```mathematica
ClearAll["Global`*"]
```

$$fk[k1\_, k2\_, r\_] := \left\{ \frac{1}{\frac{1}{k1} + \frac{1}{k2}}, \frac{1}{\frac{1}{r*k1} + \frac{1}{k2}}, \frac{1}{\frac{1}{k1} + \frac{1}{r*k2}} \right\}$$

$$frho[k1\_, \rho\_, r\_] := k1 * \left\{ \frac{\rho}{1+\rho}, \frac{r*\rho}{r+\rho}, \frac{r*\rho}{1+r*\rho} \right\}$$

```mathematica
K1 = 1; (*The contour plot does not depend on the value of k1.*)
```

$$g[k\_, r\_] := \left\{ \frac{k}{2}, \frac{r*k}{2}, \frac{k}{2} \right\}$$

```mathematica
 (*Only works when R1 is retained in the reduced model (ρ>1)*)
```
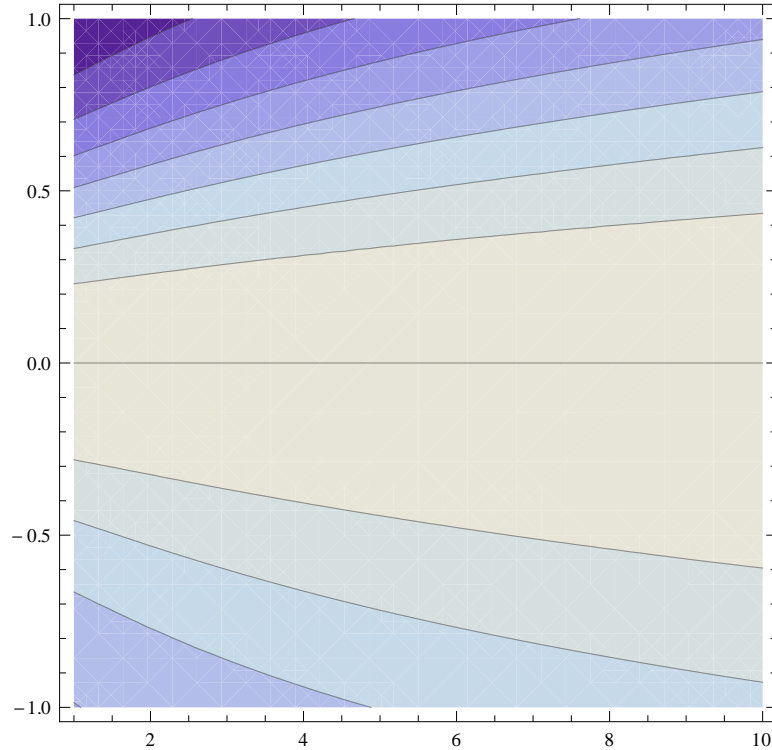
```mathematica
Fitk[ρ_, r_] := k /. Last[FindMinimum[Norm[(frho[k1, ρ, r] /. {k1 → K1}) - g[k, r]],
    {k, HarmonicMean[{k1, k1 * ρ}] /. k1 → K1}]]
```

$$RelErr[\rho\_, log10r\_] := \frac{Fitk[\rho, 10 \,\hat{}\, log10r] - HarmonicMean[\{k1, k1*\rho\}] \,/.\, k1 \to K1}{HarmonicMean[\{k1, k1*\rho\}] \,/.\, k1 \to K1}$$

```mathematica
(*Needs["PlotLegends`"]*)
ContourPlot[RelErr[ρ, log10r], {ρ, 1, 10}, {log10r, -1, 1}, AxesLabel → {"ρ", "log10r"}]
```



```mathematica
jac = Simplify[Transpose[{D[fk[k1, k2, r], k1], D[fk[k1, k2, r], k2]}]]
```

$$\left\{ \left\{ \frac{k2^2}{(k1+k2)^2}, \frac{k1^2}{(k1+k2)^2} \right\}, \left\{ \frac{k2^2 \, r}{(k2+k1 \, r)^2}, \frac{k1^2 \, r^2}{(k2+k1 \, r)^2} \right\}, \left\{ \frac{k2^2 \, r^2}{(k1+k2 \, r)^2}, \frac{k1^2 \, r}{(k1+k2 \, r)^2} \right\} \right\}$$

```mathematica
{U, S, Vh} = SingularValueDecomposition[jac /. r → 2];
```

```
S = FullSimplify[Vh, k1 > 0 && k2 > 0] /. k2 → ρ k1

fklam[k1_, k2_, r_, λ_] := λ * fk[k1, k2, r] + (1 - λ) * {k1, k2, 0};

Manipulate[ParametricPlot3D[fklam[k1, k2, r, λ] /. r → 2, {k1, 0.1, 10}, {k2, 0.1, 10},
  AxesLabel → {"J1", "J2", "J3"}, PlotRange → {{0, 10}, {0, 10}, {0, 10}}], {λ, 0, 1}]
```