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# New insights on hierarchical random measures

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# Joint work with

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# Competing risks

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- Failure time  $T$  and a cause of failure  $\Delta \in \{1, \dots, D\}$

## Examples

- ✓ D. Bernoulli (1766): disentangle the risks of dying from smallpox and from other cause
- ✓ Mortality after acute myocardial infarction due to different causes: sudden cardiovascular disease, non-sudden cardiovascular disease, non-cardiovascular disease (Andersen et al., 2002)
- ✓ Patients affected by melanoma: they have undergone a surgical excision of the tumour. The competing events are death due to melanoma and death due to other causes (Drzewiecki et al., 1980).

# Approaches to competing risks

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## Multistate models

- Markov process with
  - one transient state = 0 = "alive"
  - $D$  absorbing states =  $d$  = "death from cause  $d$ "

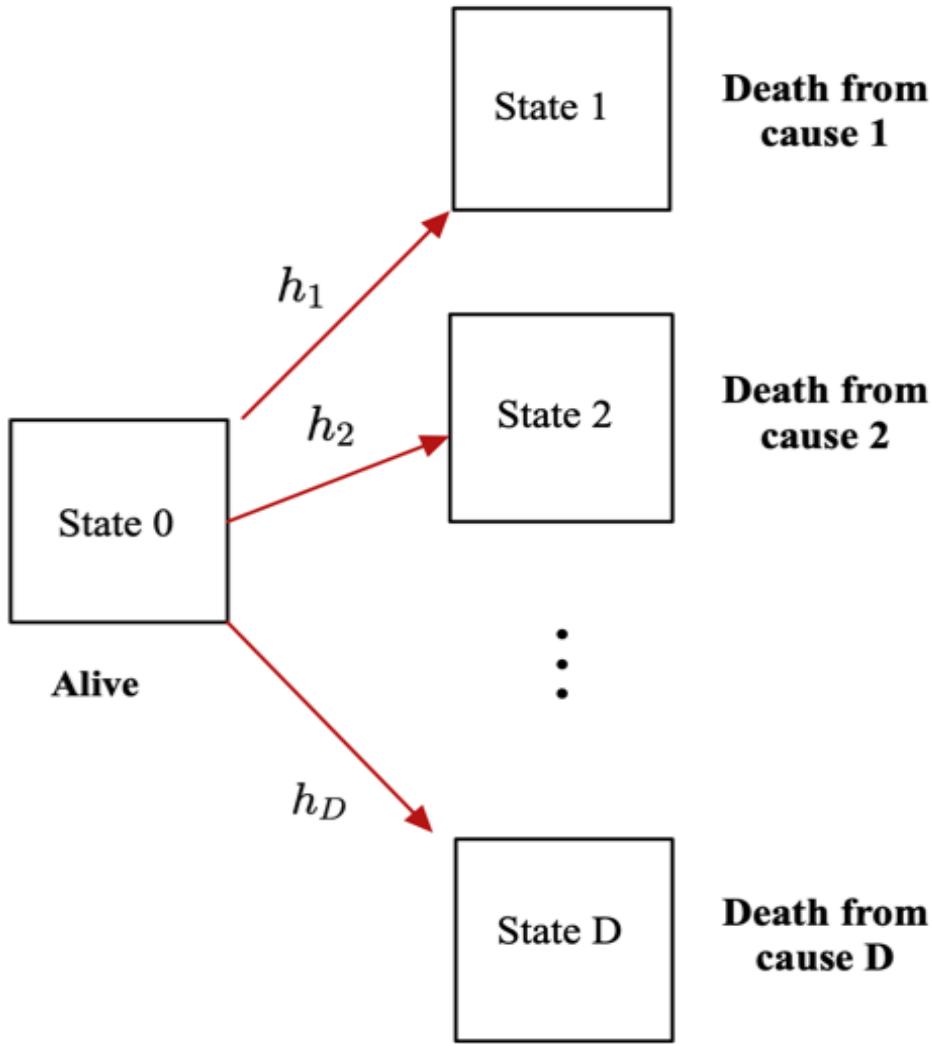
- Transition intensities from state 0 to state  $d$

$$h_d(t) = \lim_{\delta \downarrow 0} \frac{\text{Prob}[T \leq t + \delta, \Delta = d \mid T \geq t]}{\delta}$$

- $h_d$  = **cause-specific hazard**

# Multistate models

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$P_{0,d}(0, t)$  = probability of transition  
from “0” to “ $d$ ”  
between time 0 and  $t$

$P_{0,0}(0, t)$  = probability of being  
alive at time  $t$

# Multistate models: transition probabilities and likelihood

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**Survival function**

$$P_{0,0}(0, t) = \exp \left\{ - \sum_{d=1}^D \int_0^t h_d(u) \, du \right\} = \text{Prob}[T > t] = S(t)$$

**Cumulative incidence  
function**

$$P_{0,d}(0, t) = \int_0^t S(u-) h_d(u) \, du = \text{Prob}[T \leq t, \Delta = d]$$

**Observed data**

$(\tilde{T}_i, \Delta_i)_{i=1}^n$  such that

$$\begin{aligned}\tilde{T}_i &\neq T_i \text{ if } \Delta_i = 0 \\ \tilde{T}_i &= T_i \text{ if } \Delta_i = d \in \{1, \dots, D\}\end{aligned}$$

**Likelihood**

$$\mathcal{L} = \prod_{i=1}^n S(\tilde{T}_i) \prod_{d=1}^D h_d(\tilde{T}_i)^{1_{\{\Delta_i=d\}}} = \text{function of cause-specific hazards } h_d$$

# Approaches to competing risks

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## Latent time-to-event variables

- Latent failure times  $Y_1, \dots, Y_D$
- Observed data  $T = \min_{1 \leq d \leq D} Y_d$  &  $\Delta = \operatorname{argmin}_{1 \leq d \leq D} Y_d$
- Joint survival function  $S^*(t_1, \dots, t_D) = \operatorname{Prob}[Y_1 > t_1, \dots, Y_D > t_D]$
- Survival function associated with  $T \implies S(t) = S^*(t, \dots, t)$
- Cause-specific hazard

$$h_d(t) = -\frac{\partial \log S^*(t_1, \dots, t_D)}{\partial t_d} \Big|_{t_1 = \dots = t_D = t}$$

# Latent time-to-event variables

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## Caveat

- Competing risks data  $(\tilde{T}_i, \Delta_i)$  **cannot identify**
  - ✓ joint survival  $S^*$
  - ✓ marginal distribution of latent failure times
- Different  $S^*$  may lead to the same likelihood function
- Only **identifiable parameters** are the cause-specific hazards

$$S_d^*(t) = S^*(0, \dots, 0, t, 0, \dots, 0)$$

# Overview of the approaches

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## Frequentist literature

- Andersen et al. (2002), Andersen and Kneiding (2012) and Crowder (1991) on the identifiability issue
- Reviews: recent paper by Geskus (2024) and textbooks by Crowder (2012) and Geskus (2015)

## BNP literature

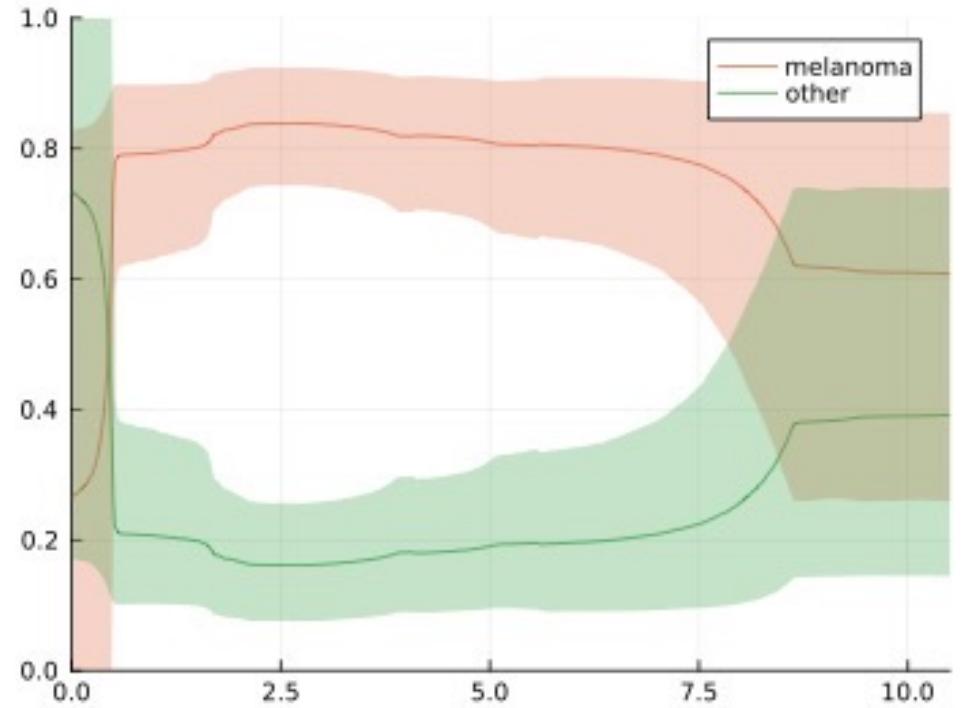
- Arf  et al. (2019) and Lau and Cripps (2022): approaches based on the beta-Stacy process and thinned CRMs, respectively
- Xu et al. (2022): dependent Dirichlet processes for estimating treatment effects with semi-competing risks

# Goals of the talk

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- (1) BNP model for the cause-specific hazard functions
- (2) Estimation of the survival function of  $T$
- (3) Estimation of the cause-specific incidence functions
- (4) Estimation of the predictive curve of competing risks

$$t \mapsto \text{Prob}[\Delta_{n+1} = d \mid T_{n+1} = t, T_{1:n}, \Delta_{1:n}]$$



Estimates of predictive curves in the melanoma dataset

# A BNP approach to competing risks

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## Bayesian model for cause-specific hazard rates

- (1) Dependence among  $h_1, \dots, h_D$
- (2) Possibility of accomodating also for covariates
- (3) Analytical tractability for posterior and predictive inference

## CRM-mixture representation

$$\tilde{h}_d(t) = \int_{\mathbb{X}} k(t, \theta) \tilde{\mu}_d(d\theta) = \sum_{\ell \geq 1} k(t, \theta_{d,\ell}) J_{d,\ell} \quad d = 1, \dots, D$$

⇒ Dependence among  $(\tilde{\mu}_1, \dots, \tilde{\mu}_D)$  ?

# A BNP approach to competing risks

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## Some notation and assumptions

- ✓ Completely random measure  $\tilde{\mu} \sim \text{CRM}(\nu)$  if for any non-negative function  $f$

$$\mathbb{E} e^{-\tilde{\mu}(f)} = \exp \left\{ - \int_{\Theta} \int_0^{+\infty} \left( 1 - e^{-sf(\theta)} \right) \nu(ds, d\theta) \right\}$$

- ✓ **Homogeneity:** Lévy intensity of the form  $\nu(ds, d\theta) = \rho(s) ds \alpha(d\theta)$

- ✓ **Assumptions:**  $\int_0^{+\infty} \min\{1, s\} \rho(s) ds < +\infty$  &  $\int_0^{+\infty} \rho(s) ds = +\infty$

$\alpha$  = diffuse and deterministic measure on  $\Theta$

or

$\alpha$  = (discrete) completely random measure on  $\Theta$

# A BNP approach to competing risks

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## Still some notation

- ✓ Laplace exponent of  $\tilde{\mu} \sim \text{CRM}(\nu)$ : for any non-negative  $f$

$$\psi(f(\theta)) = \int_0^{+\infty} \left\{ 1 - e^{-sf(\theta)} \right\} \rho(s) ds$$

- ✓ Cumulants of the functional  $\int f d\tilde{\mu}$

$$\tau_q(f(\theta)) = \int_0^{+\infty} s^q e^{-sf(\theta)} \rho(s) ds$$

# A BNP approach to competing risks

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With  $D$  cause-specific hazard rates

$$\begin{aligned}\tilde{\mu}_1, \dots, \tilde{\mu}_D \mid \tilde{\mu}_0 &\stackrel{\text{iid}}{\sim} \text{CRM}(\tilde{\nu}) \\ \tilde{\mu}_0 &\sim \text{CRM}(\nu_0)\end{aligned}$$

**Intensities**

$$\begin{aligned}\tilde{\nu}(ds, dx) &= \rho(s) ds \tilde{\mu}_0(d\theta) \\ \nu_0(ds, dx) &= \rho_0(s) ds P_0(d\theta) \quad P_0 = \text{diffuse prob. meas. on } \Theta\end{aligned}$$

## Definition:

The vector  $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_D)$  is a *hierarchical completely random measure* and is denoted as  $\tilde{\mu} \sim \text{hCRM}(\tilde{\nu}, \nu_0)$

**Observations:**

$$\begin{aligned}(\tilde{T}_1, \Delta_1), \dots, (\tilde{T}_n, \Delta_n) \mid \tilde{p} &\stackrel{\text{iid}}{\sim} \tilde{p} \\ \tilde{p} &\stackrel{\text{d}}{=} g(\tilde{\mu})\end{aligned}$$

# A BNP approach to competing risks (no censoring)

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**Prior distribution** for the incidence and survival functions

$$\begin{aligned}\tilde{P}_{0,d}(t, t + dt) &= \text{Prob}[T_i \in (t, t + dt), \Delta_i = \delta | \tilde{\mu}] \\ &= dt \exp \left\{ - \sum_{\ell=1}^D \int_0^t \int_{\Theta} k(s, \theta) \tilde{\mu}_\ell(d\theta) \right\} \int_{\Theta} k(t, \theta) \tilde{\mu}_\delta(d\theta)\end{aligned}$$

$$\tilde{P}_{0,0}(0, t) = \tilde{S}(t) = \exp \left\{ - \sum_{\ell=1}^D \int_0^t \int_{\Theta} k(s, \theta) \tilde{\mu}_\ell(d\theta) \right\}$$

**Property:**  $t \mapsto \tilde{S}(t)$  is a **proper survival function** if and only if

$$P_0 \left( \left\{ \theta \in \Theta : \int_0^{+\infty} k(s, \theta) ds = +\infty \right\} \right) = 1$$

## In terms of latent time-to-event variables

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Partially exchangeable array of latent time-to-event variables ( $D$  groups)

$\mathbf{Y}_1 = (Y_{1,1}, \dots, Y_{1,n}), \dots, \mathbf{Y}_D = (Y_{D,1}, \dots, Y_{D,n})$  such that

$$\mathbf{Y}_1, \dots, \mathbf{Y}_D \mid \tilde{\boldsymbol{\mu}} \sim \tilde{p}_1^n \times \cdots \times \tilde{p}_D^n$$

$$\tilde{p}_\delta((t, +\infty)) = \exp \left\{ - \int_0^t \int_{\Theta} k(s, \theta) \tilde{\mu}_\delta(d\theta) ds \right\} \quad \delta = 1, \dots, D$$

Hierarchical dependence structure

$$\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \dots, \tilde{\mu}_D) \sim \text{hCRM}(\tilde{\nu}, \nu_0)$$

Latent time-to-event variables **are not** the object of interest

# Key identity

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**Key formula** (Catalano et al., 2023) for Bayesian calculus when  $\alpha$  is a diffuse and fixed measure

$$\begin{aligned} \mathbb{E} \exp \left\{ - \int_{\Theta} f(\theta) \tilde{\mu}(\mathrm{d}\theta) \right\} \prod_{j=1}^k \tilde{\mu}(\mathrm{d}\theta_j)^{n_j} \\ = \exp \left\{ - \int_{\Theta} \psi(f(\theta)) \alpha(\mathrm{d}\theta) \right\} \prod_{j=1}^k \tau_{n_j}(f(\theta_j)) \alpha(\mathrm{d}\theta_j) \end{aligned}$$

What if  $\alpha$  is discrete (either random or not)?

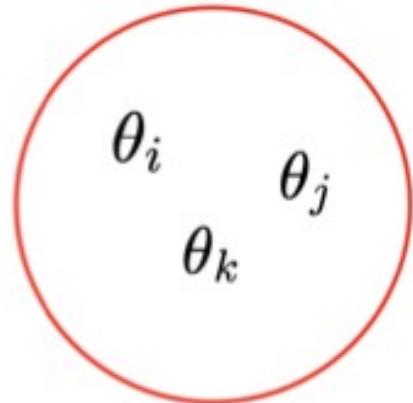
- ✓ Faa' di Bruno formula for the derivative of composite functions  
(Camerlenghi et al., 2019; Camerlenghi et al., 2021)
- ✓ augmentation of the dimension of the atoms' space, through a mark

# Marked random probability measure

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Augmentation with  $Z_i \stackrel{\text{iid}}{\sim} H$ , with  $H$  non-atomic

$$\tilde{\mu}_d = \sum_{i \geq 1} J_i \delta_{\theta_i} \quad \Rightarrow \quad \tilde{\mu}_d^a = \sum_{i \geq 1} J_i \delta_{(\theta_i, Z_i)}$$



$$\theta_i = \theta_j = \theta_k = \theta^*$$

$$\bar{J} = J_i + J_j + J_k$$

Atoms  $(\theta, Z)$  are almost surely different

$$J_i \Rightarrow (\theta_i, Z_i) \quad J_k \Rightarrow (\theta_k, Z_k)$$

$$J_j \Rightarrow (\theta_j, Z_j)$$

$$\tilde{\nu}(ds, d\theta, dz) = \rho(s) ds \tilde{\mu}_0(d\theta) H(dz)$$

# Likelihood function

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Multiplicative intensity likelihood (no censored observations)

$$\mathcal{L}(\tilde{\mu}; \text{data}) = \prod_{i=1}^n \int_{\Theta} k(T_i; \theta) \tilde{\mu}_{\Delta_i}(\mathrm{d}\theta) \exp \left\{ - \sum_{d=1}^D \int_0^{T_i} \int_{\Theta} k(s; \theta) \tilde{\mu}_d(\mathrm{d}\theta) \mathrm{d}s \right\}$$

Reparameterization, with augmented  $\tilde{\mu}^a$

$$\begin{aligned} \mathcal{L}(\tilde{\mu}^a; \text{data}) &= \prod_{i=1}^n \int_{\mathcal{Z} \times \Theta} k(T_i; \theta) \tilde{\mu}_{\Delta_i}^a(\mathrm{d}\theta, \mathrm{d}z) \\ &\quad \times \exp \left\{ - \sum_{d=1}^D \int_0^{T_i} \int_{\Theta} k(s; \theta) \tilde{\mu}_d^a(\mathrm{d}\theta, \mathrm{d}z) \mathrm{d}s \right\} \end{aligned}$$

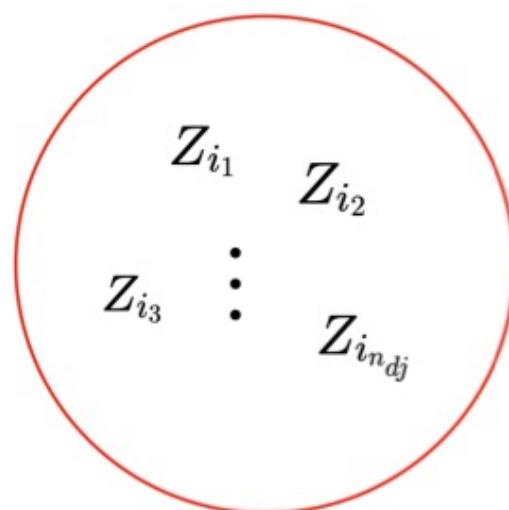
# Marginal distribution

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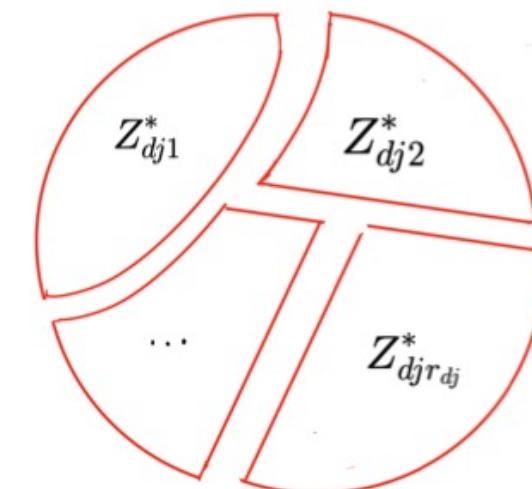
Latent variables  $\theta = (\theta_1, \dots, \theta_n)$  &  $Z = (Z_1, \dots, Z_n)$  feature

- Distinct values  $\theta^* = (\theta_1^*, \dots, \theta_k^*)$  &  $Z^* = (Z_{d,j,h}^*)_{d,j,h}$

$$\{i : \theta_i = \theta_j^*; \Delta_i = d\}$$



$$\{i : \theta_i = \theta_j^*; \Delta_i = d; Z_i = Z_{d,j,h}^*\}$$



# Marginal distribution

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Conditionally on

- the partition  $\Pi_{\boldsymbol{\theta}} = \{C_{d,j}^{\boldsymbol{\theta}} : d = 1, \dots, D; j = 1, \dots, k\}$   
into sets  $C_{d,j} = \{i : \theta_i = \theta_j^*, \Delta_i = d\}$
- the partition  $\Pi_Z = \{C_{d,j,h} : d = 1, \dots, D; j = 1, \dots, k; h = 1, \dots, r_{d,j}\}$   
into sets  $C_{d,j,h} = \{i \in C_{d,j} : Z_i = Z_{d,j,h}^*\}$
- the distinct values  $\boldsymbol{\theta}^*$  of  $\boldsymbol{\theta}$

the latent marks  $Z^*$  are independent and can be integrated out

# Marginal distribution

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**Marginal distribution** of  $(T_{1:n}, \Delta_{1:n}, \Pi_{\theta}, \Pi_Z, \theta^*)$

$$\begin{aligned} & \prod_{j=1}^k \prod_{\{i: \theta_i = \theta_j^*\}} k(T_i, \theta_i) \exp \left\{ - \int_{\Theta} \psi_0(D\psi(K_n(\theta))) P_0(d\theta) \right\} \\ & \times \underbrace{\prod_{j=1}^k \left\{ \prod_{d=1}^D \prod_{h=1}^{r_{d,j}} \tau_{q_{d,j,h}}(K_n(\theta_j^*)) \right\}}_{\Pi_Z \text{ partition into } C_{d,j,h} \text{ sets}} \underbrace{\tau_{0,r_j}(D\psi(K_n(\theta_j^*)) P_0(d\theta_j^*)}_{\text{partition of } Z \text{ associated with } \{i : \theta_i = \theta_j^*\} \text{ sets}} \end{aligned}$$

where  $K_n(\theta) = K_n(\theta, T) = \sum_{i=1}^n \int_0^{T_i} k(s, \theta) ds$

## Predictive, conditional on $\mathcal{F}_n = (T_{n+1}, \boldsymbol{T}, \boldsymbol{\Delta}, \boldsymbol{\theta}, \Pi_Z)$

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- ✓  $q_{d,j,h} = \text{card}(C_{d,j,h})$
- ✓  $r_{d,j} = \text{no. partition blocks } C_{d,j,h} \text{ of } \{i : \theta_i = \theta_j^*; \Delta_i = d\}$
- ✓  $r_j = \sum_{d=1}^D r_{d,j} = \text{no. of distinct } Z \text{ values associated with } \theta_j^*$

⇒ if  $\theta_{n+1}$  equals one of the distinct values  $\{\theta_1^*, \dots, \theta_k^*\}$  in  $\boldsymbol{\theta}$

$$\text{Prob}[\Delta_{n+1} = d, \theta_{n+1} = \theta_j^* | \mathcal{F}_n] \propto k(T_{n+1}, \theta_j^*)$$

$$\times \left\{ \underbrace{\sum_{h=1}^{r_{d,j}} \frac{\tau_{q_{d,j,h}+1}(K_{n+1}(\theta_j^*))}{\tau_{q_{d,j,h}}(K_n(\theta_j^*))}}_{\text{"old" } Z \text{ associated with } T_{n+1}} + \underbrace{\tau_1(K_n(\theta_j^*)) \frac{\tau_{r_j+1}(K_{n+1}(\theta_j^*))}{\tau_{r_j}(K_n(\theta_j^*))}}_{\text{"new" } Z \text{ associated with } T_{n+1}} \right\}$$

## Predictive, conditional on $\mathcal{F}_n = (T_{n+1}, T, \Delta, \theta, \Pi_Z)$

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$\Rightarrow$  if  $\theta_{n+1}$  equals a new value  $\theta_{k+1}^* \notin \{\theta_1^*, \dots, \theta_k^*\}$

$$\text{Prob}[\Delta_{n+1} = d, \theta_{n+1} = \theta_{k+1}^* | \mathcal{F}_n]$$

$$\propto k(T_{n+1}, \theta_{k+1}^*) \underbrace{\tau_1(K_{n+1}(\theta_{k+1}^*))}_{\text{"new" } Z \text{ value}} \underbrace{\tau_{0,1}(D\psi(K_{n+1}(\theta_{k+1}^*)))}_{\text{"new" } \theta \text{ value}}$$

- Marginalize with respect to  $\mathcal{L}(\theta_{n+1} | \mathcal{F}_n)$
- **Prediction curve:** as a function of  $T_{n+1} = t$ , for any  $d \in \{1, \dots, D\}$

$$t \mapsto \text{Prob}[\Delta_{n+1} = d | \mathcal{F}_n]$$

# Posterior characterization

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Structural conjugacy property of the proposed model

$$\tilde{\mu}_d \mid (T_{1:n}, \Delta_{1:n}, \theta_{1:n}, Z_{1:n}, \tilde{\mu}_0) \stackrel{\text{ind}}{\sim} \tilde{\mu}_d^* + \sum_{j=1}^k \sum_{h=1}^{r_{d,j}} J_{d,j,h} \delta_{\theta_j^*}$$

$$\tilde{\mu}_0 \mid (T_{1:n}, \Delta_{1:n}, \theta_{1:n}, Z_{1:n}) \sim \tilde{\mu}_0^* + \sum_{j=1}^k I_j \delta_{\theta_j^*}$$

- $\tilde{\mu}_d^*$  and  $\tilde{\mu}_0^*$  are CRMs with **non-homogeneous** Lévy intensities.
- the distributions of  $J_{d,j,h}$  and of  $I_j$  depend, respectively, on
  - the partition  $\Pi_Z$  through the frequencies  $q_{d,j,h}$
  - the frequencies  $r_j$  of distinct  $Z$  values associated with  $\theta_j^*$

# Posterior characterization

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**Root a priori**

$$\tilde{\mu}_0 \sim \text{CRM}(\nu_0)$$

$$\nu_0(ds, d\theta) = \rho_0(s) ds P_0(d\theta)$$

**Root a posteriori**

$$\tilde{\mu}_0^* \sim \text{CRM}(\nu_0^*)$$

$$\nu_0^*(ds, d\theta) = e^{-D\psi(K_n(\theta))s} \rho_0(s) ds P_0(d\theta)$$

**Leaves a priori**

$$\tilde{\mu}_d | \tilde{\mu}_0 \stackrel{\text{iid}}{\sim} \text{CRM}(\tilde{\nu})$$

$$\tilde{\nu}(ds, d\theta) = \rho(s) ds \tilde{\mu}_0(d\theta)$$

**Leaves a posteriori**

$$\tilde{\mu}_d^* | \tilde{\mu}_0^* \stackrel{\text{iid}}{\sim} \text{CRM}(\tilde{\nu}^*)$$

$$\tilde{\nu}^*(ds, d\theta) = e^{-K_n(\theta)s} \rho(s) ds \tilde{\mu}_0^*(d\theta)$$

# Estimation

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The result on the **posterior characterization** of  $(\tilde{\mu}, \tilde{\mu}_0)$  yields estimates of

- the survival function  $S(t) = \text{Prob}[T > t]$
- the cause-specific incidence function

$$\tilde{p}(\mathrm{d}t, \delta) = \text{Prob}[T \in (t, t + \mathrm{d}t), \Delta = \delta]$$

- the predictive distribution for  $\Delta_{n+1}$ , given  $\mathcal{F}_n$ , which coincides with the one deduced earlier from the marginal distribution of  $(T_{1:n}, \Delta_{1:n}, \boldsymbol{\theta}^*, \Pi_{\boldsymbol{\theta}}, \Pi_Z)$

Estimates are conditional on  $(\boldsymbol{\theta}^*, \Pi_{\boldsymbol{\theta}}, \Pi_Z)$

# Marginal sampler

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Generate  $(\theta, Z)$  values, and related partitions, through

- $\theta_{n+1} = \theta_j^*$  and  $Z_{n+1} = Z_{j,d,h}^*$  with probability proportional to

$$k(T_{n+1}; \theta_j^*) \tau_{q_{djh}+1}(K_{n+1}(\theta_j^*)) / \tau_{q_{djh}}(K_n(\theta_j^*))$$

- $\theta_{n+1} = \theta_j^*$  and  $Z_{n+1}$  = “new” with probability proportional to

$$k(T_{n+1}; \theta_j^*) \tau_1(K_{n+1}(\theta_j^*)) \tau_{0,r_j+1}(D\psi(K_{n+1}(\theta_j^*))) / \tau_{0,r_j}(D\psi(K_n(\theta_j^*)))$$

- $\theta_{n+1}$  = “new” and  $Z_{n+1}$  = “new” with probability proportional to

$$\int_{\Theta} k(T_{n+1}; \theta) \tau_1(K_{n+1}(\theta)) \tau_{0,1}(D\psi(K_{n+1}(\theta))) P_0(d\theta)$$

# Conditional sampler

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Using the posterior characterization of  $(\tilde{\mu}, \tilde{\mu}_0)$

(1) Sample

$$\tilde{\mu}_0 | (\mathbf{T}, \Delta, \boldsymbol{\theta}, \mathbf{Z}) \approx \sum_{j=1}^k V_j \delta_{\theta_j^*} + \underbrace{\sum_{h=1}^{H_0} V_h^{(0)} \delta_{\theta_h^{(0)}}}_{\text{truncated } \tilde{\mu}_0^*}$$

(2) Conditionally on  $\tilde{\mu}_0$ , sample

$$\tilde{\mu}_d | (\mathbf{T}, \Delta, \boldsymbol{\theta}, \mathbf{Z}, \tilde{\mu}_0) \approx \sum_{j=1}^k \sum_{h=1}^{r_{dj}} S_{d,j,h} \delta_{\theta_j^*} + \underbrace{\sum_{h=1}^{H_d} S_h^{(d)} \delta_{\theta_h^{(d)}}}_{\text{truncated } \tilde{\mu}_d^*}$$

Truncations  $H_0$  and  $H_d$  such that the size of discarded jumps is below a given threshold

# Illustration

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Implementation on synthetic and real data with

- Generalized gamma hCRM

$$\tilde{\mu}_0 \sim \text{CRM}(\nu_0) \quad \nu_0(ds, d\theta) = \frac{c_0}{\Gamma(1 - \sigma_0)} s^{-1 - \sigma_0} e^{-\beta_0 s} ds P_0(d\theta)$$

$$\tilde{\mu}_d | \tilde{\mu}_0 \stackrel{\text{iid}}{\sim} \text{CRM}(\tilde{\nu}) \quad \tilde{\nu}(ds, d\theta) = \frac{c}{\Gamma(1 - \sigma)} s^{-1 - \sigma} e^{-\beta s} ds \tilde{\mu}_0(d\theta)$$

- Kernel  $k(t, \theta) = \gamma(\theta) 1_{[\theta, +\infty)}(t)$  as in Dykstra and Laud (1981)

# Synthetic data

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Latent time-to-event framework with  $D = 2$  "risks"

$$Y_{i,1} \stackrel{\text{iid}}{\sim} \text{Weibull}(\text{shape} = 1.5, \text{scale} = 1)$$

$$Y_{i,2} \stackrel{\text{iid}}{\sim} \text{Weibull}(\text{shape} = 2.5, \text{scale} = 1)$$

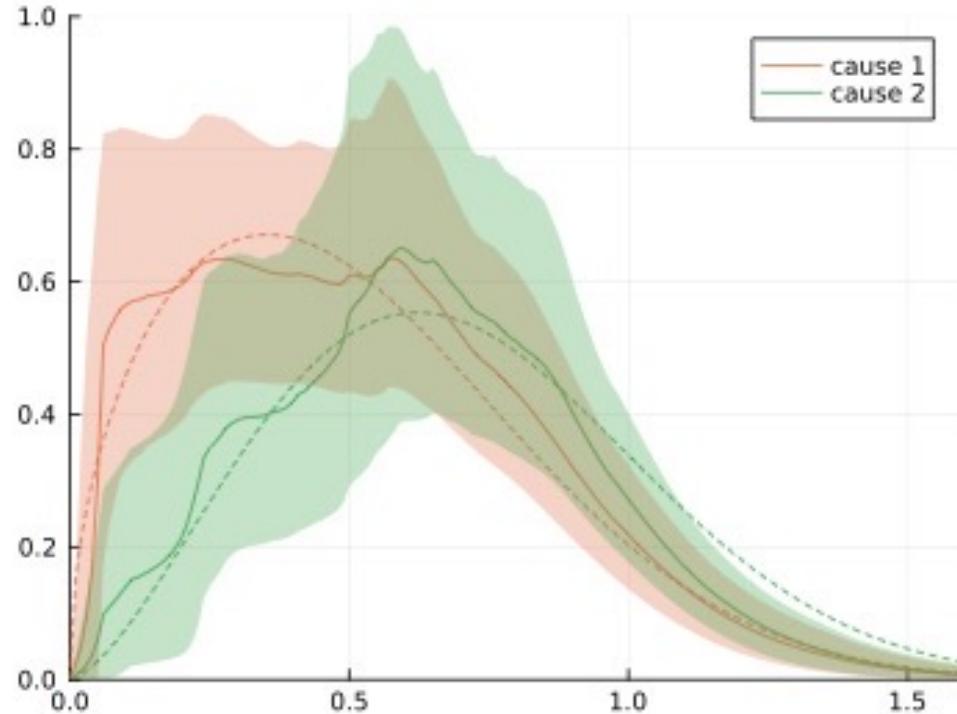
$$T_i = \min\{Y_{i,1}, Y_{i,2}\}$$

$$n = 100$$

Generating process such that causes of failure are balanced between  $d = 1$  and  $d = 2$

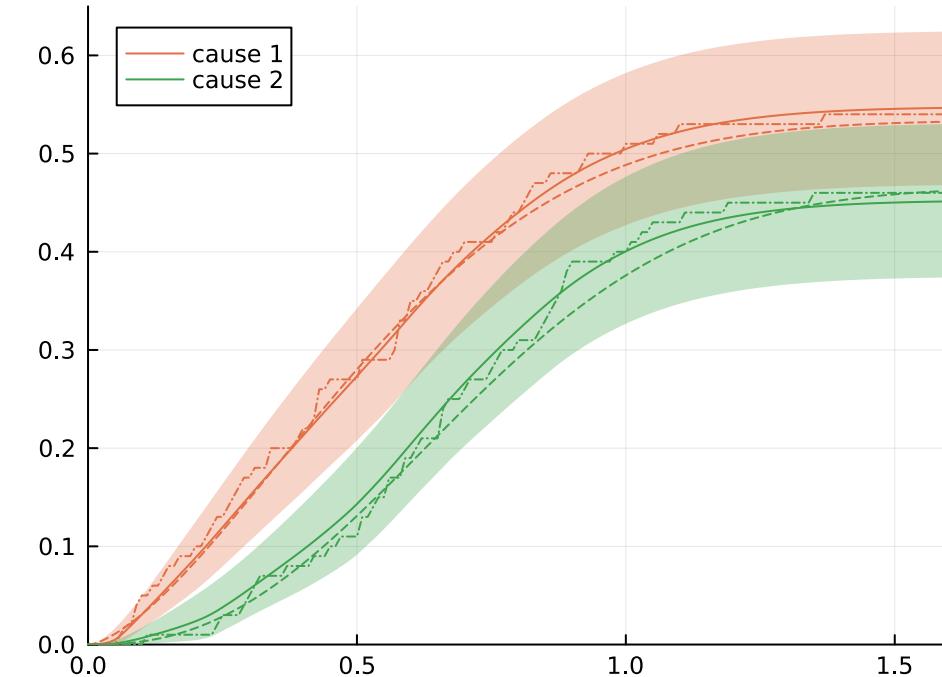
# Synthetic data

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Incidence function

$$P_{0,\delta}(t, t + dt)$$



Sub-distribution function

$$P_{0,\delta}(0, t)$$

# Comparison with independent random hazards

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Comparison with the case where

$$\tilde{\mu}_1, \dots, \tilde{\mu}_D \stackrel{\text{iid}}{\sim} \text{CRM}(\nu) \quad \nu(ds, d\theta) = \rho(s) ds c P_0(d\theta)$$

## Synthetic data

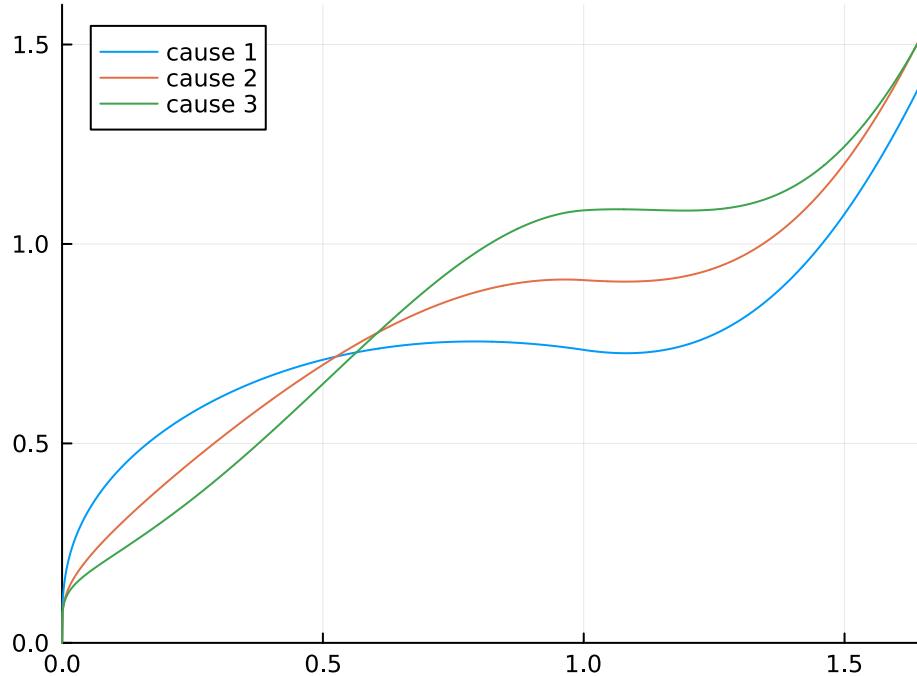
$D = 3$  latent failure times from mixtures of Weibulls with a common component

$$f_\delta = w_\delta f_\delta^* + (1 - w_\delta) f_0 \quad \delta = 1, 2, 3$$

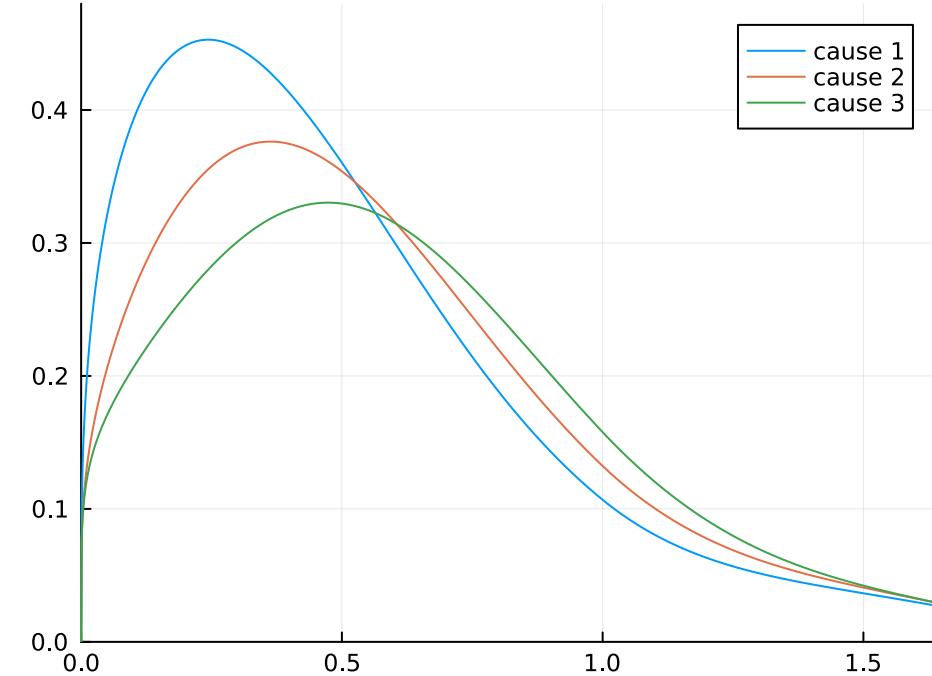
$n = 100$  observations

# Comparison with independent random hazards

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Cause-specific hazards  
 $h_1, h_2, h_3$



Cause-specific incidence functions

## Comparison with independent random hazards

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		incidence function			subdistribution		
		$d = 1$	$d = 2$	$d = 3$	$d = 1$	$d = 2$	$d = 3$
hierarchical	marginal	<b>0.1000</b>	<b>0.0890</b>	0.0940	0.1389	0.1270	0.1314
	conditional	0.1018	0.0892	<b>0.0933</b>	<b>0.1378</b>	<b>0.1247</b>	<b>0.1296</b>
independent	marginal	0.1242	0.1077	0.1015	0.1676	0.1538	0.1466
	conditional	0.1147	0.0970	0.0944	0.1496	0.1350	0.1334
frequentist					0.1958	0.1973	0.2032

Kolmogorov's distance between true and estimated quantities

## Illustration with real data

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Data of the Odense University Hospital, Denmark, on patients affected by melanoma and with

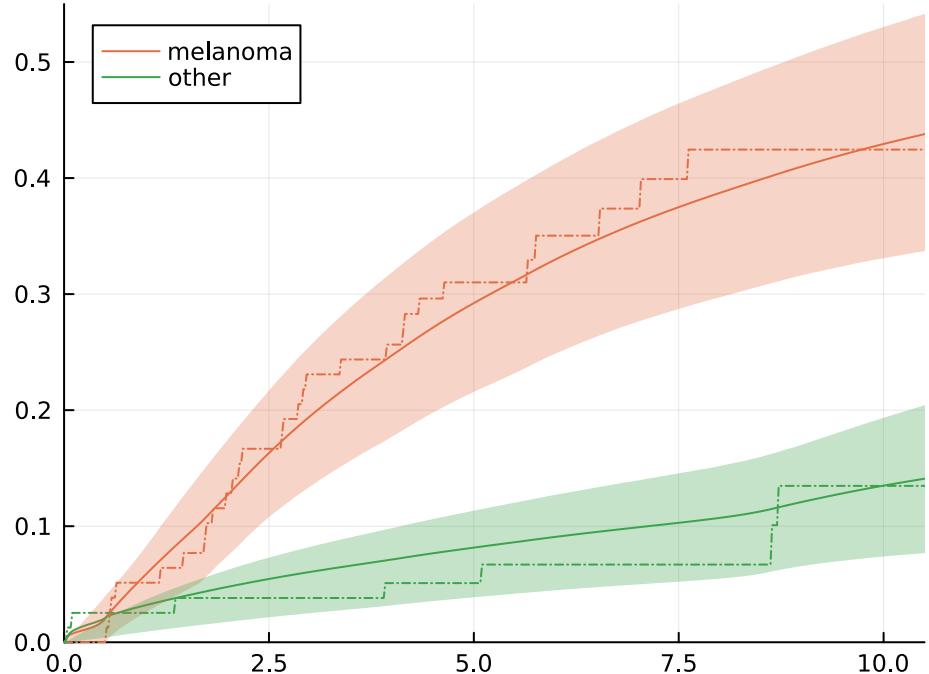
- primary (melanoma)
- competing (other causes of death)

risks ( $D = 2$ ). See also Arf  et al. (2019).

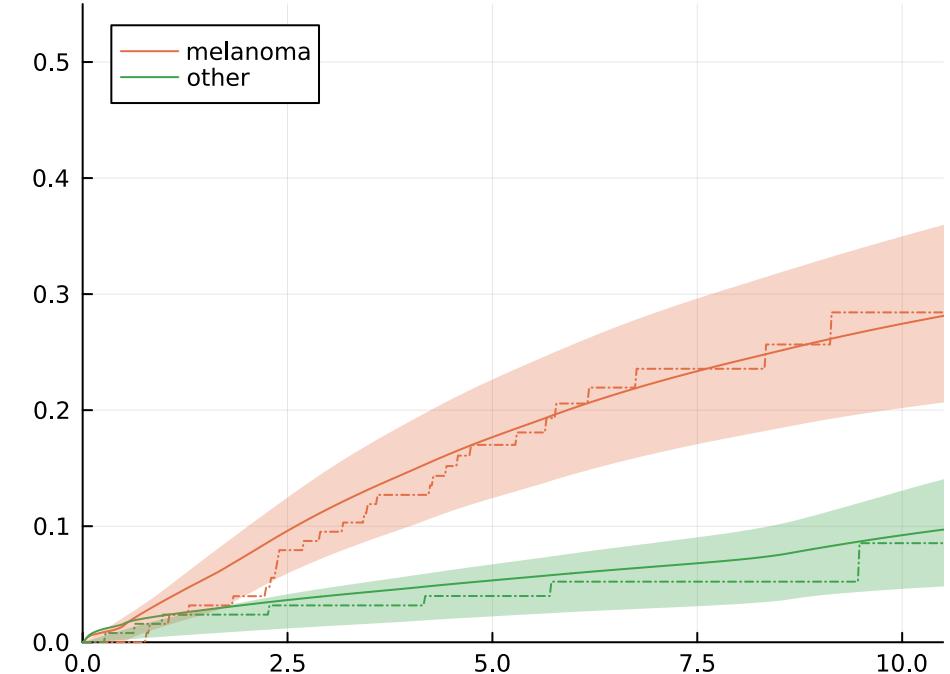
Add a categorical covariate to the model for gender.

# Illustration with real data

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Sub-distribution function for **female**  
red (melanoma), green (other cause of death)



Sub-distribution function for **male**  
red (melanoma), green (other cause of death)

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**Thank You!**