Geodesic Equation

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1 Geodesic equation

1.1 Geodesic Equation as Conventionally Defined

$\ddot{p}_k + \Gamma^k_{ij}\dot{p}_i\dot{p}_j = 0$ (Explain a bit ***)

1.2 Geodesic Equation as Used Here

Geodesic equation is a system of second-order autonomous ODEs in the parameter space.

$p''(t) = f(p, p')$.

Let $v(t) \equiv p'(t)$.

$$\begin{cases} p' = v \\ v' = f(p, v) \end{cases} \tag{1}$$

If $a(t) \equiv p''(t) = 0$: $A(t) \equiv f''(p(t)) = A_\perp + A_\parallel$

Since one way of interpreting geodesics is free motion of a point particle on a manifold, $a(t)$ needs to take a value that renders $A_\parallel = 0$.

Since $A_\parallel = P_\parallel A = J(J^TJ)^{-1}J^TA$, and any vector u in P is mapped to the prediction space as Ju, it follows that $a = -(J^TJ)^{-1}J^TA$ leads to $A_\parallel = 0$.

$$\begin{cases} p' = v \\ v' = -(J^TJ)^{-1}J^TA \end{cases} \tag{2}$$

A is usually calculated using finite difference:

$A(t) \equiv f''(p(t)) \approx \dfrac{f'(p(t+\delta t)) - f'(p(t-\delta t))}{2\delta t} \approx \dfrac{\frac{f(p(t+2\delta t))-f(p(t))}{2\delta t} - \frac{f(p(t))-f(p(t-2\delta t))}{2\delta t}}{2\delta t} = \dfrac{f(p(t+2\delta t)) + f(p(t-2\delta t)) - 2f(p(t))}{4(\delta t)^2}$

Let $\Delta t = 2\delta t$:

$A(t) \approx \dfrac{f(p(t+\Delta t)) + f(p(t-\Delta t)) - 2f(p(t))}{(\Delta t)^2} \approx \dfrac{f(p(t) + v(t)\Delta t) + f(p(t) - v(t)\Delta t) - 2f(p(t))}{(\Delta t)^2}$

New notes (2-24-16, Wed):

Let's think in this way.

$$\begin{cases} p' = v \\ v' = a \end{cases} \tag{3}$$

We need to find a that will render $A_\parallel = 0$.

Given p, v, a, $A = f''(t) \approx \dfrac{f(p(t+\Delta t)) + f(p(t-\Delta t)) - 2f(p)}{(\Delta t)^2}$, where

$$\begin{cases} f(p(t+\Delta t)) \approx f\left(p + v\Delta t + \frac{1}{2}a(\Delta t)^2\right) \approx f(p + v\Delta t) + \frac{1}{2}Ja(\Delta t)^2 \\ f(p(t-\Delta t)) \approx f\left(p - v\Delta t + \frac{1}{2}a(\Delta t)^2\right) \approx f(p - v\Delta t) + \frac{1}{2}Ja(\Delta t)^2 \end{cases} \tag{4}$$

Therefore, $A = \dfrac{1}{(\Delta t)^2}\left(f(p + v\Delta t) + \frac{1}{2}Ja(\Delta t)^2 + f(p - v\Delta t) + \frac{1}{2}Ja(\Delta t)^2 - 2f(p)\right) = A_{fd} + Ja$

$A_\parallel = P_\parallel A = \left(J(J^TJ)^{-1}J^T\right)\left(A_{fd} + Ja\right) = 0$, so $\boxed{a = -(J^TJ)^{-1}J^TA_{fd}}$.

Proof of geodesics follow isocurves when degeneracy is one.

Sketch of proof: Geodesics have constant speed on manifold; initial speed is zero, so y doesn't move along geodesic motion; which is the definition of isocurves.

1.3 Different Time Parametrizations of Geodesic Equation

The above choice of $a(t)$ leads to constant speed in prediction space, characteristic of free motion in that space. But it has the drawback that an integration of the equation corresponding to a motion towards any boundary of the manifold would run into singularity, and a geodesic motion parameterized in such a different way that the speed is constant in parameter space would avoid this problem: (emphasize time parametrization ***)

$a = a_{\parallel} + a_{\perp}$ (in the v direction ***)

Constant speed in P

$\implies \frac{\|v\|}{dt} = 0 = a_{\parallel} = 0$

$\implies a \leftarrow a - a_{\parallel} = a - \frac{a \cdot v}{v \cdot v} v$

Argument: $p(t) \to p(\tau)$ does not change the trace (? ***) of the curve in P, hence geodesic motion in prediction space is preserved.

2 Setup

2.1 Notation

$\leftarrow, \to$ P (? ***)

2.2 f

$y = f(p)$

$p \in P$, parameter in parameter space

$y \in Y$, prediction in prediction space

2.3 Linearization of f

$J \equiv Df$, Jacobian

$P_{\parallel} = J(J^T J)^{-1} J^T$, projection operator onto tangent space (explained... ***)

$P_{\perp} = I - P_{\parallel} = I - J(J^T J)^{-1} J^T$, projection operator onto normal subspace

$g \equiv J^T J$, metric tensor When g is (close to) singular, one conventional way of making it less so is $g \leftarrow g + \lambda I$.