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# THE DISTRIBUTION AND CONSUMPTION RATE OF STARS AROUND A MASSIVE, COLLAPSED OBJECT

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## ABSTRACT

We consider the steady-state distribution and consumption rate of stars orbiting a massive object at the center of a spherical, $N$-body stellar system. The distribution of stars is determined by the consumption of low angular momentum stars which pass within a small distance $r_t$ of the central mass $M$ and by the relaxation processes associated with gravitational stellar encounters. Our method employs an approximate, analytic analysis of the two-dimensional Fokker-Planck equation describing diffusion in energy $E$ and angular momentum $J$. The basic results are the following: (1) Consumption of low angular momentum stars which have entered the "loss-cone" $J \leq J_{\min} \approx (GMr_t)^{1/2}$ dominates the character of the solution inside the critical radius, $r_{\text{crit}} \gg r_t$, at which the root mean square angular momentum transferred to a star via stellar encounters in one orbital period equals $J_{\min}$. (2) The total consumption rate of stars by $M$ is roughly the number of stars inside $r_{\text{crit}}$ divided by the relaxation time at $r_{\text{crit}}$. (3) A self-consistent solution can be found in which the distribution of stars is almost isotropic for high-$J$ stars and varies only logarithmically with $J$ for low-$J$ stars. (4) The density of core stars has the following form: $n(r) \approx n(r_a)[1 + (r_a/r)^l]$, where the accretion radius $r_a \sim GM/\langle v^2\rangle \gg r_t$, $\langle v^2\rangle$ is the mean-squared velocity dispersion in the core outside $r_a$, and $l$ decreases slowly from $\sim 1.75$ at $r \gg r_{\text{crit}}$ to $\sim 1.60$ at $r \sim 10r_t$. Neglect of loss-cone effects gives a consumption rate too small by roughly the ratio $r_t/r_{\text{crit}}$ and a constant exponent $l = 1.75$ for $r \gg r_t$. These results are applied to massive black holes at the centers of globular clusters and galactic nuclei.

*Subject headings:* black holes — clusters: globular — galaxies: nuclei — stars: stellar dynamics

## I. INTRODUCTION

It has frequently been argued that massive objects, such as massive black holes or supermassive stars, reside at centers of star clusters and galactic nuclei. These objects can be powerful emitters of radiation and can exert a significant dynamical influence on surrounding stars. Such objects have also been proposed as the principal energy source in quasars (Hoyle and Fowler 1963*a*, *b*). Recently, several models involving massive black holes have been suggested to explain the globular cluster X-ray sources (Bahcall and Ostriker 1975; Hills 1975; Silk and Arons 1975). If the latter hypothesis is correct, massive black holes in cluster cores might be positively identified by the characteristic manner and rate in which they destroy stars, accrete the liberated gas, and radiate energy. Alternatively, they might also be detected by their effect on the density distribution of orbiting stars via the formation of anomalous light cusps in the cluster center or by their modification of the dynamical evolution of the cluster core (see, e.g., Peebles 1972*a* and references quoted therein).

In this paper we reexamine the steady-state distribution and rate of consumption of stars orbiting a massive object at the center of a spherical stellar cluster. The problem we consider has been previously studied by Peebles (1972*a*, *b*) and Bahcall and Wolf (1976). Motivated by their physical picture and pioneering work, we examine the manner in which stars, essentially in Keplerian orbit near the massive object, gradually diffuse inward toward the center by repeated small-angle, gravitational encounters with each other. Far from the central object of mass $M$, in the region $r > r_a$, where $r_a \sim 2GM/\langle v^2\rangle$ is the accretion radius for ambient core stars moving with rms velocity $\langle v^2\rangle^{1/2}$, the core stars are unbound to the central object and obey essentially a Maxwellian distribution law. We assume that inside $r_a$, the gravitational potential is dominated by the central mass $M$. At an extremely small radius $r_t \ll r_a$, stars are removed from the system either by tidal disruption ($r_t = r_{\text{tidal}}$), coalescence, tidal capture (e.g., Fabian, Pringle, and Rees 1975) or by gravitational-radiation energy dissipation. In steady state, the rate at which stars diffuse inward from $r_a$ equals the rate at which stars are consumed at $r_t$. A dynamical, rather than an (iso)-thermal, equilibrium is thereby achieved in the core; and it is the resulting equilibrium stellar distribution function in the region $r_t \ll r \ll r_a$ which we determine. We assume the mass of all orbiting stars equals $m \ll M$ and that the total mass of these stars within $r_a$ is much smaller than $M$.

Previous investigations (Peebles 1972*b*; Bahcall and Wolf 1976) have been based on the simplifying assumption

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that a star is not removed from the system until it has lost sufficient energy $E$ to settle in a tightly bound orbit (near $r_t$) with $E = -GM/r_t \equiv E_t$. (The quantities $E$ and $J$ will denote energy and angular momentum per unit mass throughout the paper.) Explicitly ignored in this approach, pointed out by Bahcall and Wolf (1976), is the removal of the high-energy ($E \gg E_t$), low angular momentum stars, satisfying $J \leq J_{\min}(E) \equiv [2(E + GM/r_t)]^{1/2} r_t \sim (GMr_t)^{1/2}$, which move in highly eccentric orbits at large *mean* radius $r \gg r_t$ but which move within $r_t$ at pericenter. The removal of these stars which enter the so-called loss cone in $J$-space is significant and indicates the two-dimensional nature of the problem (i.e., the two independent variables $E$ and $J$).

Our analysis incorporates the two-dimensional Fokker-Planck equation in $E$- and $J$-space, for which we employ the approximate velocity diffusion coefficients computed for the case of a Maxwellian field star distribution by Spitzer and Hart (1971) (see § III*b*). We first determine the distribution of stars as a function of $J$ for small $J$ by considering the rate at which stars of fixed $E$ enter the loss cone $J \leq J_{\min}(E)$. Integrating the two-dimensional equation over $J$ and substituting the expression for the stellar consumption rate, we obtain a Fokker-Planck equation in $E$ alone containing an energy-dependent "sink" term. We then determine the approximate steady-state solution to this equation in the "self-similar" regime $r_t \ll r \ll r_a$, and the total rate at which $M$ consumes stars.

Our basic results are the following: (1) A self-consistent solution exists for which the distribution function is essentially isotropic for large $J$, $J_{\min} \ll J \lesssim J_{\max}(E) \equiv GM/(2|E|)^{1/2}$, and decreases only logarithmically with $J$ for smaller $J$, due to capture at $J_{\min}(E)$. (2) Loss-cone effects are important inside the radius $r_{\text{crit}}$ at which $j_2$, the root mean square angular momentum transferred to a star in one orbital period, equals $J_{\min}$. (3) The total consumption rate, $F$, of stars is approximately the total number of stars inside $r_{\text{crit}}$ divided by the relaxation time at $r_{\text{crit}}$. In some cases $r_{\text{crit}} > r_a$, and the dominant contribution to $F$ comes from unbound stars. (4) The stellar density distribution in the region $r \gg r_t$ satisfies $n(r) \approx n_a[1 + (r_a/r)^l]$, where $n_a$ is the ambient density of unbound stars in the core and $l$ increases slowly (logarithmically) with $r$ from $l \approx 8/5$ at $r \approx 10r_t$ to $l \approx 7/4$ for $r \gg r_{\text{crit}}$. Since our calculation is applicable only to the self-similar regime, the numerical coefficient appearing in the second term of the density distribution cannot be accurately determined; on physical grounds it must be of order unity. See Bahcall and Wolf (1976) for a discussion of the density profile in the matching region $r \sim r_a$. Recently Frank and Rees (1976) have independently obtained results qualitatively consistent with our results (1)–(3). Result (4), in the region $r \gg r_{\text{crit}}$, agrees with the earlier calculations of Bahcall and Wolf (1976). The density profile in this region can also be determined from a simple scaling argument (Shapiro and Lightman 1976).

In § II we present a detailed, qualitative overview of the physical problem examined here, together with a brief review of previous work. In § III we outline more formally the mathematical methods, equations, and approximate solutions employed in the analysis. In § IV we illustrate our solution with specific applications.

## II. QUALITATIVE OVERVIEW

### *a) Review of the One-Dimensional Analysis*

In this section we briefly summarize the self-similar solutions obtained by Peebles (1972*a*, *b*) and Bahcall and Wolf (1976), who treat the problem as one-dimensional diffusion in energy space, and ignore loss-cone effects. Stars with energy $E \gtrsim -GM/r_a \equiv E_a$ are loosely bound or unbound to the central massive star $M$ and are essentially unaffected by its gravitational field. Stars with energy $E \leq -GM/r_t \equiv E_t$ ($|E_t| \gg |E_a|$) are consumed. Consumption leads to a slow, inward diffusion of stars toward the hole; but since the stellar relaxation time $t_r$ greatly exceeds the dynamical time $t_d$ in a cluster of many stars, this diffusion represents a small perturbation to the velocity distribution of the orbiting stars. Since the cluster is spherically symmetric and the boundary conditions depend only on $E$, we expect the distribution function to depend only on $E$. For $E$ in the range $|E_a| \ll |E| \ll |E_t|$, we assume the distribution function $f$ to be of the form (Peebles 1972*b*) $f(r, v) = K|E|^p$, where $E = \frac{1}{2}v^2 - GM/r$. From this expression one easily derives relations for the stellar density, velocity dispersion, and mean energy (see eqs. [50]). The problem then reduces to determining $p$ by considering the relaxation effects of star-star encounters and by demanding steady state.

Even though the dominant gravitational interactions between stars are small-angle scatterings, stars interact only with other neighboring stars at nearly the same radial distance from $M$, since interactions over many radial logarithmic intervals cancel because of the large mismatch of orbital frequencies. Following Peebles (1972*a*), we first consider the interactions of *typical* stars in the interval $r$ to $2r$, which move in nearly circular orbits (cf. eqs. [50]). The number of stars in this interval, $N_r$, varies roughly as

$$N_r \sim n(r)r^3 \propto r^{3/2-p} \propto |E|^{p-3/2}, \tag{1}$$

where $|E|$ refers to the mean value of $|E|$ in the interval about $r$. In one relaxation time $t_r$, a large number of these stars lose a significant fraction of their energy and move inward to lower radius. Since $t_r$ varies locally as

$$t_r \propto v^3/n \propto r^p \propto |E|^{-p} \tag{2}$$

(Chandrasekhar 1942), the inward flux of stars through a sphere at $r$, $F_{\max}(E)$, (stars s$^{-1}$), varies as

$$F_{\max}(E) \sim \frac{N_r}{t_r} \propto |E|^{2p-3/2} . \tag{3}$$

On dimensional grounds alone, $F_{\max}(E)$ gives the largest possible flux of stars at a given energy $E$ and serves as a useful local comparison for all fluxes in the problem.

A more accurate analysis, similar to the one first presented by Bahcall and Wolf (1976), yields for the net star flux, $F(E)$,

$$F(E) = -F_{\max}(E)\beta(p) , \tag{4}$$

where

$$F_{\max}(E) = F_{\max}(E_a)(E/E_a)^{2p-3/2} , \tag{5}$$

and

$$F_{\max}(E_a) \approx N_{r_a}/t_r(r_a) \approx \tfrac{1}{3}(4\pi G m)^2 \Lambda n_a{}^2 r_a{}^3 \langle v^2\rangle^{-3/2} , \tag{6}$$

where $\Lambda$ is the gravitational "Coulomb logarithm" (see § III, eq. [40d]), and the dimensionless function $\beta(p)$ is given by equation (52c) of § III. The quantity $\beta(p)$, which has several terms, results from an integration over the entire distribution function and takes into account competing contributions to $F(E)$ (the *net* flux) from both ingoing and outgoing stars.

In a one-dimensional treatment, steady state requires $F(E)$ to be independent of $E$. In the original analysis, Peebles (1972*a*, *b*) obtained $p = \frac{3}{4}$ from equation (3), which he derived from the simple scaling argument summarized above. However, $\beta(p)$ is negative for all $p > \frac{1}{4}$, corresponding to a net *outward* flux of stars in violation of the boundary conditions of the problem. This conclusion was first reached by Bahcall and Wolf (1976). The outward flux originates from the interactions in each energy interval centered on $E$ of those (few) atypical stars with low $J$ which interact with stars at much smaller characteristic radii and higher $|E|$. The density of stars at small $r$ encountered by the low-$J$ stars increases rapidly with $p$. As a result, for $p > \frac{1}{4}$, the relatively large energy dispersion imparted to such low angular momentum stars in their inward high-velocity excursions is sufficient to drive them outward at a rate which reverses the net flux at $E$.

The constancy of $F(E)$ admits another solution, as pointed out by Bahcall and Wolf (1976), corresponding to $F(E) = 0 = \beta(p)$ in the self-similar regime, giving $p = \frac{1}{4}$ (cf. eq. [52c]). By numerically solving the full, time-dependent one-dimensional diffusion equation with the appropriate boundary conditions at $E = 0$ and $E = E_t$, Bahcall and Wolf find the equilibrium flow rate

$$F(E) = \text{const.} \approx -F_{\max}(E_a)(r_a/r_t)^{2p-3/2} = -F_{\max}(E_a)(r_t/r_a) \tag{7}$$

in the limit $r_t/r_a \ll 1$. Thus $F/F_{\max} \sim (r_t/r_a) \ll 1$. This small but finite flux associated with the $p = \frac{1}{4}$ solution can be obtained by evaluating $F(E)$ near $r_t$, where "cooling" by exterior stars can no longer be balanced by "heating" from interior stars. An interesting property of the solution is that the outward flux of energy, $dE/dt$, which must equal the rate at which stars are consumed multiplied by the kinetic energy of each consumed star, is independent of $r_t$:

$$dE/dt = FE_t \sim F_{\max}(E_a)GM/r_a . \tag{8}$$

The $p = \frac{1}{4}$ solution can also be determined from a simple scaling argument (Shapiro and Lightman 1976) by carefully examining the net diffusion time scales associated with both star and energy transport in an equilibrium, $N$-body system containing a central black hole.

### *b) The Two-Dimensional Problem: Loss-Cone Effects*

In Parts (i) and (ii) of this subsection we consider the consumption and distribution of *bound* stars of *arbitrary* energy $E$ which pass within a distance $r_t$ of the central mass $M$ at perihelion. These stars enter the "loss cone" in $J$-space defined by $J \leq J_{\min}(E) = [2(E + GM/r_t)]^{1/2} r_t \approx (GMr_t)^{1/2}$ and may be subsequently removed in a dynamical time scale, $t_d$. In Part (iii) of this subsection we consider modifications to the total consumption rate $F$ of stars due to contributions from the unbound, isothermal stars in the region $r > r_a$. Such stars contribute significantly to $F$ whenever $r_{\text{crit}} > r_a$.

Since $t_d/t_r \ll 1$, Jeans's theorem applies; and we can write, for the distribution function, $f = f(E, J)$. Stars with energy $E$ and high angular momentum $J$ in the range $J_{\min} \ll J \lesssim J_{\max}(E)$, where $J_{\max}(E) = GM/(2|E|)^{1/2}$ is the angular momentum of a star with energy $E$ in circular orbit, are distributed nearly isotropically in velocity space. Since these stars do not pass near the disruption radius $r_t$, they are little influenced by its presence. For these high-$J$

stars, we then have approximately $f \approx f(E)$, independent of $J$. Stars of moderate and low $J$, on the other hand, decrease in density as $J$ decreases. The capture of stars orbiting within the loss cone produces a (moderate) density gradient in $J$-space and results in a steady influx of stars into the loss cone. In steady state, this diffusion into the loss cone and subsequent consumption is exactly compensated by an influx of new stars with high $J \gg J_{\text{min}}$.

### i) *The Loss-Cone Consumption Rate*

The time scale on which a star can survive in the loss cone is an orbital period (the dynamical time scale) given by

$$t_d = P(E) = \frac{2\pi GM}{(2|E|)^{3/2}} \,. \tag{9}$$

It is useful to define the angular momentum step size, $j_2$, associated with this time scale, where $j_2 \ll J_{\max}(E)$ is the rms angular momentum transferred to a star in one orbital period via stellar encounters. Since $J_{\max}^2(E)$ is the squared angular momentum transferred in a relaxation time $t_r$, we have the relation

$$j_2 \sim J_{\max}(t_d/t_r)^{1/2} \,. \tag{10}$$

Consider now the region around the loss cone in two-dimensional $J$-space on the velocity sphere, illustrated in Figure 1. For stars with a given energy $E$ and $J$ between $J_{\min}$ and $J_{\min} + j_2$, a fraction $\sim J_{\min}^2/(J_{\min} + j_2)^2$ enter the

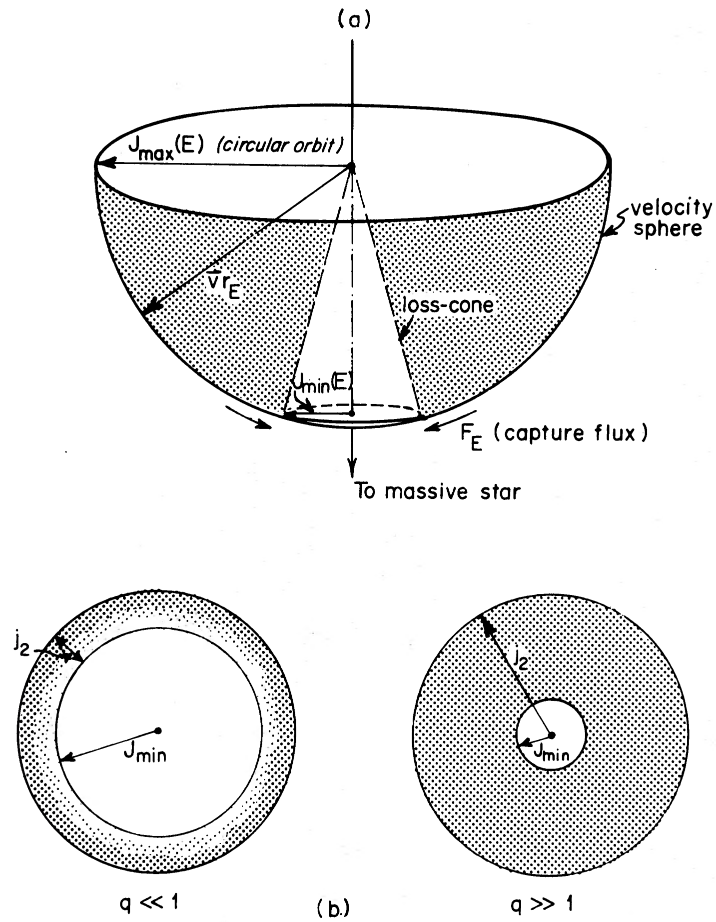


FIG. 1($a$).—Velocity distribution for ingoing stars with fixed energy $E$, radius $r_E \equiv GM/2|E|$, and speed $v = (2|E|)^{1/2}$. Stars with angular momentum $J$ in the range $J_{\min}(E) \ll J \leq J_{\max}(E)$ are distributed nearly uniformly on the velocity sphere, where $J_{\min}(E) \equiv [2(E + GM/r_t)]^{1/2} r_t$ and $J_{\max}(E) \equiv GM/(2|E|)^{1/2}$. Stars scattered into the loss-cone with $J \leq J_{\min}(E)$ may be removed from the system in an orbital period.

FIG. 1($b$).—The velocity sphere viewed from below. The quantity $j_2$ represents the dispersion in $\Delta J$ suffered by a star in one orbital period due to stellar encounters; $q(E) \equiv j_2^2/J_{\min}^2(E)$. The phase-space density of stars outside the loss-cone falls rapidly with $J$ as $J \to J_{\min}$ when $q \ll 1$ (the "diffusion" limit), but the density remains nearly uniform when $q \gg 1$ (the "pinhole" limit). In each orbital period only stars in the ring $J_{\min} < J \lesssim J_{\min} + j_2$ may enter the loss-cone.

loss cone in a time $t_d$. This fraction is an area reduction factor resulting from the two-dimensional character of $J$-space. If $j_2 \ll J_{\min}$, roughly half of the stars enter the loss cone, while if $j_2 \gg J_{\min}$, only the fraction $J_{\min}^2/j_2^2$ enter. Evidently, a key parameter in describing the rate of stellar consumption is the dimensionless variable $q(E)$, given by

$$q(E) \equiv j_2^2/J_{\min}^2 \sim (E_t/E)(t_d/t_r)\,. \tag{11}$$

The stochastic manner in which stars enter the loss cone is qualitatively different in the two limiting cases $q \ll 1$ and $q \gg 1$. For $q \ll 1$ the random-walk of stars into the loss cone may be approximated by diffusion in $J$-space, while for $q \gg 1$ the discrete size of $j_2$ is important: stars diffuse from higher $J$ only in the range $J > j_2$ and then just a fraction $\sim(J_{\min}^2/j_2^2)$ of the stars inside $J \leqslant j_2$ strike the loss cone each orbital period.

For values of $p \leq \frac{1}{4}$ which will be relevant to our solution, equations (2), (9), and (11) indicate that $q \propto |E|^{p-5/2}$ is a rapidly decreasing function of $|E|$. From equation (11), we have $q \approx t_d/t_r \ll 1$ near $r \sim r_t$ ($E \sim E_t$) while, for sufficiently small $|E|$ (and large characteristic radius), $q \gg 1$. Thus, at some critical energy $E_{\text{crit}}$ and corresponding characteristic radius $r_{\text{crit}}$, satisfying $|E_{\text{crit}}| \ll |E_t|$ and $r_{\text{crit}} \gg r_t$, $q(E_{\text{crit}}) = 1$.

In the "diffusion" limit $q \ll 1$ the consumption rate of stars by the loss cone is significant and a large fraction $\sim 1/\ln(J_{\max}/J_{\min}) \sim 1/\ln(GM/r_t|E|)$ of stars with energy $E$ are consumed in a relaxation time. (For comparison, if $J$-space were one-dimensional, the fraction would be $\sim$ unity independent of $J_{\min}$; if $J$-space were three-dimensional the fraction would be $J_{\min}/J_{\max}$.) In contrast, in the "pinhole" limit $q \gg 1$ the consumption rate of stars by the loss cone is smaller than the maximum available flux $F_{\max}(E)$ by the factor $\sim q^{-1}$. We thus find the loss-cone capture rate $F_E(E)$ (stars per unit time per unit energy) roughly satisfies

$$\begin{aligned} -F_E(E) &\sim F_{\max}(E)|E|^{-1}/\ln(GM/|E|r_t)\,, \quad q \ll 1\,, \\ &\sim q^{-1}F_{\max}(E)|E|^{-1}\,, \qquad\qquad\quad q \gg 1\,. \end{aligned} \tag{12}$$

In equation (12), the minus sign denotes removal of stars from the system. The magnitude of the loss-cone capture rate per unit energy, $F_E(E)$, is strongly peaked at the critical energy $E_{\text{crit}}$ where $q \sim 1$, since $F_E(E)$ is a small fraction of the local maximum $F_{\max}(E)|E|^{-1}$ for $q \gg 1$ and $F_{\max}(E)|E|^{-1}$ decreases rapidly with increasing $|E|$ and decreasing $q$ for $q < 1$ (see eq. [5]). Because $F_E(E)$ has a steep maximum at $E \sim E_{\text{crit}}$, the total consumption rate of stars, $F$ (stars s$^{-1}$) roughly satisfies

$$F \approx F_E(E_{\text{crit}})|E_{\text{crit}}|\,. \tag{13}$$

### ii) *The Modified Distribution Function*

Since stars are nearly isotropically distributed in velocity space, they approximately satisfy a distribution law of the form $f \propto |E|^p$ (see eq. [16] and discussion). However, the existence of a finite loss cone does make itself weakly felt in phase space for $|E| > |E_{\text{crit}}|$ (and associated characteristic radii $r < r_{\text{crit}}$) and, as shown below, results in a slow variation of the exponent $p$ with $E$.

Steady state requires that the integrated consumption rate at energy $E$ equal the incident star flux $F(E)$ (stars s$^{-1}$) of equation (4):

$$F(E) = \int_{E_t}^{E} F_E(E')dE' + F(E_t)\,. \tag{14}$$

In previous analyses which ignored loss-cone effects, stars were consumed only at $E = E_t$, so that $F_E(E) \propto \delta(E - E_t)$, giving $F(E)$ independent of $E$. For $|E| < |E_{\text{crit}}|$, $q \gg 1$, the loss-cone capture rate $F_E(E)$ is negligible ($\ll F_{\max}(E)|E|^{-1}$), implying $F(E) =$ const. and $p \approx \frac{1}{4}$, as in the one-dimensional analysis (cf. eq. [4] and discussion). However, for $|E| > |E_{\text{crit}}|$ ($r < r_{\text{crit}}$), $F(E)$ is no longer independent of $E$. In the region $|E| > |E_{\text{crit}}|$, equations (4), (12), and (14) indicate that $p$ varies slowly with energy, deviating from $\frac{1}{4}$ by a term proportional to $\sim 1/\ln(GM/r_t|E|)$. For typical values of $E_t/E_{\text{crit}} = r_{\text{crit}}/r_t \sim 10^4$–$10^7$, $p$ decreases from $\sim\frac{1}{4}$ for $|E| \ll |E_{\text{crit}}|$ to $\sim 1/10$ for $|E| \sim 0.1GM/r_t$ (cf. Table 1). A more rapid variation of $p$ would be indicative of large deviations from isotropy for high-$J$ stars and would clearly invalidate a power-law solution $f \propto |E|^p$ with $p \sim$ constant. However, for a finite loss-cone ($r_t \neq 0$), a true self-similar solution with $p \equiv$ constant cannot apply in the region $r < r_{\text{crit}}$ because the size of the loss cone is always weakly felt.

The qualitative behavior of the local consumption rate of stars between $E$ and $2E$, $\sim -F_E(E)|E|$, the (ingoing) star flux through energy $E$, $-F(E)$, and the local maximum flux at energy $E$, $F_{\max}(E)$, are all illustrated for comparison in Figure 2. The star flux $|F(E)|$ must clearly be a monotonically decreasing function of $|E|$, since stars are removed in each energy interval. Furthermore, $F(E)$ must approach $F_E(E)|E|$ for $|E|$ increasing above $|E_{\text{crit}}|$, where all rates approach the maximum flux $F_{\max}(E)$; and $F(E)$ must level off to a constant value near $F$ for $|E|$ below $|E_{\text{crit}}|$. It is significant that the constant value of $F(E) \sim F$ in the $p = \frac{1}{4}$ region is determined by the bottleneck at $r_{\text{crit}}$ rather than at $r_t$. Note that our solution extends out only to $r_a$, the inner edge of the isothermal core, which may or may not exceed $r_{\text{crit}}$; see § iii. In the event that $r_a < r_{\text{crit}}$, our solutions given in Figure 2 and Table 1

TABLE 1

THE STELLAR DISTRIBUTION FUNCTION FOR $q \gg 1$: $p(E) = d \ln f(E)/d \ln |E|$

| $\log (GM/r_t|E|)$ | $\log (r/r_t)$ | $p(E)$ |
|---|---|---|
| 1.3 | 0.92 | 0.09 |
| 2.0 | 1.64 | 0.19 |
| 3.0 | 2.65 | 0.22 |
| 4.0 | 3.65 | 0.23 |
| 5.0 | 4.65 | 0.23 |
| 6.0 | 5.65 | 0.24 |
| 7.0 | 6.65 | 0.24 |
| 8.0 | 7.65 | 0.24 |

would truncate at $r = r_a$. Also, our distribution function must be modified at radii $r$ much smaller than the radius $r_{\text{coll}} \sim r_* M/m$ where orbital velocities are comparable to the escape velocity from a typical star of mass $m$ and radius $r_*$ (Frank and Rees 1976). At radii $r \ll r_{\text{coll}}$ inelastic scatterings occur more frequently than loss-cone captures by a logarithmic factor. For typical parameters in globular clusters and elliptical galaxies (see Table 2) $r_{\text{coll}} \ll r_{\text{crit}}$ so that our calculated consumption rate is correct in any case. Typically, $r_{\text{coll}} \ll r_{\text{crit}} \ll r_a$ for globular clusters and $r_a \sim r_{\text{coll}} \ll r_{\text{crit}}$ for elliptical galaxies.

Defining $r_c \equiv \min (r_a, r_{\text{crit}})$, the *total* consumption rate of bound stars in the cluster core, $F$, roughly satisfies (cf. eqs. [5], [12])

$$F \sim F_{\max}(E_a)(r_c/r_a)/\ln (r_c/r_t) . \tag{15}$$

In the event $r_{\text{crit}} > r_a$, modifications to $F$ due to consumption of unbound stars are indicated in the following subsection iii, but there is no change in the computed distribution function (variation of $p$) of bound stars. We point out that $F$ is larger than the equilibrium flow rate obtained by Bahcall and Wolf (1976) by the large factor $r_c/r_t$ (cf. eq. [7]), even though the above distribution function approaches the one they obtain (with $p = \frac{1}{4}$) for $r \gg r_t$. Bahcall and Wolf take $r_t \sim r_{\text{coll}}$ in their analysis, and denote $F(E)$ and $F_{\max}(E_a)$ by $R(E)$ and $R_0$, respectively. The consumption rate given by equation (15) is larger by the factor $\sim (r_a/r_{\text{crit}})^{5/4}$ than the result of Hills (1975), who neglected gravitational encounters and the density cusp inside $r_a$.

Finally, we describe the $J$-dependence of the distribution function and its deviation from isotropy at low $J$. Since a strictly isotropic distribution does not permit the required diffusion of stars into the loss cone at low $J$, *some* deviation from the condition $\partial f(E, J)/\partial J = 0$ must occur. In steady state, the appropriate diffusion equation describing the influx of stars into the loss cone in two-dimensional $J$-space is given at low $J$ by

$$\text{Flux} \propto F_E/j_2^2 = -2\pi^2 J \frac{\partial f(E, J)}{\partial J} , \tag{16}$$

where the left-hand side is independent of $J$. Evidently, the constant flux of stars into the loss cone is supported by a distribution function $f$ which varies only logarithmically with $J$, despite the significance of the loss cone.

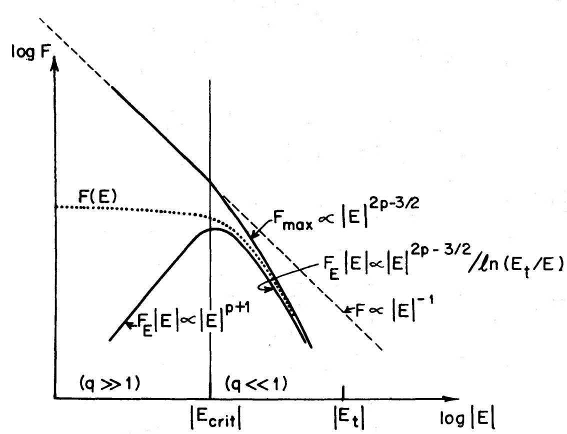


FIG. 2.—Qualitative comparison of the local consumption rate of stars between $E$ and $2E$, $-F_E(E)|E|$, the ingoing star flux through energy $E$, $-F(E)$, and the local "maximum" star flux through energy $E$, $F_{\max}(E)$. The exponent $p \approx \frac{1}{4}$ for $|E| \ll |E_t|$, but $p$ decreases (logarithmically) with $|E|$ as $|E| \to |E_t|$. Here $E_t = -GM/r_t$, where $r_t$ is the consumption radius. At $E = E_{\text{crit}}$, $q \equiv 1$. As discussed in the text, the curves apply only to stars bound to $M$.

TABLE 2

CLUSTER PARAMETERS FOR TYPICAL $N$-BODY SYSTEMS

| Object | $m$ ($M_\odot$) | $\langle v_\parallel^2\rangle^{1/2}$ (km s$^{-1}$) | $n_a$ (pc$^{-3}$) | $N$ | $r_a$ (pc) | $r_t$ (pc) | $r_{\rm crit}/r_a$ | $q(E_a)$ | $F$ ($10^{-8}$ yr$^{-1}$) |
|---|---|---|---|---|---|---|---|---|---|
| Globular cluster (M15)......... | 1 | 10 | $5\times10^4$ | $10^4$ | $0.03M_3$ | $3\times10^{-7}M_3^{1/3}$ | $0.2M_3^{-20/27}$ | $32M_3^{5/3}$ | $2M_3^{61/27}$ |
| Elliptical galaxy... | 1 | 200 | $10^5$ | $10^8$ | $0.10M_6$ | $3\times10^{-6}M_6^{1/3}$ | $70M_6^{-5/9}$ | $3\times10^{-6}M_6^{15/9}$ | $100M_6^{4/3}$ |

NOTE. $M_x \equiv M/10^x\,M_\odot$.

iii) *Consumption of Unbound Stars*

As discussed previously, the loss-cone consumption rate is determined by the bottleneck at $r_{\rm crit}$. In the event $r_{\rm crit} > r_a$, the majority of the contribution to $F$ comes from unbound stars. We thus consider here the appropriate formulae leading to $F$ in the region $r > r_a$. (*Our distribution function for bound stars inside $r_a$ is determined by consumption of bound stars and remains valid whether or not $r_{\rm crit} > r_a$.*)

In the region $r > r_a$, the local maximum flux, $F_{\rm max}$, becomes

$$F_{\rm max}(r) \sim N_r/t_r \sim (r/r_a)^3 F_{\rm max}(E_a)\,. \tag{17a}$$

The dynamical and relaxation time scales of unbound stars are, respectively, $t_d(r) \sim r/\langle v^2\rangle^{1/2}$, $t_r \sim \langle v^2\rangle^{3/2}/(G^2m^2n_a\Lambda) = $ const. The appropriate value of $q$, with $J_{\rm max}(r) \sim r\langle v^2\rangle^{1/2}$, becomes (cf. eqs. [10] and [11])

$$q \sim Gm^2n_ar^3\Lambda/(M\langle v^2\rangle r_t)\,. \tag{17b}$$

By analogy with the discussion leading to equation (12), the rate of consumption of stars between $r$ and $2r$ is again monotonically increasing with $r$ for $r < r_{\rm crit}$; but now the rate is independent of $r$ for $r_{\rm crit} < r < R$, where $R$ is the radius of the constant-density core, leading to a total consumption rate

$$F \sim F_{\rm max}(r_{\rm crit}) \ln (R/r_{\rm crit}) \sim (r_{\rm crit}/r_a)^3 F_{\rm max}(E_a) \ln (R/r_{\rm crit}) \sim n_a(r_tr_a)\langle v\rangle \ln (R/r_{\rm crit})\,. \tag{17c}$$

The last line can be obtained from equations (18), (23), and (25b) and agrees, up to the ln factor, with the simple $n\sigma v$ expression of Hills (1975).

c) *Order-of-Magnitude Relationships between Parameters*

In this section we derive approximate functional relationships between variables and parameters of the problem. More accurate expressions and numerical estimates applicable to realistic situations will be given in §§ III and IV. Let $N$ and $R$ denote the number of stars in and radius of the cluster core, respectively. The core radius $R$ is defined by the virial relation

$$\langle v^2\rangle \sim GNm/R\,, \tag{18}$$

where $\langle v^2\rangle$ is the velocity dispersion in the isothermal core. The mean free path $\lambda$ for stellar encounters in the core (exterior to $r_a$) then satisfies, approximately (cf. eq. [2]),

$$\lambda/R \sim \langle v^2\rangle^{1/2}t_r/R \sim \frac{\langle v^2\rangle^2/R}{G^2m^2(N/R^3)} \sim N\,, \tag{19}$$

where we have omitted the gravitational Coulomb logarithm. The accretion radius $r_a$, inside which the stellar distribution is affected by $M$, satisfies the relation

$$r_a/R \sim GM/\langle v^2\rangle R \sim M/Nm \ll 1\,, \tag{20}$$

where the inequality is required for steady state. The number of stars inside $r_a$, $\mathscr{N}$, satisfies the relation

$$\mathscr{N} \sim N(r_a/R)^3 \sim (M/m)^3N^{-2} \gg 1\,, \tag{21a}$$

where the inequality is required for applicability of a statistical $N$-body treatment inside $r_a$, as adopted in this paper. (See § IV for further discussion.) Since we assume that inside $r_a$ the gravitational potential is dominated by the massive object $M$, we require that

$$\mathscr{N} \ll M/m\,. \tag{21b}$$

Hence, from equations (21a) and (21b), we demand $1 \ll (M/m)^3N^{-2} \ll M/m$.

From equations (19) and (20) we may relate the mean free path at $r_a$ to $r_a$, obtaining $r_a/\lambda \sim (M/m)N^{-2} \ll 1$, where the inequality follows from equation (20).

From the functional dependence on $r$ of $t_d$ and $t_r$ (cf. eqs. [2], [9], [20], and [21]), we may express the ratio of dynamical to relaxation time scales as a function of $r < r_a$ or $|E| > |E_a|$:

$$t_d/t_r \sim (M/m)N^{-2}(E_a/E)^{3/2-p} \sim (M/m)N^{-2}(r/r_a)^{3/2-p} \, . \tag{22}$$

The flux $F_{\max}(E_a)$ satisfies

$$F_{\max}(E_a) \sim \mathcal{N}/t_r(r_a) \sim (M/Nm)^3(GNm/R^3)^{1/2} \, . \tag{23}$$

We now evaluate the dimensionless parameter $q$ and the critical radius $r_{\text{crit}}$ at which $q = 1$. From equations (11), (17b), and (22), and the above relations, we have

$$q \sim (M/m)N^{-2}(r_a/r_t)(r/r_a)^{5/2-p} \, , \quad r < r_a \, , \tag{24a}$$

$$\sim (M/m)N^{-2}(r_a/r_t)(r/r_a)^3 \, , \qquad r > r_a \, . \tag{24b}$$

To solve for $E_{\text{crit}}$ and $r_{\text{crit}}$, we set $q = 1$ and $p \approx \frac{1}{4}$:

$$E_a/E_{\text{crit}} \sim r_{\text{crit}}/r_a \sim [(r_t/r_a)(m/M)N^2]^{4/9} \, , \quad r_{\text{crit}} < r_a \, , \tag{25a}$$

$$\sim r_{\text{crit}}/r_a \sim [(r_t/r_a)(m/M)N^2]^{1/3} \, , \quad r_{\text{crit}} > r_a \, . \tag{25b}$$

If star consumption at $r_t$ is caused by ordinary tidal breakup due to mass $M$, then $r_t$ is given by

$$r_t \sim r_{\text{tidal}} \sim (M/\rho)^{1/3} \, , \tag{26a}$$

where $\rho$ is the density of a typical star. As we shall see in § IV, $r_{\text{crit}}/r_a$ might be expected to be in the range $10^{-1}$–$10^2$ for typical examples. Note that the collision radius $r_{\text{coll}}$, defined in § II*b*(ii), has a simple relation to $r_t$ if $r_t = r_{\text{tidal}}$:

$$r_{\text{coll}}/r_{\text{tidal}} \sim (M/m)^{2/3} \, . \tag{26b}$$

From equation (25) we may determine whether $r_{\text{crit}} < r_a$ or $r_{\text{crit}} > r_a$. Substitution of equations (25) and (23) into either expression (15) or (17c) then yields an approximate value for the total consumption rate of stars, $F$.

## III. METHOD AND EQUATIONS

### *a*) *Assumptions*

In this section we consider in detail the distribution and consumption of bound stars around $M$. Following Bahcall and Wolf (1976) we make the following major assumptions in our analysis:

1. A massive object, $M$, is situated at the center of a spherically symmetric, stellar cluster containing a very large number of stars, $N$, in the core.
2. The mass, $M$, is significantly less than the mass of the core of the cluster but much larger than the total mass of stars inside the accretion radius $r_a$.
3. All stars surrounding the hole have the same mass $m$, which is small compared to $M$.
4. Conditions in the central core of the cluster remain constant in time, so that a steady state can be reached in the vicinity of the black hole. The ambient core stars outside $r_a$ satisfy an isothermal distribution.
5. A star is removed whenever it passes within a distance $r_t$ of $M$.

Immediate consequences of the assumption of large $N$ are the following familiar results from stellar dynamic theory:

1*a*. Only a negligible fraction of the stars in the cluster are in binary systems (Spitzer and Hart 1971).

1*b*. The predominant collisional process between stars is local (see § II*a*), two-body, small-angle scattering in a $r^{-1}$ "Coulomb" potential.

1*c*. The dynamical time scale $t_d$ is significantly shorter than the relaxation time scale $t_r$ throughout the cluster core. Thus, in first approximation, the stellar system satisfies Liouville's theorem.

The effect of occasional close encounters between stars, which produces large deflections, is generally much smaller than the effect of more distant ones in spherical clusters without central massive objects (Chandrasekhar 1942; Spitzer 1962). Consequently, close encounters are consistently ignored in this analysis. (See, however, § IV.)

Previous investigators assume that the stellar distribution function is isotropic in velocity space. Assumption (5), however, clearly implies that the distribution function will be anisotropic, since all stars with energy $E$ and angular momentum $J \leq J_{\min}(E) \equiv [2(E + GM/r_t)]^{1/2} r_t$ will pass within the region $r \leq r_t$ in an orbital period. Thus, in general, by assumptions (5) and (1*c*), the distribution function must be written as a function of both $E$ and $J$, $f(\boldsymbol{r}, \boldsymbol{v}) = f(E, J)$.

### b) *Method and Approximations*

To determine the steady-state, stellar distribution function $f(E, J)$, we employ the two-dimensional Fokker-Planck equation in $E$- and $J$-space. The quantities $E$ and $J$ are related to the radial distance $r$, the radial velocity $v_r$, and the transverse velocity $v_t$ by the relations

$$E = \tfrac{1}{2}v_r^2 + \tfrac{1}{2}(J^2/r^2) - M/r\,, \tag{27}$$

$$J = rv_t\,. \tag{28}$$

Throughout the remainder of the paper we employ galactic units, defined by setting $G \equiv 1$. Let $N(E, J, t)$ be the total number of stars at time $t$ with energy and angular momentum in the range $dE$ and $dJ$ about $E$ and $J$, respectively. The quantity $N(E, J, t)$ is related to the distribution function $f(E, J, t)$ by the relation

$$N(E, J, t)dEdJ = \int_V f(E, J, t)dVdv\,, \tag{29}$$

where $dV$ and $dv$ are the volume elements in coordinate and velocity space, respectively. Expressing $dV$ as $4\pi r^2 dr$ and employing equations (27) and (28) to write

$$dv = 2\pi v_t dv_t dv_r = 2\pi \frac{JdJ}{r^2}\frac{dE}{v_r}\,,$$

we obtain the simple result

$$N(E, J, t) = 8\pi^2 J \int f(E, J, t)\frac{dr}{v_r} = 4\pi^2 Jf(E, J, t)P(E)\,, \tag{30}$$

where $P(E) = 2\pi M/(-2E)^{3/2}$ is the orbital period. In writing the last expression in equation (30) we utilize the result that $f$ is nearly uniform around the orbit of a star (assumption [1*c*]). Throughout the paper we will express the stellar distribution interchangeably in terms of $N(E, J, t)$ or $f(E, J, t)$, whichever is more convenient.

Let $P(E, J; \Delta E, \Delta J)$ be the probability per unit $\Delta E$ per unit $\Delta J$ that a star with energy $E$ and angular momentum $J$ undergoes a perturbation $\Delta E$ and $\Delta J$ in time $\Delta t$ via collisions with other stars in the system. Then the number density $N(E, J, t)$ is governed by the relation

$$N(E, J, t) = \int d\Delta E d\Delta J[N(E - \Delta E, J - \Delta J, t - \Delta t)P(E - \Delta E, J - \Delta J; \Delta E, \Delta J)]\,. \tag{31}$$

With the restriction to small deflections and with the condition that $N(E, J, t)$ change sufficiently slowly with $E$, $J$, and $t$, equation (31) may be expanded in powers of $(\Delta E/E)$, $(\Delta J/J)$, and $(\Delta t/t)$, leading in lowest order to the usual Fokker-Planck equation:

$$\begin{aligned}\frac{\partial}{\partial t}N(E, J, t) = &-\frac{\partial}{\partial E}[N(E, J, t)\langle\Delta E\rangle_t] + \frac{1}{2}\frac{\partial^2}{\partial E^2}[N(E, J, t)\langle(\Delta E)^2\rangle_t] - \frac{\partial}{\partial J}[N(E, J, t)\langle\Delta J\rangle_t] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial J^2}[N(E, J, t)\langle(\Delta J)^2\rangle_t] + \frac{\partial}{\partial E\partial J}[N(E, J, t)\langle(\Delta E)(\Delta J)\rangle_t]\,.\end{aligned} \tag{32}$$

The quantities $\langle\Delta E\rangle_t$ and $\langle(\Delta E)^2\rangle_t$ represent the mean values of $\Delta E$ and $(\Delta E)^2$ per unit time as functions of $E$ and $J$, with a similar definition for $\langle\Delta J\rangle_t$, $\langle(\Delta J)^2\rangle_t$ and $\langle(\Delta E)(\Delta J)\rangle_t$. Formally,

$$\begin{aligned}\langle\Delta E\rangle_t &= \int d\Delta E d\Delta J P_t(E, J; \Delta E, \Delta J)\Delta E\,, \\ \langle(\Delta E)^2\rangle_t &= \int d\Delta E d\Delta J P_t(E, J; \Delta E, \Delta J)(\Delta E)^2, \text{ etc.}\,,\end{aligned} \tag{33}$$

where $P_t(E, J; \Delta E, \Delta J)$ now denotes the collision probability per unit time in the interval $\Delta E$ and $\Delta J$.

Following the discussion of Spitzer and Shapiro (1972), we define $\Delta_p E$ and $\Delta_p J$ as the changes of $E$ and $J$ during one orbital period. Let $\langle\Delta_p E\rangle$ and $\langle(\Delta_p E)^2\rangle$ be the mean values of $\Delta_p E$ and $(\Delta_p E)^2$ averaged over many orbits, with an analogous notation for $\langle\Delta_p J\rangle$, $\langle(\Delta_p J)^2\rangle$ and $\langle(\Delta_p E)(\Delta_p J)\rangle$. We then have

$$\langle\Delta E\rangle_t = \langle\Delta_p E\rangle/P(E)\,, \qquad \langle(\Delta E)^2\rangle_t = \langle(\Delta_p E)^2\rangle/P(E)\,, \tag{34}$$

with similar relations for the other coefficients. We may write

$$\langle \Delta_p E \rangle \equiv \epsilon_1 = 2 \int_{r_p}^{r_{ap}} \langle \Delta E \rangle dr/v_r \, ; \tag{35}$$

and, since $\Delta_p E$ is the sum of repeated, uncorrelated collisions with neighboring stars during one orbit,

$$\langle (\Delta_p E)^2 \rangle = \epsilon_1^2 + \epsilon_2^2 \, , \tag{36}$$

where

$$\epsilon_2^2 \equiv 2 \int_{r_p}^{r_{ap}} \langle (\Delta E)^2 \rangle dr/v_r \, . \tag{37}$$

Thus, $\epsilon_2$ is the dispersion of $\Delta_p E$ about its mean value, $\epsilon_1$. The quantities $r_p$ and $r_{ap}$ are the values of $r$ at pericenter and apocenter, respectively ($v_r^2 = 0$ at $r = r_{ap}$ and $r_p$). The radial dependent coefficients $\langle \Delta E \rangle$ and $\langle (\Delta E)^2 \rangle$ (units of $E$ and $E^2$ per unit time, respectively) are related to the familiar (position dependent) velocity diffusion coefficients $\langle \Delta v_\parallel \rangle$, $\langle (\Delta v_\perp)^2 \rangle$, and $\langle (\Delta v_\parallel)^2 \rangle$ by the relations given in equation (5-24) of Spitzer (1962). For the average value of $\Delta_p J$ we have

$$\langle \Delta_p J \rangle \equiv j_1 = 2 \int_{r_p}^{r_{ap}} r \langle \Delta v_t \rangle dr/v_r \, ; \tag{38a}$$

and, as before,

$$\langle (\Delta_p J)^2 \rangle = j_1^2 + j_2^2 \, , \tag{38b}$$

where $j_2$, the dispersion of $\Delta_p J$, is given by

$$j_2^2 = 2 \int_{r_p}^{r_{ap}} r^2 \langle (\Delta v_t)^2 \rangle dr/v_r \, . \tag{38c}$$

The transverse velocity coefficients $\langle \Delta v_t \rangle$ and $\langle (\Delta v_t)^2 \rangle$ may be related to $\langle \Delta v_\parallel \rangle$, $\langle (\Delta v_\parallel)^2 \rangle$, and $\langle (\Delta v_\perp)^2 \rangle$ by a decomposition of $\boldsymbol{v}$ and $\Delta \boldsymbol{v}$ along $v_t$ and $v_r$ and averaging, as in Spitzer and Shapiro (1972). (Note that the coefficient multiplying the last term in eq. [21] of Spitzer and Shapiro should be $\frac{1}{2}$ instead of 1.) Finally, the cross-product term is given by

$$\langle (\Delta_p E)(\Delta_p J) \rangle = \epsilon_1 j_1 + \zeta^2 \, , \tag{39a}$$

where

$$\zeta^2 = 2 \int_{r_p}^{r_{ap}} r v_t \langle (\Delta v_\parallel)^2 \rangle dr/v_r \, . \tag{39b}$$

We note that ratios $\epsilon_1^2/\epsilon_2^2$, $j_1^2/j_2^2$, and $\epsilon_1 j_1/\zeta^2$ are all comparable to the ratio of dynamical time $t_d$ to the relaxation time $t_r$. By assumption (1*c*), these ratios are all much less than unity and the first terms in equations (36), (38b), and (39a) can therefore be ignored.

If the velocity distribution of the ambient "field stars" which a test star encounters is assumed to be Maxwellian, the diffusion coefficients become (Chandrasekhar 1942; Spitzer 1962)

$$\langle \Delta v_\parallel \rangle / v = -2 A_D l_f^3 G(x)/x \, , \tag{40a}$$

$$\langle (\Delta v_\parallel)^2 \rangle = A_D l_f G(x)/x \, , \tag{40b}$$

$$\langle (\Delta v_\perp)^2 \rangle = A_D l_f [\phi(x) - G(x)]/x \, , \tag{40c}$$

where

$$A_D = 8\pi m^2 n \ln (r/2p_0) \, , \tag{40d}$$

$$l_f^2 = 3/(2v_m^2) \, , \tag{40e}$$

and

$$x = l_f v \, . \tag{40f}$$

The functions $\phi(x)$ and $G(x)$ are tabulated in Spitzer (1962). In the above equations, $n$ and $v_m^2$ are the number density and mean square velocity of (field) stars at $r$ and $p_0 = m/v_m^2$ is the critical impact parameter for a 90° deflection of two stars moving at the mean relative velocity ($=2v_m^2$) at $r$. The argument of the natural logarithm in the above equation reflects the local origin of the predominant two-body encounters [assumption (1*b*)]. The radius $r$ is used here as a convenient, though approximate, cutoff point for the integration over the local stellar distribution. Since $v_m^2 \sim M/r$, we have $r/p_0 \sim M/m \gg 1$.

In our problem, the stellar distribution function is determined by the *mutual* interaction of neighboring stars orbiting the star $M$ in the cluster core. Hence, the "test" star and "field" star distributions are identical. Consequently, the "field" star density and velocity distribution function employed in determining the diffusion coefficients must be consistent with the solution obtained for the "test" star distribution function obtained from the Fokker-Planck equation; in this sense equation (32) is a time-dependent integro-differential equation in two variables. We shall simplify the calculation by adopting the diffusion coefficients computed from a Maxwellian distribution of field-star velocities, but we will determine the mean square velocity $v_m^2(r)$ at each $r$ self-consistently. This represents a mathematical simplification rather than a physical assumption. The *form* of the distribution function as a function of $E$ and $J$, which we determine by solving equation (32), depends essentially on the specific velocity dependence of the scattering cross section in a Coulomb potential. Since this dependence is correctly incorporated in the computation of the adopted velocity diffusion coefficients, our solution for $f(E, J)$ will provide a good first approximation to the exact solution. For example, the distribution function determined for the one-dimensional case in § III*c* gives the same power-law behavior first found by Bahcall and Wolf (1976) (i.e., $f \propto |E|^p$, $p = \frac{1}{4}$), who made no simplifying assumptions regarding the diffusion coefficients. A redetermination of the diffusion coefficients from the computed distribution function followed by a reevaluation of equation (32) might yield an improved result, but we omit here this possible refinement. In regions in which the Fokker-Planck analysis is at all applicable (cf. § IV), our simplifying assumptions regarding the diffusion coefficients will yield the correct functional behavior for the distribution function but will generate errors amounting to factors of $\sim 2$ in the computed consumption rate.

In employing equation (40), we shall adopt a further simplification, first suggested by Spitzer and Hart (1971). Since the functions $\phi(x)/x$ and $[\phi(x) - G(x)]/x$ vary slowly with $x$, we shall set $G(x)/x = 0.167$ for all $x$, equal to its value at $x = (3/2)^{1/2}$, corresponding to $v = v_m$. Following Spitzer and Hart (1971), we then take this same value of 0.167 for $(\phi - G)/2x$ and set

$$\langle(\Delta v_\parallel)^2\rangle = \tfrac{1}{2}\langle(\Delta v_\perp)^2\rangle = 0.167 A_D l_f \tag{41a}$$

and

$$\langle \Delta v_\parallel \rangle / v = -l_f^2 \langle(\Delta v_\parallel)^2\rangle\,, \tag{41b}$$

where the latter relationship ensures that in thermal equilibrium the stellar distribution relaxes to a Maxwellian.

Substituting equations (30) and (34) into (32), we cast the Fokker-Planck equation into the convenient form,

$$P(E)\frac{\partial f}{\partial t} = -\frac{\partial}{\partial E}(f\langle\Delta_p E\rangle) + \frac{1}{2}\frac{\partial^2}{\partial E^2}(f\langle(\Delta_p E)^2\rangle) - \frac{1}{J}\frac{\partial}{\partial J}(fJ\langle\Delta_p J\rangle) + \frac{1}{2J}\frac{\partial^2}{\partial J^2}(fJ\langle(\Delta_p J)^2\rangle) + \frac{1}{J}\frac{\partial^2}{\partial E \partial J}(fJ\langle(\Delta_p E)(\Delta_p J)\rangle)\,. \tag{42}$$

Consistent with our assumptions, § III*a*, the above equation is subject to the following boundary conditions:

i) The function $f(E, J) = 0$ for $J > J_{\max}(E)$, where $J_{\max}(E) \equiv M/(-2E)^{1/2}$ is the angular momentum of a circular orbit at energy $E$.

ii) No stars cross the loss cone at $J = J_{\min}(E) = [2(E + M/r_t)]^{1/2} r_t$ from *lower* angular momentum. For each value of $E$ there is a constant flux of stars $F_E$ (per unit energy interval per unit time) into the loss cone from higher $J$, where

$$F_E = 8\pi^2 \int_0^{J_{\min}} Jf(E, J)\,dJ\,. \tag{43}$$

iii) For $E \geq 0$ (stars unbound to $M$),

$$f(E) = n_a(2\pi\langle v_\parallel^2\rangle)^{-3/2}\exp(-E/\langle v_\parallel^2\rangle)\,, \tag{44}$$

where $n_a$ is the (constant) density of core stars outside the accretion radius $r_a$ and $\langle v_\parallel^2\rangle^{1/2}$ is the velocity dispersion along the line of sight of stars in the cluster core.

In subsequent sections we determine the approximate steady-state solution ($\partial f/\partial t = 0$) to equation (42) with the above boundary conditions.

Because of its complexity, equation (42) (a two-dimensional, integro-differential equation) must be solved numerically. The Monte Carlo technique for analyzing the dynamical behavior of spherical stellar systems recently developed by Spitzer and his co-workers (see Spitzer 1974 for a summary and references) is aptly suited to this task. In this paper, however, we will obtain approximate, analytic solutions to equation (42) by seeking a solution of self-similar form in the region $r_t \ll r \ll r_a$. In § III*c* we analyze the isotropic solution obtained in the limiting case $r_t \to 0$ and $r_a \to \infty$, in which case equation (42) reduces to a one-dimensional problem. In § III*d* we examine the behavior of the low-$J$, high-eccentricity stars for the case in which $r_t \neq 0$ and $r_a \to \infty$, and determine for each value of $E$ the capture rate $F_E$ and corresponding distribution function for $J/J_{\max}(E) \ll 1$. In § III*e* we utilize the results obtained in §§ III*c* and III*d* to obtain an approximate solution to the full two-dimensional equation for the region $r_t \ll r \ll r_a$ with $r_t \neq 0$. These analytic results provide a reasonable first approximation to the solution of the general problem and serve as a useful guide for future, more precise numerical work.

### *c) The One-Dimensional Problem with* $r_t \to 0$

In this section we determine the solution for the limiting case in which the consumption radius $r_t$ goes to zero, $J_{\min}(E)$ goes to zero, and the ambient isothermal region moves to infinity. The solution obtained for the distribution function in this limit will provide insight into the more general problem with $r_t \neq 0$ in the self-similar regime $r_t \ll r \ll r_a$.

In this case we assume that the distribution function is isotropic in velocity space and can be written as a function of just $E$ and $t$ (Peebles 1972*b*; Bahcall and Wolf 1976). Multiplying equation (42) by $J$ and integrating the resulting equation over all angular momentum [using the result that $f(E, t) = 0$ for $J > J_{\max}(E)$], we obtain the expression

$$\frac{\partial N(E, t)}{\partial t} = -\frac{\partial}{\partial E}[N(E, t)\langle\overline{\Delta E}\rangle_t] + \frac{1}{2}\frac{\partial^2}{\partial E^2}[N(E, t)\langle(\overline{\Delta E})^2\rangle_t] \equiv -\frac{\partial F(E)}{\partial E} \tag{45}$$

where $F(E)$ is the star flux (stars per unit time) and where, from equation (30),

$$N(E, t) = \int_0^{J_{\max}(E)} N(E, J, t)dJ = 2\pi^2 J_{\max}{}^2(E)P(E)f(E, t)\,. \tag{46}$$

In equation (45), the $J$-averaged, isotropic energy diffusion coefficients are defined by the relations

$$\langle\overline{\Delta E}\rangle_t = \int_0^{J_{\max}(E)} JdJ\langle\Delta_p E\rangle[P(E)J_{\max}{}^2(E)/2]^{-1}\,, \tag{47a}$$

and

$$\langle(\overline{\Delta E})^2\rangle_t = \int_0^{J_{\max}(E)} JdJ\langle(\Delta_p E)^2\rangle[J_{\max}{}^2(E)P(E)/2]^{-1}\,, \tag{47b}$$

and are functions of $E$ and $t$ alone. In equilibrium, $\partial N/\partial t = 0 = \partial f/\partial t$, and equation (45) yields

$$F(E) = N(E)\langle\overline{\Delta E}\rangle_t - \frac{1}{2}\frac{\partial}{\partial E}[N(E)\langle\overline{\Delta E})^2\rangle_t] = \text{const.} \tag{48}$$

Following Peebles (1972*b*) we search for a steady-state solution of the form

$$f(E) = K|E|^p\,, \tag{49}$$

where $K$ and $p$ are constants. The self-similar, power-law solution immediately implies

$$N(E) = 2\pi^2 J_{\max}{}^2(E)P(E)f(E) = 2^{-1/2}\pi^3 M^3 K|E|^{p-5/2}\,, \tag{50a}$$

$$v_m{}^2(r) = \left(\frac{6}{5+2p}\right)\frac{M}{r}\,, \qquad -\langle E\rangle = \left(\frac{2+2p}{5+2p}\right)\frac{M}{r}\,, \tag{50b}$$

and

$$n(r) = \pi 2^{5/2}\left(\frac{M}{r}\right)^{3/2+p} KB(p+1, 3/2) \equiv n_a\left(\frac{r_a}{r}\right)^{3/2+p}\,, \tag{50c}$$

where $B(m, n)$ is the standard beta function of index $m$ and $n$, and $n_a$ is the stellar density at the accretion radius $r_a$. Employing equations (35), (36) (with $\epsilon_1^2 \ll \epsilon_2^2$), (37), equation (5-24) of Spitzer (1962), and equations (40) and (50), we find for the energy diffusion coefficients

$$\langle\overline{\Delta E}\rangle_t = \frac{16An_a}{\pi}\left(\frac{r_a}{M}\right)^{p+3/2}\left(\frac{5+2p}{3}\right)\left[B(5/2-p, 3/2) - \left(\frac{2+2p}{5+2p}\right)B(3/2-p, 3/2)\right]|E|^{p+1} \tag{51a}$$

and

$$\langle(\overline{\Delta E})^2\rangle_t = \frac{16An_a}{\pi}\left(\frac{r_a}{M}\right)^{p+3/2}(\tfrac{4}{3})B(1/2-p, 5/2)|E|^{p+2}, \tag{51b}$$

where

$$A = \left(\frac{3}{2}\right)^{3/2}\left(\frac{5+2p}{6}\right)^{1/2}(0.167)8\pi m^2 \ln\left(\frac{3}{5+2p}\frac{M}{m}\right). \tag{51c}$$

In deriving equations (51a, b), it is convenient to reverse the $J$ and $r$ integrations implicit in equations (47a, b). Substitution of equations (50) and (51) into equation (48) yields (cf. eq. [4])

$$F(E) = -F_{\max}(E_a)\left(\frac{E}{E_a}\right)^{2p-3/2}\beta(p) = \text{const.}, \tag{52a}$$

$$F_{\max}(E_a) = 2An_a^2\pi M^3|E_a|^{-9/2}, \quad (E_a \equiv -M/r_a), \tag{52b}$$

$$\beta(p) = \frac{5(1-4p)}{4(3-p)}\frac{\Gamma(1/2-p)\Gamma(p+5/2)}{\Gamma(3-p)\Gamma(p+1)}, \tag{52c}$$

where

$$\Gamma(x) \equiv \int_0^\infty e^{-t}t^{x-1}dt.$$

For a net ingoing star flux satisfying the boundary conditions, we require $F(E) \leq 0$. To achieve $F(E) = $ constant, we must have either (i) $(E/E_a)^{2p-3/2} = $ const., hence $p = \frac{3}{4}$, or (ii) $\beta(p) \equiv 0$, giving $p = \frac{1}{4}$. The first root, obtained by Peebles (1972*b*), must be rejected because it leads to a huge *negative* star flux, as first pointed out by Bahcall and Wolf (1976). In fact, since $\Gamma(x) \to +\infty$ for $x \leq 0$, $\beta(p) \to -\infty$ when $3 > p > \frac{1}{2}$. Thus the appropriate solution is the second root, first obtained by Bahcall and Wolf (1976):

$$p = \tfrac{1}{4}, \quad f = K|E|^{1/4}, \quad n(r) = n_a(r_a/r)^{-7/4}. \tag{52d}$$

A time-dependent numerical solution to the Fokker-Planck equation by Bahcall and Wolf (1976) relaxes to the above root.

Although the inward particle flux $F \to 0$ as $r_t \to 0$ when $p = \frac{1}{4}$, the outward energy flux $dE/dt$ arising from consumption near the origin remains finite (see § II*a*, eq. [8]). In the absence of consumption, $F = 0 = dE/dt$ and the appropriate solution is the isothermal, Maxwell-Boltzmann distribution function, $f(E) = C\exp(-3E/v_m^2)$, where $v_m^2$ is a constant independent of $r$, and $n(r) \propto \exp(3M/rv_m^2)$. It is straightforward to show that the Boltzmann distribution indeed satisfies equation (48), since for constant $v_m^2$, equations (47a, b) give

$$|E|^{5/2}\frac{d}{d|E|}[\langle(\overline{\Delta E})^2\rangle_t|E|^{-5/2}] = 2\langle(\overline{\Delta E})\rangle_t + \frac{3\langle(\overline{\Delta E})^2\rangle_t}{v_m^2}. \tag{52e}$$

Substitution of equation (52e) into (48), together with equation (46), yields the desired result. Since this distribution implies huge densities near the center of the cluster, it cannot apply at all when consumption exists (Peebles 1972*b*).

We note finally that with $p = \frac{1}{4}$, $\langle\overline{\Delta E}\rangle_t = 0$. Thus, while stars still random-walk in energy space [i.e., $\langle(\overline{\Delta E})^2\rangle_t \neq 0$], they are not driven preferentially inward by dynamical friction.

### *d*) *The Loss-Cone Capture Rate for $r_t \neq 0$*

In this section we determine the capture rate of stars that are scattered into the loss cone [$J \leq J_{\min}(E)$] and are subsequently removed at $r = r_t \neq 0$. We are primarily concerned with the low angular momentum stars deep in the potential well of $M$ with $J$ only slightly above $J_{\min}(E)$ and $r/r_a \ll 1$. Although changes of both $E$ and $J$ during an orbital period are relatively small, a small change in $J$ can lead to capture while a small change in $E$ produces no significant effect. To determine the stellar consumption rate, diffusion in $E$-space is comparatively unimportant for the low angular momentum stars, and we may ignore derivatives with respect to $E$ in the Fokker-Planck equation, so that equation (42) becomes

$$P(E)J\frac{\partial f}{\partial t} = -\frac{\partial}{\partial J}(fJ\langle\Delta_p J\rangle) + \frac{1}{2}\frac{\partial^2}{\partial J^2}(fJ\langle(\Delta_p J)^2\rangle) . \tag{53}$$

Assuming a steady-state, equation (53) gives

$$F_E = N(E, J)\langle\Delta J\rangle_t - \frac{1}{2}\frac{\partial}{\partial J}[N(E, J)\langle(\Delta J)^2\rangle_t] , \tag{54}$$

where $F_E$ equals the flux of stars (per unit energy interval per unit time) into the loss cone from higher angular momentum and is independent of $J$. Thus $F_E = F_E(E)$.

Since the diffusion of the low-$J$ stars into the loss cone results from encounters primarily with stars possessing higher angular momentum [i.e., "typical" stars with $J \gg J_{\min}(E)$], we may adopt the distribution function $f = K|E|^p$ in a first approximation in evaluating the "field" star parameters, i.e., $n(r)$ and $v_m(r)$, in the diffusion coefficients. From equations (38) (with $j_1^2 \ll j_2^2$), equations (20) and (21) of Spitzer and Shapiro (1972), and equations (40) and (50), we obtain

$$\langle\Delta J\rangle_t = \frac{j_1}{P(E)} = \frac{An_a}{\pi}\left(\frac{r_a}{M}\right)^{p+3/2} B(5/2 - p, 1/2)|E|^{p-1}/J \tag{55}$$

and

$$\langle(\Delta J)^2\rangle_t \approx j_2^2/P(E) = 2J\langle\Delta J\rangle_t . \tag{56}$$

In deriving the quantity $j_1$ given above for the low-$J$ stars, we have neglected the contribution from the first term in the expression for $\langle\Delta v_t\rangle$ given by equation (20) of Spitzer and Shapiro (1972). For the high-$J$ stars, $J \lessapprox J_{\max}(E)$, this term serves to isotropize the distribution function in velocity space. By ignoring the term for stars satisfying $J \ll J_{\max}(E)$, however, we are making an error only of order

$$(v_t/v_m)^2 \sim [J/J_{\max}(E)]^2 \ll 1 .$$

Substituting equations (55) and (56) into equation (54) and integrating, we obtain

$$N(E, J) = -\frac{2F_E P(E)}{j_2^2} J \ln J + CJ , \tag{57a}$$

where $C$ is an arbitrary constant. Solving for $C$ in terms of $N(E, J_{\min})$, we may re-write equation (57a) in the form

$$N(E, J) = N(E, J_{\min})\frac{J}{J_{\min}}\left[1 - \frac{2F_E P(E) J_{\min}}{j_2^2 N(E, J_{\min})}\ln(J/J_{\min})\right] . \tag{57b}$$

The flux $F_E$ can be determined from the "surface" boundary condition in $J$-space, namely, that stars with energy $E$ scattered into the range $0 \leq J \leq J_{\min}(E)$ are removed in an orbital period. Employing the probability distribution function for the perturbations $\Delta_p E$ and $\Delta_p J$, $P(E, J; \Delta_p E, \Delta_p J)$, we may write

$$-F_E = \int_{J_{\min}(E)}^{J_{\max}(E)} dJ\frac{N(E, J)}{P(E)}\left\{\int_{-J}^{-[J - J_{\min}(E)]} d\Delta_p J \int_{-\infty}^{\infty} d\Delta_p E P(E, J; \Delta_p E, \Delta_p J)\right\} . \tag{58}$$

We define

$$P(E, J; \Delta_p J) \equiv \int_{-\infty}^{\infty} d\Delta_p E P(E, J; \Delta_p E, \Delta_p J) ,$$

where $P(E, J; \Delta_p J)$ is the probability per unit $\Delta_p J$ that a star undergoes a change in angular momentum from $J$ to $J + \Delta_p J$ in an orbital period. We shall assume that the dispersion, $j_2$, in $\Delta_p J$ about its mean value, $j_1 \ll j_2$, is Gaussian, which gives

$$P(E, J; \Delta_p J) = \frac{1}{(2\pi)^{1/2} j_2}\exp[-(\Delta_p J)^2/2j_2^2]\left\{\frac{J + \Delta_p J}{j_2 \exp[-(J^2/2j_2^2)]/(2\pi)^{1/2} + \frac{1}{2}J[1 + \operatorname{erf}(J/2^{1/2}j_2)]}\right\} . \tag{59}$$

The term preceding the curly brackets in equation (59) gives the (Gaussian) probability of undergoing a step size $\Delta_p J$ in angular momentum from $J$ to $J + \Delta_p J$ in an orbital period; the numerator of the term inside the curly brackets expresses the fact that the number of angular momentum states between $J' = J + \Delta_p J$ and $J' + dJ'$ is proportional to $J'$; the denominator ensures the normalization condition $\int_{-J}^{\infty} d\Delta_p J P(E, J; \Delta_p J) = 1$. Substituting equation (59) into equation (58), reversing the order of integration, and assuming $J_{\max}(E) \gg [J_{\min}(E), j_2]$ (i.e., $E \gg -M/r_t$), we obtain the dimensionless integral

$$-F_E = \frac{J_{\min}}{\pi^{1/2}P(E)}\int_0^{\infty} \exp(-x^2)dx\left[\left[\int_{\max(1,\sqrt{2q}x)}^{\sqrt{(2q)}x+1} dy N(E, yJ_{\min}) \frac{1-(2q)^{1/2}x/y}{(q/2\pi)^{1/2}\exp[-(y^2/2q)]/y + \frac{1}{2}\{1+\mathrm{erf}[y/(2q)^{1/2}]\}}\right]\right], \tag{60}$$

where $q \equiv j_2^2/J_{\min}^2$. The above equation yields the following limiting expressions:

$$-F_E = \frac{j_2 N(E, J_{\min})}{(2\pi)^{1/2}P(E)}, \qquad q \ll 1,$$

$$= \frac{gJ_{\min}N(E, J_{\min})}{P(E)}, \quad q \gg 1; \tag{61}$$

$$g = \frac{1}{2}\int_0^{\infty} dy \frac{e^{-y}}{\{e^{-y} + (\pi y)^{1/2}[1 + \mathrm{erf}(y^{1/2})]\}} \approx 0.180. \tag{62}$$

In obtaining the above expression in the limit $q \gg 1$, we have assumed that the distribution function is nearly isotropic for $J_{\min} \leq J \leqslant j_2$. Equation (57) is not strictly applicable for this range of angular momentum in this limit, since the random-walk of stars into the loss cone cannot be described by a diffusion equation (i.e., the Fokker-Planck equation). Rather, equation (31) must be solved directly for $N(E, J)$, as sketched in the Appendix. In the "pinhole" limit, $q \gg 1$, the distribution is even more isotropic than in the "diffusion" limit, $q \ll 1$, since entrance into the loss cone does not require a density gradient in the angular momentum distribution. Inserting equation (61) into equation (57b), we find

$$N(E, J) = N(E, J_{\min})\frac{J}{J_{\min}}\left[1 + \left(\frac{2}{\pi}\right)^{1/2} q^{-1/2}\ln\left(\frac{J}{J_{\min}}\right)\right], \quad q \ll 1,$$

$$= N(E, J_{\min})\frac{J}{J_{\min}}\left[1 + (2g)q^{-1}\ln\left(\frac{J}{J_{\min}}\right)\right], \qquad q \gg 1. \tag{63}$$

Equation (63) indicates that as a result of consumption the distribution function deviates from isotropic behavior. However, the deviation is not drastic, since it is described by a slowly varying (logarithmic) function of $J$. For the high-$J$ stars with $J_{\min}(E) \ll J \leqslant J_{\max}(E)$, the omitted terms in the velocity diffusion coefficients drive the distribution function even closer to isotropy. Defining

$$N(E) = \int_{J_{\min}(E)}^{J_{\max}(E)} N(E, J)dJ,$$

and approximating $N(E, J)$ by equation (63) even for the high-$J$ stars, we obtain

$$N(E) \approx N(E, J_{\min})J_{\min}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{16}\left(\frac{M}{r_t|E|}\right)q^{-1/2}\ln\left(\frac{1}{4}\frac{M}{r_t|E|}\right), \quad q \ll 1,$$

$$\approx N(E, J_{\min})J_{\min}\frac{1}{8}\left(\frac{M}{r_t|E|}\right), \qquad q \gg 1. \tag{64}$$

The factor of 4 in the logarithm term in the case $q \ll 1$ above is not reliable due to the above approximation. Equations (63) and (61) then give, finally,

$$-F_E = \left[\frac{8|E|r_t}{M\ln(M/4|E|r_t)}\right]q\frac{N(E)}{P(E)}, \quad q \ll 1,$$

$$= \left(\frac{8|E|r_t}{M}\right)g\frac{N(E)}{P(E)}, \qquad q \gg 1. \tag{65}$$

Equation (65) relates the capture rate in each energy interval $dE$ about $E$ to the total number of stars in that interval. The number density $N(E)$ is determined by solving the full two-dimensional Fokker-Planck equation (see § III*e*).

*e) Approximate Solution of the Two-Dimensional Fokker-Planck Equation*

As discussed in the previous section, the distribution function $f(E, J)$ in the range $J_{\min}(E) \ll J \lesssim J_{\max}(E)$ is essentially isotropic. Consequently, it is appropriate to search for a steady-state solution to the general Fokker-Planck equation in which the distribution function again has the power-law form $f(E) = K|E|^p$. A solution obeying this self-similar form will again apply only in the range $r_t \ll r \ll r_a$, in which case $-M/r_t \ll \langle E \rangle \ll -M/r_a$ and $J_{\max}(E) \gg J_{\min}(E)$. However, the analysis will correctly account for the consumption of bound stars moving into a finite loss cone in $E$- and $J$-space.

Employing equations (55) and (56), we may write for the dimensionless parameter $q(E) \equiv j_2{}^2/J_{\min}{}^2$ which appears in the capture rate,

$$q(E) = \frac{J\langle \Delta J \rangle_t P(E)}{(E + M/r_t)r_t{}^2}$$

$$= 3^{3/2}\left(\frac{10 + 4p}{3}\right)^{1/2} \pi(0.167) B(\tfrac{5}{2} - p, \tfrac{1}{2}) \ln\left(\frac{3}{5 + 2p}\frac{M}{m}\right) \frac{m^2}{M^2} n_a r_a{}^3 \left(\frac{r_a}{r_t}\right) \left|\frac{E r_a}{M}\right|^{p - 5/2} . \tag{66}$$

Substituting equation (66) into equation (65), which gives the dependence of $F_E$ on $E$, we can then solve the Fokker-Planck equation. Multiplying equation (42) by $J$, integrating from $J_{\min}(E)$ to some arbitrary $J > J_{\max}(E)$, and assuming $J_{\max}(E) \gg J_{\min}(E)$, we find

$$\frac{\partial N(E, t)}{\partial t} = -\frac{\partial}{\partial E}[N(E, t)\langle(\overline{\Delta E})\rangle_t] + \frac{1}{2}\frac{\partial^2}{\partial E^2}[N(E, t)\langle(\overline{\Delta E})^2\rangle_t] + N(E, J_{\min}, t)\langle \Delta J \rangle_t$$

$$- \frac{1}{2}\frac{\partial}{\partial J}[N(E, J_{\min}, t)\langle(\Delta J)^2\rangle_t] - \frac{\partial}{\partial E}\left[N(E, J_{\min}, t)\frac{\langle \Delta_p E \Delta_p J \rangle}{P(E)}\right] , \tag{67}$$

where

$$N(E, t) = \int_{J_{\min}}^{J_{\max}} N(E, J, t) dJ ,$$

and where the diffusion coefficients are defined in equations (51), (55), (56), and (39a, b). Since $\langle \Delta_p E \Delta_p J \rangle \sim J\epsilon_2{}^2/E \sim J\epsilon_2(t_d/t_r)^{1/2}$, $\langle \overline{\Delta E} \rangle_t \sim \epsilon_1/P(E)$, and $N(E, J_{\min}) \sim N(E)J_{\min}/J_{\max}{}^2$, the ratio of the last term to the first term on the right-hand side of equation (67) is of order $J_{\min}/J_{\max} \ll 1$. Consequently we ignore the cross-product term. Substituting equation (54) into (67), we obtain

$$\frac{\partial}{\partial t} N(E, t) = -\frac{\partial}{\partial E}[N(E, t)\langle \overline{\Delta E} \rangle_t] + \frac{1}{2}\frac{\partial^2}{\partial E^2}[N(E, t)\langle(\overline{\Delta E})^2\rangle_t] + F_E , \tag{68}$$

which, in steady state, reduces to the simple equation

$$F(E) \equiv N(E)\langle \overline{\Delta E} \rangle_t - \frac{1}{2}\frac{\partial}{\partial E}[N(E)\langle(\overline{\Delta E})^2\rangle_t] = \int_{E_t}^{E} F_E dE + F(E_t) , \tag{69}$$

where $F(E_t)$ is an arbitrary integration constant. Equation (68) differs from equation (45), appropriate when disruption occurs only at the origin, by the addition of a "sink term" on the right-hand side. The two equations are identical when $r_t \to 0$.

For the case $q \ll 1$, we substitute equations (65), (66), and the expression for $F(E)$ given by equation (52) into equation (69), obtaining

$$F_{\max}(E_a)\left(\frac{E}{E_a}\right)^{2p - 3/2} \bar{\beta}(p) = \text{const.} , \tag{70}$$

where

$$\bar{\beta}(p) = \frac{\Gamma(p + 5/2)\Gamma(1/2 - p)}{\Gamma(p + 1)\Gamma(3 - p)}\left[\frac{5(1 - 4p)}{4(3 - p)} + \frac{(1 - 2p)(3 - 2p)}{4(3/2 - 2p)\ln(M/4|E|r_t)}\right] .$$

To ensure a net inward flux of stars, we again require $\bar{\beta}(p) = 0$ as the appropriate solution to equation (70), yielding the following cubic equation for $p$:

$$p = \tfrac{1}{4} - (\tfrac{1}{2} - p)\left[\frac{(\frac{1}{4} - p)}{(3 - p)} + \frac{(3/2 - p)}{4(\frac{3}{4} - p)\ln(M/4|E|r_t)}\right] , \quad q \ll 1 ,$$

$$\approx \frac{1}{4} - \frac{11}{64}\frac{1}{\ln(M/4|E|r_t)} , \quad |E| \ll M/r_t . \tag{71}$$

The above (cubic) equation for $p$ gives $p$ as a slowly varying function of $E$. For $|E| \ll M/r_t$, $p \to \frac{1}{4}$ as before; as $|E|$ increases, $p$ decreases below $\frac{1}{4}$. The value of $p$ is tabulated as a function of $M/r_t|E|$ in Table 1 for $q \ll 1$. The slow, logarithmic deviation of $p$ from a constant justifies the approximate power-law behavior of the distribution function we have assumed. The error in our calculated $p$ is largest at large $|E|$ and small $r$, where $p$ varies most rapidly, and is $\lesssim 10\%$ at $|E| \sim 1/10|E_t|$ or $r \sim 10r_t$. This results in a corresponding error of $\lesssim 1\%$ in the exponent appearing in the density profile below.

For the case $q \gg 1$, we obtain similarly

$$p = \tfrac{1}{4} + \mathcal{O}\left(\frac{1}{q}\right), \quad q \gg 1\,. \tag{72}$$

Since $p$ is monotonic and slowly varying with $q$, equations (71) and (72) indicate that $p$ is very near $\frac{1}{4}$ for $q$ near unity.

The stellar density in the region $r > r_t$ may be expressed approximately as

$$n(r) \approx n_a[1 + (r_a/r)^{3/2+p(r)}]\,, \tag{73}$$

where $p(r)$ is obtained from equation (71) by substituting $\langle E \rangle = [(2 + 2p)/(5 + 2p)]M/r$ for $E$ in the logarithm. The above expression, up to numerical factors, yields the correct asymptotic behavior in the regions $r \ll r_a$ and $r \gg r_a$. See discussion of result (4) of § I. The value of $p(r)$ is tabulated as a function of $r/r_t$ in Table 1. For $r \gg r_t$, $n(r) \propto r^{-7/4}$ as before, but as $r$ approaches $r_t$, the density increases less rapidly. This behavior reflects the decrease in the incident stellar flux as $r$ decreases, resulting in a depletion on orbiting stars.

The parameter $q(E)$ given in equation (66) may be evaluated for $p = \frac{1}{4}$, yielding

$$q = (6.25) \ln\left(\frac{6}{11}\frac{M}{m}\right)\left(\frac{m}{M}\right)^2 (n_a r_a^3)(r_a/r_t)|E r_a/M|^{-9/4}\,. \tag{74}$$

In the event $q(E_a) < 1$ ($r_{\rm crit} > r_a$), equation (17b) is the appropriate expression for $q$ (see § II*b*[iii]).

The total capture rate of *bound* stars in the system, $F$, may be obtained by integrating equation (65) for $F_E$ over energy. Since $F_E \propto |E|^{-2}$ for $q \ll 1$ and $F_E \propto |E|^{1/4}$ for $q \gg 1$ (cf. Fig. 2), the largest contribution to the stellar capture rate originates from stars with energy $E \sim E_c$, where $E_c = \min(E_a, E_{\rm crit})$. For typical globular cluster parameters $E_a \sim E_{\rm crit}$; but, in general, $q$ need not exceed unity inside $r_a$. We thus find

$$F = \int_{E_t}^{E_a} F_E dE = \frac{2^{1/2}\pi}{B(p+1, 3/2)}\left[\frac{q(E_c)}{(3/2 - 2p)\ln(M/4|E_a|r_t)}\right](r_t/r_a)(n_a r_a^3)(M/r_a^3)^{1/2}|E_c r_a/M|^{p+1}\,. \tag{75}$$

The above expression for $F$ was obtained by substituting the expression for $F_E$ in the limit $q \ll 1$ and is therefore exact whenever $q(E_a) \ll 1$. The appropriate value of $p$ to insert in equation (75) is $p$ evaluated at $E_c$ (see Table 1). If $p(E_c) \approx \frac{1}{4}$, we have

$$F = 9.0\,\frac{q(E_c)}{\ln(M/4|E_c|r_t)}(r_t/r_a)(n_a r_a^3)(M/r_a^3)^{1/2}|E_c r_a/M|^{5/4}\,. \tag{76}$$

In the case $q(E_a) > 1$ ($r_{\rm crit} > r_a$), an approximate expression for $F$, obtained by consideration of the consumption of unbound stars, is given in equation (17c) (see § II*b*[iii]).

## IV. DISCUSSION AND APPLICATIONS

Applications to two types of $N$-body systems—globular clusters and elliptical galaxies—are presented in Table 2. Data for the core of the adopted globular cluster M15 are taken from Peterson and King (1975), who tabulate $n_a$ and $\langle v_\parallel^2 \rangle^{1/2}$. Data for the compact core of an elliptical galaxy are only crude estimates. See Bahcall and Wolf (1976) for a detailed analysis of the globular cluster data.

In evaluating the table we adopt for the consumption radius $r_t$ the tidal disruption radius $r_{\rm tidal}$ given in equation (26), and assume a mean stellar density $\rho \sim 1\ \mathrm{g\ cm^{-3}}$. Compact stars or red giants would have different values of $r_{\rm tidal}$. In the case $r_{\rm crit}/r_a > 1$, the approximate formulae for $r_{\rm crit}$ and $F$ of equations (25b) and (17c) were used in the table; in the case $r_{\rm crit}/r_a < 1$, the more exact expressions given in equations (74) and (76) were used.

As the total number of stars $N$ becomes large, our results derived from the Fokker-Planck equation become asymptotically correct. However, for finite $N$, in the innermost regions of the cusp, the number of stars is small and a Fokker-Planck analysis is no longer applicable, as pointed out by Bahcall and Wolf (1976). Within a radius $r_0 \approx r_a(n_a r_a^3)^{-4/5}$, the cusp solution indicates that there are no stars. For globular clusters $r_* \sim (r_a/10) \times (M/10^3\,M_\odot)^{-12/5}$. Since $r_{\rm crit} \gtrsim r_0$ for this example, the consumption rate obtained from the Fokker-Planck equation is still a good approximation.

In a subsequent paper, we hope to consider in detail (some of) the following relevant topics: (1) A more accurate determination of $F$ and $r_{\rm crit}$ when $r_{\rm crit}/r_a > 1$. (2) An analysis of the effects of gravitational radiation on stars which orbit close to the central mass $M$ (Ostriker 1976). (3) The fate of stars and gas from disrupted stars inside $r_t$ and the nature of the radiation of such gas as it accretes onto $M$. (4) An analysis of the effects of large-angle scattering in the core, which is likely to be significant when the number of stars in each radial interval is small. (5) The stellar distribution and consumption rate of stars when their number inside $r_a$ is not very large. (6) The secular effect of a continual consumption of stars near the center on the dynamical evolution of the cluster as a whole.

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## APPENDIX

As discussed in § III*d*, it is not appropriate to use the Fokker-Planck (diffusion) equation to determine the distribution function in the range $J \lesssim j_2$ when $q = j_2{}^2/J_{\min}{}^2(E) \gg 1$. In this regime we must solve equation (31) directly to obtain $N(E, J)$, since an expansion of this integral equation in powers of $\Delta J/J$ ($\gg 1$) is not valid.

Defining $\Delta_p E \equiv E' - E$ and $\Delta_p J \equiv J' - J$, we may write equation (31) for *arbitrary q* in the form

$$N(E, J, t) = \int dE'dJ'N(E', J', t - P(E))P(E', J'; E' - E, J - J')\,, \tag{A1}$$

where $P(E', J'; \Delta_p E, \Delta_p J)$ is the probability distribution function for the changes $\Delta_p E$ and $\Delta_p J$ and $P(E)$ is the orbital period. In steady-state $N[E', J', t - P(E)] = N(E', J', t)$. If we ignore small changes in $E$ for stars with $J \lesssim j_2$ (see § III*d*), we may write

$$P(E', J'; E - E', J - J') = \delta(E - E')P(E, J; \Delta_p J)\,, \tag{A2}$$

where $P(E, J; \Delta_p J)$ is given by equation (59). We then have

$$N(E, J) = \int_{J_{\min}(E)}^{J_{\max}(E)} dJ'N(E, J') \frac{J' \exp[-(J' - J)^2/2j_2{}^2]}{(2\pi)^{1/2} j_2\{j_2 \exp(-J^2/2j_2{}^2)/(2\pi)^{1/2} + J[1 + \operatorname{erf}(J/2^{1/2}j_2)]/2\}}\,. \tag{A3}$$

It is convenient to define the following quantities:

$$x \equiv (J - J_{\min})/j_2$$

$$y \equiv (J' - J_{\min})/j_2$$

$$N(E, J) = N(E, xj_2 + J_{\min}) \equiv H(x)\,. \tag{A4}$$

Equation (A3) then becomes

$$\{\exp[-(x + q^{-1/2})^2/2] + (\pi/2)^{1/2}(x + q^{-1/2})\{1 + \operatorname{erf}[(x + q^{-1/2})/2]\}\}H(x) = \int_0^\infty dy(y + q^{-1/2}) \exp[-(x - y)^2/2]H(y)\,. \tag{A5}$$

For $q \gg 1$, equation (A5) reduces to

$$\{\exp(-x^2/2) + (\pi/2)^{1/2}x[1 + \operatorname{erf}(x/\sqrt{2})]\}H(x) = \int_0^\infty dyy[\exp -(x - y)^2/2]H(y)\,. \tag{A6}$$

For large $x$ ($J \gg j_2$), a Taylor-series expansion of $H(y)$ may be used in equations (A5) and (A6), and the Fokker-Planck result, equation (57b), will again apply. Employing equation (57b) and (A4), the general solution for arbitrary $x > 0$ can be written in the form

$$H(x) = H(0)[xq^{1/2} + 1][1 + \gamma(x)\ln(xq^{1/2} + 1)]$$

where $\gamma(x)$ is $\approx$ constant for large $x$. From equation (63) we know that $\gamma(x) \equiv (2\pi)^{1/2}q^{-1/2}$ for all $x > 0$ when $q \ll 1$ while $\gamma(x) \sim q^{-1} \ll 1$ when $q \gg 1$. In the latter case, $\gamma(x)$ must be determined by solving equation (A6) for small $x$. Nevertheless, since $\gamma \ll 1$ in this limit, the distribution function is very nearly isotropic, even for low-$J$ stars near the loss cone.

## REFERENCES

Bahcall, J., and Ostriker, J. P. 1975, *Nature*, **256**, 23.
Bahcall, J., and Wolf, R. A. 1976, *Ap. J.*, **209**, 214.
Chandrasekhar, S. 1942, *Principles of Stellar Dynamics* (Chicago: University of Chicago Press).
Fabian, A., Pringle, J., and Rees, M. J. 1975, *M.N.R.A.S.*, **172**, 15p.
Frank, J., and Rees, M. J. 1976, *M.N.R.A.S.*, **176**, 633.
Hills, J. G. 1975, *Nature*, **254**, 295.
Hoyle, F., and Fowler, W. A. 1963*a*, *M.N.R.A.S.*, **125**, 169.
———. 1963*b*, *Nature*, **197**, 533.
Ostriker, J. P. 1976, private communication.
Peebles, P. J. E. 1972*a*, *Gen. Rel. and Grav.*, **3**, 63.
———. 1972*b*, *Ap. J.*, **178**, 371.
Peterson, C. J., and King, I. R. 1975, *A.J.*, **80**, 427.
Shapiro, S. L., and Lightman, A. P. 1976, *Nature*, **262**, 743.
Silk, J., and Arons, J. 1975, *Ap. J.* (*Letters*), **200**, L131.
Spitzer, L. 1962, *Physics of Fully Ionized Gases* (2d ed.; New York: Interscience).
———. 1975, in *IAU Symposium No. 69*, *Dynamics of Stellar Systems*, ed. A. Hayli (Dordrecht: Reidel), p. 3.
Spitzer, L., and Hart, M. H. 1971, *Ap. J.*, **164**, 399.
Spitzer, L., and Shapiro, S. L. 1972, *Ap. J.*, **173**, 529.

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