

RESEARCH ARTICLE

Dir-GLM: A Bayesian GLM with data-driven reference distribution

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Abstract

The recently developed semi-parametric generalized linear model (SPGLM) offers more flexibility as compared to the classical GLM by including the baseline or reference distribution of the response as an additional parameter in the model. However, some inference summaries are not easily generated under existing maximum-likelihood based inference (GLDRM). This includes uncertainty in estimation for model-derived functionals such as exceedance probabilities. The latter are critical in a clinical diagnostic or decision-making setting. In this article, by placing a Dirichlet prior on the baseline distribution, we propose a Bayesian model-based approach for inference to address these important gaps. We establish consistency and asymptotic normality results for the implied canonical parameter. Simulation studies and an illustration with data from an aging research study confirm that the proposed method performs comparably or better in comparison with GLDRM. The proposed Bayesian framework is most attractive for inference with small sample training data or in sparse-data scenarios.

KEY WORDS

Ordinal regression, Nonparametric Bayes, Exceedance probabilities, Skewed Dirichlet, Dependent Dirichlet process.

1 | INTRODUCTION

We introduce a Bayesian model-based approach to a recently developed semi-parametric extension of the classical generalized linear model (GLM¹). Rathouz et al² and Wurm et al³ proposed a semi-parametric generalized linear model (SPGLM) for regression with a scored ordinal outcome or other discrete distributions with finite support, for example, a count of symptoms or of items endorsed. In those earlier works, the SPGLM model is referred to as generalized linear density ratio model (GLDRM). The main novelty of the proposed framework is that the model-based approach facilitates full predictive inference, including clinically important exceedance probabilities and inference for design questions. We are introducing Bayesian Dirichlet ordinal regression as a novel statistical data analysis technique.

The defining characteristic of the SPGLM is a nonparametric baseline or reference distribution f_0 that replaces a parametric specification in the classical GLM such as binomial or Poisson distribution. Keeping f_0 nonparametric instead, the analyst needs to only specify the linear predictor and link function, even avoiding a variance function. Model specification is less onerous than even with quasilikehood (QL) models, while still yielding a valid likelihood function. Beyond the initial proposal by Rathouz et al², which focused primarily on the finite support case, Huang et al⁴ went on to mathematically characterize the SPGLM in the infinite support case, and Maronge et al^{5,6} have shown how the model can be used to handle outcome-dependent or generalized case-control sampling.

Despite these developments, there are still many important gaps in the literature on inference procedures that have been developed for these models. These include inference for application-driven functionals of the fitted models such as exceedance probabilities, which are crucial in clinical diagnostic,⁷ natural hazard detection,^{8,9} financial risk management,¹⁰ or in general, any decision-making setting. For these quantities, approach based on maximum likelihood estimation cannot easily generate

Abbreviations: Dir, Dirichlet; SPGLM, semi-parametric generalized linear model, GLM, generalized linear model.

inference, as well as a full description of all relevant uncertainties, as would be needed for a design problem. These comments are elaborated below using a motivating case study in Section 2. The main aim of this paper is to develop a Bayesian approach to inference under the SPGLM. We will refer to the proposed Bayesian extension of the model as the Dir-GLM.

Under a Bayesian approach, the baseline distribution f_0 of the GLM becomes one of the unknown parameters. A complete Bayesian inference model therefore requires a prior probability model $p(f_0)$ on f_0 . Prior models on random probability measures are known as nonparametric Bayesian models (BNP) ^{11,12}; the most widely used one being the Dirichlet process (DP) prior. ^{13,14} One characterization of the DP prior is as a limit of symmetric Dirichlet priors on finite sample spaces. ¹⁵ In fact, the DP prior reduces to a Dirichlet distribution in the case of a finite sample space, as is the case in the SPGLM development here. This formulation allows for data-driven learning about the baseline distribution in the sense that f_0 is estimated based on observed data by full posterior inference, instead of being restricted to a specific parametric family as in the case of classical GLM.

Many BNP models in the recent literature introduce variations of the DP prior to dependent DP (DDP) models for families of dependent random probability measures. ^{16,17,18} The dependency here is introduced through a set of predictors x ; exploring this dependence structure is of substantial interest in biomedical and public health studies. In particular, DDP family include models under which the dependent random probability measures share the same atoms but dependent weights through x (common atoms DDP) and others that share the same weights but dependent atoms through x (common weights DDP). The proposed Dir-GLM is a special case of a common atoms DDP prior for a finite sample space.

The rest of the article is organised as follows. Section 2 motivates the problem, and provides a detailed data description for the “Assets and HEAlth Dynamics among the oldest old” (AHEAD) study, ¹⁹ which is later used in an illustrative analysis. In Section 3, we fully specify and develop our Dir-GLM model followed by theoretical results, and a discussion on model properties and induced prior distributions. Section 4 outlines a Markov chain Monte Carlo (MCMC) algorithm for posterior simulation within the Dir-GLM framework. In Section 5, we illustrate our method on various simulated datasets and examine frequentist operating characteristics using replicated simulation. Section 6 reports inference for the AHEAD study dataset, covering performance on both small and large samples. In Section 7, we conclude with a discussion and future directions. Proofs and additional data illustrations are included in the Supplementary Information.

2 | THE AHEAD STUDY

The AHEAD study ¹⁹ collected nationwide longitudinal data from an initial sample of over 7,000 participants aged 70 years and older at baseline, including their spouses and household partners. This study’s primary goals involved tracking the transitions in physical, functional, and cognitive health and investigating how changes in late-life physical and cognitive health relate to patterns of dissaving and income flow. For the analysis here, we focus specifically on the baseline data collected in 1993; descriptions of the variables are provided in Table 1. The complete data sample size is $n = 6,441$.

TABLE 1 Description of the variables from the AHEAD study.

Variable	Type	Description
numiadl	Ordinal	Number of instrumental activities of daily living tasks for which the individual has some difficulty, ranging from 0 to 5.
age	Continuous	Age (in years) at the interview of the individual.
sex	Nominal	Sex of the individual (0 = male, 1 = female).
iwr	Ordinal	Immediate word recall is a measure of cognitive function. Number of words (out of 10) that the individual can list immediately after hearing them read.
netwc	Ordinal	Categorical values of net worth. (1 = lowest bracket, ..., 5 = highest bracket).

In Figure 1, we present the data for the variables *numiadl* (difficulty count of daily activities), *iwr* (immediate word recall), and *age*. For this graphic, we have coded *age* into 5 ordinal levels. We observe evidence for a positive association between the variables *numiadl* and *age*, while *numiadl* shows a negative association with *iwr*. Hence, modeling the scored ordinal variable *numiadl* based on *age*, *iwr* and other variables is of particular interest. Some options for modeling this response, with their limitations, include: A Poisson model, the problem here being that the distribution is limited to only 0 through 5, and has considerable overdispersion at 0. Similarly, one could use a binomial distribution; this model would also suffer from overdispersion. Of course, a quasiliikelihood approach could be taken, but this would not yield full distributional inferences (such

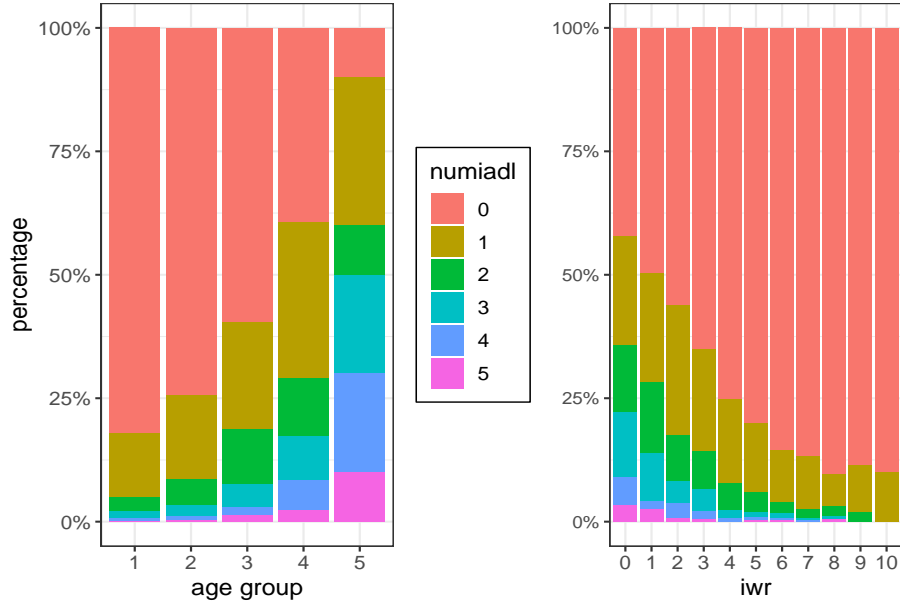


FIGURE 1 Variables *numiadl* (difficulty count of daily activities), *iwr* (immediate word recall), and *age* from the AHEAD data set; age group (years): 1 (70-75), 2 (76-82), 3 (83-89), 4(90-96), and 5 (96+).

as exceedance probabilities) because the QL model is essentially a model for the mean. Finally, ordinal data regression could be used; our concern with this is that the regression parameters (in for, e.g., ordinal logistic regression) are for log-cumulative odds ratios, which can be challenging to interpret to applied audiences; our model provides regression parameters for the mean response, interpreted in terms of the analyst-chosen link function.

Rathouz et al² analysed the same data using maximum likelihood in the SPGLM, as stated in (1)–(3) below, yielding valid inferences for regression parameters β . It has been challenging, however, to extract both estimates and uncertainty quantifications for either the centering distribution or for model-derived quantities such as exceedance probabilities $p(y \geq y_0 | x)$, which depend jointly on β and the baseline density. The main reason for this limitation is that the joint sampling variance-covariance matrix would be needed for all model parameters, a quantity that has been difficult to extract. Even if available, manipulating it via, e.g., the “delta method” may prove challenging, and may fail to provide sufficiently good approximation to valid inferences for reasonable sample sizes. An alternative to addressing these critical gaps is a Bayesian approach, which always provides full uncertainty quantification for any model-derived quantity or event of interest via the joint posterior sample of all the model parameters.

The proposed semiparametric Bayesian approach implements this goal and prepares the model for future extensions to multivariate or repeated measurement models by introducing appropriate random effects and other latent variables. The concept behind the proposed Dir-GLM is straightforward. Existing semi-parametric frequentist methods^{2,3} lack a Bayesian perspective, while parametric Bayesian approaches²⁰ are restricted by their parametric nature. To overcome the limitations of both, and simultaneously employ their respective strengths, we combine these two approaches to develop a novel semi-parametric Bayesian model known as Dir-GLM.

3 | MODEL DEVELOPMENT

In the upcoming discussion, we initially consider f_0 to be a probability measure on \mathcal{R} , without further constraints. Later we focus on ordinal regression, which involves finite discrete probability measures. But we start with a discussion of a more general context.

3.1 | The Dir-GLM Model

Consider a GLM

$$y \sim f(y | x) = \exp \{ \theta y - b(\theta) \} f_0(y) \quad (1)$$

with response $y \in \mathcal{Y} \subset \mathcal{R}$ and a p -dimensional covariate vector x . Here, f_0 is the baseline density and $b(\cdot)$ is the log-normalization constant

$$b(\theta) = \log \int_{\mathcal{Y}} \exp(\theta y) f_0(y) dy. \quad (2)$$

In the classical GLM, the baseline distribution $f_0(y)$ is assumed to be in a parametric family—in the proposed semi-parametric model it becomes an unknown parameter. As in the classical GLM, $\eta = x^T \beta$ is a linear predictor, g is a link function, and $\mu = E(y | x) = g^{-1}(\eta)$. For fixed f_0 , the expectation μ implicitly determines θ by the equation

$$\mu = E(y | x) = b'(\theta) = \frac{\int_{\mathcal{Y}} y \exp(\theta y) f_0(y) dy}{\int_{\mathcal{Y}} \exp(\theta y) f_0(y) dy}, \quad (3)$$

where we note that $b'(\theta)$ is a strictly increasing function of θ . The free parameters in the model are $\beta \in \mathcal{R}^p$ and $f_0 \in \mathcal{F}_0$, with \mathcal{F}_0 defined below. By contrast, $\theta = \theta(\beta, f_0; x)$ is a derived parameter based on (3). We can write the solution for θ as a function of μ , denoted as $b'^{-1}(\mu; f_0)$, which additionally depends on f_0 , in addition to depending on β , implicitly through $\mu = x^T \beta$.

Here, we consider y to be a scored ordinal outcome, by which we mean an ordinal outcome in which the mean response of (given) scores s_ℓ is of interest. In this case: $\mathcal{Y} = \{s_1, \dots, s_k\}$, where $s_\ell < s_{\ell+1}$, and the integration in (2) and (3) becomes a finite sum. In this setting f_0 is restricted to $\mathcal{F}_0 := \{f : f(s_\ell) \in [0, 1] \text{ and } \sum_{\ell=1}^k f(s_\ell) = 1\}$, i.e., the $k-1$ dimensional simplex. For the remaining discussion we restrict attention to this case of scored ordinal outcomes.

Extending the SPGLM to a Bayesian inference model requires prior distributions on the parameters, $\beta = (\beta_1, \dots, \beta_p)$ and $f_0 = \{f_0(s_1), \dots, f_0(s_k)\}$. We use

$$\beta \sim N_p(0, \sigma_\beta^2 \mathcal{I}_p), \text{ and } f_0 \sim \text{Dir}(\alpha H) \equiv \text{Dir}(\alpha H(s_1), \dots, \alpha H(s_k)), \quad (4)$$

where $\alpha > 0$ is a user-specified concentration parameter, and $H(s_\ell)$ is the probability mass at s_ℓ under a user-specified centering distribution H , implying $E(f_0) = (H)$. The Dirichlet prior on f_0 can equivalently be represented as $f_0(\cdot) = \sum_{\ell=1}^k u_\ell \delta_{s_\ell}(\cdot)$, where $u_\ell \propto \tilde{u}_\ell$ (i.e. $u_\ell = \tilde{u}_\ell / \sum_{\ell'=1}^k \tilde{u}_{\ell'}$) with $\tilde{u}_\ell \sim \text{Gamma}(\alpha H(s_\ell), \phi)$, independently across ℓ , for an arbitrary scale parameter ϕ . This implies the following model for $f(y | x)$ in (1)

$$f(y | x) = \exp \{ \theta y - b(\theta) \} \sum_{\ell=1}^k u_\ell \delta_{s_\ell}(y) = \sum_{\ell=1}^k [\exp \{ \theta s_\ell - b(\theta) \} u_\ell] \delta_{s_\ell}(y) = \sum_{\ell=1}^k w_{x\ell} \delta_{s_\ell}(y), \quad (5)$$

where $w_{x\ell} = \exp \{ \theta s_\ell - b(\theta) \} u_\ell$ varies with x (recall that θ is a function of $\mu = x^T \beta$ and f_0). That is, $w_{x\ell}$ is also a function of β and f_0 . Let, w_x denote the vector of weights, $w_x = (w_{x1}, \dots, w_{xk})$ with $\sum_{\ell=1}^k w_{x\ell} = 1$. We refer to the distribution of w_x as a tilted Dirichlet distribution. In general, a tilted Dirichlet random vector $v = (v_\ell; \ell = 1, \dots, k)$ is a modified Dirichlet random vector $\tilde{v} \sim \text{Dir}(c)$ with hyperparameter $c = (c_\ell; \ell = 1, \dots, k)$ that is exponentially tilted as $v_\ell \propto e^{\theta s_\ell} \tilde{v}_\ell$ with respect to given scores $s_\ell \in s = (s_{\ell'}; \ell' = 1, \dots, k)$ to fix a mean $\mu = \sum_{\ell} s_\ell v_\ell / k$ (with respect to the same scores). That is, we use a Dirichlet distribution to generate random weights for a scored ordinal random variable y with probability function $p(y = s_\ell) = v_\ell$, subject to a given mean μ (of y). The desired mean is achieved by exponential tilting of a Dirichlet random vector. We write $v \sim t\text{Dir}(c, \mu, s)$.

We next discuss the (likelihood) identifiability under the proposed model. Equation (4), without additional restrictions, would suffer from a lack of likelihood identifiability. A given pair (f_0, θ_0) could always be replaced by another arbitrary pair (f_1, θ_1) with $f_1 \propto f_0 \cdot \exp(cy)$ and $\theta_1 = (\theta_0 - c)$, leaving the joint likelihood invariant. Rathouz et al² introduce a constraint on f_0 as $E_{f_0}(y) = \sum_{\ell=1}^k s_\ell f_0(s_\ell) = \mu_0$ for a user-specified μ_0 . Let then f_0^* denote the f_1 in that set of equivalent models with $E_{f_0^*}(y) = \mu_0$. Let $\mathcal{F}_0^* = \{f \in \mathcal{F}_0 : E_f(y) = \mu_0\}$ denote the restricted set of distributions on \mathcal{Y} with the prescribed expectation μ_0 . A (prior) probability measure on \mathcal{F}_0 implicitly defines a prior probability model on the restricted space \mathcal{F}_0^* .

Finally, some comments about the choice of hyper-parameters. The prior model (4) includes three hyper-parameters: α, σ_β^2 , and H . As default choices, we suggest $\alpha = 1$ and $\sigma_\beta^2 = 1$, with the latter possibly being modified based on the units of x . In

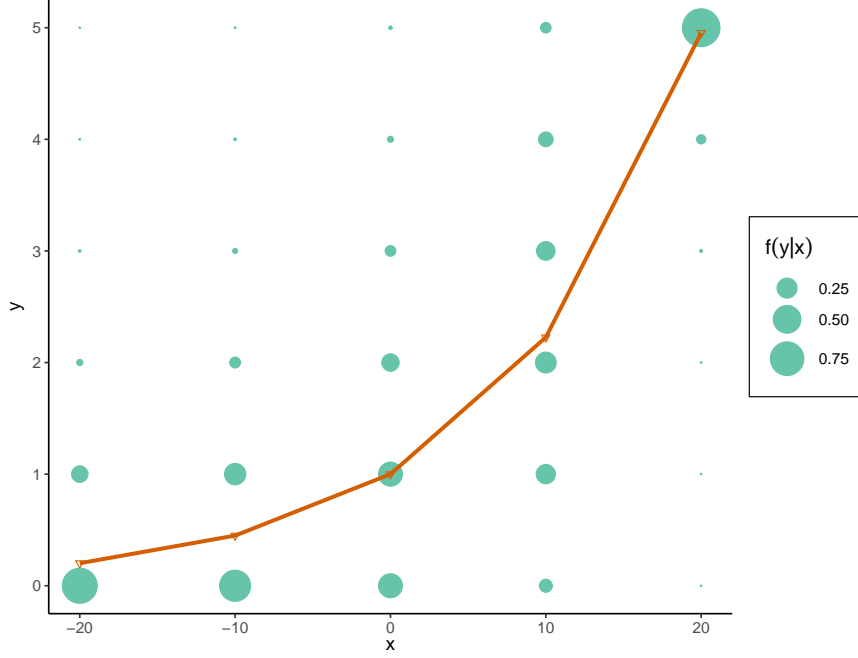


FIGURE 2 Ordinal regression model (1)–(3). $E(y | x)$ is plotted against x (orange line) together with $f(y | x)$ (green bullets), where the size of each bullet is proportional to $f(y | x)$.

practice, the choice of μ_0 can be based on prior judgement, or alternatively one can use any measure of central tendency of y , such as the mean or median. Additionally, we choose the hyper-parameter H to favor f_0 centered around the selected μ_0 , in the following sense. Select H to be equal or close to the frequency distribution of y such that the mean of H is μ_0 . Then, $E_H(y) = E_H\{E_{f_0}(y | f_0)\} \approx \mu_0$.

In summary, equation (5) together with the prior model (4) and the restriction on f_0 define an ordinal regression in the form of a family of tilted Dirichlet distributions for w_x . We refer to this Bayesian ordinal regression model as Dir-GLM. The main addition to the SPGLM² is the Dirichlet prior on f_0 to define the tilted Dirichlet regression (Dir-GLM).

3.2 | Model Properties and Implied Prior for θ

The regression coefficients β in the proposed model maintain the same interpretation as in a traditional GLM. Specifically, β is linked to the mean response via: $g\{E(y|x)\} = \eta_x = x^T \beta$, where g is analyst-chosen link function. Consequently, β_j can be interpreted as the change in mean response (on the g -scale) for a unit change in x_j keeping others (x_{-j}) fixed. Under alternative choices of η_x the interpretation of β would change accordingly.

The proposed Dir-GLM model has the following property. The proof can be found in Appendix A of the Supplementary Information.

Result 1. *The exponentially tilted Dirichlet distribution, $tDir(c, \mu, s)$ with $c = \{\alpha H(s_\ell); \ell = 1, \dots, k\}$, $s = \{s_\ell; \ell = 1, \dots, k\}$, is outside the Dirichlet parametric family; however, it remains a valid probability distribution defined over the $(k - 1)$ dimensional simplex. In the context of (1), exponential tilting transforms $f_0(s_\ell)$ to $f(s_\ell | x) \propto \exp(\theta s_\ell) f_0(s_\ell)$.*

Figure 2 illustrates the ordinal regression model (1)–(3), specifying g as the log link and $\mu_0 = 1$. The figure plots $E(y | x)$ against x (orange line) together with $f(y | x)$ (green bullets). The increasing conditional expectation of the regression mean curve requires $f(y | x)$ to become increasingly left-skewed with increasing x . For $x = 0$, $E(y | x)$ becomes exactly equal to 1, which results in (1) representing an i.i.d. sample from f_0 . However, when x moves away from 0 in either direction, the distribution of y departs from f_0 , and becomes more and more left- or right-skewed (relative to f_0) due to the tilting factor. Thus, the tilting parameter provides a flexible way to introduce and control skewness (relative to f_0) in the Dirichlet distribution, and introduce the desired regression on x .

For larger $|x|$, the prior distribution of the derived parameter θ tends to become comparatively flatter (less informative). Conversely, for x closer to 0, the prior distribution appears to be concentrated around 0 (more informative). In particular, for $x = 0$ the prior distribution on θ is degenerate at 0. The following Result 2 characterizes the implied prior on θ for x close to 0. The proof can be found in Appendix A of the Supplementary Information.

Result 2. *Under the Dir-GLM with prior model (4), with the additional constraint that μ is bounded away from the two endpoints s_1 and s_k , and $f_0 \in \mathcal{F}_0^* = \{f_0^* \in \mathcal{F}_0 : E_{f_0^*}(y) = \mu_0\}$, if g and μ_0 are chosen such that as $\|x\|_2 \rightarrow 0$, $\mu = g^{-1}(x^T \beta) \xrightarrow{P} \mu_0$, then the derived parameter $\theta = \theta(x; \beta, f_0)$ has the following properties*

- (a) $\theta \xrightarrow{P} 0$, as $\|x\|_2 \rightarrow 0$, and
- (b) θ is asymptotically normal with mean zero, as $\|x\|_2 \rightarrow 0$.

The first part of Result 2 demonstrates that, under the specified assumptions, inference essentially reduces to a density estimation problem as the norm of x approaches zero or, equivalently, for unrelated x and y . The second part refines this argument, stating that more extreme exponential tilting becomes increasingly less probable, like normal tail probabilities.

4 | POSTERIOR SAMPLING AND INFERENCE

We implement posterior inference using Markov chain Monte Carlo (MCMC) posterior simulation with Metropolis-Hastings (MH) transition probabilities.

Let $\{x_i, y_i\}_{i=1}^n$ denote the observed data. For short, we write f_ℓ for $f_0(s_\ell)$. Given currently imputed parameters $(\beta, f_1, \dots, f_k) = (\beta^{(t)}, f_1^{(t)}, \dots, f_k^{(t)})$, MH transition probabilities are defined using the following proposals $\tilde{\beta}$ and \tilde{f}_0 .

For $\tilde{\beta}$, we define a random walk proposal using the Fisher information matrix,

$$\tilde{\beta} \sim \mathcal{N}_p(\beta, \rho \mathcal{I}^{-1}(\beta; f_1, \dots, f_k) 1_A(\tilde{\beta})), \quad (6)$$

where $A = \{\beta \in \mathcal{R}^p : g^{-1}(x_i^T \beta) \in [s_1, s_k]\}$, $\rho \in (0, 1]$ is a tuning parameter, and $\mathcal{I}^{-1}(\beta; f_1, \dots, f_k)$ is the inverse Fisher information matrix associated with β . That is,

$$\mathcal{I}(\beta; f_1, \dots, f_k) = \sum_{i=1}^n \frac{x_i x_i^T}{(g'(\mu_i))^2 b''(\theta_i)},$$

$g'(\mu_i) = g'(g^{-1}(x_i^T \beta))$, and $b''(\theta_i) = \text{Var}(y_i | x_i)$ is given by

$$b''(\theta_i) = \frac{\sum_{\ell=1}^k s_\ell^2 \exp(\theta_i s_\ell) f_\ell}{\sum_{\ell=1}^k \exp(\theta_i s_\ell) f_\ell} - \left[\frac{\sum_{\ell=1}^k s_\ell \exp(\theta_i s_\ell) f_\ell}{\sum_{\ell=1}^k \exp(\theta_i s_\ell) f_\ell} \right]^2.$$

The value of $\theta_i = \theta_i(\beta, f_1, \dots, f_k; x_i)$ is determined as in (3) through numerical solution of

$$\mu_i = g^{-1}(x_i^T \beta) = \frac{\sum_{\ell=1}^k s_\ell \exp(\theta_i s_\ell) f_\ell}{\sum_{\ell=1}^k \exp(\theta_i s_\ell) f_\ell}.$$

Wurm and Rathouz³ discuss the evaluation of θ in detail. We summarize their discussion in Appendix H of the Supplementary Information.

For $\tilde{f}_0 = (\tilde{f}_1, \dots, \tilde{f}_k)$ we use an independence chain proposal based on a weighted empirical distribution:

$$(\tilde{f}_1, \dots, \tilde{f}_k) \sim \text{Dir} \left(\alpha H(s_1) + \sum_{i=1}^n \omega_i 1_{\{y_i = s_1\}}, \dots, \alpha H(s_k) + \sum_{i=1}^n \omega_i 1_{\{y_i = s_k\}} \right) \quad (7)$$

Here, $\omega_i = \frac{c_i}{\sum_{i=1}^n c_i}$, where $c_i = [\exp\{\theta_i y_i - b(\theta_i)\}]^{-1}$. The motivation for incorporating the weights $\{\omega_i\}_{i=1}^n$ in the parameters of the above Dirichlet proposal is as follows: If y_i were an i.i.d. sample from f_0 , then (7) with $\omega_i = 1$ would define the conditional

posterior. The weights are adjusted for the lack of i.i.d. sampling (i.e., sampling is condition on x) by inverse probability weighting with the additional tilting factor.

We use the two proposals $\tilde{\beta}$ and \tilde{f}_0 to define the following MH transition probabilities. For β we proceed either jointly or one at a time based on the proposals (6). For f_0 , we implement posterior MCMC simulation in the unrestricted (equivalence) class \mathcal{F}_0 , and later post-process the f_0 posterior samples to project them onto the restricted class \mathcal{F}_0^* . The posterior updating of f_ℓ can be performed either jointly based on the proposal (7), or sequentially using the implied proposals. For the latter, we first define: $g_{-\ell} := \frac{f_{-\ell}}{1-f_\ell}$, where $f_{-\ell}$ represents the f_0 vector after excluding f_ℓ . The implied proposal for f_ℓ is: $\tilde{f}_\ell \sim \text{Beta}(\delta_\ell, \sum_{j \neq \ell} \delta_j)$, where δ_ℓ is the ℓ th shape parameter in the proposal (7), and $\tilde{f}_{-\ell} := g_{-\ell}(1 - \tilde{f}_\ell)$. Lastly, the tuning parameter ρ in (6) can be adjusted to achieve a well-mixing MCMC chain.

5 | SIMULATION STUDY

We validate the proposed method, and compare its operating characteristics to GLDRM.^{2,3} For this we conducted a variety of simulation experiments in which we quantify the degree of (un)biasedness in parameter estimates, coverage probabilities, and statistical efficiency in both small and large sample regimes. Furthermore, we focus on performance in estimating exceedance probabilities. These are often of special interest in clinical diagnostic or decision-making settings.

5.1 | Data generation

For the scored ordinal response y_i , we take $\mathcal{Y} = \{0, 1, \dots, 5\}$ as the support. The covariate vector is considered as $x_i = (x_{1i}, x_{2i}) \in \mathcal{R}^2$, where $x_{1i} = 1$ and $x_{2i} \stackrel{iid}{\sim} \mathcal{N}(0, 1)$. We then generate $\{y_i\}_{i=1}^n$ conditional on $\{x_i\}_{i=1}^n$ using regression model (1)–(3), with mean specified as: $\log \mu_i = \log E(y_i | x_i) = \beta_0 + \beta_1 x_{2i}$. The simulation truth for the regression parameter is $\beta = (\beta_0, \beta_1) = (-0.7, 0.2)$. For the centering distribution parameter f_0 , we use these two choices:

- (a) **Scenario 1:** Truncated Poisson with a right truncation at 5, i.e., $f_0(s) = p_0(s) / \sum_{\ell=0}^5 p_0(\ell)$, $s \in \mathcal{Y}$. Here p_0 is the pmf of a Poisson distribution with mean $\lambda = 1$.
- (b) **Scenario 2:** Zero-inflated truncated Poisson, i.e., $f_0(0) = 3p_0(0) / [3p_0(0) + \sum_{\ell=1}^5 p_0(\ell)]$, and $f_0(s) = p_0(s) / [3p_0(0) + \sum_{\ell=1}^5 p_0(\ell)]$, $s = 1, \dots, 5$.

These two scenarios are designed to reflect common settings for data with health and behavioral outcomes, namely bounded count or count-like responses, and inflation at one end of the scale. We use \mathcal{D}_n to refer the observed data $\{x_i, y_i\}_{i=1}^n$. For a small sample setting, we consider $n = 25$, and for a large sample setting, $n = 250$. Repeatedly simulating data under this design generates R replicated data sets. We consider $R = 2,000$.

5.2 | Competing methods

We fit both competing inference models for each simulated data set as follows. Considering the prior distributions as in (4) with $\alpha = 1$ and H as the empirical distribution of y , we fit the proposed Dir-GLM model by MCMC posterior simulation for β and f_0 using the transition probabilities described in (6) and (7). We use tuning parameter $\rho = 1$, and a total of 5,000 MCMC samples are generated for each replicate. We discard the first 2,000 iterations as initial burn-in and use the remaining $B = 3,000$ Monte Carlo samples for the following results. We fit the GLDRM model using maximum likelihood (ML) estimation.^{2,3}

We use the following measures for a comprehensive comparison of the two inference methods. For quantifying frequentist bias (or lack thereof): Est_a and Est_m , which respectively are average and median of the parameter estimates across the replicates. For this comparison, we tilt both the posterior samples and ML estimate of f_0 to have a common reference mean μ_0 . We use μ_0 equal to the simulation ground truth (Section 5.1). We report posterior means as estimates under Dir-GLM and maximum likelihood estimates for GLDRM. For statistical efficiency in frequentist terms, we report $RRMSE$ (relative root mean square error) and RL (relative length) for uncertainty intervals. Both are relative to GLDRM. We report $RRMSE_m$, defined as the ratio of median (across replicates) RMSE for Dir-GLM versus the median RMSE for GLDRM. As an additional measure, we provide $RRMSE_a$, which is same as $RRMSE_m$ with the only difference being use of averages instead of medians across replicates.

Similarly, the measures RL_m and RL_a , based on relative length of the uncertainty intervals, are reported. Although both types of result summaries are reported in the tables, for more robust conclusions, we discuss the comparison using median-based summaries only. Frequentist coverage probabilities (CP) of posterior credible intervals and ML based confidence intervals are computed. These 95% credible or confidence intervals (CI) are constructed using symmetric quantiles based on posterior samples for Dir-GLM and Wald method for GLDRM.

5.3 | Results

Table 2 compares Dir-GLM estimates for the regression parameters β_j to corresponding GLDRM estimates in both small and large sample settings and under both simulation scenarios. As expected, both GLDRM and Dir-GLM estimators have some bias in the small sample case, whereas both are approximately unbiased in the large sample regime. However, the Dir-GLM estimator is more efficient in all settings for the small sample case, with a 11% to 21% lower RMSE and 7% to 12% lower CI length. For large samples, both estimators have comparable efficiency with the Dir-GLM estimator being slightly more efficient. Frequentist coverage probabilities for the regression parameters based on both methods are close to the 0.95 nominal level. Table 3 reports the same for $f_0(s_\ell)$, $l = 0, \dots, 5$, in small sample setting. For $f_0(0), \dots, f_0(3)$, both methods report similar performance as for the regression parameters in terms of frequentist (un)biasedness and statistical efficiency. However, the Dir-GLM estimator performs favorably in estimating $f_0(4)$ and $f_0(5)$, underscoring the limitation of maximum likelihood estimation of $f_0(s_\ell)$ in sparse-data scenarios when only a limited number of observations are available for $y = s_\ell$. Also, GLDRM does not naturally provide uncertainty quantification for $f_0(s_\ell)$ whereas the proposed Dir-GLM model provides the posterior credible intervals as a by-product of MCMC simulation. Overall, the proposed Dir-GLM model compares favorably with the GLDRM model.

TABLE 2 Small and Large sample simulation results for regression parameters.

n	S	Par	Method	True	Est _a	RRMSE _a	RL _a	Est _m	RRMSE _m	RL _m	CP
25	1	β_0	GLDRM	-0.7	-0.78	1.00	1.00	-0.74	1.00	1.00	0.97
			Dir-GLM		-0.76	0.82	0.91	-0.74	0.89	0.93	0.97
		β_1	GLDRM	0.2	0.20	1.00	1.00	0.19	1.00	1.00	0.93
			Dir-GLM		0.16	0.79	0.92	0.17	0.83	0.93	0.97
	2	β_0	GLDRM	-0.7	-0.81	1.00	1.00	-0.77	1.00	1.00	0.97
			Dir-GLM		-0.77	0.75	0.87	-0.75	0.85	0.90	0.97
		β_1	GLDRM	0.2	0.18	1.00	1.00	0.18	1.00	1.00	0.94
			Dir-GLM		0.14	0.73	0.86	0.14	0.79	0.88	0.97
250	1	β_0	GLDRM	-0.7	-0.71	1.00	1.00	-0.71	1.00	1.00	0.97
			Dir-GLM		-0.71	1.00	0.99	-0.71	0.98	0.99	0.96
		β_1	GLDRM	0.2	0.20	1.00	1.00	0.20	1.00	1.00	0.96
			Dir-GLM		0.20	0.99	0.99	0.20	1.00	0.98	0.96
	2	β_0	GLDRM	-0.7	-0.71	1.00	1.00	-0.71	1.00	1.00	0.97
			Dir-GLM		-0.71	0.98	0.99	-0.71	0.97	0.99	0.96
		β_1	GLDRM	0.2	0.20	1.00	1.00	0.20	1.00	1.00	0.96
			Dir-GLM		0.20	0.97	0.98	0.20	0.96	0.98	0.96

Notes: Est_a, RRMSE_a and RL_a (Est_m, RRMSE_m and RL_m) are average (median) of Est(s), RRMSE(s) and RL(s) obtained from $R = 2,000$ simulated data sets. Abbreviations: n – Sample size; S – Scenario as per Section 5.1; Par – Parameter; Est – Estimate; RRMSE – Relative (w.r.t. GLDRM) root mean square error; CP – Coverage probability with α level 0.95; RL – Relative length (w.r.t. GLDRM) for credible/confidence interval.

Finally, we compare estimation of selected exceedance probabilities. For the GLDRM, we estimate exceedance probabilities by plugging in maximum likelihood estimates:

$$p(y \geq y_0 | x) \hat{=} p(y \geq y_0 | x; \beta^{(mle)}, f_0^{(mle)}) . \quad (8)$$

For Dir-GLM, we compute posterior predictive probabilities for the exceedance events defined as:

$$p(y \geq y_0 | x) = \int p(y \geq y_0 | x, \beta, f_0) p(\beta, f_0 | \mathcal{D}_n) d\beta df_0 \hat{=} (1/B) \sum_{b=1}^B p(y \geq y_0 | x, \beta^{(b)}, f_0^{(b)}) , \quad (9)$$

TABLE 3 Small sample simulation results for f_0 .

n	Scenario	Parm	Method	Truth	Est _a	RRMSE _a	Est _m	RRMSE _m	CP
25	1	$f_0(0)$	GLDRM	0.367	0.291	1.00	0.307	1.00	N/A
			Dir-GLM		0.364	0.59	0.368	0.71	0.94
		$f_0(1)$	GLDRM	0.368	0.454	1.00	0.414	1.00	N/A
			Dir-GLM		0.386	0.58	0.369	0.86	0.92
		$f_0(2)$	GLDRM	0.185	0.220	1.00	0.236	1.00	N/A
			Dir-GLM		0.168	0.62	0.159	0.60	0.95
		$f_0(3)$	GLDRM	0.062	0.034	1.00	0.000	1.00	N/A
			Dir-GLM		0.058	0.52	0.051	0.40	0.92
		$f_0(4)$	GLDRM	0.015	0.001	1.00	0.000	1.00	N/A
			Dir-GLM		0.018	1.00	0.014	0.34	0.97
	2	$f_0(5)$	GLDRM	0.003	0.000	1.00	0.000	1.00	N/A
			Dir-GLM		0.006	2.44	0.005	0.67	0.98
		$f_0(0)$	GLDRM	0.471	0.397	1.00	0.409	1.00	N/A
			Dir-GLM		0.458	0.58	0.461	0.66	0.89
		$f_0(1)$	GLDRM	0.232	0.288	1.00	0.259	1.00	N/A
			Dir-GLM		0.246	0.58	0.240	0.83	0.88
		$f_0(2)$	GLDRM	0.172	0.236	1.00	0.240	1.00	N/A
			Dir-GLM		0.178	0.60	0.164	0.58	0.91
		$f_0(3)$	GLDRM	0.085	0.070	1.00	0.062	1.00	N/A
			Dir-GLM		0.076	0.58	0.070	0.38	0.88
		$f_0(4)$	GLDRM	0.031	0.007	1.00	0.000	1.00	N/A
			Dir-GLM		0.029	0.77	0.019	0.51	0.90
		$f_0(5)$	GLDRM	0.009	0.000	1.00	0.000	1.00	N/A
			Dir-GLM		0.010	1.44	0.006	0.46	0.93

Notes: Abbreviations are same as Table 2. N/A – Not available.

where $(\beta^{(b)}, f_0^{(b)})$ denotes the b th posterior Monte Carlo sample. The comparison is performed for $y_0 = 2$ (moderate case) and 4 (severe case) given different values of x . For the moderate case, both GLDRM and Dir-GLM estimators have comparable performance in terms of frequentist (un)biasedness, with the Dir-GLM estimator being slightly more efficient. Performance of the GLDRM estimator deteriorates as we move from moderate to severe exceedance events. In the case of severe exceedance events, the Dir-GLM compares favorably. The graphics are provided in Appendix B (Figures 5 and 6) of the Supplementary Information.

5.4 | Additional simulation studies

We provide additional simulation results in the Supplementary Information. Figure 3 in Appendix B shows the Rao–Blackwellized marginal posterior density²¹ estimate with a 95% credible band for the regression parameters. Figure 4 in Appendix B demonstrates consistency of the Dir-GLM estimates as a function of the sample size. Table 1 in Appendix C reports large sample simulation results for f_0 .

6 | THE AHEAD STUDY

6.1 | Model Fit

We fit both inference models for the AHEAD data set following the same setup as in the simulations reported in Section 5.2. The only change being the use of 10,000 MCMC iterations for the Dir-GLM. We discard the first 3,000 samples as initial transient, and use the remaining $B = 7,000$ posterior Monte Carlo samples for the reported inference. We evaluate the Dir-GLM fit by comparing observed frequencies $\hat{f}(y = y_0 \mid x_{age})$ with the posterior estimates of $p(y = y_0 \mid x_{age})$ for a future subject. We use $x_{age} = 70, 80, 90$ and $y_0 = 0, 1, \dots, 5$, and conduct a χ^2 goodness-of-fit test, which yields a p-value exceeding 0.4. This result suggests that the proposed model adequately fits the data, providing no significant evidence against it.

6.2 | Predictive Inference

A comparison of prediction under Dir-GLM versus GLDRM is implemented in two ways. First, we perform a small sample study in which both models are fitted based on a small training data set of size $n = 100$, randomly sampled from the complete AHEAD data of size 6,441. We then assess prediction accuracy on the held-out test data set of size 6,341. For comparison, we focus on estimating probabilities of exceedance events at moderate and severe difficulty in daily activities, i.e. $p(y \geq y_0 | x)$ with $y_0 = 2$ and 4, respectively. We estimate these quantities using (8) and (9). The AUC (area under the curve) values for moderate exceedance are comparably similar: 0.70 for GLDRM and 0.71 for Dir-GLM. In the case of severe exceedance, the AUC value is 0.75 for GLDRM and an improved 0.79 for Dir-GLM. The ROC plot highlights that Dir-GLM consistently improves over GLDRM. This performance difference is largest for the severe exceedance event. In fact, in sparse data cases, AUC under the GLDRM can get close to 0.50. The ROC plots are provided in Appendix D (Figures 7 and 8) of the Supplementary Information.

Second, we compare the precision in estimation using confidence and credible intervals for the regression coefficients. Figure 3 shows these uncertainty intervals respectively for GLDRM and Dir-GLM. In addition, Table 4 reports the relative interval lengths (w.r.t. GLDRM) i.e., ratio of the interval length for Dir-GLM versus the GLDRM. Notably, in the case of small training samples, the intervals are considerably narrower for Dir-GLM as compared to GLDRM, with a reduction ranging from 6% to 69%. This discrepancy reduces to a negligible difference with large training samples. This underscores that GLDRM estimates can exhibit substantial uncertainty, particularly in scenarios involving small sample sizes, whereas Dir-GLM estimates offer narrower uncertainty intervals. In addition, we provide point estimates and uncertainty intervals for the model parameters in Appendix E (Tables 2 and 3) of the Supplementary Information.

TABLE 4 AHEAD study. Small and large training sample performance using 95% credible/confidence intervals of the regression parameters. The table shows the relative interval lengths for Dir-GLM (w.r.t. GLDRM).

Training sample size	Relative uncertainty interval lengths							
	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
small	0.31	0.94	0.76	0.89	0.36	0.38	0.42	0.38
large	0.99	0.98	1.06	1.00	1.00	0.97	1.01	0.97

Note: See also Figure 3.

6.3 | Uncertainty in exceedance probability estimates

In decision-making problems, accurately quantifying uncertainty in exceedance probability estimates is an important consideration. However, inference under GLDRM does not naturally report such uncertainties. In contrast, the Dir-GLM readily provides uncertainty quantification for any function of the model parameters as a byproduct of the generated posterior Monte Carlo samples. We focus here on both moderate and severe exceedance probabilities for different ages, i.e. $p(y \geq y_0 | x_{age})$, $y_0 = 2, 4$, and $x_{age} = 70, 80, 90$. For short, we write model parameters as $\phi = (\beta, f_0)$. The b th posterior Monte Carlo sample for $p(y \geq y_0 | x_{age}, \phi)$ is obtained as

$$p(y \geq y_0 | x_{age} = t, \phi^{(b)}) = \frac{\sum_{x: x_{age}=t} p(y \geq y_0 | x, \phi^{(b)})}{\sum_x 1_{\{x: x_{age}=t\}}(x)},$$

where $\phi^{(b)} = (\beta^{(b)}, f_0^{(b)})$ denotes the b th sample from the posterior. Note that in this case, the design with respect to x is considered to be fixed. Figure 4 shows a kernel density estimate of the implied posterior distribution on severe exceedance probabilities, $p(y \geq 4 | x_{age}, \phi)$. A peaked density indicates less posterior uncertainty about $p(y \geq 4 | x_{age}, \phi)$. As expected, with an increase in age, exceedance probability estimates increase non-linearly but with a decrease in precision. This pattern is comparatively less prominent in case of moderate exceedance (see Figure 9 of the Supplementary Information).

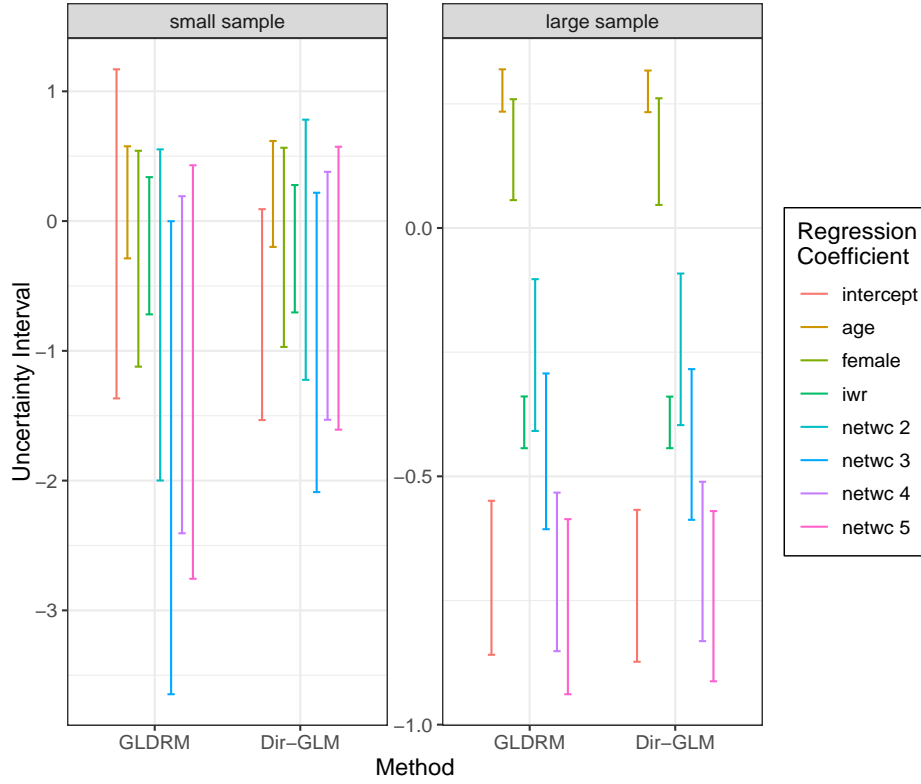


FIGURE 3 AHEAD study. Credible/Confidence intervals for regression parameters by methods based on small sample and large sample (full data set). *iwr* (immediate word recall), and *netwc* j (net worth category j), $j = 1, 2, \dots, 5$.

7 | DISCUSSION

We have proposed a flexible Bayesian model-based approach to inference under a recently developed semi-parametric extension of the classical generalised linear model (SPGLM). The major contributions of the developed Dir-GLM are, first, it allows us to estimate and provide critically needed uncertainty quantification for the centering distribution and model-derived quantities such as exceedance probabilities, which may be important in clinical diagnostic or other decision-making settings. Second, it shows promise in small sample and sparse-data scenarios over the existing maximum likelihood-based methods. Third, the proposed methodology remains valid for any finite-supported response and under a meaningful definition of the mean response. Fourth, it provides a launching pad for important future work; see last paragraph.

Our simulation study shows that the proposed Dir-GLM compares favourably with maximum likelihood inference (GLDRM) in terms of (un)biased parameter estimates, coverage probabilities, and statistical efficiency in both small and large sample regimes, with considerable benefits in small sample settings. We have also discussed an application of the proposed method for inference in the AHEAD study. Results show that Dir-GLM improves predictive performance, in case of exceedance events, over GLDRM, and this performance difference is largest for sparse-data scenarios. Credible intervals for model parameters are more precise under the Dir-GLM, again, especially in small sample settings. In addition, under the Dir-GLM, posterior inference for moderate and severe exceedance probabilities show that with increasing age, exceedance probabilities increase non-linearly but with a decrease in precision for the estimate. This pattern is most prominent for severe exceedance, as expected, since data are sparser for older ages and for higher exceedance thresholds.

For future work, building upon this current methodology, there are multiple research directions one could explore. One is a Bayesian inference approach for multi-variate or repeated measures SPGLM that will include random effects (or other latent variables); a Bayesian approach is especially attractive for this application because, under the ML paradigm, similar extensions are likely to be prohibitively difficult to implement using adaptive quadrature, the default computational approach for non-linear random effects regression models. Second, the proposed Dir-GLM will serve as a foundation for a semi-parametric (continuous response) setting. Third, the SPGLM, married with our Bayesian approach to inference, unlocks opportunities to balance

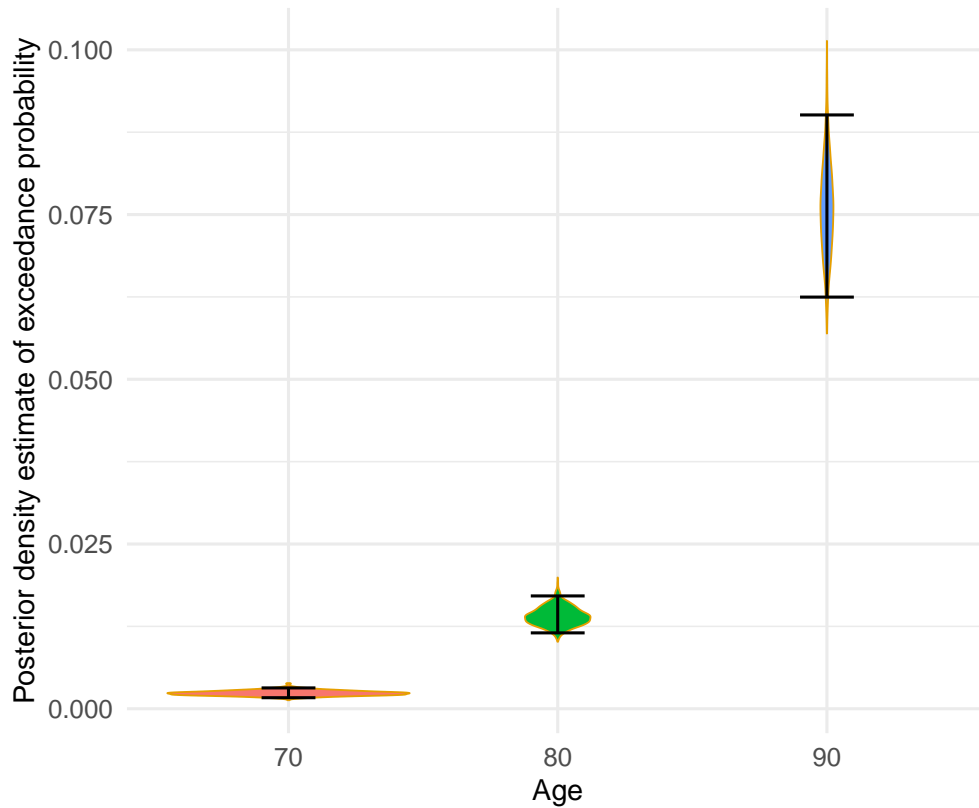


FIGURE 4 AHEAD study. Severe exceedance probabilities $\pi = p(y \geq 4 \mid x_{age}, \mathcal{D}_n)$ for $x_{age} = 70, 80$, and 90 . The vertically plotted density summarizes uncertainty about the probability of the exceedance event as a function of parameters $\phi = (\beta, f_0)$. That is, let $\pi_\phi = p(y \geq 4 \mid x_{age}, \phi)$ (with $\pi = \int \pi_\phi dp(\phi \mid \mathcal{D}_n)$). The violin plots show posterior distribution of π_ϕ ; a peaked density indicates less posterior uncertainty about π_ϕ .

cost-effectiveness and precision in clinical study designs for finding specific population subgroups significantly affected by a particular condition, disease, or treatment response in cases where the response distribution follows a form not well captured by existing parametric models. This could be achieved by finding the minimum sample size required,^{22,23,24,25} in setting up clinical designs, while maintaining a permitted error level for our model-derived quantity of interest.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings in this paper are available in the University of Michigan Health and Retirement Study (HRS)²⁶ repository at <https://hrsdata.isr.umich.edu/data-products/1993-ahead-core>. For accessing the curated data, kindly direct request to P. J. Rathouz (paul.rathouz@austin.utexas.edu).

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