system mass, stiffness and damping matrix can not be estimated correctly from these modal data.

A.1.3. SINGLE DEGREE OF FREEDOM SYSTEM: EXAMPLE

Consider a single degree of freedom system with:

$$M = 2\,kg,\ C = 4\frac{N}{m/s},\ K = 5000\frac{N}{m}.$$

The system equation in the Laplace domain is:

$$\left(Mp^2 + Cp + K\right)X(p) = F(p)$$
$$\left(2p^2 + 4p + 5000\right)X(p) = F(p),$$

where $Z(p) = (2p^2 + 4p + 5000)$ is the dynamic stiffness.

The transfer function is the inverse of the dynamic stiffness,

$$H(p) = \frac{1/M}{p^2 + (C/M)p + (K/M)} = \frac{1/2}{p^2 + 2p + 2500}.$$

The system poles, i.e. the roots of the characteristic equation $(p^2 + 2p + 2500)$, are:

$$\lambda_{1,2} = -(C/2M) \pm \sqrt{(C/2M)^2 - (K/M)}$$
$$\lambda_{1,2} = -1 \pm \sqrt{1 - 2500} = -1 \pm j49.9900\,rad/s.$$

The undamped natural frequency $\Omega_1 = \sqrt{K/M} = 50\,rad/s = 7.9577\,Hz$.

The critical damping $C_c = 2M\sqrt{K/M} = 200\frac{N}{m/s}$ and the damping ratio $\zeta_1 = C/C_c = 0.02$ or 2%.

The residue $A_1 = \frac{1/M}{j2\omega_1} = -j5.001\times10^{-3}\,s/kg$. Hence, the partial fraction formulation of the transfer function is:

$$H(p) = \frac{A_1}{(p - \lambda_1)} + \frac{A_1^*}{(p - \lambda_1^*)} = \frac{-j5.001\times10^{-3}}{(p - (-1 + j49.9900))} + \frac{j5.001\times10^{-3}}{(p - (-1 - j49.9900))}.$$

A.1.4. MULTIPLE DEGREE OF FREEDOM SYSTEM: EXAMPLE

A.1.4.1. General viscous damping

Consider a two degree of freedom system, as depicted in figure a.1.11, where the masses, dampers and stiffnesses have following values:

$$M_1 = M_2 = 2\,kg,$$
$$C_1 = 3\frac{N}{m/s},\ C_2 = 1\frac{N}{m/s}, C_3 = 4\frac{N}{m/s},$$
$$K_1 = 4000\frac{N}{m},\ K_2 = 2000\frac{N}{m},\ K_3 = 4000\frac{N}{m}.$$

These values yield following system equation in the Laplace domain:

$$[Z(p)]\{X(p)\} = \left(p^2\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} + p\begin{bmatrix}4 & -1\\-1 & 5\end{bmatrix} + \begin{bmatrix}6000 & -2000\\-2000 & 6000\end{bmatrix}\right)\{X(p)\} = \{F(p)\}$$

The transfer function matrix is:

$$[H(p)] = [Z(p)]^{-1} = \left[p^2\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} + p\begin{bmatrix}4 & -1\\-1 & 5\end{bmatrix} + \begin{bmatrix}6000 & -2000\\-2000 & 6000\end{bmatrix}\right]^{-1} \quad \text{or}$$

$$[H(p)] = \frac{adj([Z(p)])}{|Z(p)|} = \frac{\begin{bmatrix}2p^2 + 5p + 6000 & p + 2000\\p + 2000 & 2p^2 + 4p + 6000\end{bmatrix}}{(2p^2 + 4p + 6000)(2p^2 + 5p + 6000) - (p + 2000)^2}$$

The system poles and corresponding modal vectors are the eigenvalues and eigenvectors of the following eigenvalue problem (see eq. a.1.29, $(p[A] + [B])\{Y\} = \{0\}$):

$$\left(p\begin{bmatrix}0 & 0 & 2 & 0\\0 & 0 & 0 & 2\\2 & 0 & 4 & -1\\0 & 2 & -1 & 5\end{bmatrix} + \begin{bmatrix}-2 & 0 & 0 & 0\\0 & -2 & 0 & 0\\0 & 0 & 6000 & -2000\\0 & 0 & -2000 & 6000\end{bmatrix}\right)\{Y\} = \{0\}$$

This results in:

$$\lambda_1 = -0.87501 + 44.7135j = 44.722^{\angle 91.246^\circ} rad/s$$

$$\begin{Bmatrix} \lambda_1\{\psi\}_1 \\ \{\psi\}_1 \end{Bmatrix} = \begin{Bmatrix} 4.0321\times10^{-1} + 7.0693\times10^{-1} \\ 1.1937\times10^{-2} + 7.0682\times10^{-1} \\ 1.5802\times10^{-2} - 3.9942\times10^{-4} \\ 1.5796\times10^{-2} - 5.7609\times10^{-4} \end{Bmatrix} = \begin{Bmatrix} 7.0694\times10^{-1\ \angle 89.671^\circ} \\ 7.0692\times10^{-1\ \angle 89.032^\circ} \\ 1.5807\times10^{-2\ \angle -1.4479^\circ} \\ 1.5807\times10^{-2\ \angle -2.0886^\circ} \end{Bmatrix} \quad \text{and}$$

$$\lambda_2 = -1.3750 + 63.2296j = 63.245^{\angle 91.121^\circ} rad/s$$

$$\begin{Bmatrix} \lambda_2\{\psi\}_2 \\ \{\psi\}_2 \end{Bmatrix} = \begin{Bmatrix} -9.1825\times10^{-2} + 7.0097\times10^{-1} \\ 1.0291\times10^{-1} + 6.9954\times10^{-1} \\ 1.1112\times10^{-2} + 1.2106\times10^{-3} \\ -1.1094\times10^{-2} - 1.3864\times10^{-3} \end{Bmatrix} = \begin{Bmatrix} 7.0696\times10^{-1\ \angle 97.461^\circ} \\ 7.0707\times10^{-1\ \angle -81.631^\circ} \\ 1.1178\times10^{-2\ \angle 6.2173^\circ} \\ 1.1180\times10^{-2\ \angle -172.88^\circ} \end{Bmatrix}$$

and the complex conjugates: $\lambda_1^*,\ \begin{Bmatrix} \lambda_1^*\{\psi\}_1^* \\ \{\psi\}_1^* \end{Bmatrix};\ \ \lambda_2^*,\ \begin{Bmatrix} \lambda_2^*\{\psi\}_2^* \\ \{\psi\}_2^* \end{Bmatrix}$.

Equation a.1.35 expresses the transfer function matrix in terms of residues and system poles:

$$[H(p)] = \sum_{r=1}^{N} \left(\frac{[A]_r}{(p-\lambda_r)} + \frac{[A]_r^*}{(p-\lambda_r^*)} \right) \qquad \text{(a.1.35)}$$

where:

$$[A]_r = \frac{adj([Z(\lambda_r)])}{\prod_{s=1, s\neq r}^{2N} E(\lambda_r - \lambda_s)} \quad \text{or}$$

$$[A]_r = P_r . adj([Z(\lambda_r)]) .$$

In this example:

$$[A]_1 = (-3.1266\times10^{-8} - 1.3978\times10^{-6}j)\begin{bmatrix} 1998.6+67.1j & 1999.1+44.7j \\ 1999.1+44.7j & 1999.4+22.4j \end{bmatrix}$$

$$[A]_1 = \begin{bmatrix} 3.1263\times10^{-5} - 2.7958\times10^{-3}j & -2.9326\times10^{-9} - 2.7958\times10^{-3}j \\ -2.9326\times10^{-9} - 2.7958\times10^{-3}j & -3.1267\times10^{-5} - 2.7958\times10^{-3}j \end{bmatrix} \quad \text{and}$$

$$[A]_2 = (-3.1266\times10^{-8} + 9.8824\times10^{-7}j)\begin{bmatrix} 1999.1-31.6j & 1998.6+63.2j \\ 1998.6+63.2j & -1997.7-94.8j \end{bmatrix}$$

$$[A]_2 = \begin{bmatrix} -3.1263\times10^{-5} - 1.9766\times10^{-3}j & 2.9326\times10^{-9} + 1.9771\times10^{-3}j \\ 2.9326\times10^{-9} + 1.9771\times10^{-3}j & 3.1267\times10^{-5} - 1.9772\times10^{-3}j \end{bmatrix}$$

and the complex conjugates $[A]_1^*$ and $[A]_2^*$.

Equation a.1.45, $[A]_r = Q_r\{\psi\}_r\{\psi\}_r^t$, defines the scaling constant Q_r for the modal vector $\{\psi\}_r$. In this example equation a.1.45 becomes:

$$[A]_1 = (0.69019 - 11.168j)\begin{bmatrix} 2.4956\times10^{-4} - 1.2624\times10^{-5}j & 2.4939\times10^{-4} - 1.5413\times10^{-5}j \\ 2.4939\times10^{-4} - 1.5413\times10^{-5}j & 2.4920\times10^{-4} - 1.8200\times10^{-5}j \end{bmatrix}$$

$$[A]_2 = (-3.6504 - 15.393j)\begin{bmatrix} 1.2202\times10^{-4} + 2.6905\times10^{-5}j & -1.2160\times10^{-4} - 2.8836\times10^{-5}j \\ -1.2160\times10^{-4} - 2.8836\times10^{-5}j & 1.2115\times10^{-4} + 3.0760\times10^{-5}j \end{bmatrix}$$

and the complex conjugates $[A]_1^*$ and $[A]_2^*$.

The orthogonality conditions (a.1.68 and a.1.69) for this example are:

$$[\Phi]^t[A][\Phi] = 10^{-2}\begin{bmatrix} .55127+8.9201j & 0 & 0 & 0 \\ 0 & -1.4585+6.1504j & 0 & 0 \\ 0 & 0 & .55127-8.9201j & 0 \\ 0 & 0 & 0 & -1.4585-6.1504j \end{bmatrix}$$

$$[\Phi]^t[B][\Phi] = \begin{bmatrix} 3.9933-.16844j & 0 & 0 & 0 \\ 0 & 3.8688+1.0068j & 0 & 0 \\ 0 & 0 & 3.9933+.16844j & 0 \\ 0 & 0 & 0 & 3.8688-1.0068j \end{bmatrix}$$

The reader can check that these values comply with equations a.1.86, $[{}^{\backslash}Q_{\backslash}] = [{}^{\backslash}a_{\backslash}]^{-1}$, and a.1.74, $[{}^{\backslash}b_{\backslash}] = -[{}^{\backslash}a_{\backslash}][{}^{\backslash}\Lambda_{\backslash}]$.

A.1.4.2. Proportional viscous damping

A two degree of freedom system, as depicted in figure a.1.11, with mass, damping and stiffness values:

$$M_1 = M_2 = 2kg ,$$
$$C_1 = 3\frac{N}{m/s},\ C_2 = 2\frac{N}{m/s}, C_3 = 3\frac{N}{m/s},$$
$$K_1 = 4000\frac{N}{m},\ K_2 = 2000\frac{N}{m},\ K_3 = 4000\frac{N}{m},$$

is a proportionally damped system. The coefficients α and β of equation a.1.57, $[C] = \alpha[M] + \beta[K]$, are:

$$\alpha = -\frac{1}{2}\ 1/s,\ \ \beta = \frac{1}{1000}\ s .$$

The corresponding eigenvalues and eigenvectors are:

$$\lambda_1 = -0.7500 + 44.7151j = 44.721^{\angle 90.961^\circ} \, rad/s$$

$$\begin{Bmatrix} \lambda_1 \{\psi\}_1 \\ \{\psi\}_1 \end{Bmatrix} = \begin{Bmatrix} -2.1660\times10^{-1} + 6.7293\times10^{-1} \\ -2.1660\times10^{-1} + 6.7293\times10^{-1} \\ 1.5126\times10^{-2} + 4.5903\times10^{-3} \\ 1.5126\times10^{-2} + 4.5903\times10^{-3} \end{Bmatrix} = \begin{Bmatrix} 7.0693\times10^{-1\,\angle 107.84^\circ} \\ 7.0693\times10^{-1\,\angle 107.84^\circ} \\ 1.5807\times10^{-2\,\angle 16.881^\circ} \\ 1.5807\times10^{-2\,\angle 16.881^\circ} \end{Bmatrix} \quad \text{and}$$

$$\lambda_2 = -1.7500 + 63.2213j = 63.246^{\angle 91.586^\circ} \, rad/s$$

$$\begin{Bmatrix} \lambda_2 \{\psi\}_2 \\ \{\psi\}_2 \end{Bmatrix} = \begin{Bmatrix} 2.0629\times10^{-1} + 6.7626\times10^{-1} \\ -2.0629\times10^{-1} - 6.7626\times10^{-1} \\ 1.0598\times10^{-1} - 3.5563\times10^{-1} \\ -1.0598\times10^{-1} + 3.5563\times10^{-1} \end{Bmatrix} = \begin{Bmatrix} 7.0702\times10^{-1\,\angle 73.063^\circ} \\ 7.0702\times10^{-1\,\angle -106.96^\circ} \\ 1.1179\times10^{-2\,\angle -18.549^\circ} \\ 1.1179\times10^{-2\,\angle 161.45^\circ} \end{Bmatrix}$$

and the complex conjugates: $\lambda_1^*, \begin{Bmatrix} \lambda_1^* \{\psi\}_1^* \\ \{\psi\}_1^* \end{Bmatrix}; \; \lambda_2^*, \begin{Bmatrix} \lambda_2^* \{\psi\}_2^* \\ \{\psi\}_2^* \end{Bmatrix}$.

As stated in section A.1.2.6. the modal vectors $\{\psi\}_r$ are normal modal vectors: the phases of the elements are equal (e.g. mode 1) or differ exactly 180° (e.g.mode 2). If they are properly rescaled they equal the modal vectors for the undamped case (section A.1.4.3). The system poles are complex. Their modulus equals the modulus of the system poles of the undamped system (section A.1.4.3). This is not true for the general viscous damping case.

Other quantities as the residues, modal a and modal b, modal scaling factors can be calculated similarly as for the general viscous damping case.

A.1.4.3. No damping

A two degree of freedom system, as depicted in figure a.1.11, without damping and with mass and stiffness values:

$$M_1 = M_2 = 2kg,$$
$$K_1 = 4000\frac{N}{m}, \; K_2 = 2000\frac{N}{m}, \; K_3 = 4000\frac{N}{m},$$

yields an eigenvalue problem as stated by equation a.1.55 ($\left(p^2[M]+[K]\right)\{X\} = \{0\}$). The purely imaginary system poles, and corresponding modal vectors are:

$$\lambda_1 = 44.721j \; rad/s, \quad \{\psi\}_1 = \begin{Bmatrix} 0.7071 \\ 0.7071 \end{Bmatrix},$$

$$\lambda_2 = 63.246j \; rad/s, \quad \{\psi\}_2 = \begin{Bmatrix} 0.7071 \\ -0.7071 \end{Bmatrix}$$

and the complex conjugates $\lambda_1^*, \{\psi\}_1^*; \; \lambda_2^*, \{\psi\}_2^*$.

As stated before, the modal vectors and the modulus of the system poles are identical to these of the proportionally damped case.

Equations a.1.77 and a.1.78 ($[\psi]^t[M][\psi] = [{}^{\backslash}m_{\backslash}]$ and $[\psi]^t[K][\psi] = [{}^{\backslash}k_{\backslash}]$) define the modal masses and stiffnesses:

$$\begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix}^t \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix}^t \begin{bmatrix} 6000 & -2000 \\ -2000 & 6000 \end{bmatrix} \begin{bmatrix} .7071 & .7071 \\ .7071 & -.7071 \end{bmatrix} = \begin{bmatrix} 4000 & 0 \\ 0 & 8000 \end{bmatrix}.$$

According to equations a.1.80 and a.1.56 the frequency response function matrix is:

$$[H(j\omega)] = \sum_{r=1}^{N} \frac{\{\psi\}_r \{\psi\}_r^t}{m_r\left(\omega_r^2 - \omega^2\right)} = \sum_{r=1}^{N} \frac{j2\omega_r Q_r \{\psi\}_r \{\psi\}_r^t}{\left(\omega_r^2 - \omega^2\right)},$$

$$[H(j\omega)] = \frac{0.5\begin{Bmatrix} .7071 \\ .7071 \end{Bmatrix}\begin{Bmatrix} .7071 \\ .7071 \end{Bmatrix}^t}{2000 - \omega^2} + \frac{0.5\begin{Bmatrix} .7071 \\ -.7071 \end{Bmatrix}\begin{Bmatrix} .7071 \\ -.7071 \end{Bmatrix}^t}{4000 - \omega^2}.$$

The corresponding modal scale factors, according to equation a.1.91, are:

$$Q_1 = -5.5902\times10^{-3} j \quad \text{and} \quad Q_2 = -3.9528\times10^{-3} j.$$

A.1.5. CONCLUSIONS

This chapter covered the basic theory of modal analysis. It showed that the dynamics of a structure are completely described by the modal parameters. The chapter explained that these modal parameters can be derived from the knowledge of the mass, stiffness and damping matrices of a model of the structure (the analytical approach) or from the measurement of frequency response functions on the structure (the experimental approach). A small numerical example illustrated the theory.