

# 15.415x Foundations of Modern Finance

---

Leonid Kogan and Jiang Wang

MIT Sloan School of Management

## Lecture 12: Options, Part 1



# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods

## Concept Check

The holder of the American Put option has the right to sell the underlying asset at the strike price at any time before or at the maturity date of the option.?

**TRUE**

Explanation

This is the definition of the American Put option.

# Introduction to option types

We begin our discussion with the most common types of options, so-called plain vanilla options, which trade both on exchanges and over-the-counter.

## ■ Option types:

gives its Owner or Holder (buyer) the right to buy...

- **Call:** The right to buy an asset (the **underlying asset**) for a given price  $K$  (strike price, or exercise price) on or before a given date (expiration date, or maturity date).

gives its Owner, Holder, or writer (seller) the right to sell...

- **Put:** The right to sell an asset for a given price on or before the expiration date.

To buy an **Insurance** is a **type** of Long in a **Put Option** (we think the car will be damaged) with an **Option Premium** (**Cost**) equals to the Insurance Policy of the Car (paid annually - maturity).

Here the buyer of the Put has the Right to "Sell" (on this case the right to put a Claim) but not the Obligation, in the other hand the Seller of the Put has the Obligation to "Buy" the asset (in this case to pay the claim)

In a Put, the higher the Risk is perceived the higher the cost (Option Premium).

## ■ Exercise styles:

- **European:** Owner can exercise the option only on expiration date.

- **American:** Owner can exercise the option on or before expiration date.

A **"call" option** is a contract between two parties to exchange a stock at a "strike" price by a predetermined date. One party, the buyer of the "call", has the right, but not an obligation, to buy the stock at the strike price by the future date, while the other party, the seller of the call, has the obligation to sell the stock to the buyer at the strike price if the buyer exercises the option.

A **"put" option** is a contract between two parties to exchange a stock at a "strike" price, by a predetermined date. One party, the buyer of the "put", has the right, but not an obligation, to sell the stock at the strike price by the future date, while the other party, the seller of the put, has the obligation to buy the stock from the buyer at the strike price if the buyer exercises the option.

# Introduction to option types

- Key elements in defining an option:
  - Underlying asset and its price  $S$ ,
  - Exercise price (**strike price**)  $K$ ,
  - Expiration date (**maturity date**)  $T$  (today is 0),
  - European or American.

# Key concepts

## ■ Introduction: option types

## ■ Payoffs of European options.

Option Payoff and Net Payoff, P&L Diagram

- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods

### Concept Check 1

The buyer of a European Put option on a stock benefits more if the stock price at the maturity date of the option is lower.?

**TRUE**

Explanation:

The payoff of the put option is negatively related to the stock price. If on the maturity date the stock price is below the strike price, the holder of the option prefers the stock price to be as low as possible. The lower it is, the higher is the payoff. If on the maturity date the stock price is above the strike price, the holder of the option is indifferent because the payoff is zero. Altogether, the buyer of a European Put prefers the stock price to be lower.

The writer (seller) of the option has indeed an obligation to act however the buyer of the option decides.

### Concept Check 2

Consider a European Call option. In the region where the underlying price is below the strike price, the net payoff of the option is negative.?

**TRUE**

Explanation:

In this region, the payoff of the option is zero, and the net payoff is negative, reflecting the initial price of the option.

## Example: a European call option

- A European call option on IBM with exercise price  $K$  \$100. (at Maturity)
- It gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at \$100 on the expiration date.
- The option's payoff depends on the share price of IBM on the expiration date.



European options on stocks are typically settled by physical delivery: the buyer of the option receives shares of the stock if the option is exercised. We convert the value of this transaction into dollars and think about the payoff of the option at expiration as the corresponding cash flow.

Note that this is **their right** and **not their obligation**.  
The buyer may choose not to exercise the option, if doing so would lead to a loss.  
As a result, the **pay for this option at maturity cannot be negative**.

In this sense, the call option is clearly **different from a forward** agreement, which is a binding **obligation** to buy the underlying at maturity.  
The **payoff of the forward may be positive or negative**.

## Example: a European call option

IBM Price at T	Action	Payoff
< 80	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
Price of the underlying asset at Maturity $S_T$	Exercise	$S_T - 100$

K: Strike Price

### ■ Observations:

- The payoff of a Call option is never negative; sometimes, it is positive.
- Actual payoff depends on the price of the underlying asset:

$$CF_T(\text{call}) = \max[0, S_T - K]$$

Note: Option Premium (Costs) in dotted-line

# Payoff diagrams (strike price = 100)

(or RISK GRAPH or P&L Diagram, Payout & Loses Diagram)

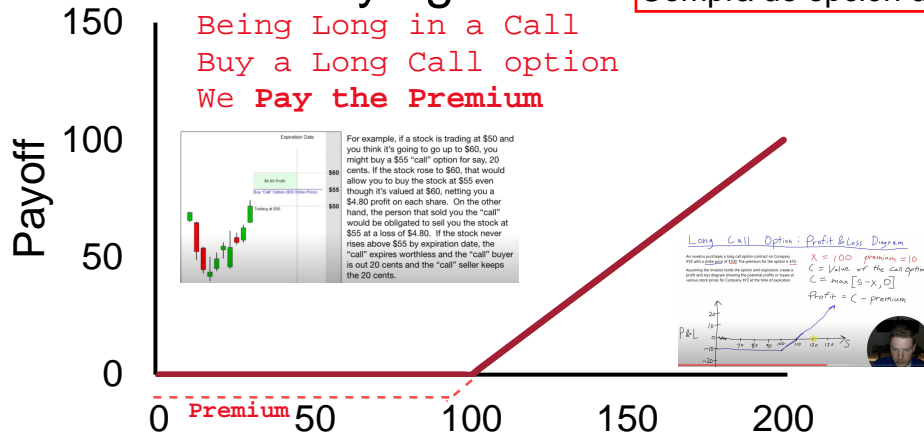
payoff of a long position in the call option, with a strike price of \$100.

$$CF \text{ (Payoff)} = (S_t - K)$$

$$\text{Net Profit} = CF - PV(\text{Premium})$$

## Buying a call

Compra de opción de compra:



Nota: aqui el Net Profit siempre se va a ver afectado por el Premium

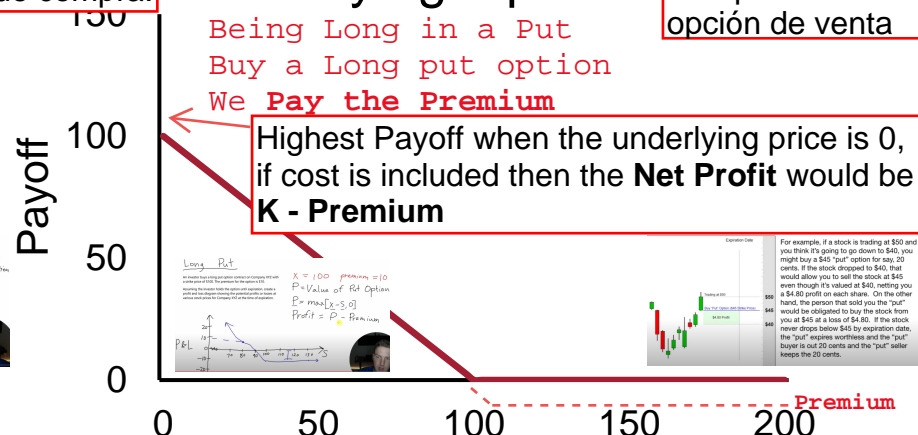
$$CF \text{ (Payoff)} = (K - S_t)$$

$$\text{Net Profit} = CF - PV(\text{Premium})$$

Recall that the put option gives the owner the right to sell the underlying at the strike price.

## Buying a put

Compra de opción de venta

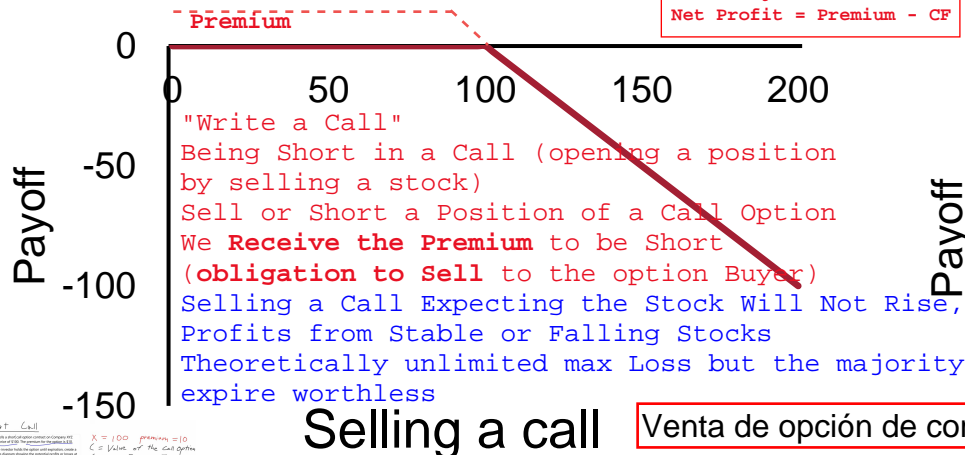


Underlying Asset Price

Asset Price

$$CF = -(S_t - K)$$

$$\text{Net Profit} = \text{Premium} - CF$$



## Selling a call

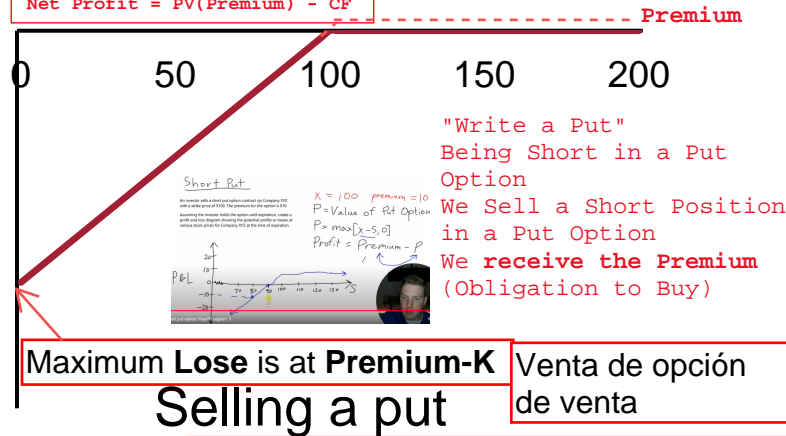
Venta de opción de compra

The payoff of a short position is the negative of the long position payoff. The plots are then the mirror images of the long position payoff plots.

Asset Price

$$CF \text{ (Payoff)} = -(K - S_t)$$

$$\text{Net Profit} = PV(\text{Premium}) - CF$$



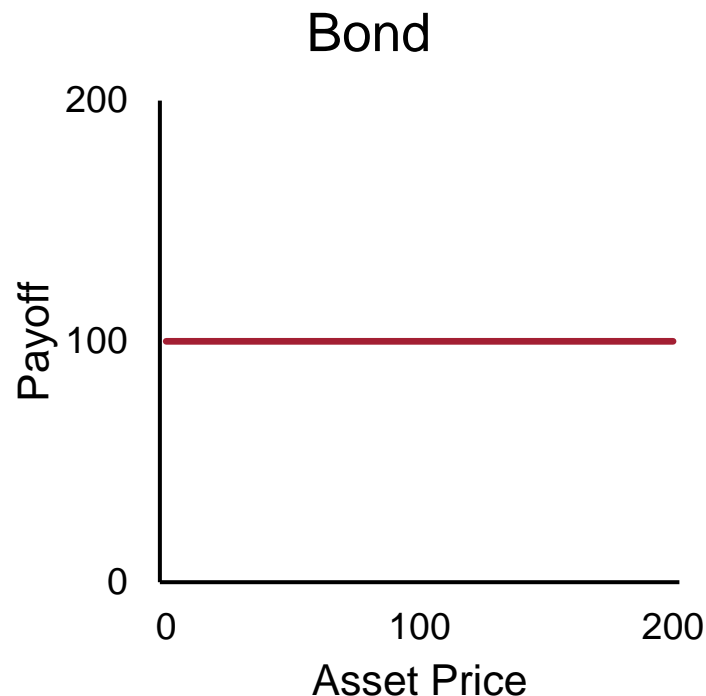
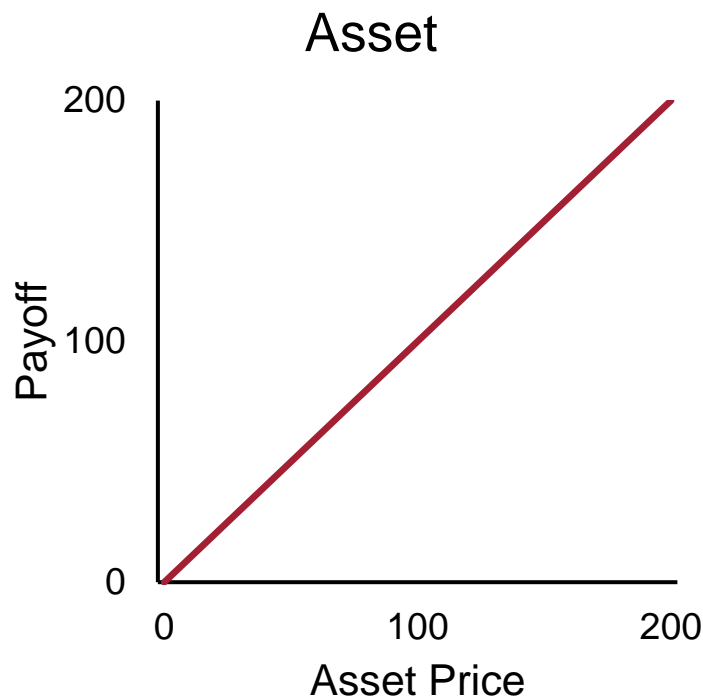
## Selling a put

Venta de opción de venta

The writer (seller) of the option receives the Option Premium and has indeed an obligation to act however the buyer of the option decides.

# Payoff diagrams

- The underlying asset and the bond (with face value \$100) have the following payoff diagrams



Going forward, we will often combine options with the **underlying asset positions** and **bonds**. On the left, the plot the payoff of the underlying asset as a function of itself.(a 45-degree line). On the right, we show the payoff of the **risk-free bond**, which is a constant **independent** of the underlying price.

# Net option payoff

It is important to distinguish the payoff of the option, which is never negative, from its net payoff.  
The **net payoff** of the **option includes** its **cost**: If the option expires unexercised, its net **payoff is negative**.

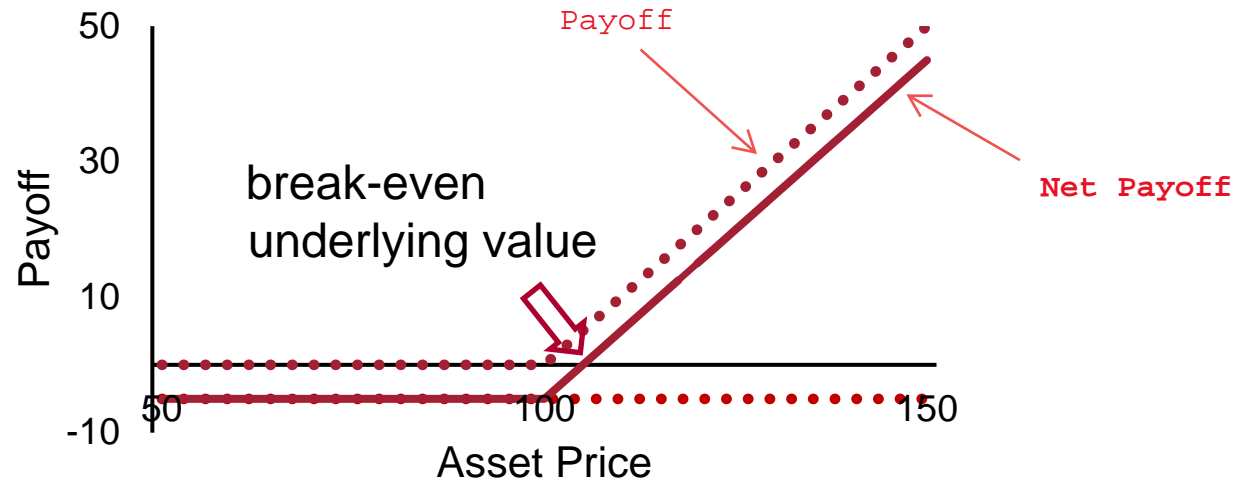
- The net payoff of an option **must include its cost**. "Option Premium"
- Example: a European call on IBM shares with an exercise price of \$100 and maturity of three months is trading at \$5. Suppose this call is currently trading at \$5.
- The 3-month interest rate, not annualized, is 1%.
- At maturity, the call's net payoff is as follows.

IBM Price	Action	Payoff	Net payoff
< 80	Not Exercise	0	-5.05
80	Not Exercise	0	-5.05
90	Not Exercise	0	-5.05
100	Not Exercise	0	-5.05
110	Exercise	10	4.95
120	Exercise	20	14.95
$S_T$	Exercise	$S_T - 100$	$S_T - 100 - 5.05$

At maturity, the call's net payoff is equal to its payoff minus the **future value of the option price**, which is \$5.05.

# Net option payoff

the net payoff of the option is **negative** for the values of the underlying **below** the **strike price**, and for some values above the strike price until the **break-even** point where the **payoff becomes positive**.



The break-even point is given by  $S_T$  at which Net Payoff is zero:

$$\begin{aligned}\text{Net payoff} &= \max[S_T - K, 0] - C(1 + r)^T \\ &= S_T - 100 - (5)(1 + 0.01) \\ &= 0\end{aligned}$$

or

The **price** of IBM **must be above** this threshold, above \$105.05, in order for the holder of the option to make a **profit**.

$$S_T = \$105.05$$

The break-even point is that value of the underlying that exceeds the strike by the future value of the option price.

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
  - Protective Put, Bull Spread, Straddle
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods

## Concept Check

Consider an investor holding a bull spread on 100 shares of a stock. This investor can make unlimited gains if the stock price rises before the maturity date.?

**FALSE**

Explanation:

The payoff of the bull spread is limited from above, in contrast to a straight Call option.

# Option strategies: protective put

\*Shown Strategy Payoffs  
not Net Payoffs

Posicion de Venta Protectora

Buy the underlying stock, and buy a put with a strike price of \$50:  
at same Maturity Date

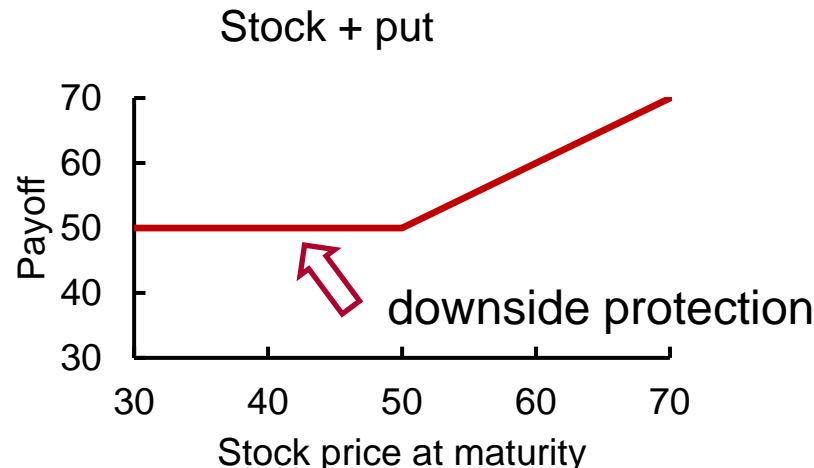


This trade consists of two positions, a **long position in the underlying stock**, and a **long position in the put option**.

The strategy offers the buyer **participation in the upside**.

This position appreciates in value as the stock price rises.

At the same time, the **put option limits potential losses**, creating a **floor** in the payoff.

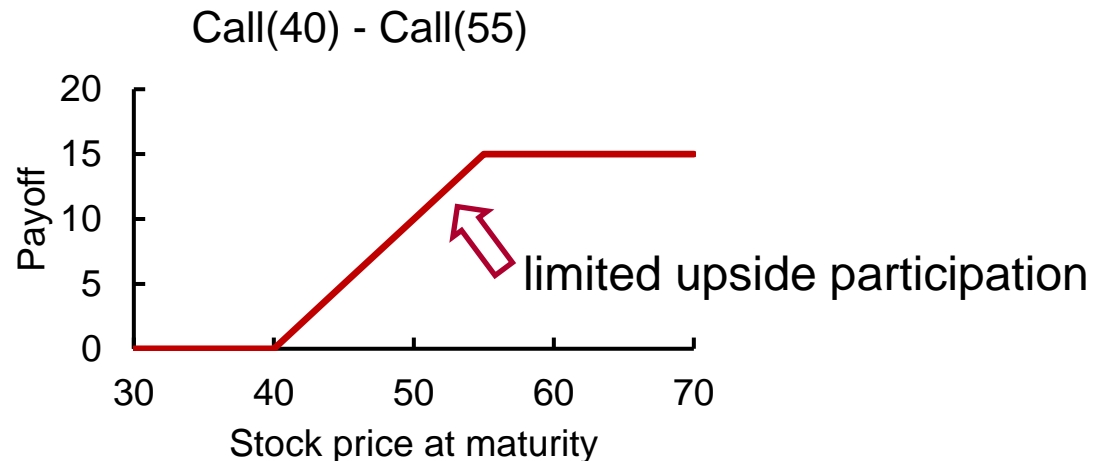
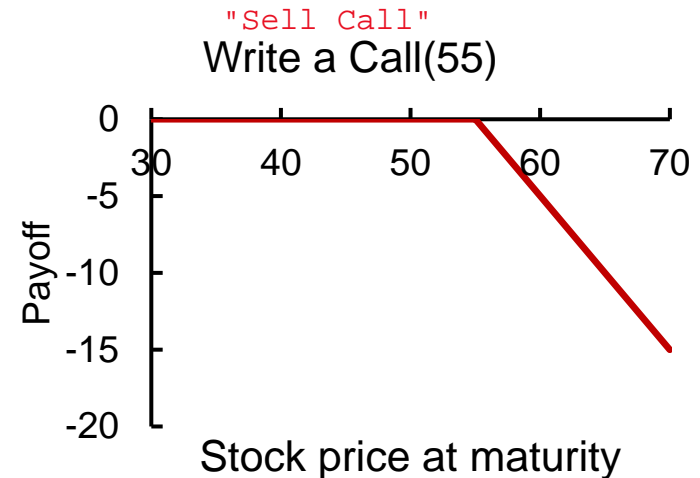
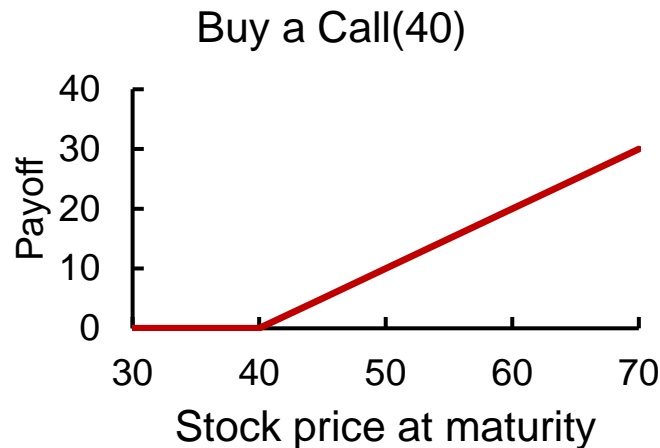


# Option strategies: bull spread or Collar

Diferencial Alcista o Margen Alcista

\*Shown Strategy Payoffs  
not Net Payoffs

Buy a call with a strike price of \$40 and write a call with a strike price of \$55:  
at same Maturity Date



We **buy** the call with the lowest strike, \$40, and **sell** the call with the highest strike, \$55.

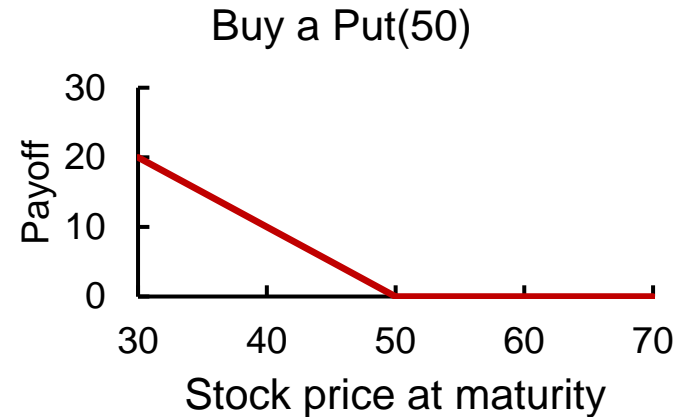
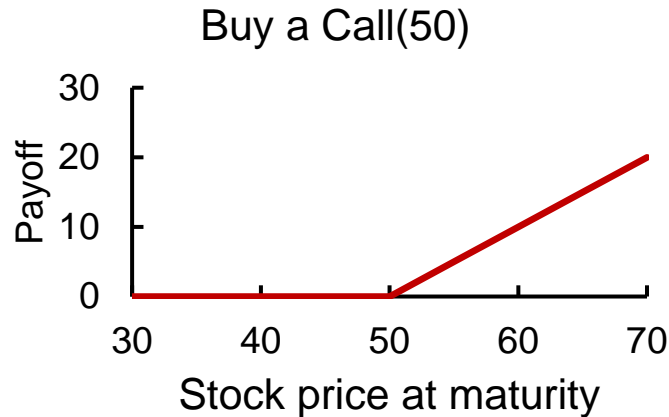
The resulting position, **offers** some **upside exposure**, which is kept above \$55.

Compare this to the straight call with the \$40 strike, this position **sacrifices participation** in further upside above \$55 in exchange for the lower initial cost of \$40.

# Option strategies: **straddle**

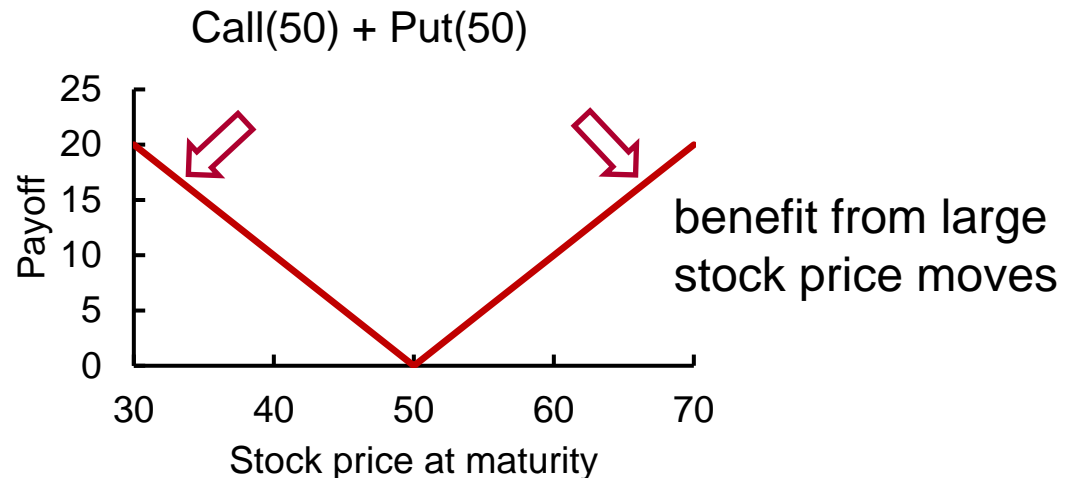
\*Shown Strategy Payoffs  
not Net Payoffs

Buy a call with a strike price of \$50, and buy a put with a strike price of \$50:  
at same Maturity Date



The payoff for this position is relatively high when the stock price is far away from the strike price, on either side.

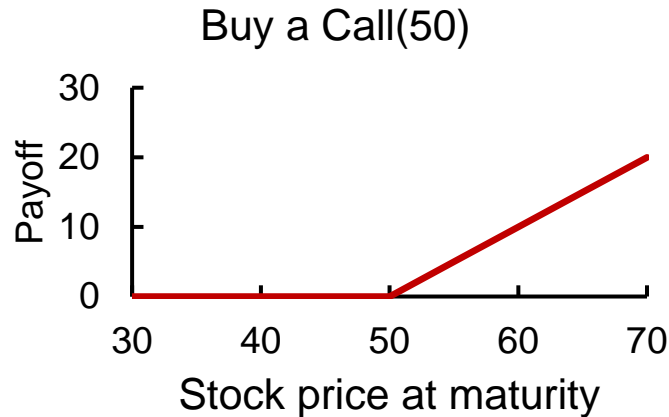
These are bets on higher future **volatility** of the underlying price.



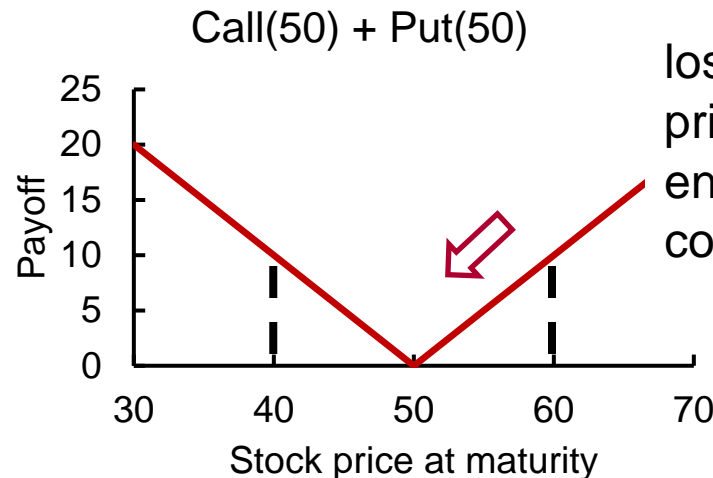
# Option strategies: straddle

\*Shown Strategy Payoffs  
not Net Payoffs

Buy a call with a strike price of \$50, and buy a put with a strike price of \$50:  
at same Maturity Date



The buyer of the straddle loses money at maturity if the underlying price fails to move far enough away from the strike price to avoid the initial costs (Net Payoff)



lose money if the stock price does not move far enough to cover the initial cost of Call(50) + Put(50)

(Net Payoff)

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options

Warrant, Convertible Bond, Callable Bond

## Concept Check

A firm's debt is equivalent to a riskless bond + a short put on the firm's assets.

**TRUE**

### Explanation

Consider the payoff of the firm's debt. If the assets of the firm fall below the face value of debt at debt maturity (the firm defaults), the payoff of debt equals the value of the firm's assets. If the firm's assets are worth more than the debt value, then the payoff of debt equals its face value.

In other words, if we denote the face value of debt by  $F$ , the value of firm's asset by  $A$ , and the payoff of firm's debt at maturity by  $D$ , then  $D = \min\{F, A\}$ .

Decompose the formula into two parts:  $D = \min\{F, A\} = F + \min\{0, A - F\} = F - \max\{0, F - A\}$ , where we use the property of  $\min$  and  $\max$  functions:  $\min\{a, b\} = -\max\{-a, -b\}$  for any numbers  $a$  and  $b$ .

The debt is hence equivalent to a portfolio of a riskless bond with a face value of  $F$ , and a short position in a Put option, where the underlying is  $A$  and the strike price is  $F$ .

# Corporate securities as options

- Consider a firm with debt in its capital structure. (in this case, a single Bond)

Balance sheet, market values			
Assets	\$30	\$25	Bond
		\$5	Equity
Total	\$30	\$30	

- Firm bond has a face value of \$50. 

We assume that the bond pays no coupons and simply promises \$50 at maturity.
- Default is likely: if the firm's assets are worth less than \$50 when the bond matures, the firm will be unable to afford its debt. 

the firm will have to default on its debt.
- In that event, the assets are turned over to the bondholders, and the equity is worth zero. 

If that happens, bondholders will own the firm, and the equity value will be zero.

# Corporate securities as options

- Consider the value of the stock, and a call on the underlying assets of the firm with an exercise price of \$50:

Stock Price		Payoff of the Call Option
Asset value	Value of the stock	Value of a call on the assets, strike = \$50
⋮	⋮	⋮
\$20	\$0	\$0
\$40	\$0	\$0
\$50	\$0	\$0
\$60	\$10	\$10
\$80	\$30	\$30
\$100	\$50	\$50
⋮	⋮	⋮

The payoff of a Debt is equal to the firm's assets minus the payoff of its equity.  
i.e. Asset Value \$60 - Value of Stock (Equity) \$10 = \$50 Bond Price (Debt)

- The stock gives the same payoff as a call option written on firm's assets.
- Equity is essentially a call option on the firm's assets, strike price equal to face value of debt. (Bond in this case above)

The **payoff of a Debt** is equal to the firm's assets minus the payoff of its equity. i.e. Asset Value \$60 - Value of Stock (Equity) \$10 = \$50 Bond Price (Debt)

A firm's **debt** is equivalent to a riskless bond + a short put on the firm's assets.

- The **payoff of a Debt** is equal to the firm's assets minus the payoff of its equity. i.e. Asset Value \$60 - Value of Stock (Equity) \$10 = \$50 Bond Price (Debt)  
A firm's **debt** is equivalent to a riskless bond + a short put on the firm's assets.

$$D = A - E = A - \max[0, A - F]$$

# Corporate securities as options

Warrants are closely related to call options on the firm's stock. What makes them different is that when the warrants are exercised, new shares are issued, which dilutes the value of the firm's shares. This **dilution effect must be taken into account when valuing warrants.**

- **Warrant**: Call options on the firm's stock, with stock dilution as a result of exercise.
- **Convertible bond**: A portfolio combining straight bonds and a call on the firm's stock with the exercise price related to the conversion ratio.
- **Callable bond**: A portfolio combining straight bonds and a short position in a call on these bonds.

Convertible bonds are bonds that can be converted into equity at a pre specified conversion ratio.

From the perspective of the buyer, a callable bond is a portfolio combining straight bonds with a short position in a call written on these bonds.  
A callable bond gives the **issuer** the right to **buy the bond back** at a specified price.

**Stock dilution**, also known as **equity dilution**, is the decrease in existing **shareholders' ownership percentage** of a company as a result of the company issuing new **equity**. New equity increases the total **shares outstanding** which has a dilutive effect on the ownership percentage of existing shareholders. This increase in the number of shares outstanding can result from a primary market offering (including an initial public offering), employees exercising stock options, or by issuance or conversion of **convertible bonds, preferred shares or warrants** into stock. This dilution can shift fundamental positions of the stock such as ownership percentage, voting control, earnings per share, and the value of individual shares.

I have read some on the internet and one distinguishing feature between stock "**option**" and stock "**warrant**" seems to be, that:

- stock options are created by participants e.g. traders on the secondary market
- stock warrants are created by the company which generally also issues the underlying stock
- if a stock option is exercised, the number of shares outstanding does not change because the share is merely exchanged between two parties
- if a stock warrant is exercised, the number of shares does change, because the company is either giving out a share (be it from treasury stock or from previously authorized but unissued stock) increasing the shares outstanding, or the company is repurchasing a share into treasury stock decreasing the shares outstanding
- This makes stock options non-dilutive (or "dilution-neutral?") when exercised
- and makes stock warrants either dilutive or anti-dilutive?

In 15.516x Financial Accounting, we speak of stock option compensation. In that context, are we actually more correctly speaking of stock **warrant** compensation? Because as an employee, if I exercise whatever that thing might be named, I'm actually getting a share from the company and not from another trader?

In a sea of conflicting usage of these terms, is there a well-rounded definition for both terms?

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options (Model-Free Pricing based on restrictions)

Concept Check

0/1 point (ungraded)

Consider two European Call options on the same underlying asset with the same time to maturity and different strikes:  $K_1 < K_2$ . Then, the price of the first option is at least as high as the price of the second option.

☒ True  
✓


☐ False

Explanation

The value of the Call option at maturity (with an underlying  $S_T$  and strike  $K$ ) is  $\max\{0, S_T - K\}$ . For any given  $S_T$  and two strikes  $K_1 < K_2$ ,  $\max\{0, S_T - K_1\} \geq \max\{0, S_T - K_2\}$ .

Hence, the terminal payoff is always larger for the option with a lower strike price. The No Arbitrage condition then implies that the price of the first option is at least as high as the price of the second option.

# Preliminaries

- For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.
- Notation:
  - $S$  : Price of stock now;
  - $S_T$  : Price of stock at  $T$ ;
  - $B$  : Price of discount bond of par \$1 and maturity  $T$  ( $B \leq 1$ ); 
  - $C$  : Price of a European call with strike  $K$  and maturity  $T$ ;
  - $P$  : Price of a European put with strike  $K$  and maturity  $T$ .
- For our discussion:
  - Consider only European options, exercised only at maturity.
  - Assume the underlying stock pays no dividends.

We maintain the assumption that interest rates are positive. So the price of the bond is less than 1 (para que haya un crecimiento en el precio del Bond hasta 1)

# Basic properties of options

- If an option is exercised now, the resulting cash flow is called its exercise value. (exercise Value of the Option)

**C:** ■ For a call, its exercise value is  $S - K$ , where  $S$  is the current stock price;

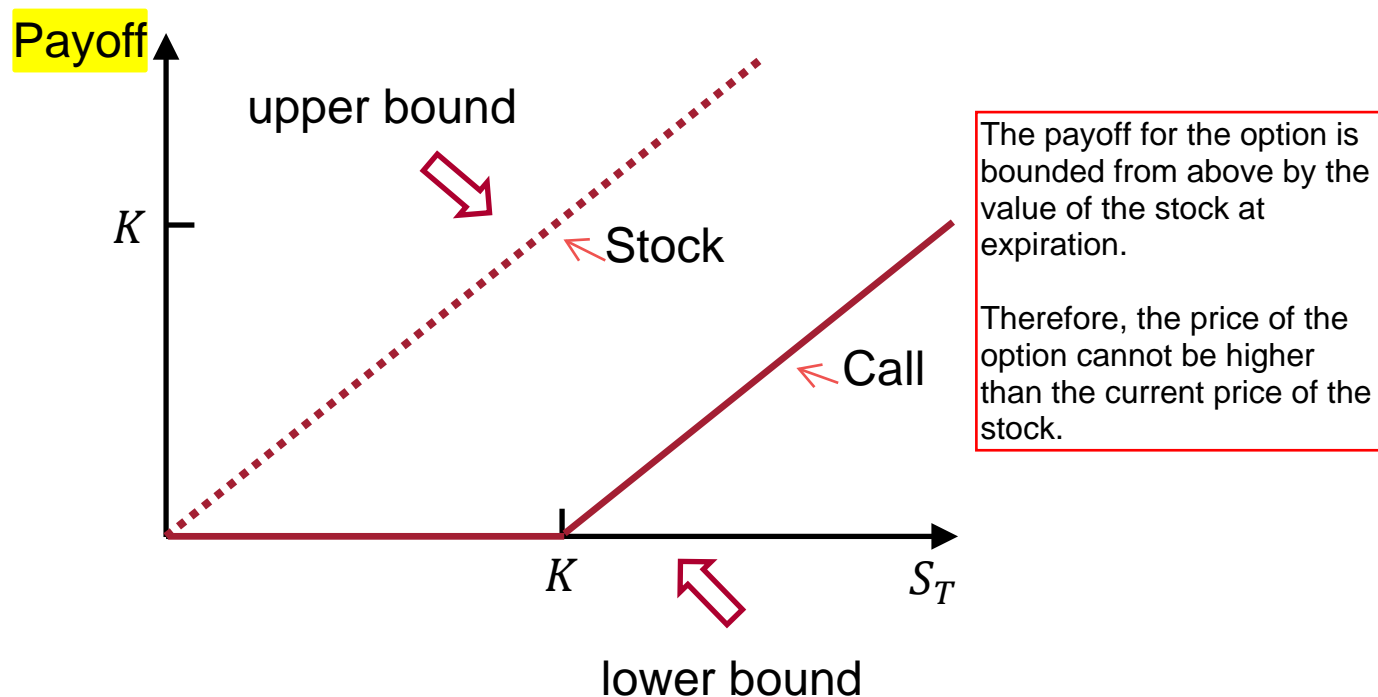
**P:** ■ For a put, its exercise value is  $K - S$ .

- An option is deemed to be:  
se considera que una opcion esta:
  - **In the money** (ITM) if its exercise value is positive;
  - **At the money** (ATM) if its exercise value is zero;
  - **Out of the money** (OTM) if its exercise value is negative.

# Price bounds for European Call Options

## ■ European options on a non-dividend paying stock.

1. The payoff of call can never be negative:  $C \geq 0$ . therefore the price of the option is at least zero
2. The payoff of stock dominates that of call:  $C \leq S$ .



## Price bounds (cont'd)

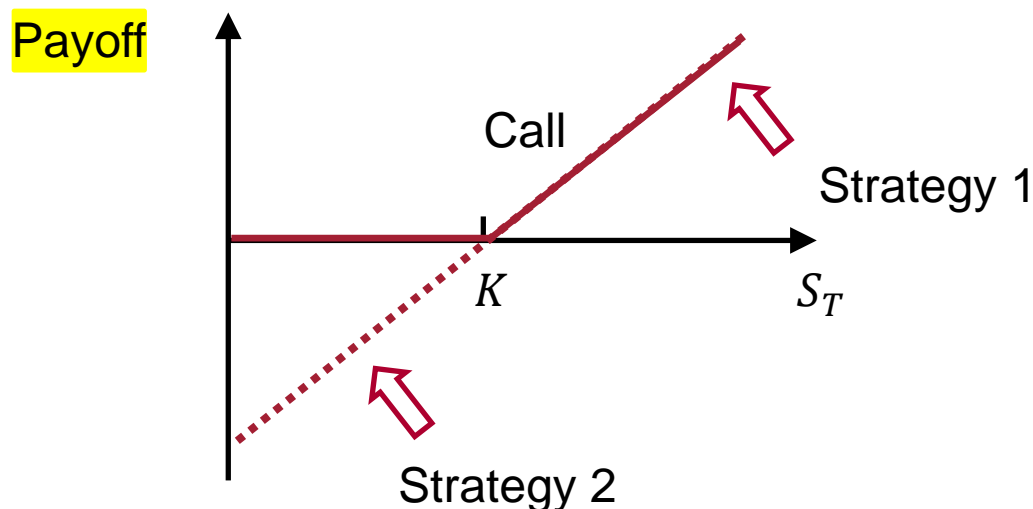
### Call Exercise minimum Value

3. Lower bound:  $C \geq S - KB$

$C$ : Option Price

$KB$ : Present Value of the Strike Price

- Strategy 1: Buy a call;
- Strategy 2: Buy a share of stock by borrowing  $KB$ .
- The payoff of Strategy 1 dominates that of Strategy 2.



- Since  $C \geq 0$ , we have:

$$C \geq \max[0, S - KB]$$

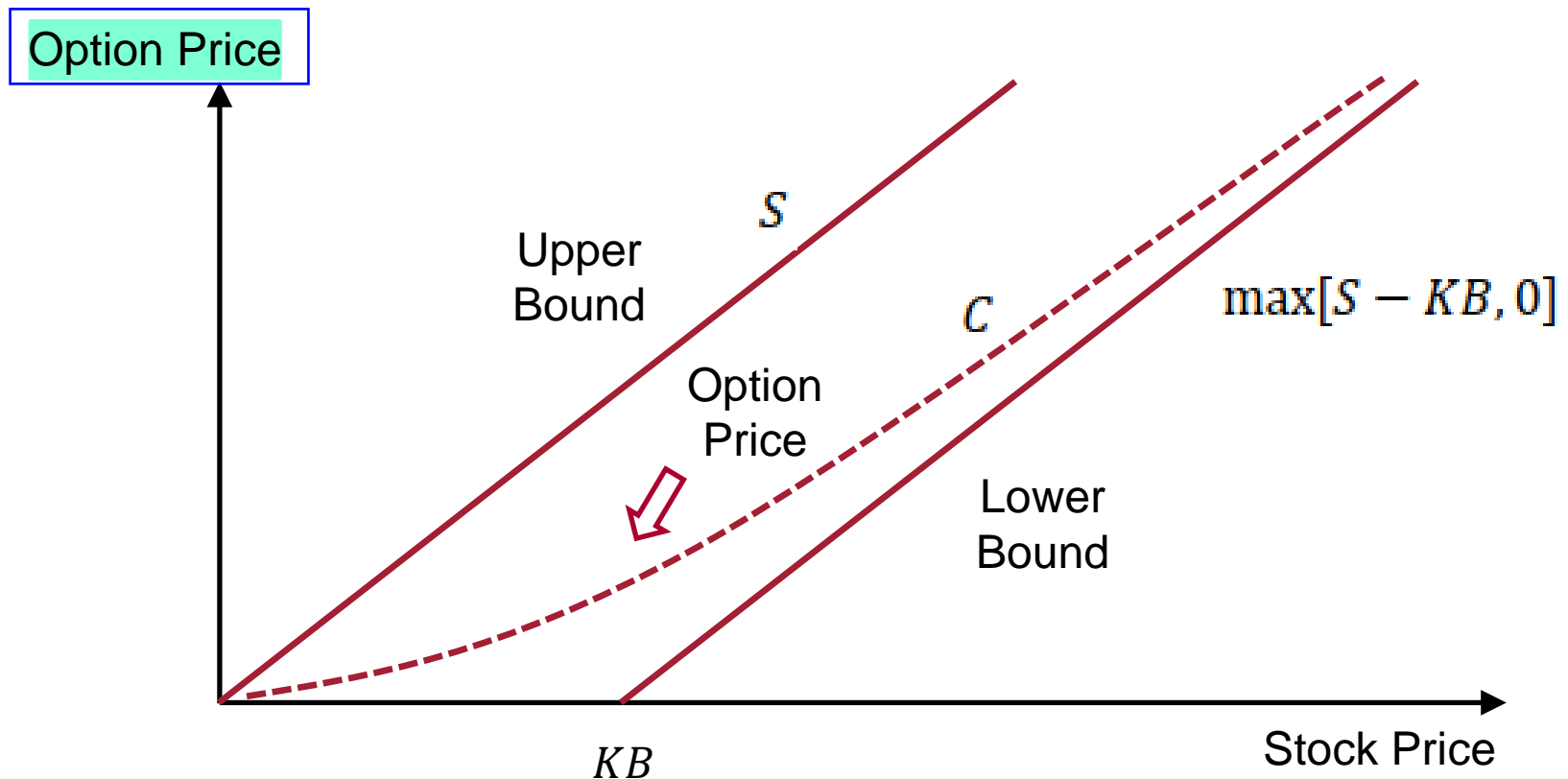
The price of the call has to be at least as high as the price of the stock, minus the present value of the strike

## Price bounds (cont'd)

## European Call Price Boundaries

4. Combining the above, we have:

$$\max[S - KB, 0] \leq C \leq S$$



The call price is bounded above by the stock price. And it is bounded from below by the maximum of 0 and the stock price minus the present value of the strike.

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options

## Concept Check

1/1 point (ungraded)

Consider a market where the stock cannot be sold short. Then, the Put-Call Parity for European options may fail to hold for options on this stock.

☒ True

☐ False



## Explanation

We used an arbitrage argument to establish the Put-Call Parity relation. If the stock cannot be sold short, then it is possible that  $C + BK < P + S$ . Without short-sale restrictions, market participants could take advantage of this inequality: short Portfolio 2 and buy Portfolio 1 to lock in arbitrage profits equal to  $P + S - C - BK$ . Short sale restrictions prevent this trade, which allows the Put-Call Parity to be violated without creating an arbitrage opportunity.

# Put-call parity for European options

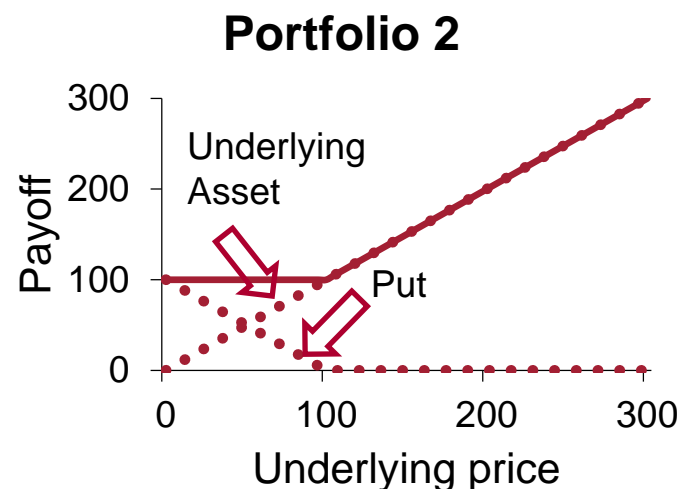
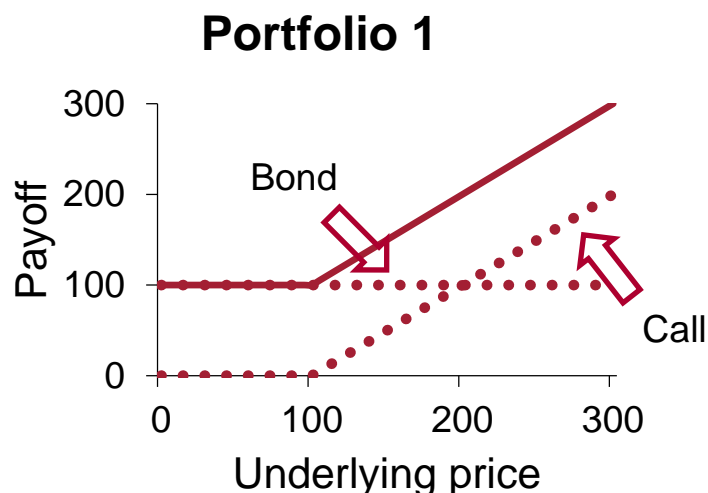
This result is based on the straightforward arbitrage argument and is a highly robust relation between option prices in the data.

We maintain the assumption that the underlying asset, the stock, **does not pay dividends**.

Portfolio 1: A call with strike  $K=\$100$  and a bond with par of  $\$100$ ;

Bond with a par value equal to the strike.

Portfolio 2: A put with strike  $\$100$  and a share of the underlying asset.



$$C + BK = P + S$$

If this equity doesn't hold then there is an arbitrage opportunity

Their payoffs are identical, so must be their prices:

This is called the **put-call parity**.

It connects prices of the European put and call options with the **same strike price**.

The Law of One Price implies that the initial value of the two portfolios must be the same.

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods

## Concept Check 1

Assume that the risk-free rate is constant over time and positive. Consider a stock paying no dividends over the next year, and an American call option on this stock maturing in 6 months. Assume you can trade all assets without restrictions at their market prices. Alice holds the option and expects the price of the stock to fall below market's expectations going forward. Then, it may be optimal for Alice to exercise her call options prior to their maturity.

### FALSE

Explanation: The answer is False, as we established using arbitrage arguments — our arguments hold regardless of the beliefs of the holder of the option. The essential point is that the value of the exercised option under the assumptions of the question is always lower than its market price. Even if Alice is pessimistic about the future prospects of the underlying stock relative to other investors, she is better off selling the option, rather than exercising it early.

# American options, early exercise

## American Calls

- American options are worth more than their European counterparts.
- Without dividends and with positive interest rates ( $B_t \leq 1$ ), never exercise an American call early.

American options cannot be worth less than their European counterparts since the holder of the option can always choose not to exercise the option until maturity.

- If exercise at  $t < T$ , collect  $S_t - K$

The payoff from immediate exercise is no more than the market price of the option.

Suppose we exercise an American call early. In that case, the payoff of the option at the point of exercise is equal to the difference between the stock price and the strike.

But If sell the option at  $t$  instead, collect at least the price of a European call, which is

European call is no less than the difference between the stock price and the present value of the strike.

$$C(S_t, K, T - t) \geq \max[0, S_t - KB_t] \geq S_t - K$$

stock price minus strike price, which is the payoff from immediate exercise of the American call.

- Better to sell than exercise, thus early exercise is never optimal!

again, without dividends and positive interest rates.

- By the law of one price:

$$c(S_t, K, T - t) = C(S_t, K, T - t)$$

To see why this is the case, note that the market value of the American call is at least as high as the market value of the European call, with the same strike and the same time to maturity.

If it is not optimal to exercise an American call before maturity, then its market value must be the same as the value of the corresponding European call.

# American options, early exercise

## American Puts

- Without dividends, it can be optimal to exercise an American put early.
- Example. A put with a strike of \$10 on a stock with price of zero.
  - Exercise now gives \$10 today;
  - Exercise later gives \$10 later. or less (due to positive interest rates)
- Better to exercise now (assuming positive interest rate).

In contrast to the American calls, it may be optimal to exercise an **American put** even if the stock does not pay dividends. Suppose we have a put option with a strike of \$10. Suppose the current stock price is 0. If we exercise the option immediately, we collect \$10. If we wait and exercise later, we receive no more than \$10, and at a later date. We conclude that in this case, under positive interest rates, it is optimal to exercise the option immediately.

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period (Model-Based Pricing)  
Replicating Portfolio
- Binomial model: multiple periods

## Concept Check

The price of a European **Put** option in the binomial model depends on the magnitude of the up and down moves in the stock price.

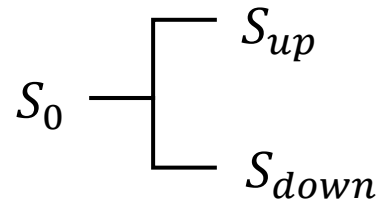
## TRUE

Explanation: The price does depend on the magnitude of the stock price moves, up (***u***) and down (***d***), as we established using an arbitrage argument.

# Option pricing models

## Binomial Model

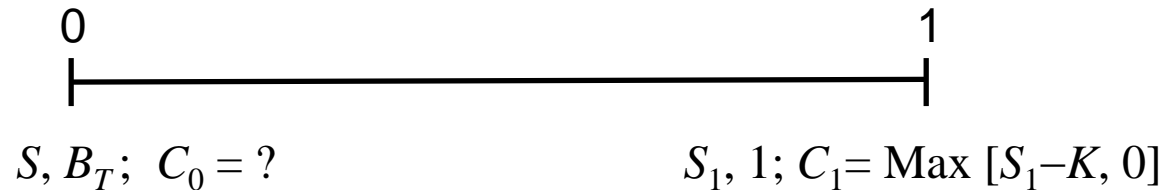
- In order to have a complete option pricing model, we need to make additional assumptions about the price process of the underlying asset (stock).
- We assume that prices do not allow arbitrage.
- A benchmark model – price follows a binomial process.



So far, we have been focused on model free results. These are restrictions on option prices that must hold, regardless of the properties of the underlying asset. We now want to refine these results to make more precise statements about option prices.

# Binomial option pricing model

## Single Period



We can only trade today at time zero.

### ■ Determinants of option value:

1. price of underlying asset  $S$ ,
2. strike price  $K$ ,
3. time to maturity  $T$ ,
4. interest rate  $r$ ,
5. **volatility of underlying asset  $\sigma$ .**

Return volatility of the underlying asset.  
We want to highlight that option prices are **highly sensitive** to stock return **volatility**, which is one of the **key inputs** into the **model**.

# A European call on a stock

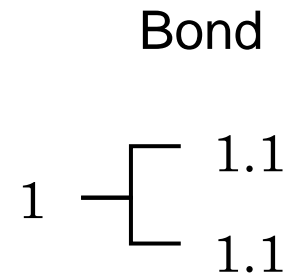
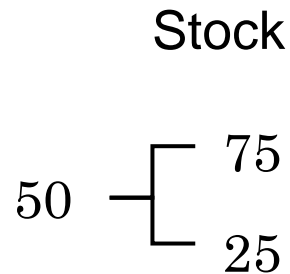
Note: Same principles apply to **Put** options.  
See Concept Check before.

- Current stock price is \$50;
- There is one period to go;
- Stock price will either go up to \$75 or go down to \$25;
- There are no cash dividends;
- The strike price is \$50;
- One period borrowing and lending rate is 10%.

We assume that the one period interest rate is 10%.  
One can borrow or lend without constraints  
at this risk-free rate.

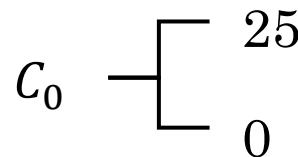
# A European call on a stock

- The stock and bond present two investment opportunities:



$r = 10\%$  from problem statement

- The option's payoff at expiration is:



$$C_1 = \text{Max} [S_1 - K, 0]$$

$K = \$50$  from problem statement

- What is  $C_0$ , the value of the option today?

Premium to pay/charge

Our objective is to figure out the price of the option at time zero.

# Replicating portfolio

- Form a portfolio of stock and bond that **replicates** the call's payoff:

(at time 1)

- $a$  shares of the stock;

Then with the low front price, the initial price of the option must be equal to the cost of this replicating portfolio.

- $b$  dollars in the riskless bond.

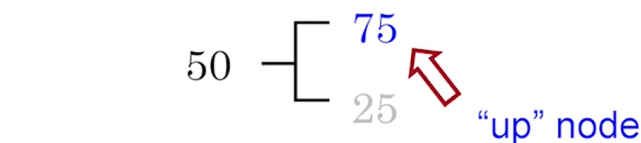
$b$  units of the riskless bond.  
Recall that each unit is \$1.

$$S \cdot a + b = C_0$$

such that:

$$75a + 1.1b = 25$$

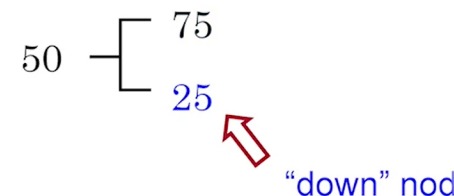
$$25a + 1.1b = 0$$



“up” node

“down” node

- Unique solution:  $a = 0.5$  and  $b = -11.36$ .



# Replicating portfolio

- Replication strategy:

To replicate the option, we must **borrow** \$11.36, and **buy** half a share of the stock.

- buy half a share of stock and sell \$11.36 worth of bond;
- payoff of this portfolio is identical to that of the call;
- market value of the call must equal the current cost of this “replicating portfolio” which is  $S_t * a + b = C_0$

$$(50)(0.5) - 11.36 = 13.64 \text{ Premium to pay/charge}$$

- Number of shares needed to replicate one call option is called the option's hedge ratio or delta
- In the above problem, the option's delta is  $a = 0.5$ .

$$S_t * a + b = P_0$$

For a Put Option Price

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods (Model-Based Pricing)  
Replicating Portfolio

## Concept Check

To price the option by arbitrage arguments in the multi-period binomial model, we need to assume that all investors can trade each period.

### FALSE

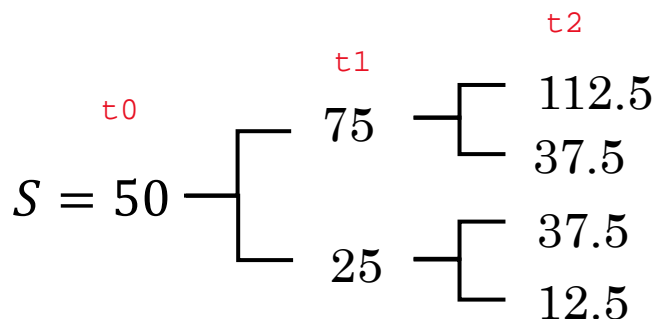
Explanation: We don't need all investors to engage in arbitrage activities. It is sufficient that some of the investors can trade in each period, and can thus arbitrage the difference between the price of the option and the cost of the replicating portfolio.

# Binomial option pricing model

## Multiple Periods

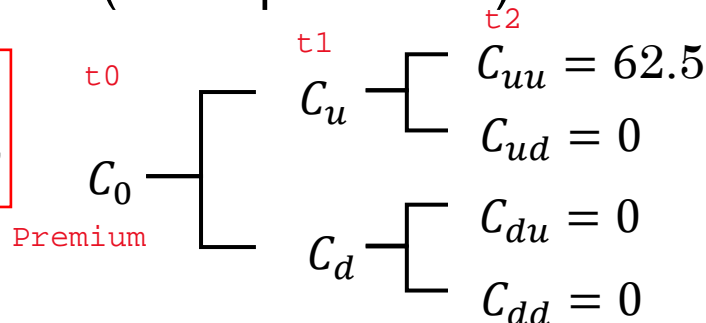
### ■ Multiple periods:

Suppose that the stock price starts at \$50, as before, and rises or falls by 50% each period for two periods.



### ■ European call price process (strike price = 50):

At maturity, which is time 2, the option pays \$62.5 in the highest node, where the stock price is \$112.50, and nothing in other nodes, where the stock price is below the strike.



$$C_2 = \text{Max}[S_2 - K, 0]$$

$$C_{uu} = \text{Max}[112.5 - 50, 0] = 62.5$$

$$C_{ud} = \text{Max}[37.5 - 50, 0] = 0$$

$$C_{du} = \text{Max}[37.5 - 50, 0] = 0$$

$$C_{dd} = \text{Max}[12.5 - 50, 0] = 0$$

### ■ The terminal value of the call is known.

### ■ $C_u$ and $C_d$ denote the option value next period when the stock price goes up and goes down, respectively.

### ■ Compute the time-0 value working backwards: first $C_u$ and $C_d$ and then $C_0$ .

## Period 1, “up” node

Start with Period 1:

- Suppose the stock price goes up to \$75 in period 1.
- Construct the replicating portfolio at node ( $t = 1$ , up):

$$112.5a + 1.1b = 62.5$$

$$37.5a + 1.1b = 0$$

- Unique solution:  $a = 0.833, b = -28.4$ .
- The cost of this portfolio:  $(0.833)(75) - 28.4 = 34.075$ .
- By Law of One Price,  $C_u = 34.075$  – same as the initial cost of the replicating portfolio.

## Period 1, “down” node

- Suppose the stock price goes down to \$25 in period 1. Repeat the above for node ( $t = 1$ , down):

$$112.5a + 1.1b = 0$$

$$37.5a + 1.1b = 0$$

- The replicating portfolio:  $a = 0$ ,  $b = 0$ .
- The call value at the lower node next period is  $C_d = 0$ .

## Period 0

- Now go back one period, to period 0:
- The option's value next period is either 34.075 or 0:

$$C_0 \begin{cases} C_u = 34.075 \\ C_d = 0 \end{cases} \quad \leftarrow \text{Payoffs at end of period } t1$$

- If we can construct a replicating portfolio in time 0 price  
of the option next period, then the cost of this replicating portfolio must  
equal the option's present value.

# Period 0

Replicating Portfolio at time 0

$$C_0 \begin{cases} C_u = 34.075 \\ C_d = 0 \end{cases}$$

- Find  $a$  and  $b$  so that:

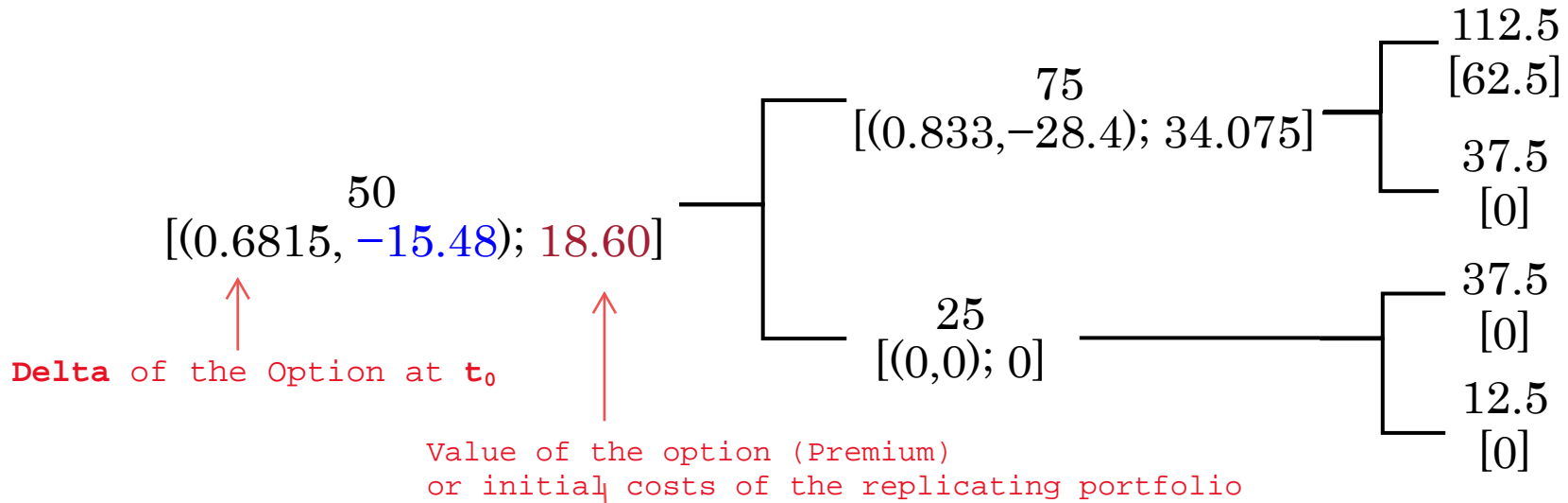
$$75a + 1.1b = 34.075$$

$$25a + 1.1b = 0$$

- Unique solution:  $a = 0.6815, b = -15.48$ .
- The cost of this portfolio:  $(0.6815)(50) - 15.48 = 18.60$ .
- The present value of the option must be  $C_0 = 18.60$ .

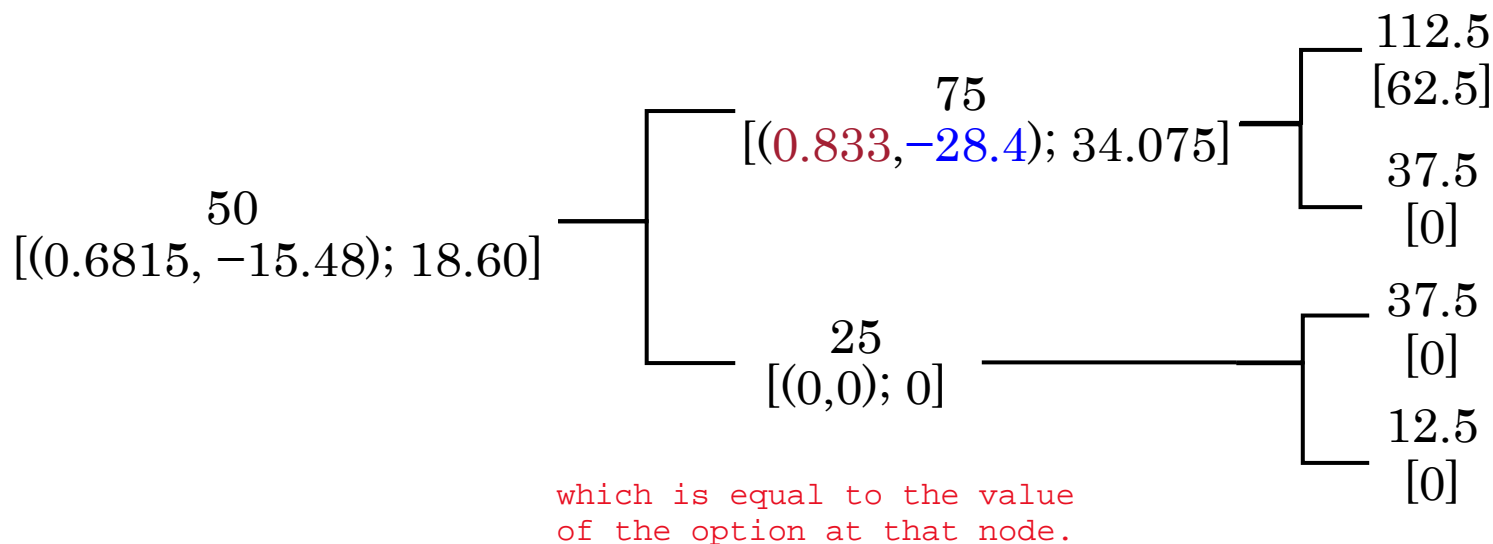
Premium or initial costs in the case of the replicating portfolio

# Option replication



- Period 0: Spend **\$18.60** and borrow **\$15.48** at 10% interest rate to buy 0.6815 shares of the stock.

# Option replication



- Period 1, “up node”: the portfolio value is 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%.
  - One period later, the payoff of this portfolio exactly matches that of the call: 62.5 for  $S_{uu} = 112.5$  and 0 for  $S_{ud} = 37.5$ . matches the payoff of the call option
- Period 1, “down node”: the portfolio becomes worthless. Close out the position. we have no stocks or bonds in the replicating portfolio.
  - The portfolio payoff one period later is zero.

# Binomial option pricing model

- Bottom line:
  - Replication strategy gives payoffs identical to those of the call.
  - Initial cost of the replication strategy must equal the call price.

The bottom line is that no matter what happens to the stock price between time 0 and option maturity, our replication strategy generates the same terminal value as the option. By the law of one price, the initial cost of the replicating portfolio must equal the market price of the option.

# Binomial option pricing model

- What we have **used** to calculate option's value:

- current stock price, stock returns up and down, is a measure of stock volatility
- magnitude of possible future changes of stock price — volatility,
- interest rate, risk-free interest rate
- strike price,
- time to maturity.

# Binomial option pricing model

## ■ What we have **not used**:

We **did not use** the probabilities of the up and down nodes in the stock price. And we did not use any information about investor's risk tolerance.

We did not need this information because our pricing argument relies on option replication.

We replicate the option state by state, no matter what happens to the stock price.

As a result, we do not need to know how likely one state is over the other or how investors feel about gains and losses.

- probabilities of upward and downward movements,

- investor's attitude towards risk.

risk tolerance

## ■ Questions on the Binomial Model:

- What is the length of a period?

what should be the length of each time step?

- Price can take more than two possible values.

How can we reconcile this with the binomial model?

- Trading takes place continuously.

The binomial model has discrete time steps,

Is the binomial model a good description of stock returns?

- Response: The length of a period can be arbitrarily small.

All of these concerns can be addressed by **shortening the time step in the model**.

We will explore this in the second part of the lecture.

As we reduce the time step in the binomial model, we will derive the celebrated **Black-Scholes-Merton option pricing model**.

# Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods