are interested in other ways of defining risk aversion, check Mas-Colell et al. (1995, p. 186-187). We consider another one here. Define the **certainty equivalent** of a lottery as the certain amount of money that provides the same utility as the lottery itself. That is, the individual must be indifferent between receiving the certainty equivalent for sure and facing the lottery.

Definition 3.4. *The* ***certainty equivalent*** *of a lottery with CDF F for an individual with utility function u is defined as the solution to the equation*

$$u(c(F,u)) = \int u(x)dF(x). \tag{3.5}$$

Intuitively, if an individual is risk averse, his certainty equivalent must be less than the expected value of the lottery, as he prefers receiving the expected value for sure rather than facing the lottery. To capture this intuition we can define the **risk premium** of a lottery as the difference between the expected value of the lottery and its certainty equivalent.

Definition 3.5. *The* ***risk premium*** *of a lottery with CDF F for an individual with utility function u is*

$$\pi(F,u) = \int xdF(x) - c(F,u). \tag{3.6}$$

You should show in Exercise 3.2 that an individual is risk averse if and only if the risk premium is non-negative for each lottery.

We now have a notion of risk aversion, but not a quantitative measure, which we will develop next. Again, we start from intuition, how could we measure risk aversion? The risk premium might be a starting point, the higher the risk premium, the more risk averse the individual, as he requires a lower certainty equivalent to face the lottery. Consider two individuals with utility function u and v. If for each lottery F, the risk premium of the first individual is higher than that of the second, i.e. $\pi(F,u) \geq \pi(F,v)$, we can say that the first individual is more risk averse than the second. However, such condition boils down to comparing certainty equivalents:

$$\pi(F,u) \geq \pi(F,v) \iff c(F,u) \leq c(F,v) \text{ for each lottery } F,$$

you should show this. We therefore have the following definition.

Definition 3.6. *An individual with utility function u is* ***more risk averse*** *than an individual with utility function v if for each lottery F*

$$c(F,u) \leq c(F,v).$$

We now want to develop a measure of risk aversion that is related to the rate at which the certainty equivalent changes as we change the lottery. Consider a lottery over monetary outcomes that pays $x+\varepsilon$ with probability $1/2$ and $x-\varepsilon$ with probability $1/2$, call it F_ε. By Definition 3.4

$$u\big(c(F_\varepsilon,u)\big) = \tfrac{1}{2}u(x+\varepsilon) + \tfrac{1}{2}u(x-\varepsilon). \tag{3.7}$$

Since both sides of Equation (3.7) are twice differentiable in ε and $u'(c(F_\varepsilon, u)) > 0$ since u is increasing, the implicit function theorem implies that $c(F_\varepsilon, u)$ is twice differentiable in a neighborhood of 0. Differentiating (3.7) with respect to ε gives

$$u'\big(c(F_\varepsilon, u)\big)\, c'(\varepsilon) = \tfrac{1}{2}u'(x+\varepsilon) - \tfrac{1}{2}u'(x-\varepsilon).$$

Evaluating at $\varepsilon = 0$,

$$u'(x)\, c'(0) = 0 \quad\Longrightarrow\quad c'(0) = 0, \qquad c(0) = x.$$

Differentiating again with respect to ε,

$$u''\big(c(F_\varepsilon, u)\big)\big(c'(\varepsilon)\big)^2 + u'\big(c(F_\varepsilon, u)\big)c''(\varepsilon) = \tfrac{1}{2}u''(x+\varepsilon) + \tfrac{1}{2}u''(x-\varepsilon).$$

Evaluating at $\varepsilon = 0$ and using $c'(0) = 0$ and $c(0) = x$, we obtain

$$u'(x)c''(0) = u''(x) \quad\Longrightarrow\quad c''(0) = \frac{u''(x)}{u'(x)}.$$

The ratio between the second and first derivative of the utility function is the **Arrow-Pratt coefficient of absolute risk aversion**. It is not by chance that it appears here. As we noticed already, risk aversion is related to the concavity of the utility function, which is captured by its second derivative. In principle we could use the second derivative alone as a measure of risk aversion, but this would not be satisfactory, as multiplying the utility function by a positive constant would change the second derivative but not risk aversion. Dividing the second derivative by the first derivative solves this problem, as multiplying the utility function by a positive constant multiplies both derivatives by the same constant, leaving their ratio unchanged. The simplest modification of the measure to address such issue is to divide the second derivative by the first derivative, leading to the Arrow-Pratt coefficient.

Definition 3.7. *The **Arrow-Pratt coefficient of absolute risk aversion** for an individual with utility function u at outcome x is*

$$r(x, u) = -\frac{u''(x)}{u'(x)}.$$

Hence, we just showed that the limit of the second derivative of the certainty equivalent as $\varepsilon \to 0$ is exactly $-r(x, u)$. In the exercises, you are asked to show the following equivalence between certainty equivalents and the Arrow-Pratt coefficient.

Proposition 3.1. *An individual with utility function u is **more risk averse** than an individual with utility function v if and only if for each x*

$$r(x, u) \geq r(x, v),$$

where $r(x, u)$ and $r(x, v)$ are the Arrow-Pratt coefficients of absolute risk aversion for individuals with utility functions u and v.

Things to read. This section mostly draws from Mas-Colell et al. (1995, ch. 6.C). Alternatives treatments can be found in Kreps (1988, ch. 6) and Kreps (2013, ch. 6).

3.3 Exercises

Exercise 3.1. Check that the CDF in Figure 3.1 satisfies the three properties of a CDF.

Exercise 3.2. Show that, if an individual with expected utility preferences and utility function u is risk averse, his risk premium is non-negative for each lottery.

Exercise 3.3. Prove Proposition 3.1. (If you are stuck, check Kreps (2013) or exercises 6.C.6 and 6.C.7 in Mas-Colell et al. (1995).)

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Lecture 4

Applications and stochastic dominance

4.1 Applications

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4.2 Stochastic dominance

Things to read. leggi Mas-Colell et al. (1995).

4.3 Exercises

Exercise 4.1. ciao ciao

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Lecture 5

State space representation

5.1 States

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Things to read. leggi Mas-Colell et al. (1995)

5.2 Exercises

Exercise 5.1. ciao ciao

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