Lecture 6, Tuesday Oct 5

Student Presentation – Jonah

Quick overview of last time -

- Nonlin preview
- Ch. 4, generalized inverse, data and model resolution matrices

New –

- Hints for this week's homework
- Inversion example – tomography
- Singular value decomposition (Ch. 7)

Homework 5 – constrained inversion, due today

Homework 6 – Earth density problem, due Tues Oct 12

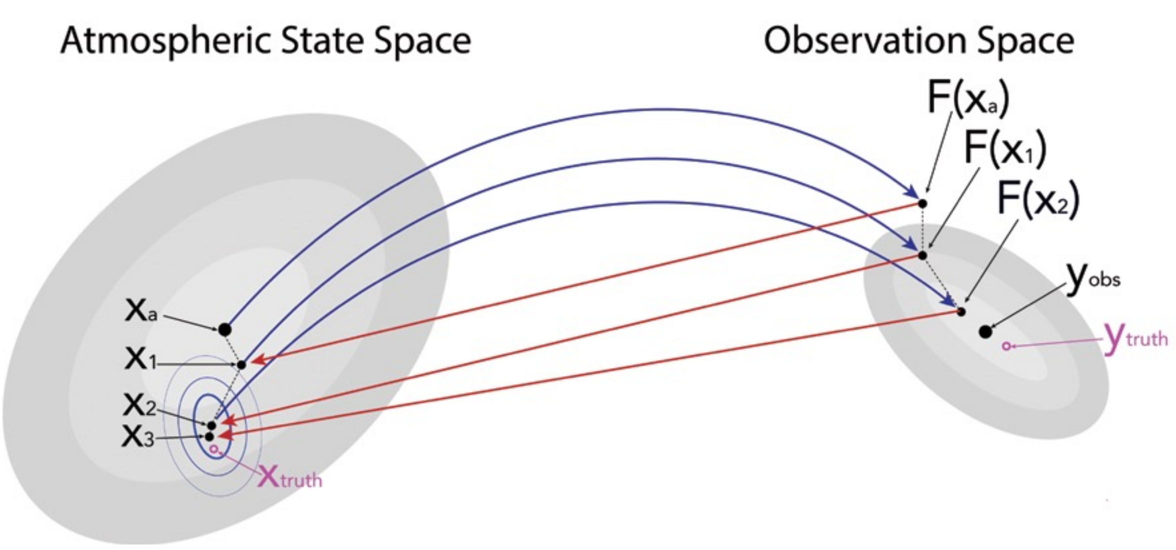
Start recording!

*reminder from Lindsay

Student presentation – Jonah

Optimal Estimation Retrievals and Their Uncertainties

What Every Atmospheric Scientist Should Know



Maahn et al., 2020

Some review from last time

Taylor Series:

$$f(m_0) \;+$$

$$d = f(m) = \left.\frac{df}{dm_1}\right|(m - m_0)$$

$$+ \left.\frac{df}{dm_2}\right|(m - m_0)$$

(partial derivatives)

$$+ \quad \frac{d^2 f}{dm_1^2}(m - m_0)^2 + \cdots$$

$$= f(m_0) + \sum_{j=1}^{m} \frac{df(m)}{dm_j}(m - m_0)$$

$$(f(m_0) = initial\ guess)$$

$$= f(m_0) + G_0(m - m_0)$$

$$f(m) - f(m_0) = G_0(m - m_0)$$

$$\Delta d = G\Delta m \quad iterate\ with\ new\ m_0$$

Definition C.3. Given a vector-valued function of a vector, $\mathbf{F}(\mathbf{x})$, where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix} \tag{C.5}$$

the **Jacobian** of $\mathbf{F}$ is

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} . \tag{C.6}$$

Some authors use the notation $\nabla\mathbf{F}(\mathbf{x})$ for the Jacobian. Notice that the rows of $\mathbf{J}(\mathbf{x})$ are the gradients (C.1) of the functions $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})$.

Aster, Borchers, Thurber (2019)
Parameter Estimation and Inverse Problems

Steps for nonlinear iterative inversions:

1) make initial guess of model parameters, m_0
2) calculate partial derivatives evaluated at m_0, this gives the matrix G
3) Since $G\Delta m = \Delta d$

   solve for $\Delta m = (G^T G)^{-1} G^T \Delta d$
4) solve $m_1 = m_0 + \Delta m$

   let this be the new model, go to step (1) and iterate

   stop when residual $\sum \left(d_{pred} - d_{obs} \right)^2$

   is small enough.

(Chapter 4)

Generalized Inverse $\mathbf{G}^{-g}$

$$\mathbf{m}^{est} = \mathbf{G}^{-g}\mathbf{d}^{obs}$$

looks like a matrix inverse
except

$M \times N$, not necessarily square
and
$\mathbf{G}\mathbf{G}^{-g}$ and $\mathbf{G}^{-g}\mathbf{G}$ don't have to $= \mathrm{I}$

$$\mathbf{m}^{est} = \mathbf{G}^{-g}\mathbf{d}^{obs}$$

$$\mathbf{m}^{est} = \boxed{[\mathbf{G}^T\mathbf{G}]^{-1}\mathbf{G}^T}\mathbf{d} \qquad = \mathbf{G}^{-g}\mathbf{d}^{obs}$$

$$\mathbf{m}^{est} = \boxed{\mathbf{G}^T[\mathbf{G}\mathbf{G}^T]^{-1}}\mathbf{d} \qquad = \mathbf{G}^{-g}\mathbf{d}^{obs}$$

$$\mathbf{m}^{est} = \boxed{[\mathbf{G}^T\mathbf{G} + \varepsilon^2\mathbf{I}]^{-1}\,\mathbf{G}^T}\mathbf{d} \qquad = \mathbf{G}^{-g}\mathbf{d}^{obs}$$

Data Resolution Matrix

$$G m = d$$

$$m^{est} = G^{-g} d$$

how well do predicted data match observed data?

$$d^{pre} = G m^{est} = \underbrace{G G^{-g}}_{N} d^{obs} = N d^{obs}$$

if $N = I$, $d^{pre} = d^{obs}$ Good! we've fit data perfectly!

for $d = n \times 1$

$N = G G^{-g}$ is an $n \times n$ square matrix and is called the data resolution matrix

$N = GG^{-g}$ is an $n \times n$ square matrix
and is called the data resolution matrix

overdetermined (LS)

$$G^{-g} = (G^TG)^{-1}G$$
$$N = G(G^TG)^{-1}G$$

underdetermined (min length)

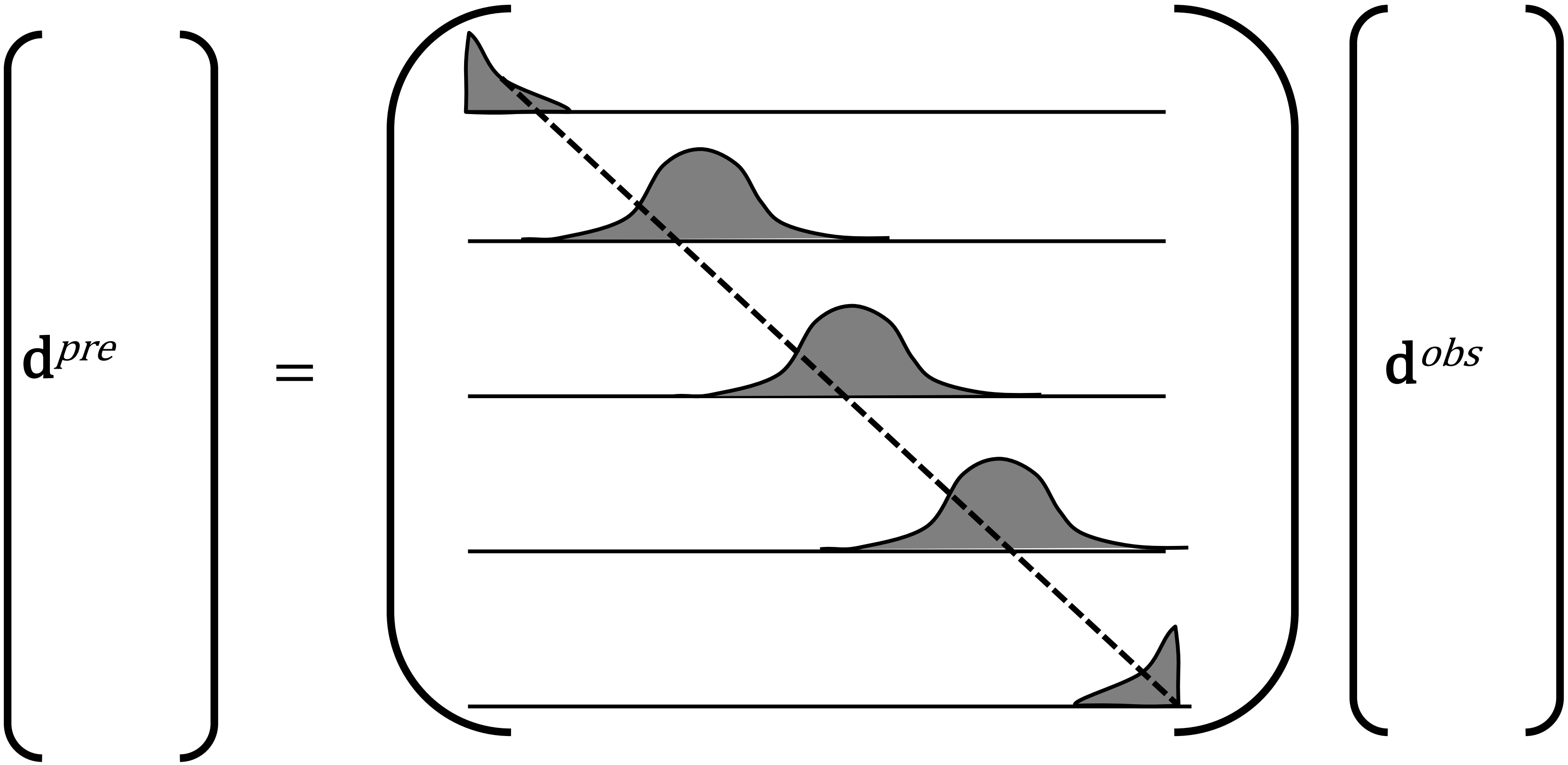
$$G^{-g} = G^T(GG^T)^{-1}$$
$$N = GG^T(GG^T)^{-1} = I$$

ie line fit ~~goes~~ through 1 point goes through that point

damped

$$G^{-g} = (GG^T + \varepsilon^2 I)^{-1}G$$
$$N = G(GG^T + \varepsilon^2 I)^{-1}G$$

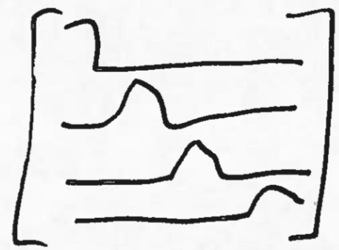
$$\mathbf{d}^{\mathbf{pre}} = \mathbf{N}\mathbf{d}^{\mathbf{obs}}$$



The closer **N** is to **I**, the closer d_i^{pre} is to d_i^{obs}

K

$$d^{pre} = N d^{obs}$$

$d^{pre} =$  d^{obs}

rows of N describe how well data can be predicted, or <u>resolved</u>.

if rows have sharp maximum centered on diagonal, then data are well resolved.

if graphs broad, data poorly resolved.

Resolution matrix shows how much weight each observation has in influencing the predicted value.

Model Resolution Matrix

$Gm = d$

imagine $G m^{true} = d^{obs}$

try to find out how close

m^{est} is to m^{true}

$$m^{est} = G^{-g} d^{obs} = \underbrace{G^{-g} G}_{R} m^{true} = R\, m^{true}$$

R is the model resolution matrix

R is the model resolution matrix

overdetermined (LS)

$$R = G^{-g} G = (G^T G)^{-1} G^T G = I$$

underdetermined

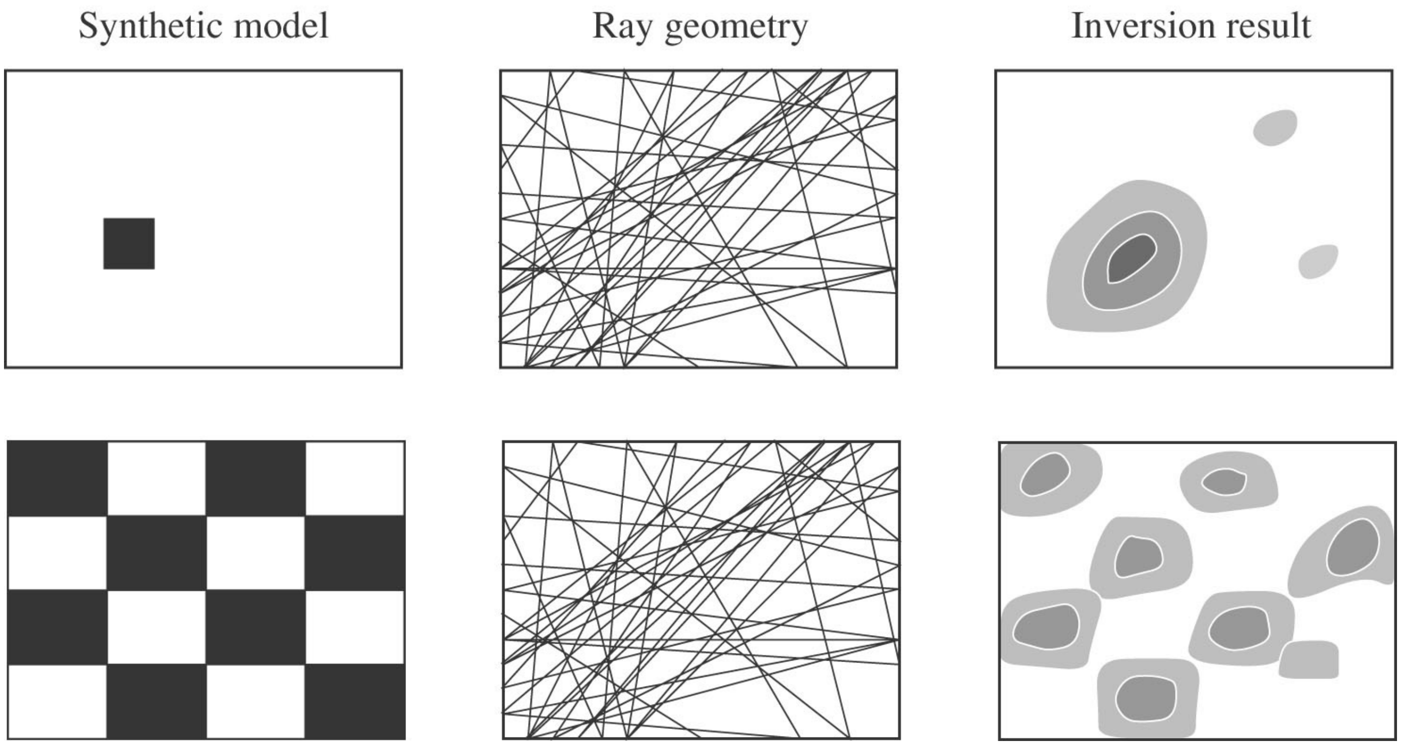
$$R = G^{-g} G = G^T (G G^T)^{-1} G$$

useful for telling how well model parameters are resolved

the simple least squares solution
minimizes the spread of data resolution
and
has zero spread of the model resolution

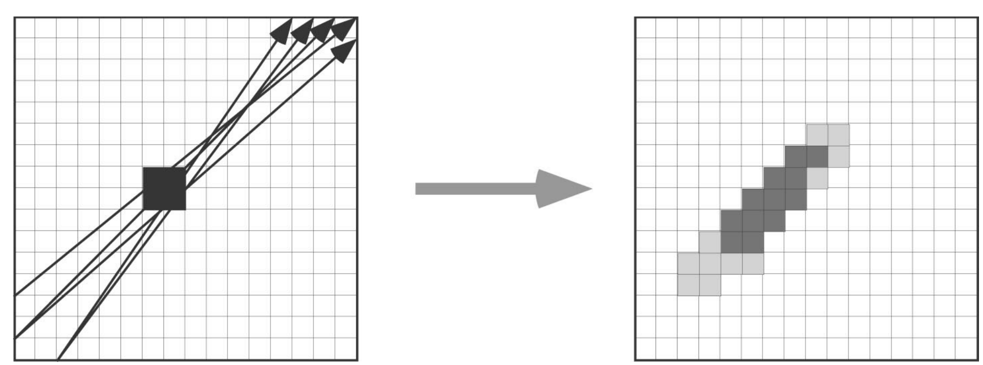
the minimum length solution
minimizes the spread of model resolution
and
has zero spread of the data resolution

Synthetic models – another way to test model resolution

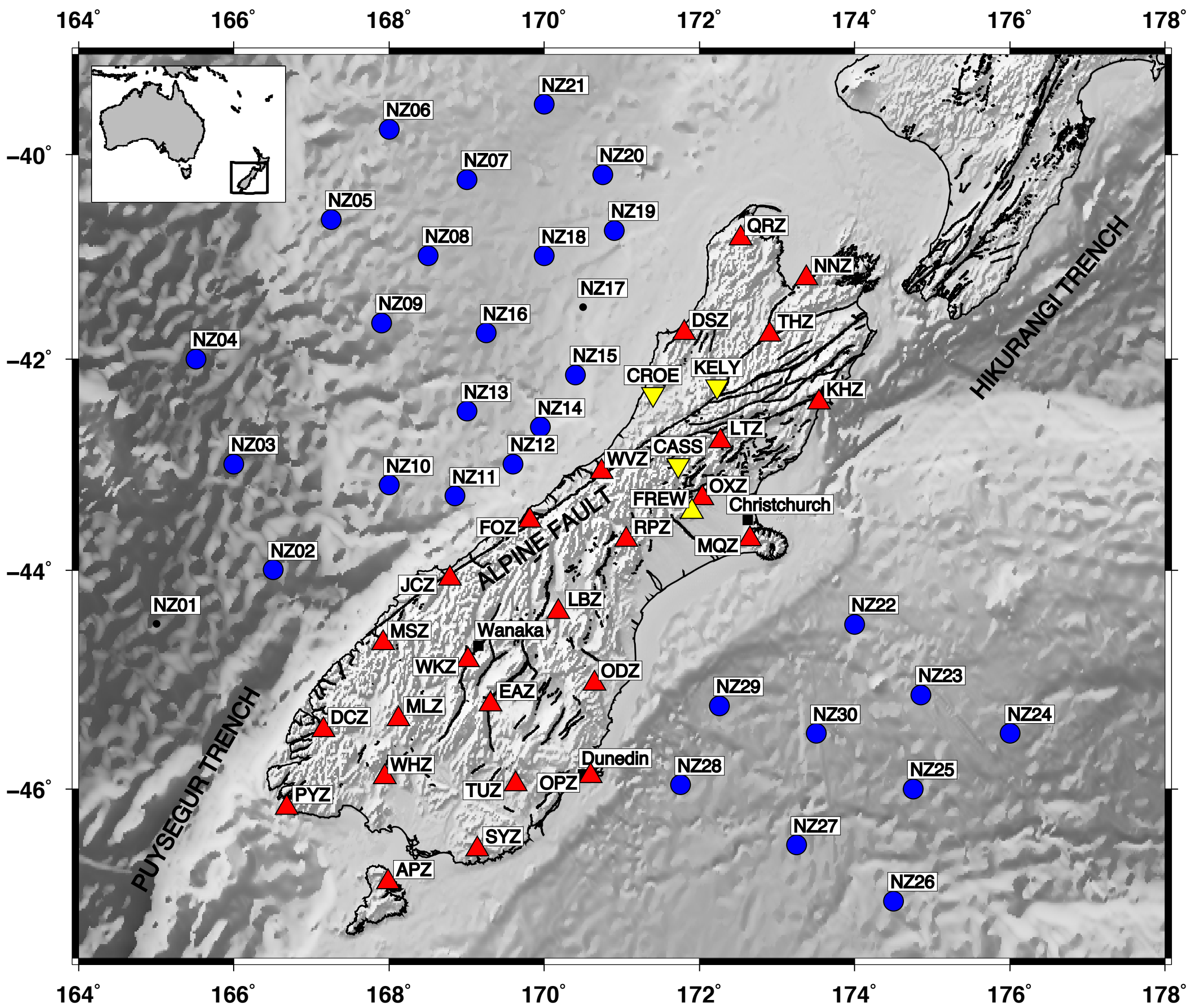


'feature recovery'

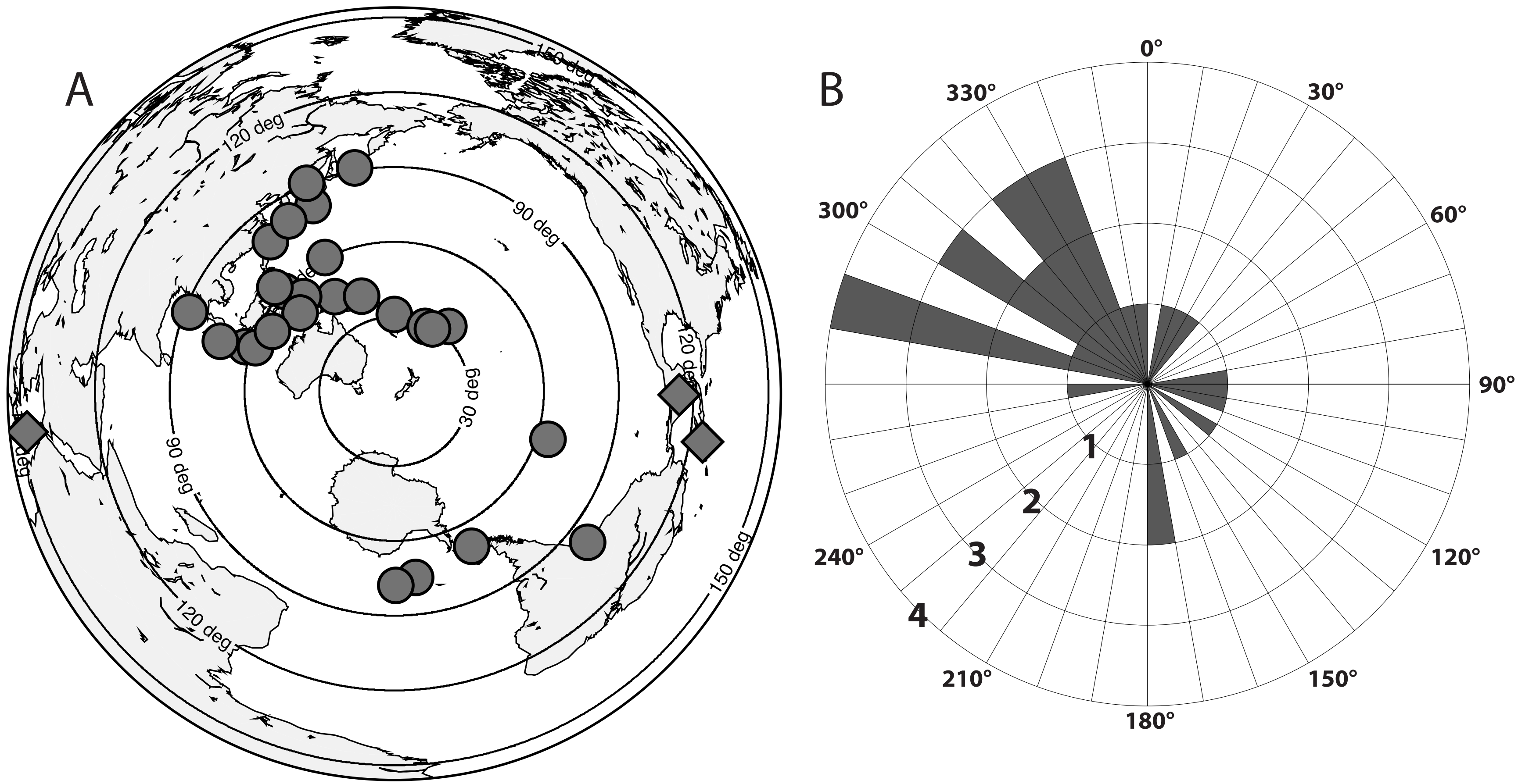
'checkerboard test'



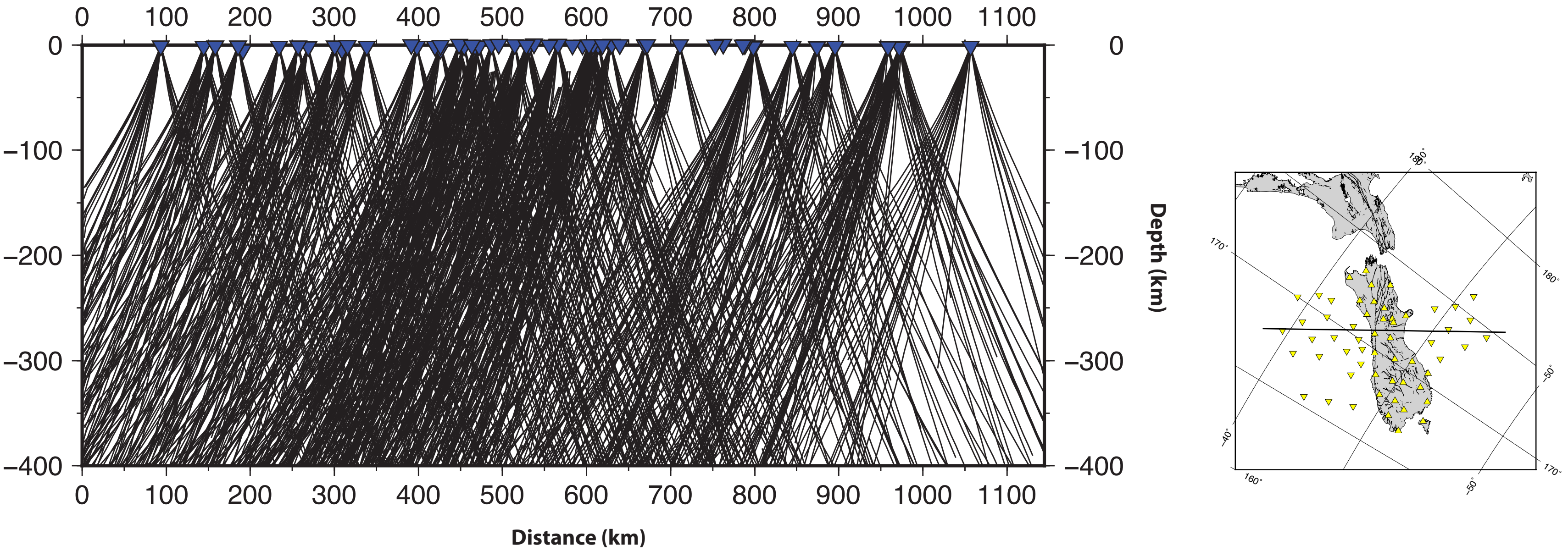
Sample inverse problem
New Zealand mantle tomography, Dan Zietlow, CU (now at NCAR)



Zietlow et al., JGR, 2016

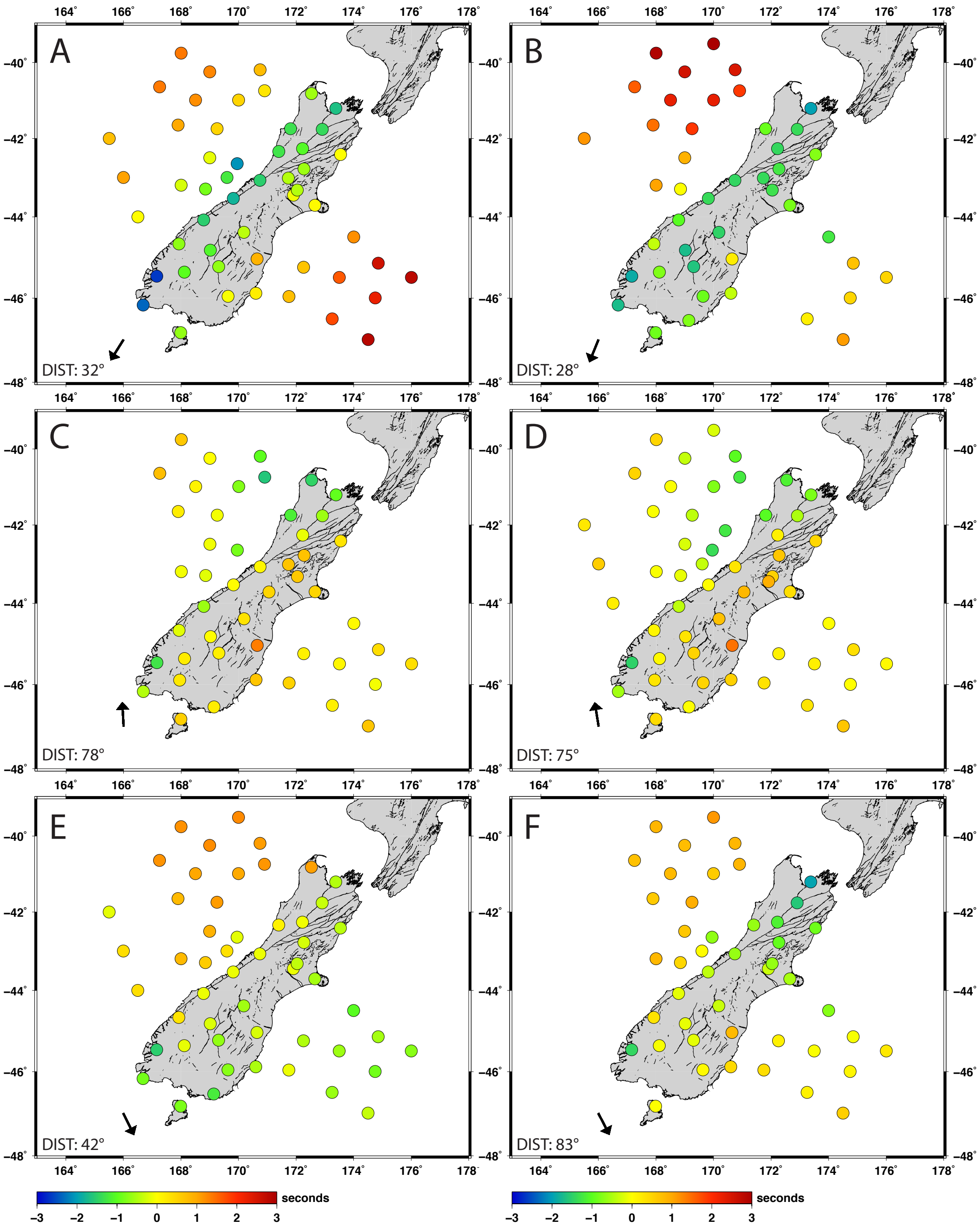


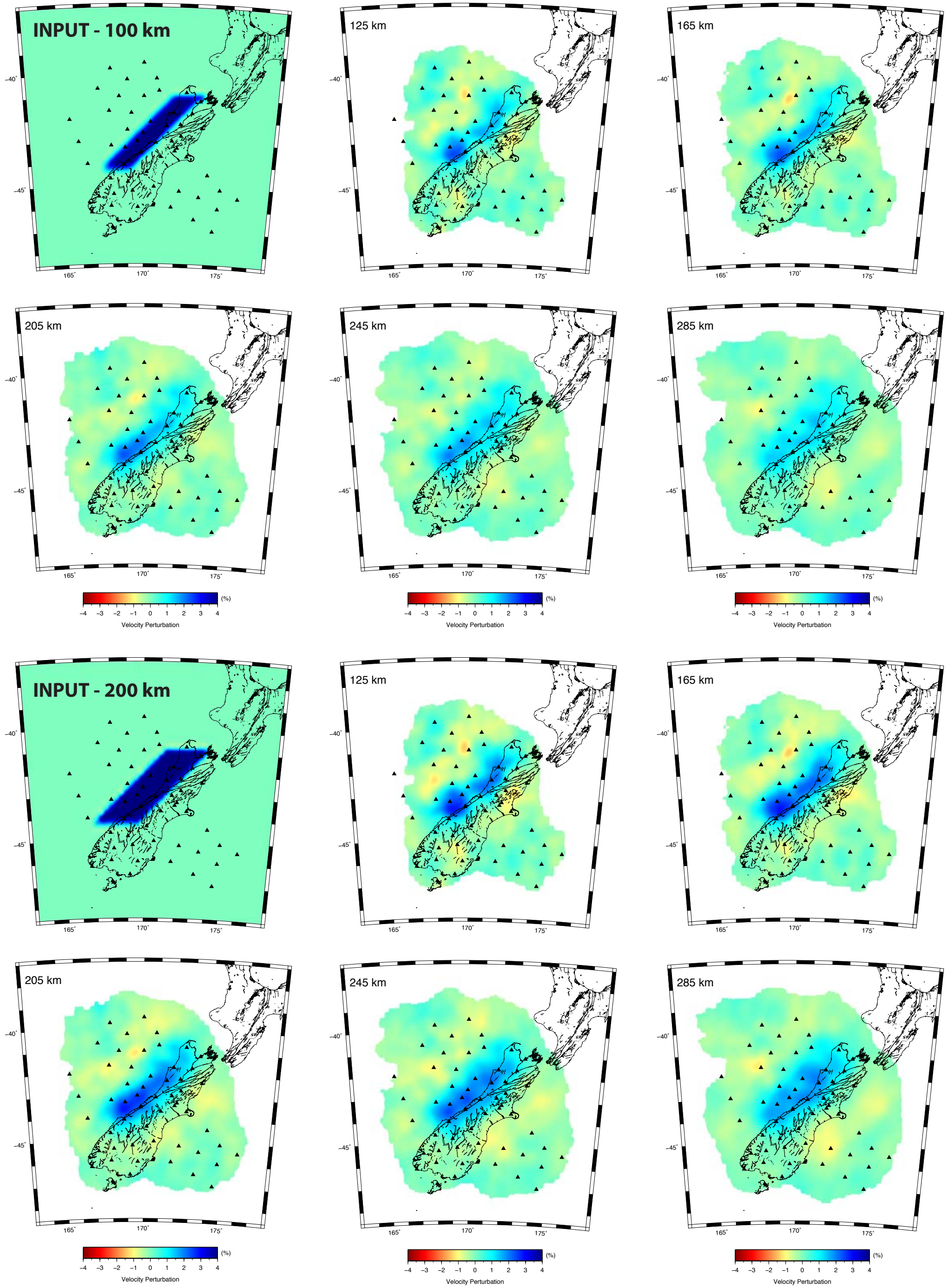
A
150 deg
120 deg
90 deg
30 deg
90 deg
120 deg
150 deg
B
0°
30°
60°
90°
120°
150°
180°
210°
240°
300°
330°
1
2
3
4



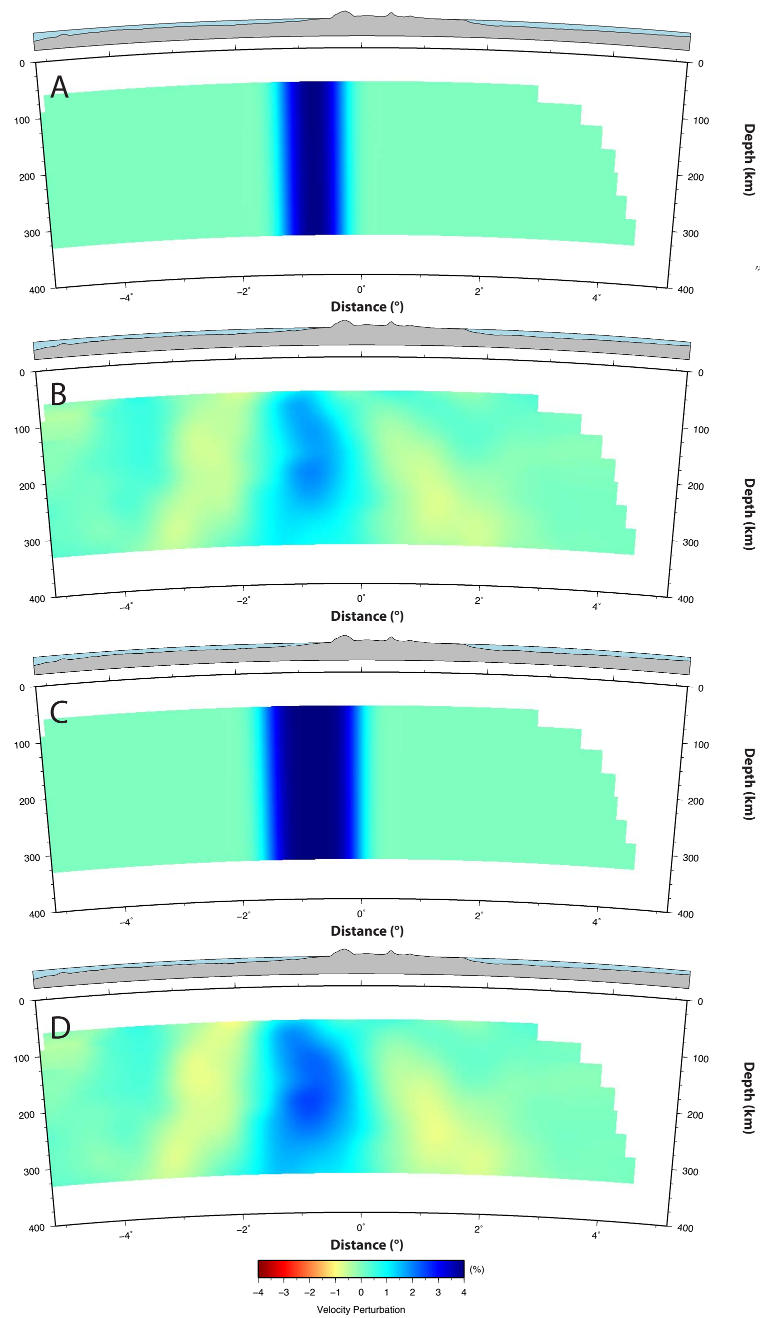
0
100
200
300
400
500
600
700
800
900
1000
1100
0
−100
−200
−300
−400
Depth (km)
Distance (km)
170°
180°
−40°
−50°
160°

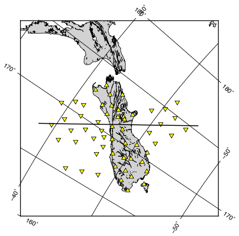
Travel times





Feature recovery tests



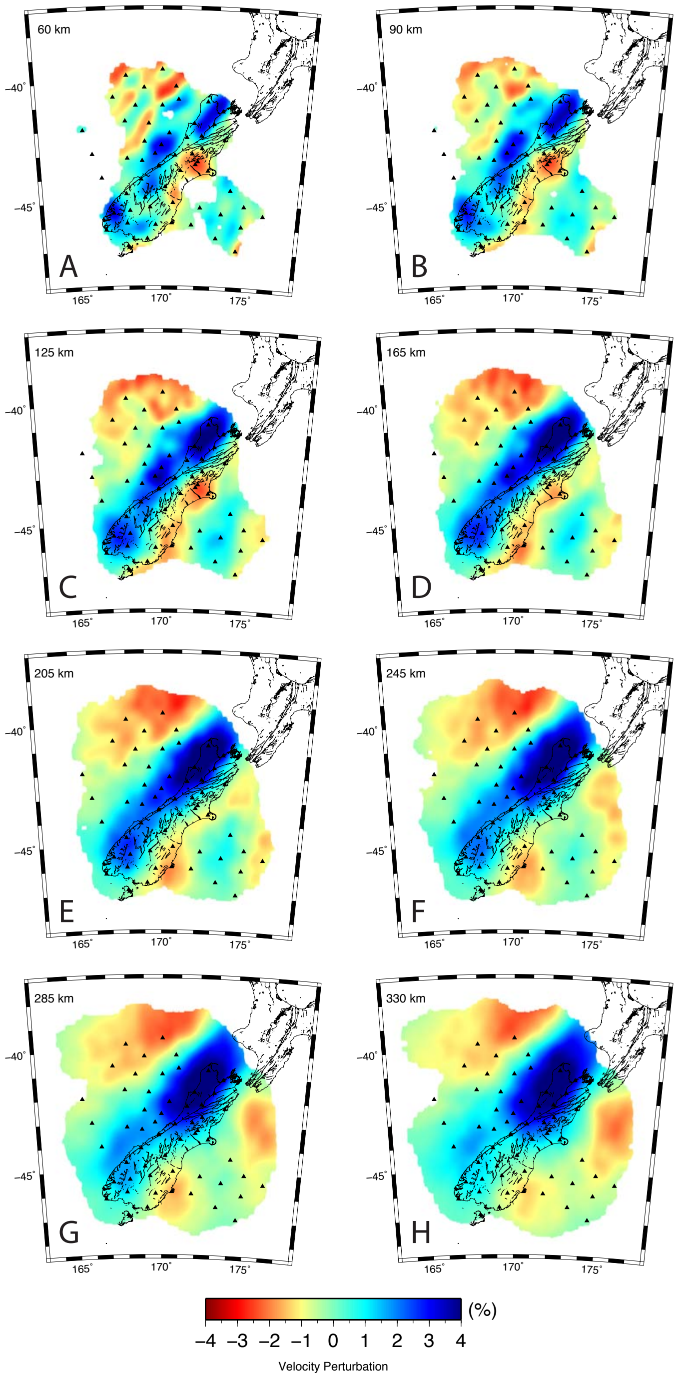


Feature recovery tests

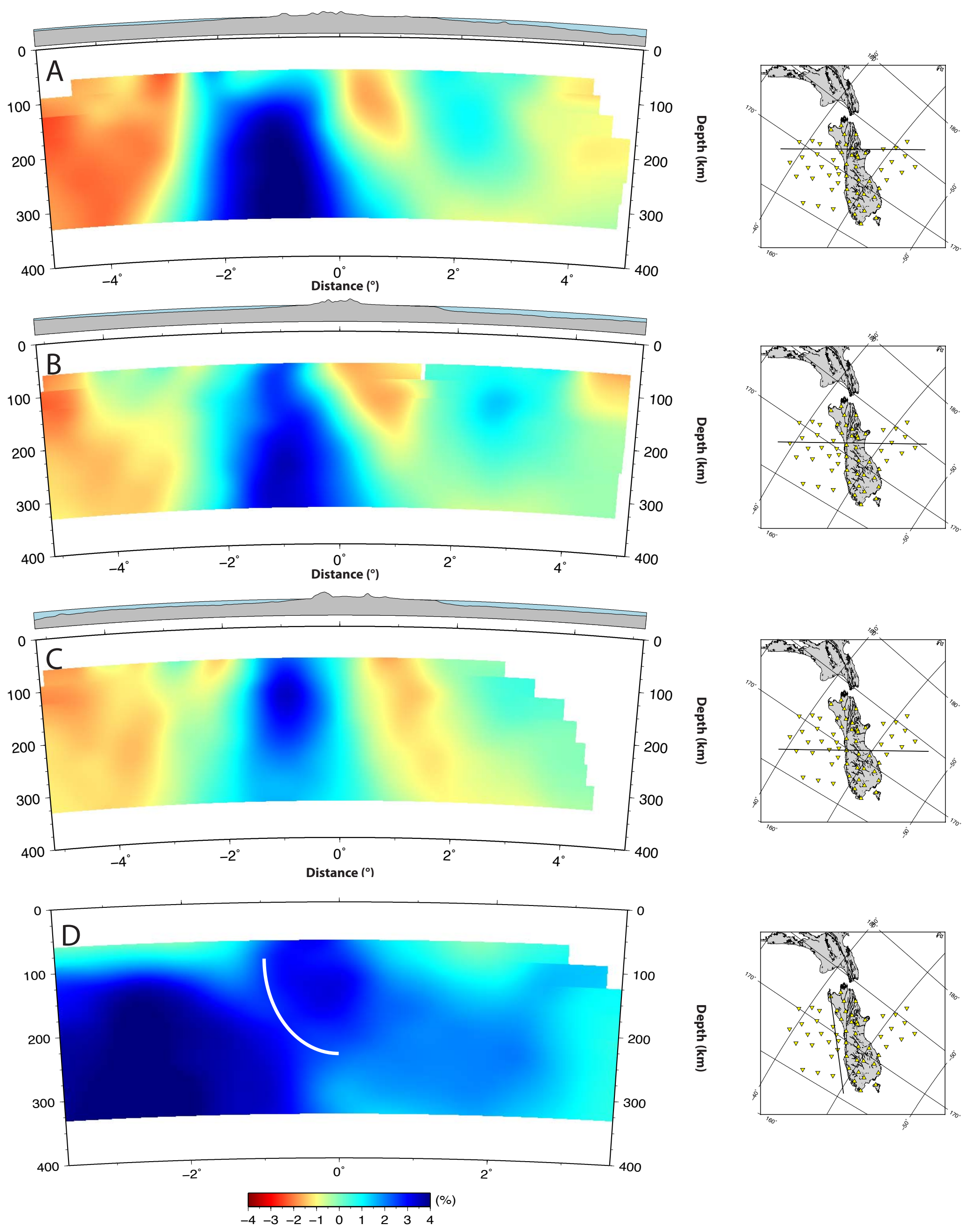
Cost function for the inversion

$$\boldsymbol{E} = \|A\boldsymbol{m} - \boldsymbol{d}\|^2 + \gamma\|L\boldsymbol{m}\|^2 + \varepsilon\|\boldsymbol{m}\|^2$$

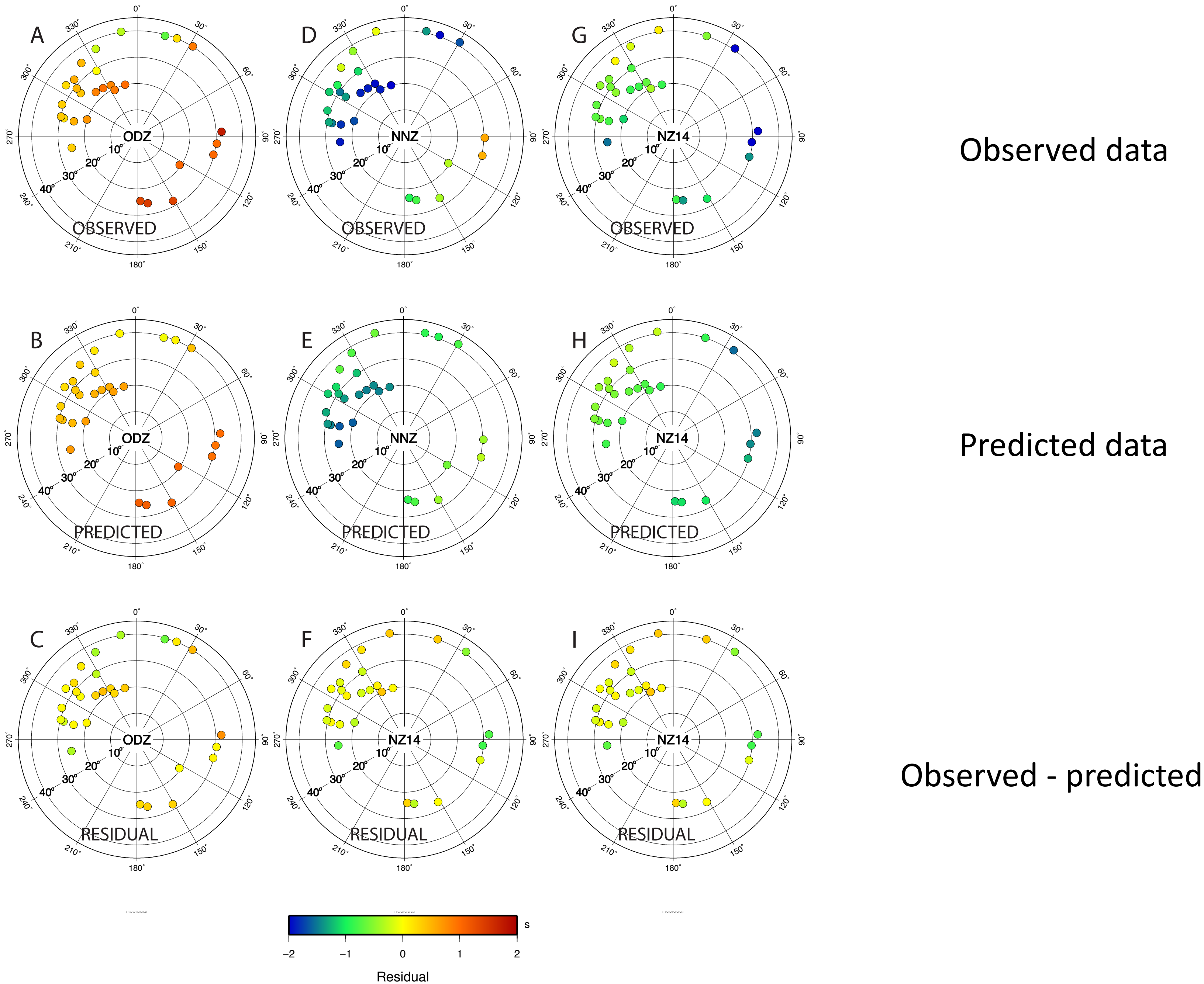
where ***A*** is a matrix that contains the partial derivatives relating travel time residuals to the model parameters, ***m*** is a vector containing the perturbations to the velocity model, ***d*** are the differential travel time data, ***γ*** is the smoothing parameter, ***L*** is the smoothing matrix (similar to a Laplacian smoothing matrix) where the weight of smoothing between neighboring nodes decreases with inverse distance, and ***ε*** is the damping parameter



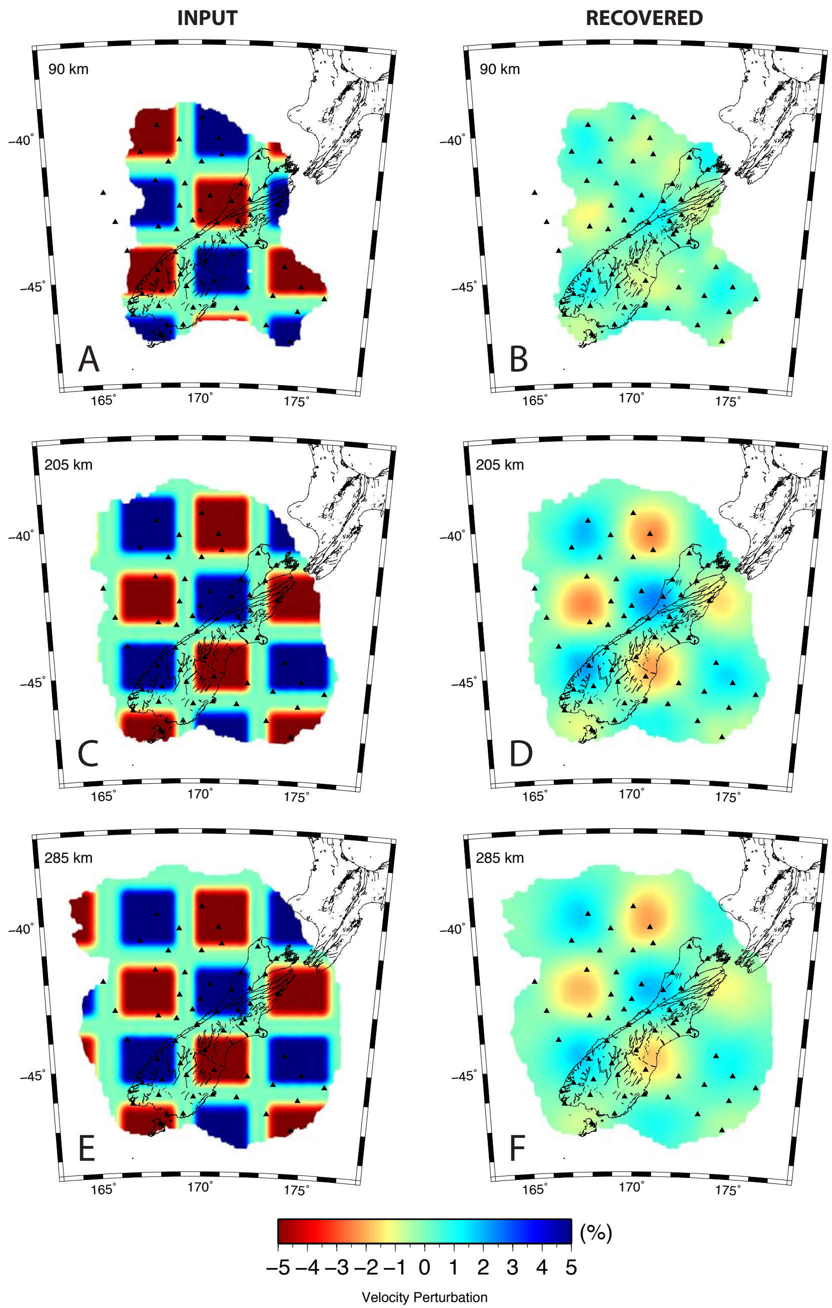
Tomography results



Tomography results



A
ODZ
OBSERVED
D
NNZ
OBSERVED
G
NZ14
OBSERVED
Observed data
B
ODZ
PREDICTED
E
NNZ
PREDICTED
H
NZ14
PREDICTED
Predicted data
C
ODZ
RESIDUAL
F
NZ14
RESIDUAL
I
NZ14
RESIDUAL
Observed - predicted
0°
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
10°
20°
30°
40°
s
−2
−1
0
1
2
Residual



Checkerboard tests

GEOL/PHYS 6670 Inverse Theory Homework 6
Due on Tuesday, October 12

1. Given mass (M_e) and moment of inertia (Ie) of the Earth, use methods of geophysical inverse theory to solve for the density of the Earth's crust, mantle, and core. State any assumptions made, your reason for choice of inverse method, and show and clearly describe your work. Include copies of computer code written for this problem.
2. (optional) Calculate the uncertainties of the model parameters (crustal density, mantle density, core density).

Important equations.

For a a spherically symmetric Earth model, the mass M_e, and moment of inertia I_e are determined by the radial density $\rho(r)$, where

$$M_e = \int_0^{R_e} \left(4\pi r^2\right) \rho(r)\ dr \tag{3.57}$$

and

$$I_e = \int_0^{R_e} \left(\frac{8}{3}\pi r^4\right) \rho(r)\ dr\ . \tag{3.58}$$

Using $R_e = 6.3708 \times 10^6$ m as the radius of a spherical Earth, and supposing that from astronomical measurements we can infer t $M_e = 5.973 \pm 0.0005 \times 10^{24}$ kg and $I_e = 8.02 \pm 0.005 \times 10^{37}$ kg $\cdot$, I put some hints at the bottom of the HW 6 handout

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Some hints (please think about the problem on your own before looking at the hints):

How many model parameters (3 – density of crust, density of mantle, density of core)
How many data parameters (2 – mass and moment of inertia)
Is the problem underdetermined or overdetermined? (underdetermined)
Since underdetermined, simple least squares won't work. Consider using minimum length solution.
Minimum length solution minimizes model length. You might end up with negative crustal densities, which don't make sense. Consider guessing initial model parameters and minimize $(m-\langle m\rangle)^T(m-\langle m\rangle)$ instead of m^Tm.

Hints on model uncertainty calculation. You can use Menke eqn 2.17b, where M is G^{-g}, and assuming uncorrelated data you can let (cov d) = $\sigma_d^2 I$.

- Earth density homework (HW 6)
  - Negative crustal densities? Think why that might be
  - Minimum length minimizes model length
  - Works better to use minimum length where instead of minimizing $m^T m$ you minimize $(m-\langle m \rangle)^T(m-\langle m \rangle)$. To do this guess initial model parameters and then solve.

SVD

Singular Value Decomposition

Menke Chapter 7

Singular Value Decomposition

eigenvalues & eigenvectors

eigenvalue problem

(do examples on board)

consider

eigenvalue, eigenvector

$$\underset{=}{A}\,\underline{x} = \lambda\,\underline{x}$$

matrix, vector, scalar, vector

This will have a solution if

$$(\underset{=}{A} - \lambda \underset{=}{I})\,\underline{x} = 0$$

eigenvalues are roots of

$$|(\underset{=}{A} - \lambda \underset{=}{I})| = 0$$

$$(\det(\underset{=}{A} - \lambda \underset{=}{I}) = 0)$$

for general nonsquare matrix M

$$\underline{\underline{M}} = \underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^T$$

← eigenvectors of $M^T M$

matrix with eigenvalues of $M^T M$ and MM^T

eigenvectors of MM^T

and

$$\underline{\underline{M}}^{-1} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}^{-1}\,\underline{\underline{U}}^T$$

Why SVD is useful

Strang,

"Singular values" are the square roots of the r positive eigenvalues of $A^T A$, which are also the r positive eigenvalues of AA^T. These square roots are the only nonzero entries in the $m \times n$ diagonal matrix Σ

$$\underset{m \times n}{A} = \underset{m \times m}{Q_1} \; \underset{m \times n}{\Sigma} \; \underset{n \times n}{Q_2^T}$$

pseudoinverse when A not invertible

Do rest on checkboard, but include copy of notes in the slides example, find eigenvalues and eigenvectors of $\underline{\underline{M}} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$

so I want λ & $\underline{x}$ such that

$$\underline{\underline{M}}\,\underline{x} = \lambda \underline{x}$$

$$(\underline{\underline{M}} - \lambda \underline{\underline{I}})\,\underline{x} = 0$$

so $\det(M - \lambda I) = \det \begin{pmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{pmatrix} = 0$

recall

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

so

$$\det \begin{pmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 30$$

$$= \lambda^2 - 3\lambda + 2 - 30 = \lambda^2 - 3\lambda - 28$$

$$= (\lambda - 7)(\lambda + 4) = 0$$

$\lambda_1 = 7$, $\lambda_2 = -4$

for $\lambda_1 = 7$

$$(M - \lambda I)x = 0$$

$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - 7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-6x_1 + 6x_2 = 0$$
$$5x_1 - 5x_2 = 0$$

$$\Rightarrow$$

$$\underline{v_1} = \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is eigenvector corresponding to eigenvalue $\lambda_1 = 7$ and every other eigenvector belonging to $\lambda_1 = 7$ is a multiple of $\underline{v}$

next $\lambda_2 = -4$ gives

$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 5 & 6 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$5x_1 + 6x_2 = 0$$

$$x_1 = \frac{-6}{5}x_2$$

$$\underline{v}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underset{\substack{\uparrow \\ \text{any constant}}}{C}\begin{pmatrix} -6/5 \\ 1 \end{pmatrix}$$

an aside:

eigenvalue shorthand ew german: eigenwert

eigenvector shorthand ev german: eigenvektor

some useful matrix identities

AB not necessarily $= BA$

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

for orthogonal matrix

$$Q^{-1} = Q^T$$

if A is real and symmetric,
it can be written as

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^T$$

transpose of Q

eigenvectors of A

eigenvalues of A

and

$$A^{-1} = (Q \Lambda Q^T)^{-1}$$

$$= Q^{T^{-1}} \Lambda^{-1} Q^{-1}$$

$$= Q \Lambda^{-1} Q^T$$

for general nonsquare matrix M

$$\underline{\underline{M}} = \underline{\underline{U}}\, \underline{\underline{\Lambda}}\, \underline{\underline{V}}^T$$

eigenvectors of M^TM

matrix with eigenvalues of M^TM and MM^T

eigenvectors of MM^T

and

$$\underline{\underline{M}}^{-1} = \underline{\underline{V}}\, \underline{\underline{\Lambda}}^{-1} \underline{\underline{U}}^T$$

linking SVD to data & model

$$\underset{n \times m}{\underline{\underline{G}}} \; \underset{m \times 1}{\underline{m}} = \underset{n \times 1}{\underline{d}}$$

$$\underset{m \times n}{G^T} \; \underset{n \times m}{G} \rightarrow m \times m$$

dimensions of model

$$\underset{n \times m}{G} \; \underset{m \times n}{G^T} \rightarrow n \times n$$

dimensions of data

$$G = U \Lambda V^T$$

V^T → eigenvectors of $G^T G$ dimensions of model space

U → eigenvectors of $G G^T$ dimensions of data space

Λ — $(\text{eigenvalues})^{1/2}$ of $G G^T$ and $G^T G$ referred to as 'singular values' of matrix G

Theorem: The eigenvalues of a real symmetric matrix are all real and the eigenspaces corresponding to the distinct eigenvalues are all orthogonal.

(previous example was not symmetric)

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example:

$$\text{let } \underline{\underline{M}} = \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix} \qquad \text{symmetric}$$

$$\text{set } \det\left[M - \lambda I \right] = 0$$

$$\det \begin{bmatrix} 4-\lambda & 8 \\ 8 & 4-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)^2 - 64 = 0$$

$$16 - 8\lambda + \lambda^2 - 64 = 0$$

$$\lambda^2 - 8\lambda - 48 = 0$$

eigenvalues $\lambda = 12 \text{ or } -4$

eigenvectors are then (next page)

eigenvectors are

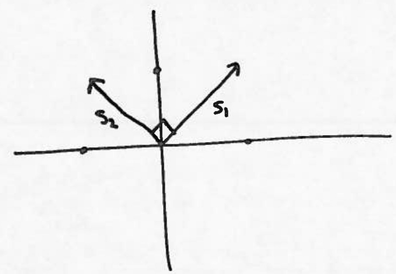
for $\lambda = 12$

$$(M - 12I)\,\underline{s_1} = 12\,\underline{s_1} \quad \leftarrow \lambda$$

$$\begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \underline{s_1} = 12\,\underline{s_1}$$

gives $s_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $s_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



eigenvectors are orthogonal

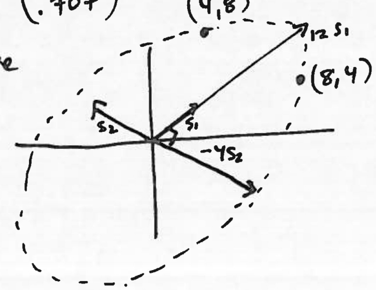
(2 vectors are orthogonal if their dot product is zero $x^T y = 0$ ie $x_1 y_1 + x_2 y_2 + \ldots x_n y_n = 0$)

normalize s_1 & s_2 and mult by corresponding eigenvalues

$s_1 = \begin{pmatrix} .707 \\ .707 \end{pmatrix}, \lambda_1 = 12$

$s_2 = \begin{pmatrix} -.707 \\ .707 \end{pmatrix}, \lambda_2 = {}^{-}4$

eigenvectors give axes and eigenvalues give magnitudes



$M = \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$

consider columns of M as vectors

$\begin{pmatrix} 4 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ & $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ lie on the ellipse with eigenvectors as major axes

Derivation of Singular Value Decomposition

Menke section 7.7, p 138

consider a nonsquare $N \times M$ matrix G

form an $(N+M) \times (N+M)$ square symmetrix matrix from $\underset{N\times M}{G}$ & $\underset{M\times N}{G^T}$

$$S = \begin{pmatrix} 0 & G \\ G^T & 0 \end{pmatrix}$$

symmetric so will have real eigenvalues and orthonormal eigenvect

solve the eigenvalue problem $\underline{\underline{S}}\,\underline{w} = \lambda \underline{w}$

partition $\underline{w}$ into a part $\underline{u}$ of length N and part $\underline{v}$ of length M

$$① \quad \begin{pmatrix} 0 & G \\ G^T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

multiply out

$$② \quad \begin{aligned} G v &= \lambda u \\ G^T u &= \lambda v \end{aligned}$$

multiply both sides of ① by $\begin{pmatrix} 0 & G \\ G^T & 0 \end{pmatrix}$

$$③ \quad \begin{pmatrix} 0 & G \\ G^T & 0 \end{pmatrix}\begin{pmatrix} 0 & G \\ G^T & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \underbrace{\begin{pmatrix} 0 & G \\ G^T & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}}_{\to\, = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \text{ from } ①} = \lambda^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

10

④ $$\begin{pmatrix} GG^T & 0 \\ 0 & G^TG \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

multiply out

⑤ $$GG^T u = \lambda^2 u$$
$$G^TG v = \lambda^2 v$$

thus v & u are eigenvectors of GG^T and G^TG with the same eigenvalues

There can be at most $p \leq \min(N,M)$ nonzero eigenvalues

can form an $M \times M$ matrix with v_i's as columns and call it V

$$V = \begin{bmatrix} v_1 & v_2 & \cdots \\ \downarrow & \downarrow & \end{bmatrix}$$

also an $N \times N$ matrix U

$$U = \begin{bmatrix} u_1 & u_2 & \cdots \\ \downarrow & \downarrow & \end{bmatrix}$$

form $p \times p$ matrix $\Lambda_p = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix}$

Truncated SVD

truncated SVD

$A = U \Lambda V^T$

$A^{-1} = V \Lambda^{-1} U^T$

let U, V, Λ be 4×4

get rid of smallest λ, what happens?

$$A = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix} \begin{bmatrix} V_1 \longrightarrow \\ V_2 \longrightarrow \\ V_3 \longrightarrow \\ V_4 \longrightarrow \end{bmatrix}$$

4×4 4×4 V^T

4×3 omit $\lambda 4$ 3×3 3×4

$A^{-1} = V \Lambda^{-1} U^T$

$$= \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & 1/\lambda_3 & \\ & & & 1/\lambda_4 \end{bmatrix} \begin{bmatrix} u_1 \longrightarrow \\ u_2 \longrightarrow \\ u_3 \longrightarrow \\ u_4 \longrightarrow \end{bmatrix}$$

U^T

4×4 4×4 4×4

4×3 3×3 3×4

Truncated SVD

Truncate too soon –
solution lacks details from model vectors associated with small singular values

Keep too many small singular values –
solution unstable in presence of noise

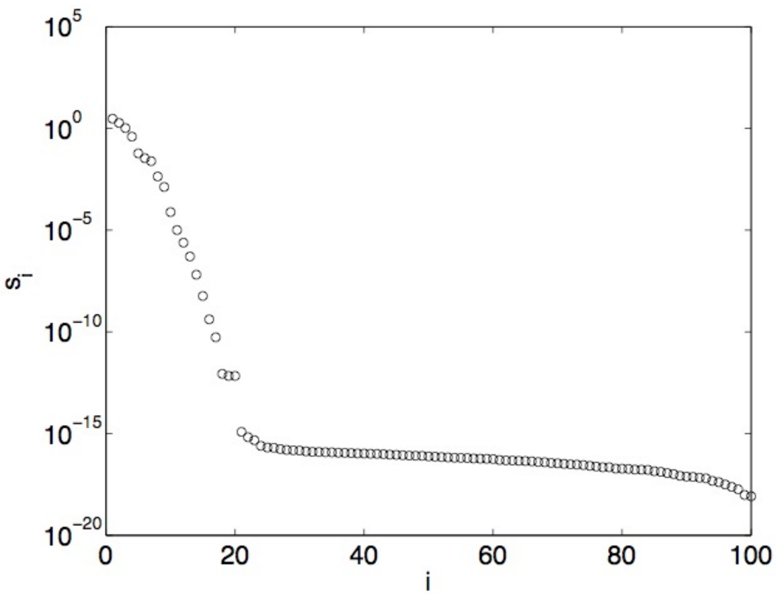


Figure 4.26: Singular values of $\mathbf{G}$ for the Shaw example ($n = 100$).

EQ location example with SVD

have earthquake P waves at 6 stns (Table 6.1)

from Lay & Wallace

guess solution

$x_0 = 21, y_0 = 21, z_0 = 12, t_0 = 30$

determine $\underline{\underline{G}}$, $G_{ij} = \frac{\partial d_i}{\partial m_j}$

$$t_i = t_0 + \frac{\left[(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2\right]^{1/2}}{v}$$

Lay and Wallace

Lay 9.
Wallace

TABLE 6.1 Earthquake Location Examples: Station Location

Station	x	y	P-wave time	$d_i - d_o$
1	2.0	31.0	40.02	5.77
2	3.0	−5.0	42.76	6.93
3	50.0	58.0	41.07	2.70
4	55.0	47.0	40.38	2.72
5	81.0	3.0	45.05	4.05
6	−9.0	−18.0	45.75	7.02

$$\mathbf{G} = \begin{bmatrix} 0.1331 & -0.0700 & 0.0841 & 1.000 \\ 0.0917 & 0.1325 & 0.0612 & 1.000 \\ -0.1030 & -0.1315 & 0.0426 & 1.000 \\ -0.1318 & -0.1004 & 0.0465 & 1.000 \\ -0.1621 & 0.0486 & 0.0324 & 1.000 \\ 0.1021 & 0.1327 & 0.0408 & 1.000 \end{bmatrix}. \tag{6.56}$$

$\partial t/\partial x \quad \partial t/\partial y \quad \partial t/\partial z \quad \partial t/\partial t_0$

$$\Lambda = \begin{bmatrix} 2.452 & 0 & 0 & 0 \\ 0 & 0.342 & 0 & 0 \\ 0 & 0 & 0.2101 & 0 \\ 0 & 0 & 0 & 0.0199 \end{bmatrix} \tag{6.57}$$

$$\mathbf{V} = \begin{bmatrix} -0.0117 & 0.7996 & -0.5854 & -0.1327 \\ 0.0018 & 0.5981 & 0.7961 & 0.0915 \\ 0.0512 & 0.0524 & -0.1528 & 0.9855 \\ 0.9986 & 0.0056 & -0.0005 & 0.0522 \end{bmatrix}.$$

$$(6.58)$$

$$\underline{\underline{D}} = \underline{\underline{u}}\,\underline{\underline{u}}^T$$

$$\mathbf{D} = \begin{bmatrix} 0.8120 & 0.2605 & 0.0987 & 0.1328 & -0.2265 & 0.0775 \\ 0.2605 & 0.5887 & -0.2663 & -0.0232 & 0.2741 & 0.1662 \\ 0.0987 & -0.2663 & 0.6142 & 0.3450 & 0.0160 & 0.1922 \\ 0.1328 & -0.0232 & 0.3450 & 0.3906 & 0.2881 & -0.1333 \\ -0.2265 & 0.2741 & 0.0160 & 0.2881 & 0.6951 & -0.0467 \\ -0.0776 & 0.1662 & 0.1922 & -0.1333 & -0.0467 & 0.8992 \end{bmatrix}. \quad (6.59)$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{m} = \begin{bmatrix} 8.268 \\ 9.704 \\ 9.063 \\ 4.480 \end{bmatrix}. \quad (6.60)$$

TABLE 6.2 Comparison of Hypocentral Guess, Solutions After Various Iterations, and True Model Values

Parameter	Initial guess	1st iteration	3rd iteration	6th iteration	True value
x	21.0	29.3	29.9	30.0	30.0
y	21.0	30.7	30.2	30.2	30.0
z	12.0	21.1	9.1	8.9	8.0
t	30.0	34.5	34.9	35.0	35.0

Insight from SVD

EQ SVD, 2

eigenvalues of $\underline{\underline{G}}^T\underline{\underline{G}}$ given by $\underline{\underline{\Lambda}}$

From looking at eigenvectors and eigenvalues can gain insight

biggest eigenvalue, 2.452, corresponds to first column of $\underline{\underline{V}}$

biggest value of 1st column of $\underline{\underline{V}}$ is the 4th value, corresponding to t_0.

so t_0 is important, stable, well resolved

smallest eigenvalue, .0199, corresponds to 4th column of $\underline{\underline{V}}$. There the 3rd value, z, dominates. Thus estimate of change of depth least stable part of this inverse.

Data resolution matrix, $\underline{\underline{D}} = \underline{\underline{U}}\,\underline{\underline{U}}^T$, lets you see how important the observations are.

Diagonal elements give importance of each station. Off-diagonal elements give measure of influence other stations have on a given value.

inversion predicts change to $\underline{m}$, iterate.

$$\Lambda = \begin{bmatrix} 2.452 & 0 & 0 & 0 \\ 0 & 0.342 & 0 & 0 \\ 0 & 0 & 0.2101 & 0 \\ 0 & 0 & 0 & 0.0199 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.0117 & 0.7996 & -0.5854 & -0.1327 \\ 0.0018 & 0.5981 & 0.7961 & 0.0915 \\ 0.0512 & 0.0524 & -0.1528 & 0.9855 \\ 0.9986 & 0.0056 & -0.0005 & 0.0522 \end{bmatrix}.$$

(6.58)

$\underline{\underline{D}} = \underline{\underline{U}}\,\underline{\underline{U}}^T$

Data resolution matrix

$$D = \begin{bmatrix} 0.8120 & 0.2605 & 0.0987 & 0.1328 & -0.2265 & 0.0775 \\ 0.2605 & 0.5887 & -0.2663 & -0.0232 & 0.2741 & 0.1662 \\ 0.0987 & -0.2663 & 0.6142 & 0.3450 & 0.0160 & 0.1922 \\ 0.1328 & -0.0232 & 0.3450 & 0.3906 & 0.2881 & -0.1333 \\ -0.2265 & 0.2741 & 0.0160 & 0.2881 & 0.6951 & -0.0467 \\ -0.0776 & 0.1662 & 0.1922 & -0.1333 & -0.0467 & 0.8992 \end{bmatrix}$$

Lay and Wallace

Next time,

Gradient descent

Back propagation

Some modifications to syllabus – guest lecture on machine learning on Oct 19