Optimization: Basic Concepts

Optimization

Centro de Investigación en Matemáticas A.C.

What is optimization? I

- Roughly speaking, it refers to the task of finding the best solution to a given function under some constraints
- Optimization is part of almost everything that we do:
  - Personnel schedules,
  - Routing planning,
  - Shopping,
  - Etc...

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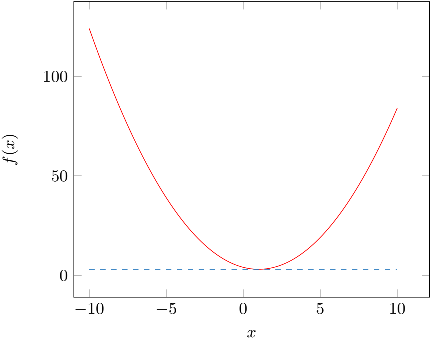
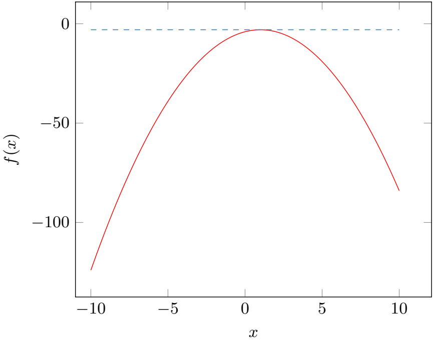
What is optimization? II



Minimization



Maximization

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Formal Definition

Formally...

$$
\begin{aligned}
\min \;\; & f_i(\mathbf{x}), & i = \{1, \ldots, m\} \\
\text{Subject to: } & g_j(\mathbf{x}) \leq 0, & j = \{1, \ldots, p\} \\
& h_k(\mathbf{x}) = 0, & k = \{1, \ldots, q\}
\end{aligned}
$$

where:

- $\mathbf{x}$ is an n-dimensional vector of decision variables
- $f_i(\mathbf{x})$ is called objective function
- $g_j(\mathbf{x})$ and $h_k(\mathbf{x})$ are called inequality and equality constraints, respectively is an equality constraint

The objective function

- It expresses the main goal of the problem which is either to be minimized or maximized
- Problems can have:
  - no objective function,
  - a single objective function,
  - multiple objective functions

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Minimization or maximization?

An optimization problem can be written as a minimization problem or as a maximization problem. These two problems are easily converted to the other form:

$$\begin{aligned} \min f(\mathbf{x}) \quad &\Longleftrightarrow \quad \max -f(\mathbf{x}) \\ \max f(\mathbf{x}) \quad &\Longleftrightarrow \quad \min -f(\mathbf{x}) \end{aligned}$$

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Decision Variables

- Decision variables represent the parameters that need to be determined to solve the problem
- They control the value of the objective function
- Defining the set of variables to a problem is one of the most difficult and crucial steps when formulating an optimization problem

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Constraints

- Constraints narrow the admissible values of the decision variables
- They can be
  - implicit:

$$0 \leq x_i \leq 10$$

  - explicit:

$$x_1 + x_2 - x_3 = 0$$

$$x_1 - x_3 \leq 0$$

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Feasible Solution and Feasible Region I

- A solution that meets the constraints is known as **feasible solution**.

$$[5, 5, 5]$$
$$[9, 5, 5]$$
$$[9, 0, 9]$$
$$[3, 1, 4]$$

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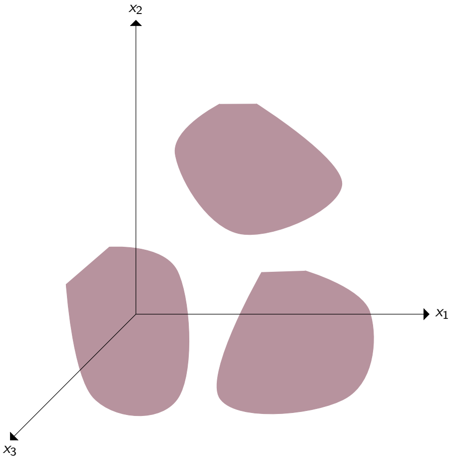
$[9, 5, 5]$	✖
$[9, 0, 9]$	✔
$[3, 1, 4]$	✔

Feasible Solution and Feasible Region II

- The set of all feasible solutions is known as the **feasible region**



Local and Global Optima I

Definition

A solution $\mathbf{x}^*$ is called local optimum if and only if $\forall \mathbf{x} \in \mathbf{X}_{\mathbf{vecindad}} : \; f(\mathbf{x}^*) \lhd f(\mathbf{x})$

Definition

A solution $\mathbf{x}^*$ is called global optimum if and only if $\forall \mathbf{x} \in \mathbf{X} : \; f(\mathbf{x}^*) \lhd f(\mathbf{x})$

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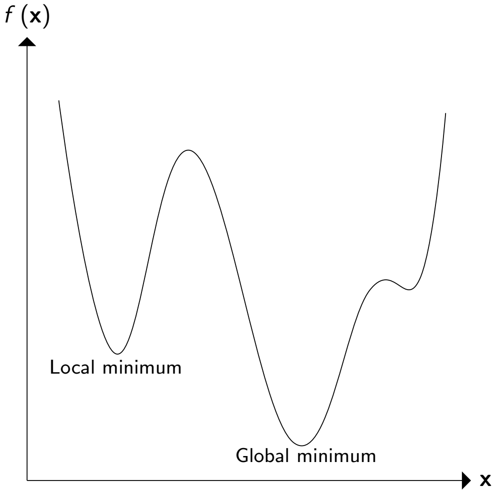
Local and Global Optima II



Classification I

Optimization problems can be categorized based on:

- their constraints
  - Constrained optimization problems
  - Unconstrained optimization problems
- their functions;
  - Separable optimization problems
  - Non-separable optimization problems
- the sort of solution;
  - Continuous optimization problems
  - Combinatorial optimization problems

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Classification II

- their decision variables;
  - Static optimization problems
  - Dynamic optimization problems
- the values on decision variables;
  - Integer programming
  - Real-valued optimization
  - Mixed-integer optimization

Classification III

- the sort to equations;
  - Quadratic programming
  - Geometric programming
  - Linear programming
  - No-linear optimization
- the number of objectives
  - Single objective optimization
  - Multi-objective optimization
  - Many objective optimization

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Black-box Optimization I

- It refers to a problem setup in which an optimization algorithm is supposed to optimize an objective function in a so-called black-box fashion
- Problems of this type regularly appear in practice:
  - When optimizing parameters of an unknown model
  - When modeling is too complex

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Black-box Optimization II

There are a number of algorithms for black-box optimization

- Random Search,
- Pattern Search,
- Nelder-Mead Simplex,
- Evolutionary Algorithms,
- Particle Swarm Optimization,
- Simulating Annealing Search,
- Bat Search,
- Bee Search,
- ...

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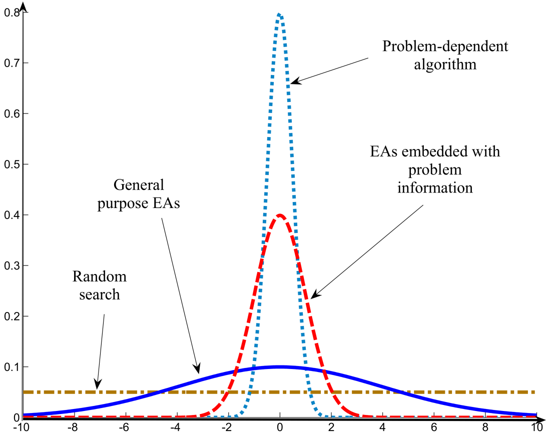
No Free Lunch Theorem



One kind of interpretation of No Free Lunch theorem

Retrieved from Xinjie Yu and Mitsuo Gen (2010). Introduction to Evolutionary Algorithms. Springer-Verlag.

The Exploration–Exploitation Dilemma I

- **Exploration** refers to exploring unknown regions with the aim of gaining new knowledge
- **Exploitation** refers to delving in what it is known with the aim of getting something close to what it is expected

Should we explore or exploit?

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The Exploration–Exploitation Dilemma II

- An "intelligent search" requires the proper balance of exploration and exploitation
- The proper balance of exploration and exploitation depends on how regular our environment is
- If our environment is rapidly changing, then our knowledge quickly becomes obsolete and we cannot rely as much on exploitation
- However, if our environment is highly consistent, then our knowledge is dependable and it may not make sense to try very many new ideas

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Thanks for your attention!

Questions?

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