Calculate Historical Volatilities and Correlations - Methods Comparison

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January 13, 2025

Abstract

Risk Estimation, Historical Volatility, Exponential Weighted Moving Average

Introduction

Estimating asset volatilities and correlations using historical data is at the same time a relatively classic but still openly debated challenge. This article will then present and compare different methods usable to estimate risk metrics using historical data. The influence of the parametrization of each method will also be discussed to understand their implication.

After presenting the dataset used in the analysis, our starting point will be to present the common method used to calculate volatility directly as standard deviation of historical data. If this method allows a first estimation that can be useful for quick analysis, it presents some limitations as well that will be discussed in the part 2 of this article. To balance these limitations, the part 3 will introduce weighting scheme more particularly using the Exponential Weighted Moving Average method.

The purpose of this analysis is to challenge risk estimation process used for end-usage such as calibration of Economic Scenario Generator or construction of Strategic Asset Allocation. In this way, we are interested to find the right trade-off between having smooth and consistent estimation over time but also estimation representing the most likely scenario to happen in the next 10 years or 30 years.

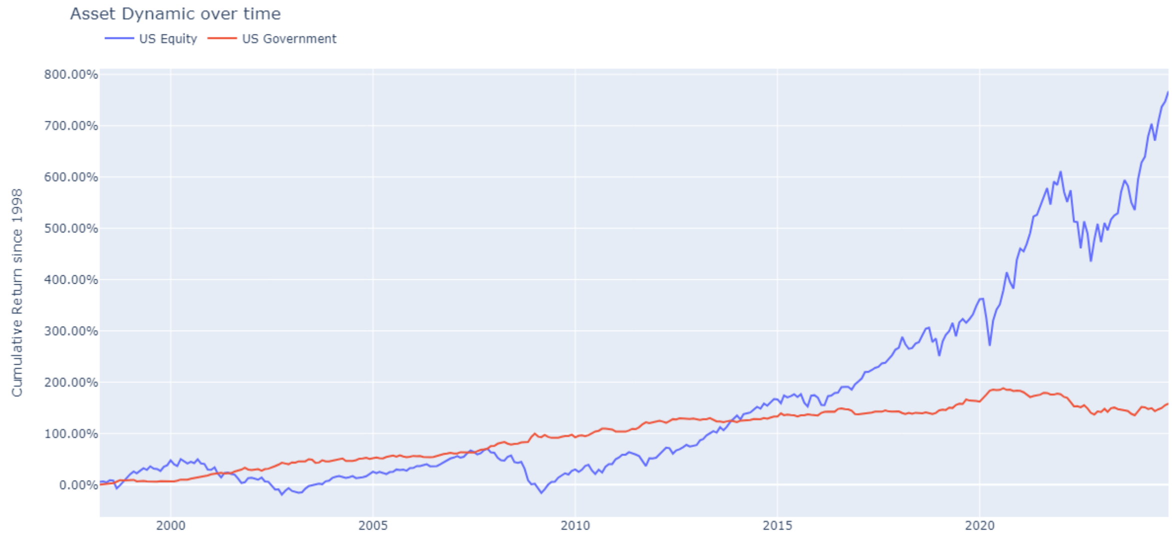
1 Example case: US Government vs US Equity

Considered as core assets in portfolio allocation, US Equity and US Government Bonds are very present in multiple asset allocation. Also, thanks to the depth of their historical time series, the assets allow us to have a larger perspective of volatility and correlation estimation.

To collect historical data associated to these 2 asset classes, we use the following indexes:

- MSCI USA for US Equity
- Bloomberg US Treasury for US Government Bonds

The following graph shows the cumulative returns of assets between early 1998 and August 2024. For the following analysis, we kept monthly data corresponding to the last business day of each month.



The interest of this period is the fact that it contains different market phases for both Equity and Government Bonds. More particularly, the period contains the following market stress / crisis:

- the Dot-com bubble (2000-2002),
- the Financial crisis (2007-2008),
- the Covid-19 recession (2020),
- the increase of inflation and nominal rates (2021)

Neverthless following an article published by AQR team in 2023 [1], these last twenty years are far from representing the exhibited correlation between bonds and stocks over the last century.

2 Historical Volatility without weighing scheme

The *standard* historical volatility model use a constant weighing scheme, meaning that all the historical observations contribute the same to the estimation of volatility.

At a date $t+1$, the formula associated to the historical covariance between 2 assets i and j using the information available until date t can be expressed as:

$$\sigma^2_{i,j}(t+1|t) = \sum_{k=1}^{t} \frac{(r_i(k) - \bar{r}_i(k)) \times (r_j(k) - \bar{r}_j)}{t} \tag{1}$$

And then the variance of the asset i can be calculated as:

$$\sigma^2_i(t+1|t) = \sum_{k=1}^{t} \frac{(r_i(k) - \bar{r}_i)^2}{t} \tag{2}$$

where

- $r_i(k)$ return calculated at the date k,

[1] see Reference [6] A Changing Stock–Bond Correlation: Drivers and Implications - Alfie Brixton, Jordan Brooks, Pete Hecht, Antti Ilmanen, Thomas Maloney, and Nicholas McQuinn - Journal of Portfolio Management, March 2023.

- $\bar{r}_i = \sum_{k=1}^{t} \frac{r_i(k)}{t}$ average return of asset i until date t,

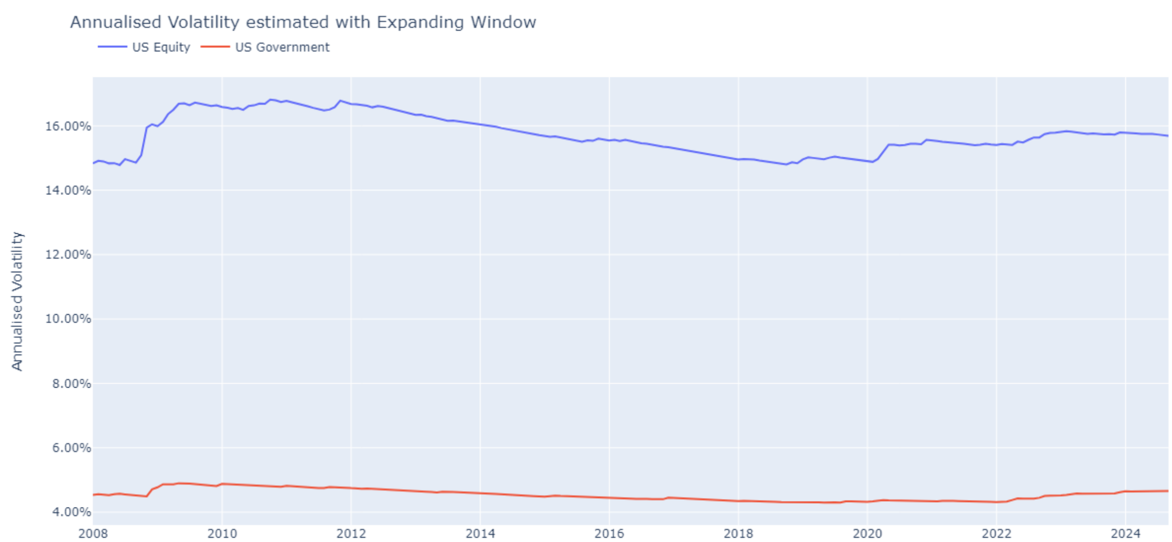
Note: in the following section, we will expressed volatility is an annualized term. This means that even if we estimate volatility with monthly return, we scale its value to be interpreted as an annual volatility using the following formula: $\sigma_i^2(t+1|t)_{Annualized} = \sigma_i^2(t+1|t)_{Monthly} \times 12$, where 12 represents the number of monthly returns in one year.

2.1 Expanding Windows

Using expanding windows in historical volatility estimation means than all the dates prior to t are used.

For US Equity and US Government Bonds assets, the historical volatility estimation using expanding window provides the following results [2] over time:

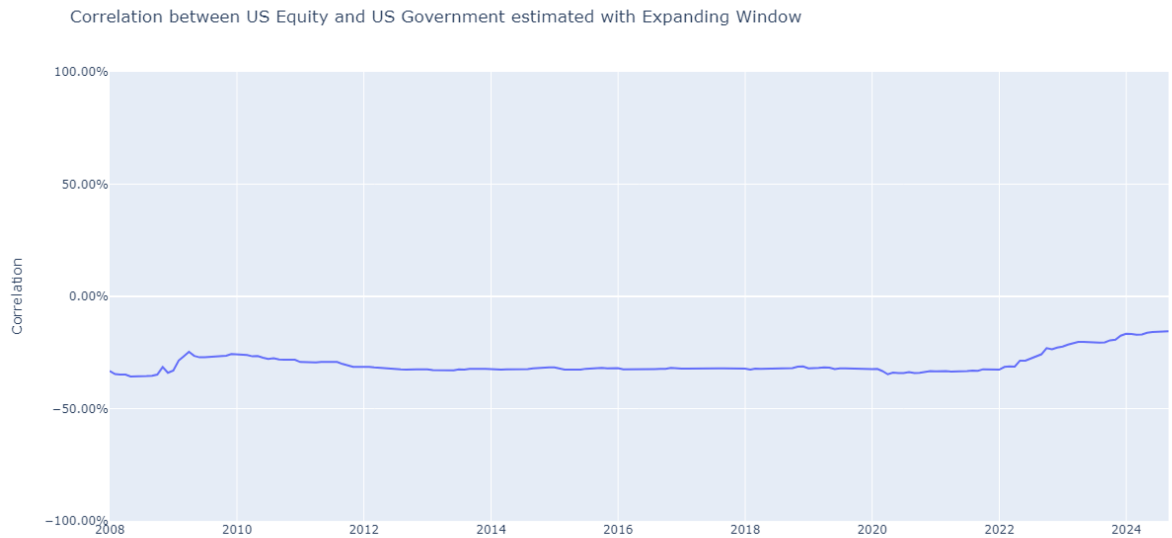
- the volatility of US Equity stay historically around 15% with a peak during the financial crisis at 17%
- the volatility of US Government Bonds is rather stable around 5%



In the expanding version, the window size increases over time and then the volatility estimation gets naturally smoother when the number of observations increase.
The limit of this is that the larger window implies also that adding new data have lower impact on the risk estimation. For example, the jump of volatility observed at the end of the period during the covid-crisis is much lower than the one observed during the financial crisis.

For correlation between Equity and Government, similar remarks could be done with an estimation comprised between -35% and -15%. The increase at the end of the period could be explained by a regime change in rates environment, where the increase of rates tends to correlate positively equity and government bonds.This is consistent with the explanation provided in the article of AQR about the sentivity of Equity / Government bonds correlation to surprises in growth changes and inflation changes.

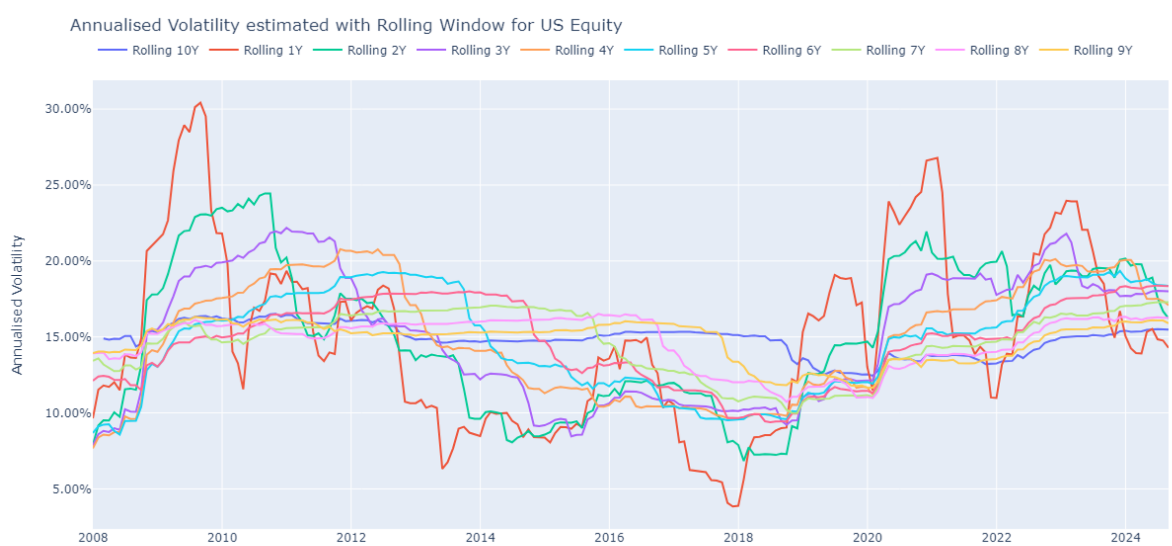
[2] In this graph as in the following, we start the visualization of the volatilities and correlation 10 years after the first observed return. This is done because historical risk estimation methods required enough data to obtain stable results. Indeed, estimating an asset volatility using only a few months of returns will not make sense for us.



2.2 Rolling Windows

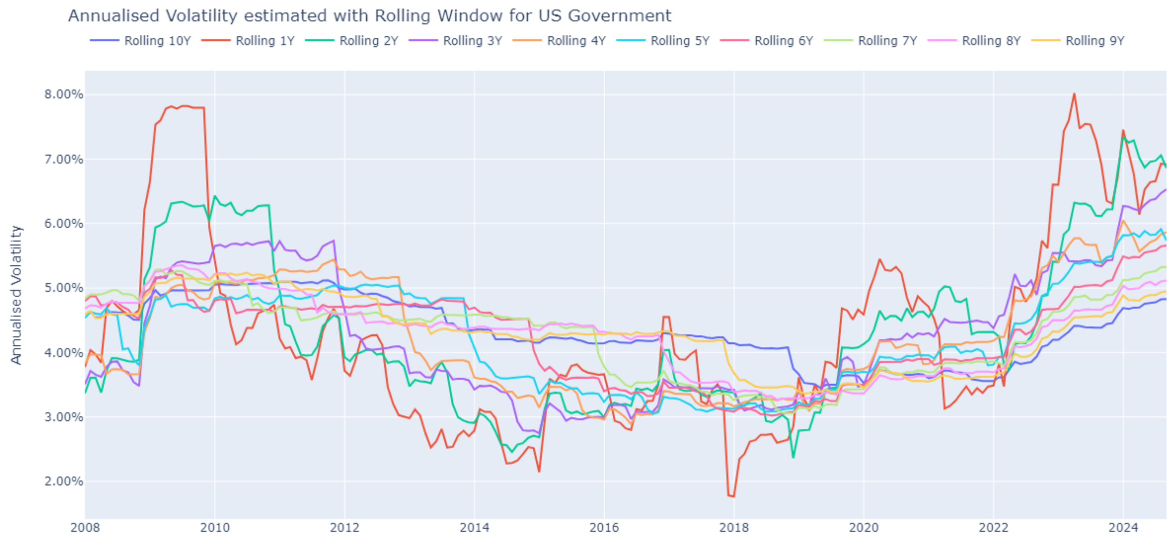
Estimation of volatility using rolling windows mean that only a certain part of the historical data is used for the calculation. The size of the rolling windows helps then to control the historical depth used.

For example, using small rolling window for example 1 year (in red in the graph below) will provide a more reactive estimation than a larger window such as 10 years (in purple in the graph below). Indeed, with monthly data, 1 year window means that volatility is estimated using only the last 12 monthly returns while 10 years rolling windows correspond means 120 monthly returns.
So, with constant weight scheme the influence of the addition of one new monthly return is considerable higher for smaller window, and therefore, volatility estimation tends to rougher over time.

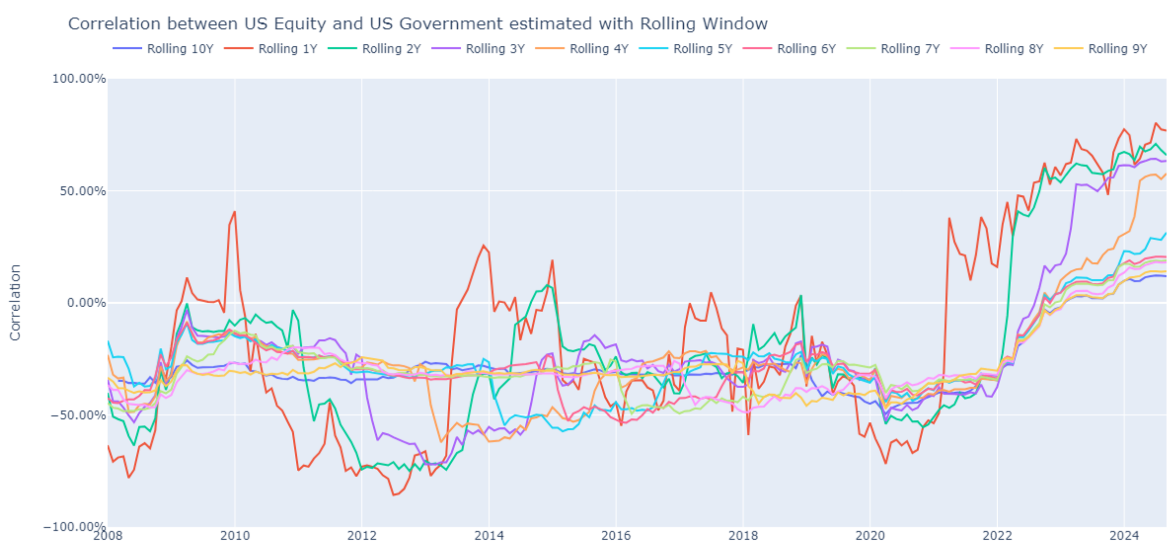


For US Equity as for US Government, we can then see directly the impact of each market scenario in volatility estimation with 1 year rolling window, such as the sudden jump after start of financial crisis

(reaching around 30% for Equity and 8% for Government) or even the during the Covid-19 recession for Equity (27%) and the increase of rates for Government bonds (8%).
At the opposite, with the 10 years rolling window, the volatility estimation is smoother over time: between 12.5% and 16.5% for Government bonds and between 3.5% and 5.5% for Government bonds.



For correlation between Equity and Government, similar behavior can be noticed. The estimation using 1 year rolling window is then relatively rough over time with value comprise between -85% and +80%, while the estimation using 10 years rolling window is more stable with value between -46% and 12%. In both cases however, we can see the same trends to have increase of correlation following major market event such as financial crisis but also and more particularly the change of rates policy since 2021.
These observations support the important of the choice for the size of the rolling window, as there is a clear trade off to do between having a smooth and consistent estimation of risk over time but also be able to identify rapidly change of market behavior such as the one occurring in 2021.



3 Exponential Weighing Moving Average

The logic of the Exponential Weighing Moving Average model (noted EWMA after) is to introduce a weighting scheme so that the weight of each observation is exponentially decreasing over time.

This may help to reach a certain trade-off between giving more importance to market returns occurring in the most recent month (and then be able to identify new regime) while keeping a more consistent estimation than in the methodology using rolling window.

3.1 Estimation of risk metrics

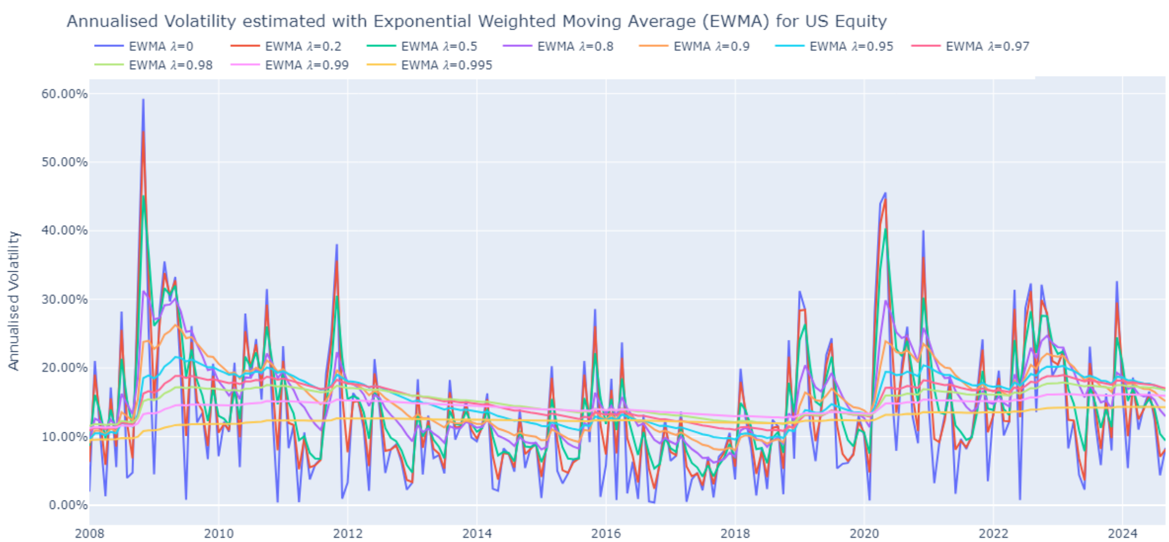
The covariance and variance are then calculated iteratively such as:

$$\sigma_{i,j}^2(t+1|t) = \lambda \times \sigma_{i,j}^2(t|t-1) + (1-\lambda) \times r_i(t) \times r_j(t) \tag{3}$$

$$\sigma_i^2(t+1|t) = \lambda \times \sigma_i^2(t|t-1) + (1-\lambda) \times r_i^2(t) \tag{4}$$

λ is called decay factor of the EWMA model. It is used to determine how many weights we want to give to recent observation compared to the past one. Its value is comprised between 0 and 1.

In the graph below, we can observe the influence of the decay factor on the estimation of volatility:



First observation here is that using low values of decay factor, such as λ such as 0 (in purple), 0.2 (in red) or 0.5 (in green), results in unstable volatility estimation with peak varies from one month to another.
At the opposite, higher value of λ such as 0.995 (in yellow), 0.99 (in pink) or 0.98 (in light green) provide smoother estimation of volatility over time.

3.2 Weighting Scheme

To understand better the influence of decay factor on volatility estimation, we can rewrite the variance equation to highlight its implicit weighting scheme such as:

$$
\begin{aligned}
\sigma_i^2(t+1|t) &= \lambda \times \sigma_i^2(t|t-1) + (1-\lambda) \times r_i^2(t) \\
&= \lambda \times \left[\lambda \times \sigma_i^2(t-1|t-2) + (1-\lambda) \times r_i^2(t-1)\right] + (1-\lambda) \times r_i^2(t) \\
&= \lambda^t \times (1-\lambda) \times r_i^2(1) + \cdots + (1-\lambda) \times r_i^2(t) \\
&= \sum_{k=0}^{t-1} \lambda^k \times (1-\lambda) \times r_i^2(k) \\
&= \sum_{k=0}^{t-1} \omega_{t-k} \times r_i^2(t-k)
\end{aligned}
$$

The weight of the observation at the date $t-k$ is noted $\omega_{t-k} = \lambda^k \times (1-\lambda)$ and we can reindex it as $\omega_k = \lambda^{t-k} \times (1-\lambda)$ is set as the weight of the observation at the date k.

In this way, we can effectively verify that the method gives more importance / weight to the most recent observations. Indeed as $0 \leq \lambda \leq 1$, $0 \leq 1-\lambda \leq 1$ and $\omega_k = \lambda^{t-k} \times (1-\lambda) \geq \omega_{k-1} \geq ... \geq \omega_1$ ie the weight associated to the date t is higher than the weight of the date $t-1$ and ultimately the weight of the first date.

The weights of each observation across time deduced from the decay factor parameter can be represented in the following table:

	λ=0	**λ=0.2**	**λ=0.5**	**λ=0.8**	**λ=0.9**	**λ=0.95**	**λ=0.97**	**λ=0.98**	**λ=0.99**	**λ=0.995**
2024/08	100.00%	80.00%	50.00%	20.00%	10.00%	5.00%	3.00%	2.00%	1.00%	0.50%
2024/07	0.00%	16.00%	25.00%	16.00%	9.00%	4.75%	2.91%	1.96%	0.99%	0.50%
2024/06	0.00%	3.20%	12.50%	12.80%	8.10%	4.51%	2.82%	1.92%	0.98%	0.50%
2024/05	0.00%	0.64%	6.25%	10.24%	7.29%	4.29%	2.74%	1.88%	0.97%	0.49%
2024/04	0.00%	0.13%	3.12%	8.19%	6.56%	4.07%	2.66%	1.84%	0.96%	0.49%
...	...	...	...	...	...	...	...	...	...	...
1998/07	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.04%	0.10%
1998/06	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.04%	0.10%
1998/05	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.04%	0.10%
1998/04	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.04%	0.10%
1998/03	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.04%	0.10%

A special case can be observed with $\lambda = 0$, as this version of EWMA is equivalent to an estimation with a rolling window of 1 month: all the risk is determined based on the last observed monthly return.

Small decay factor such as $\lambda = 0.2$ give large weights to the most recent observations, the last monthly returns is then representing already 80% of the weights associated the different observation, while large decay factor such as $\lambda = 0.995$ give much smaller weights for last monthly returns (0.5%).

3.3 Cumulative Weights and Half-Life Time

To confirm that, we can calculate the cumulative weights of the observations over past dates and defined the half-life time as the number of past observations needed to correspond to 50% of the total weight:

$$\tau = argmin_k \sum_{k=0}^{t-1} \lambda^k \times (1-\lambda) - 50\% \tag{5}$$

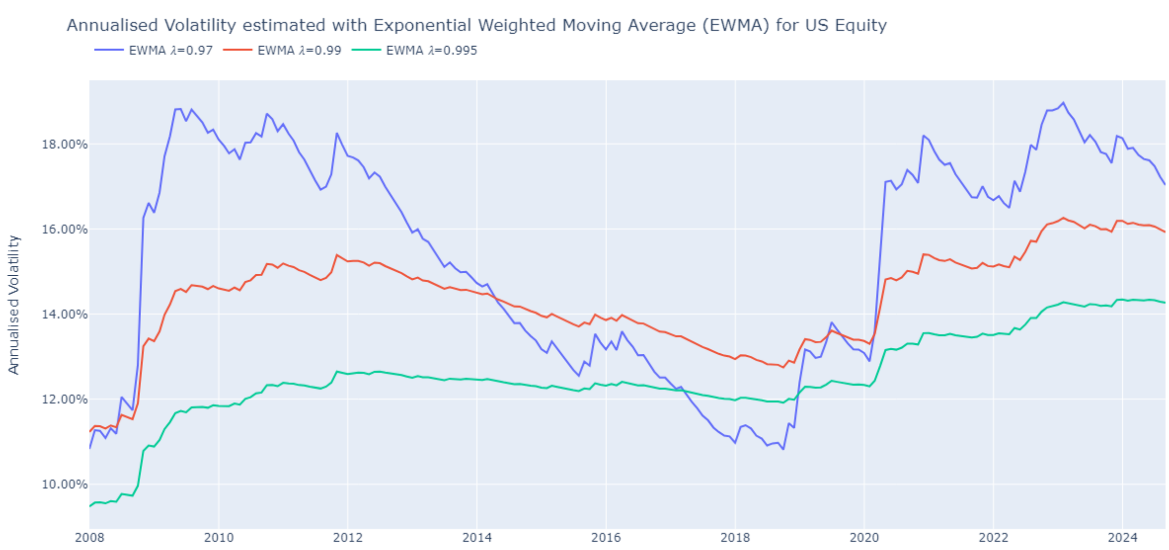
The following table shows the first historical date so that cumulative weights reaching 50%. For small decay factors, such as $\lambda = 0$ to 0.9, the monthly returns occurring in the last year represent already more than half of the cumulative weights, while large value of decay factor continues to provide 50% of weights or more to older observations.

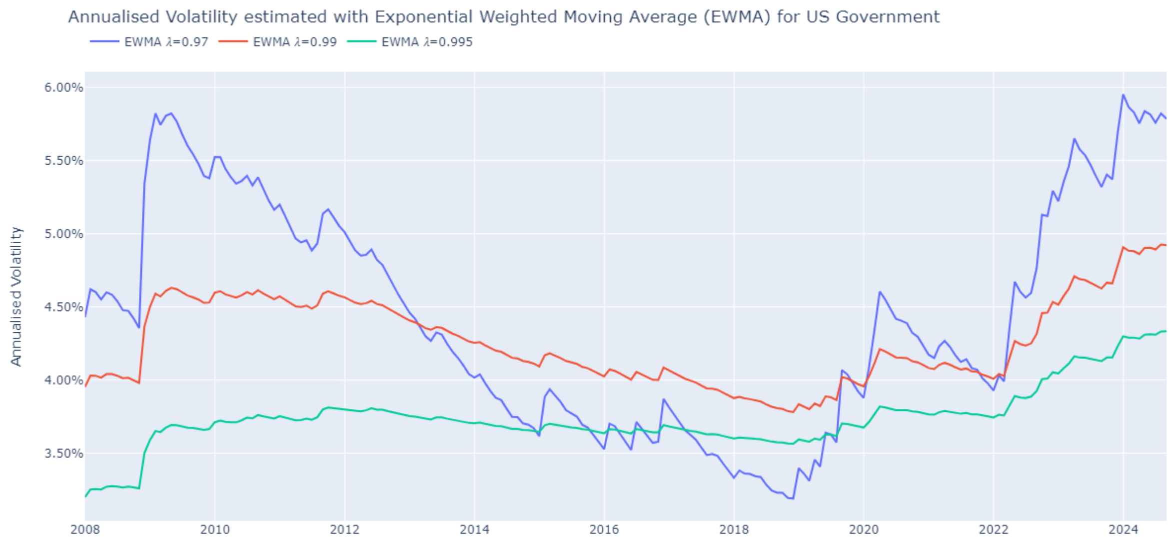
	Date reaching Half-Time	Number Dates before reaching Half-Time
λ=0	2024/08	1
λ=0.2	2024/08	1
λ=0.5	2024/08	1
λ=0.8	2024/06	3
λ=0.9	2024/02	7
λ=0.95	2023/07	14
λ=0.97	2022/10	23
λ=0.98	2021/11	34
λ=0.99	2018/12	69
λ=0.995	2013/03	138

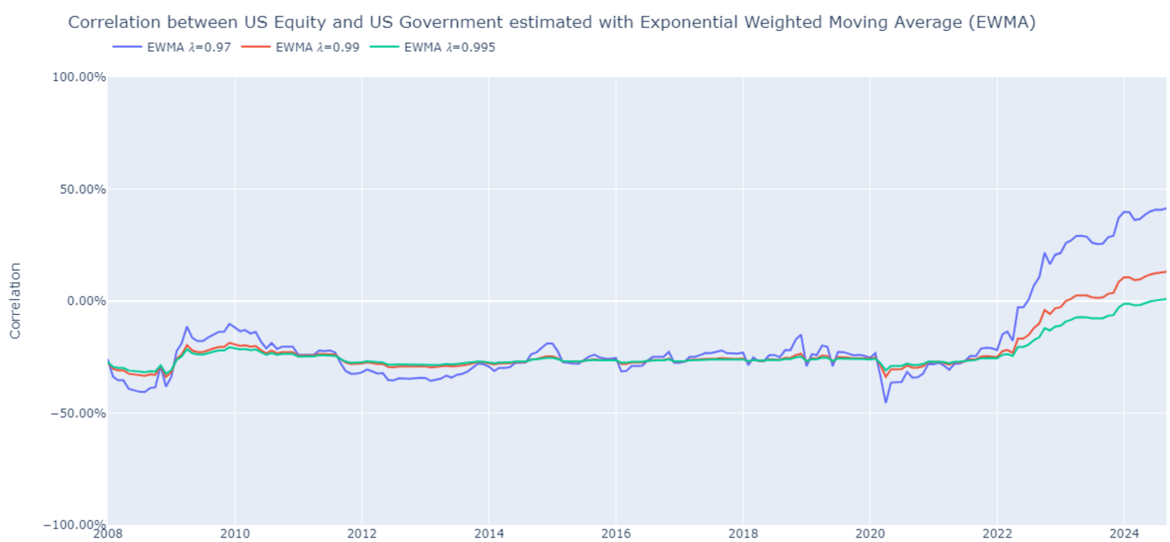
3.4 Selection of Decay Factor

Based on these different indicators, we decide to focus on certain value of decay factor to simplify the visualization of volatilities and correlation in the comparative section of this article. The estimation using EWMA with decay factors of $\lambda = 0.97$, 0.99 and 0.995 have then been selected, due to their ability to be a trade-off between giving more weights to recent observation while keeping a long-term vision.

We can visualize then more clearly the estimation of volatilities and correlation for these factors in the following graphs:







4 Comparison of methods and Conclusions

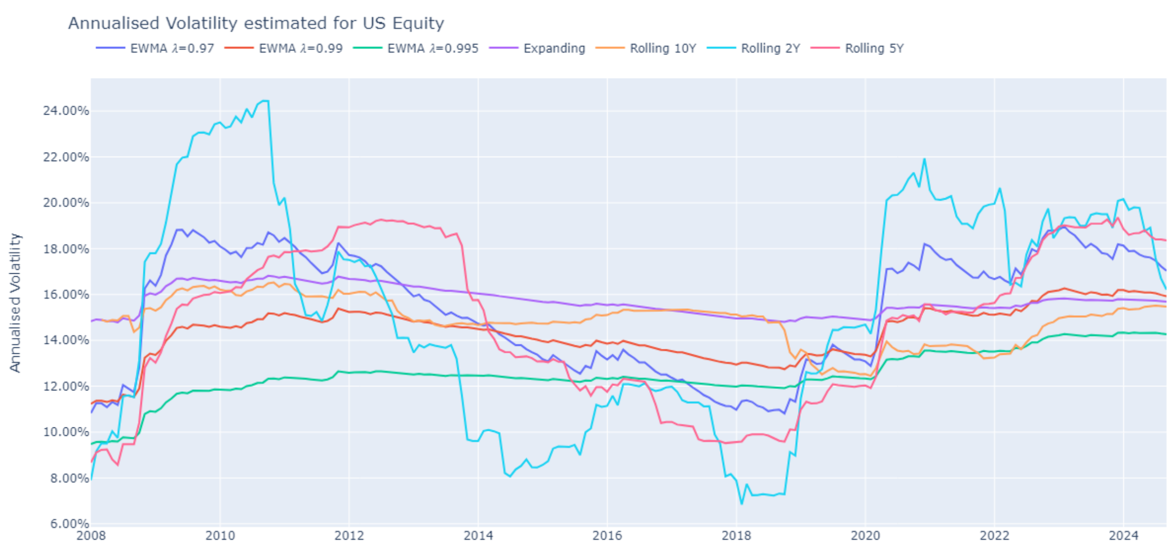
To conclude the article comparing different methods of volatilities and correlations, we can first analyze the following summary of estimation for US Government and US Equity at the end of the period.

The first conclusion is then that the numbers are different across methods but also across different parametrization of the same method. For example, for US Government Bonds, the rolling 10 years methods provide an estimation of 4.83%, which is quite close to the estimation of 4.66% using expanding method since early 1998 and the estimation using EWMA with decay factor of 0.995 (4.33%). For correlation, the difference tends to be more important due to the larger impact of the estimation period.

	US Equity	US Government	Correlation
Method			
Expanding	15.68%	4.66%	-15.57%
Rolling 2Y	16.22%	6.86%	65.91%
Rolling 5Y	18.36%	5.74%	31.35%
Rolling 10Y	15.48%	4.83%	11.91%
EWMA λ=0.97	17.03%	5.79%	41.37%
EWMA λ=0.99	15.92%	4.92%	13.07%
EWMA λ=0.995	14.27%	4.33%	0.83%

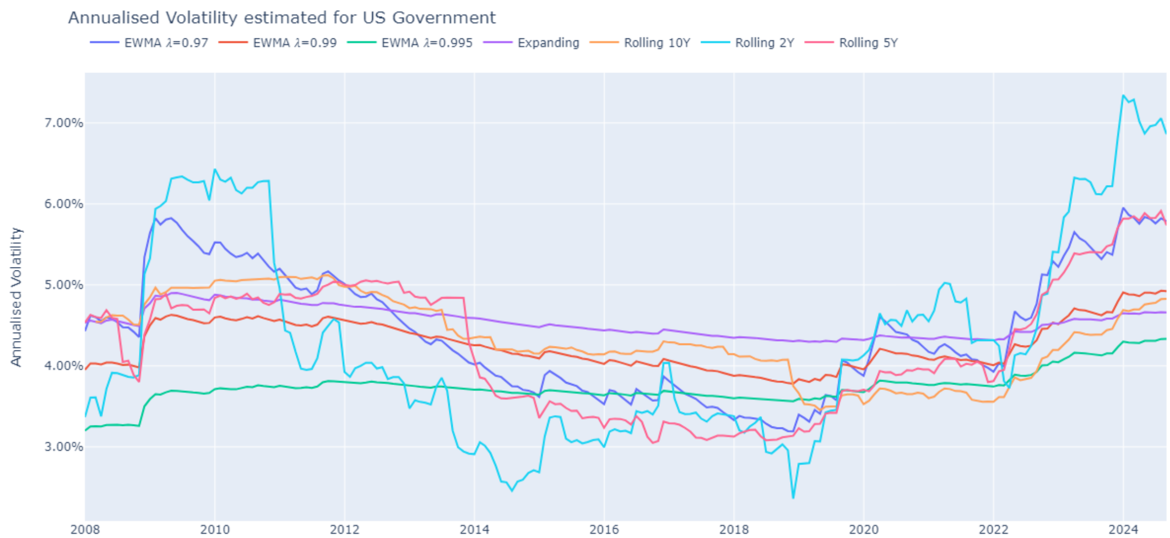
The main difference relies then on the choice of the parameters (size of the rolling window and decay factor for EWMA). As discussed in the article, this choice of parameter corresponds to a trade-off between having a stable estimation of the metrics across time and having an estimation able to adapt to current market condition.
In the following, we will focus on application using long term estimation of risk such as forecast simulation or Strategic Asset Allocation. In this way, we will tend to exclude too rough estimation over time while still seeking to reflect switch in market regimes such, as the one linked to rate environment that occurred in 2020.

This can be visualized properly in the following graphs of historical estimation of volatilities and correlations with the different methods:

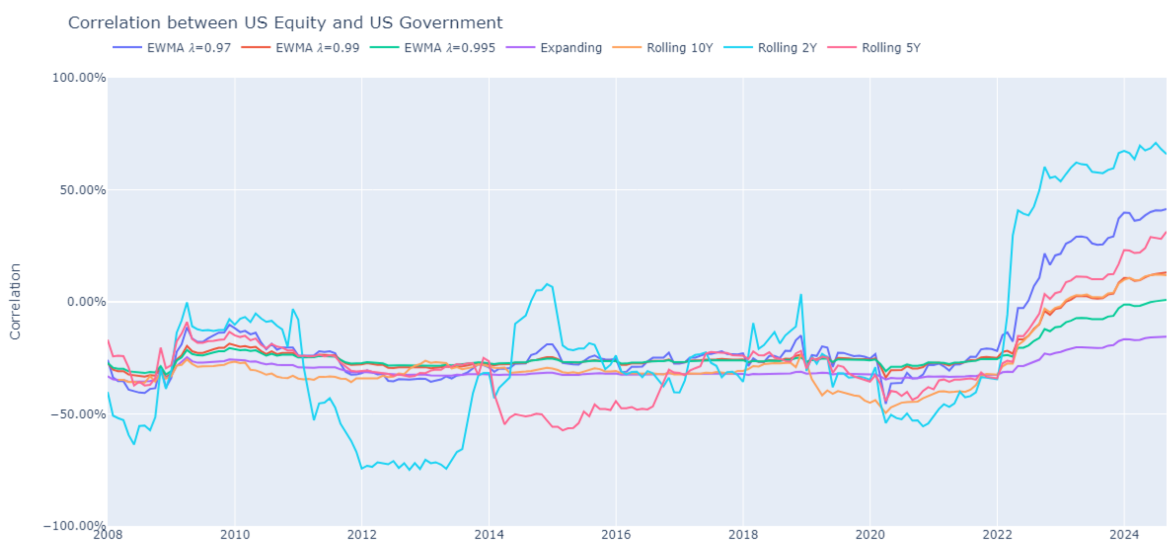


On US Equity volatility, the parametrization using a rolling window of 2 years (clear blue in graph) and 5 years (pink in graph) rolling window or even EWMA with smaller decay factor (such as 0.97, blue in graph) provide too bumpy estimations. At the opposite, estimation using expanding window (in purple here) give much less importance to recent observations and then tends to miss change of regime in the market.

To analyze that, we can for example check more particularly the period between 2018 and 2022. Indeed, this period correspond to a succession of a relatively calm market, followed by a sudden equity crisis linked to covid-19 followed by quick recovery and later a phase of rates increases. During this period, the estimation of US Equity volatility with rolling 2 years window goes from 7% (2018), 13% (2019) to 20% (2021). If the trend of volatility increase during the period is real, the amplitude is a bit exaggerated to be used in application such as calibration or Strategic Asset Allocation, so we may exclude choice as the 2 or 5 years rolling window. In the expanding method however, the impact of this periods is too minor due to the marginal weight of these months: between 15% and 16% over the period.



Similarly for US Government volatility and correlation between Equity and Government Bonds, the estimation with 2 and 5 years rolling window and the EWMA with 0.97 decay factor are a bit to irregular over time to be used for use cases such as calibration or Strategic Asset Allocation.



The consistent methodology for these type of use cases will then be rather the long-term rolling window such as 10 years and the EWMA with decay factor such as 0.99 or 0.995. The main difference between both is the advantage of EWMA method to have an explicit weighting scheme to pick up switch of market regime faster than rolling window.

References

[1] *RiskMetricsTM—Technical Document*, J.P. Morgan and Reuter, December 1996.

[2] *Return to RiskMetrics: The Evolution of a Standard*, Mina Jorge and Xiao Jerry Yi, April 2001.

[3] *Exponentially Weighted Moving Models*, Eric Luxenberg and Stephen Boyd, Department of Electrical Engineering, Stanford University, April 2024

[4] *What should the value of lambda be in the exponentially weighted moving average volatility model?*, Bernard Bollen, January 2014.

[5] *The Stock-Bond Correlation*, Czasonis, M., M. Kritzman, and D. Turkington, Journal of Portfolio Management, February 2021.

[6] *A Changing Stock–Bond Correlation: Drivers and Implications*, Alfie Brixton, Jordan Brooks, Pete Hecht, Antti Ilmanen, Thomas Maloney, and Nicholas McQuinn, Journal of Portfolio Management, March 2023.