FOURIER SERIES

Purpose

- Fourier series used to analyze periodic signals.
- The harmonic content of the signals is analyzed with the help of Fourier series.
- Fourier series can be developed for continuous time as well as discrete time signals.

Types of Fourier series

Depending upon the representation, these are three types of fourier series :

i) Trigonometric Fourier series.

ii) Compact trigonometric Fourier series or polar Fourier series.

iii) Exponential Fourier series.

Trigonometric Fourier Series

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k)\cos k\omega_0 t + \sum_{k=1}^{\infty} b(k)\sin k\omega_0 t$$

where

$$a(0) = \frac{1}{T}\int_{<T>} x(t)dt$$

$$a(k) = \frac{2}{T}\int_{<T>} x(t)\cos k\omega_0 t\, dt$$

$$b(k) = \frac{2}{T}\int_{<T>} x(t)\sin k\omega_0 t\, dt$$

- Here $\int_{<T>}$ indicates integration over one time period.
- And $\omega_0 = \frac{2\pi}{T}$, where *'T'* is period of the signal *x(t)*. This form of Fourier series is also called quadrature Fourier series.

Compact Trigonometric Fourier Series

The trigonometric Fourier series can be represented in compact form. It is also called compact or polar Fourier series.

Defining equations

$$x(t) = D(0) + \sum_{k=1}^{\infty} D(k)\cos(k\omega_0 t + \phi(k))$$

where $$D(0) = a_0 = \frac{1}{T}\int_{<T>} x(t)dt$$

$$D(k) = \sqrt{a(k)^2 + b(k)^2} \text{ and } \phi(k) = -\tan^{-1}\left(\frac{b(k)}{a(k)}\right)$$

Exponential Fourier Series

Defining equations

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \quad \text{(synthesis equation)}$$

where $$X(k) = \frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt \quad \text{(analysis equation)}$$

- Here $X(k)$ are called Fourier series coefficients.
- $x(t)$ and $X(k)$ are represented by the Fourier series (FS) pair as,

$$x(t) \xleftrightarrow{FS} X(k)$$

Convergence of Fourier Series - Dirichlet Conditions

The Fourier series is convergent if the signal $x(t)$ satisfies some conditions. These conditions are called Dirichlet conditions.

i) **Single valued property :** $x(t)$ must have only one value at any time instant within the interval T_0.

ii) **Finite discontinuities :** $x(t)$ should have at the most finite number of discontinuities in the interval T_0. Because of this, the signal can be represented mathematically.

iii) **Finite peaks :** The signal $x(t)$ should have finite number of maxima and minima in the interval T_0.

iv) **Absolute integrability :** The signal $x(t)$ should be absolutely integrable, i.e. $\int_{<T_0>} |x(t)| < \infty$. This is because the analysis equation integrates $x(t)$.

- Above conditions are sufficient but not necessary conditions for Fourier series representation.
- Most of physical signals satisfy above conditions.

Properties of Fourier Series

Linearity

If $x(t) \xleftrightarrow{FS} X(k)$ and $y(t) \xleftrightarrow{FS} Y(k)$

then $$\boxed{z(t) = ax(t) + by(t) \xleftrightarrow{FS} Z(k) = aX(k) + bY(k)}$$.

Proof : From equation (3.2.3) we can write $Z(k)$ as,

$$Z(k) = \frac{1}{T}\int_{<T>} z(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{<T>} [a\,x(t) + b\,y(t)]\, e^{-jk\omega_0 t}\, dt$$

$$= a\,\frac{1}{T}\int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt + b\,\frac{1}{T}\int_{<T>} y(t)\, e^{-jk\omega_0 t}\, dt = a\,X(k) + b\,Y(k)$$

Significance : This property is used to analyze signals which are represented as linear combination of other signals.

Time Shift or Translation

If $x(t) \xleftrightarrow{FS} X(k)$ then,

$$\boxed{z(t) = x(t - t_0) \xleftrightarrow{FS} Z(k) = e^{-jk\omega_0 t_0} X(k)}$$

Proof : Fourier coefficients of $x(t - t_0)$ will be,

$$Z(k) = \frac{1}{T}\int_{<T>} x(t - t_0)\, e^{-jk\omega_0 t}\, dt$$

Put $t - t_0 = m$. Limits of integration will shift by t_0. But again the integration is over one period. Hence limits can be kept same. i.e.,

$$Z(k) = \frac{1}{T}\int_{<T>} x(m)\, e^{-jk\omega_0 (m + t_0)} = \left[\frac{1}{T}\int_{<T>} x(m)\, e^{-jk\omega_0 m}\right] \cdot e^{-jk\omega_0 t_0}$$

The quantity inside the square brackets is $X(k)$. Hence,

$$Z(k) = e^{-jk\omega_0 t_0} X(k)$$

Frequency Shift

If $x(t) \xleftrightarrow{FS} X(k)$ then,

$$\boxed{z(t) = e^{jk_0 \omega_0 t}\ x(t) \xleftrightarrow{FS} Z(k) = X(k-k_0)}$$

$$Z(k) = \frac{1}{T} \int_{<T>} z(t)\, e^{-jk_0 \omega_0 t}\, dt \text{ by definition}$$

$$= \frac{1}{T} \int_{<T>} [e^{jk_0 \omega_0 t}\ x(t)] e^{-jk\omega_0 t} \text{ by putting for } z(t)$$

$$= \frac{1}{T} \int_{<T>} [x(t)] e^{-j(k-k_0)\omega_0 t} = X(k-k_0)$$

Scaling

If $x\ (t) \xleftrightarrow{FS} X(k)$

then, $\boxed{z(t) = x(at) \xleftrightarrow{FS} Z(k) = X(k)}$

Proof : $X(k) = \frac{1}{T} \int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt$

- Since $x(t)$ is periodic, then $z(t) = x(at)$ is also periodic. And if 'T' is the period of $x(t)$, then period of $z(t)$ will be $\frac{T}{a}$.
- Similarly if frequency of $x(t)$ is ω_0. The frequency of $z(t) = x(at)$ will be $a\omega_0$, since 't' is multiplied by factor 'a'.

Therefore Fourier coefficients of $z(t)$ can be written as,

$$Z(k) = \frac{1}{\left(\frac{T}{a}\right)} \int_{\left\langle \frac{T}{a} \right\rangle} z(t)\, e^{-jk(a\omega_0)t}\, dt = \frac{a}{T} \int_{\left\langle \frac{T}{a} \right\rangle} x(at)\, e^{-jka\omega_0 t}\, dt$$

Put $at = m$, then $dt = \frac{1}{a} dm$, then above equation becomes,

$$Z(k) = \frac{a}{T} \int_{\left\langle \frac{T}{a} \right\rangle} x(m)\, e^{-jk\omega_0 m} \cdot \frac{1}{a} dm = \frac{1}{T} \int_{\left\langle \frac{T}{a} \right\rangle} x(m)\, e^{-jk\omega_0 m}\, dm = X(k)$$

Comment

Fourier coefficients of $x(t)$ and $x(at)$ are same, but spacing between frequency components change form ω_0 to $a\omega_0$.

Time Differentiation

If $\quad x(t) \xleftrightarrow{FS} X(k)$

then, $\quad \boxed{\dfrac{dx(t)}{dt} \xleftrightarrow{FS} j\,k\,\omega_0\ X(k)}$ ... (3.3.5)

Proof :

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \text{ By definition of exponential Fourier series} \qquad \text{... (3.3.6)}$$

Differentiating with respect to 't',

$$\frac{d\,x(t)}{d\,t} = \sum_{k=-\infty}^{\infty} X(k)\, jk\,\omega_0\, e^{jk\omega_0 t}$$

$$\therefore \qquad \frac{d\,x(t)}{d\,t} = \sum_{k=-\infty}^{\infty} [\,jk\,\omega_0 X(k)]\, e^{jk\omega_0 t}$$

We know that $x(t) \xleftrightarrow{FS} X(k)$. Comparing above equation with equation (3.3.6) we get,

$$\frac{d\,x(t)}{d\,t} \xleftrightarrow{FS} jk\,\omega_0 X(k)$$

Convolution in Time

If $x(t) \xleftrightarrow{FS} X(k)$ and $y(t) \xleftrightarrow{FS} Y(k)$

then, $$\boxed{z(t) = x(t) * y(t) \xleftrightarrow{FS} Z(k) = T\,X(k)\,Y(k)}$$

Proof : We know that,

$$Z(k) = \frac{1}{T} \int_{<T>} z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{<T>} [x(t) * y(t)]\, e^{-jk\omega_0 t}\, dt$$

$x(t) * y(t) = \int_{<T>} x(\tau)\, y(t-\tau)\, d\tau$. This convolution is performed over one period for periodic signals. Putting this convolution in equation (3.3.8) we get,

$$Z(k) = \frac{1}{T} \int_{<T>} \int_{<T>} x(\tau)\, y(t-\tau)\, d\tau\, e^{-jk\omega_0 t}\, dt$$

Interchanging the order of integrations,

$$Z(k) = \frac{1}{T} \int_{<T>} x(\tau) \int_{<T>} y(t-\tau)\, e^{-jk\omega_0 t}\, d\tau\, dt$$

Put $t - \tau = m$. Therefore $dt = dm$. Since integration is over one period, this substitution will just shift the integrating limits. But it will be again over one period only. Hence we can write,

$$Z(k) = \frac{1}{T} \int_{<T>} x(\tau) \int_{<T>} y(m)\, e^{-jk\omega_0(\tau+m)}\, d\tau\, dm$$

$$= \frac{1}{T} \int_{<T>} x(\tau) \int_{<T>} y(m)\, e^{-jk\omega_0\tau} \cdot e^{-jk\omega_0 m}\, d\tau\, dm$$

$$= \frac{1}{T} \int_{<T>} x(\tau)\, e^{-jk\omega_0\tau}\, d\tau \int_{<T>} y(m)\, e^{-jk\omega_0 m}\, dm$$

$$= \frac{1}{T} [T\,X(k)] \cdot [T\,Y(k)] = T\; X(k)\; Y(k)$$

Significance : Convolution of two periodic signals results in multiplication of their Fourier coefficients and period T.

Multiplication or Modulation Theorem

If $x(t) \xleftrightarrow{FS} X(k)$ and $y(t) \xleftrightarrow{FS} Y(k)$

then, $\boxed{z(t) = x(t)\, y(t) \xleftrightarrow{FS} Z(k) = X(k) * Y(k)}$

Proof :

$$Z(k) = \frac{1}{T} \int_{<T>} z(t)\, e^{jk\omega_0 t}\, dt \text{ By definition}$$

$$= \frac{1}{T} \int_{<T>} [x(t)\, y(t)]\, e^{jk\omega_0 t}\, dt \quad \text{Putting for } z(t)$$

By synthesis equation, $x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$. Putting this expression for $x(t)$ in above equation,

$$Z(k) = \frac{1}{T} \int_{(T)} \sum_{m=-\infty}^{\infty} X(m)\, e^{jm\omega_0 t} \cdot y(t)\, e^{-jk\omega_0 t}\, dt$$

Note that index of summation is changed in above equation to differentiate between two indices of 'k' and 'm'. Interchanging the order of integration and summation,

$$Z(k) = \sum_{m=-\infty}^{\infty} X(m) \left[\frac{1}{T} \int_{(T)} y(t)\, e^{-j(k-m)\omega_0 t}\, dt \right]$$

The quantity inside the bracket indicates Fourier coefficients $y(k-m)$. Hence above equation will be,

$$Z(k) = \sum_{m=-\infty}^{\infty} X(m)\, y(k-m)$$

i.e. $\quad Z(k) = X(k) * Y(k)$

Parseval's Theorem

If $x(t)$ is the periodic power signal with Fourier coefficients $X(k)$, then average power in the signal is given by $\sum_{k=-\infty}^{\infty} |X(k)|^2$. i.e.,

$$\boxed{\text{Power, } P = \sum_{k=-\infty}^{\infty} |X(k)|^2}$$

Proof : The power in the signal is given as,

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)^2 \, dt \text{ By definition}$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \text{ for periodic signal}$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt$$

We have, $$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \qquad \text{By synthesis equation}$$

$$\therefore \qquad x^*(t) = \left[\sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \right]^* \qquad \text{By taking conjugates of both sides}$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)\, e^{-jk\omega_0 t}$$

Putting above expression of $x^*(t)$ in equation (3.3.11),

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \sum_{k=-\infty}^{\infty} X^*(k)\, e^{-jk\omega_0 t} \, dt$$

Here $\int_{-T/2}^{T/2} = \int_{<T>}$ i.e. integration over one period of $x(t)$. Interchanging the order of summation and integration,

$$P = \sum_{k=-\infty}^{\infty} X^*(k) \cdot \frac{1}{T} \int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)\, X(k) = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

Significance : Power of the signal can be obtained by squaring and adding the magnitudes of Fourier coefficients.