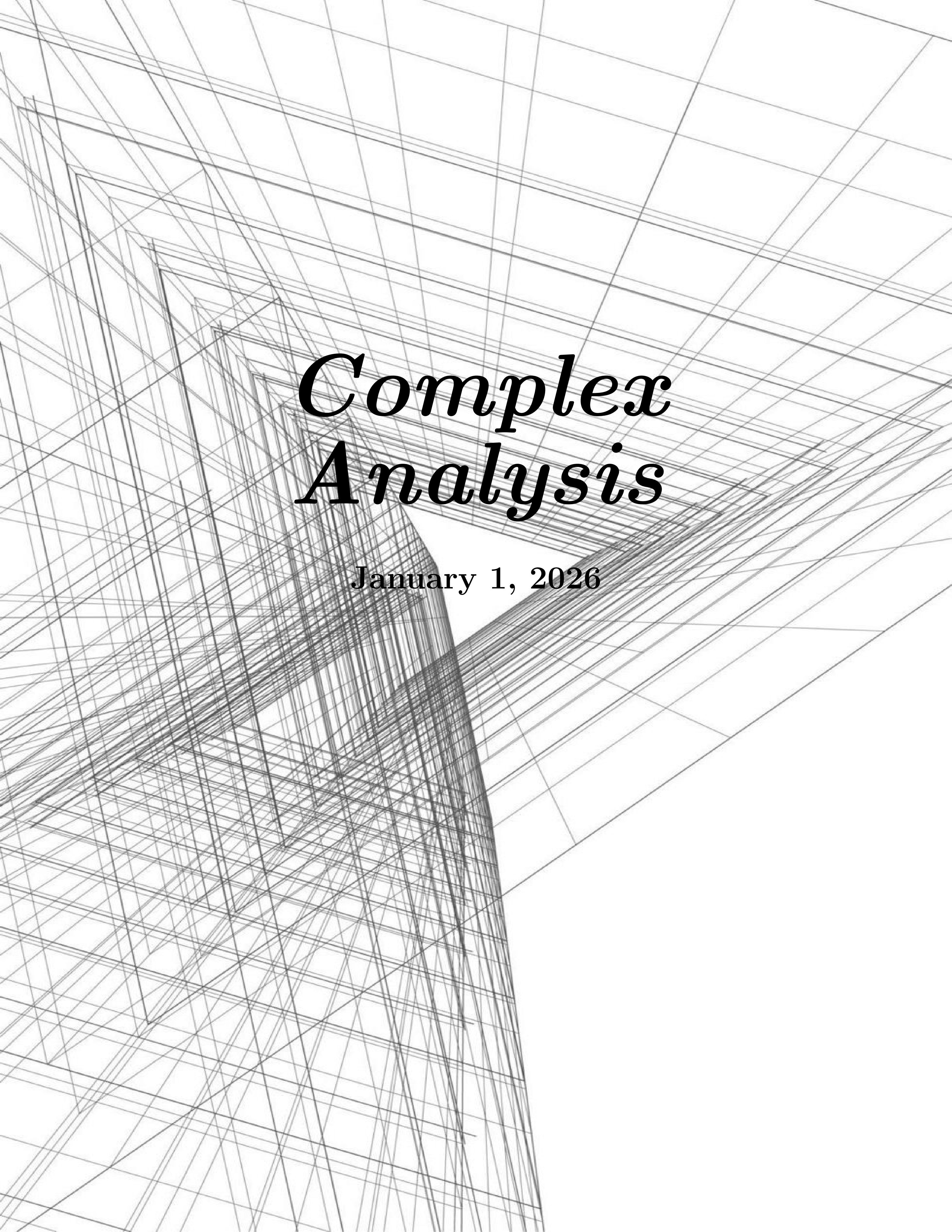

Complex Analysis

January 1, 2026

Contents

Chapter 1

Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

The complex field $\mathbb{C}$ is defined to be $\mathbb{R} \times \mathbb{R}$, with $a + ib$ stands for (a, b). The addition and multiplication is defined as follows.

Definition 1.1.1: Complex Field

The complex field $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$, with addition and multiplication:

- **Addition:** $(a + ib) + (c + id) = (a + c) + (b + d)i.$
- **Multiplication:** $(a + ib)(c + id) = (ac - bd) + (ad + bc)i.$

We can verify that $(\mathbb{C}, +, \cdot)$ indeed is a field. Its additive identity being 0 and multiplicative identity 1.

Remark:
We can also define the complex field in a more abstract way. Showing only properties of the field that has to be matched.
Let $\mathbb{F}$ be a field. Then $\mathbb{C}$ is a subfield of $\mathbb{F}$ that matches the following properties.

- $\mathbb{R}$ is a subfield of $\mathbb{F}$. (isomorphic)
- $\exists \alpha \in \mathbb{F}, \alpha^2 + 1 = 0$. Let one of them be i.
- $\mathbb{C}$ is the subfield generated by $(\mathbb{R}, i)$.

We can prove that $\mathbb{C}$ does not depend on the choice of $\mathbb{F}$ and i. The existence of such a field is constructed by 1.1.1 above.

There are also isomorphic representations of the complex field. For example,

- $\left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \cong \mathbb{C}.$

- $\mathbb{R}[x]/\left(x^2+1\right)\mathbb{R}[x] \cong \mathbb{C}$.

1.1.2 Square Roots

Now let's consider some properties that are special about the complex field. The original intuition of constructing $\mathbb{C}$ is that not all reals have square roots. We denote the square root of -1 to i so as to fill the blank. We now state that square root is closed under $\mathbb{C}$.

Theorem 1.1.1: Square Roots of Complex Numbers

$$\forall a \in \mathbb{C}, \exists b \in \mathbb{C}, a = b^2$$

Proof. We can explicitly find the square roots using the result that every non-negative reals have square roots.

Let $\alpha + i\beta \in \mathbb{C}$, we find

$$(x+iy)^2 = \alpha + i\beta$$

that is,

$$\begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta \end{aligned}$$

Form there we get

$$\left(x^2+y^2\right)^2 = \left(x^2-y^2\right)^2 + 4x^2y^2 = \alpha^2 + \beta^2$$

giving

$$x^2 + y^2 = \sqrt{\alpha^2+\beta^2}$$

that is,

$$\begin{cases} x^2 = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2+\beta^2}\right) \\ y^2 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2+\beta^2}\right) \end{cases}$$

Only two result matches, as we can see.

□

1.1.3 Conjugation and Absolute Value

Define $a - bi$ to be conjugation of $a + bi$. And define

$$\operatorname{Re} a = \frac{a+\overline{a}}{2}, \operatorname{Im} a = \frac{a-\overline{a}}{2i}$$

Theorem 1.1.2: Conjugations

Let $R(a_1, \ldots, a_n)$ be any rational operation applied to $a_1, \ldots, a_n \in \mathbb{C}$, then

$$\overline{R(a_1, \ldots, a_n)} = R\left(\overline{a_1}, \ldots, \overline{a_n}\right) \tag{1.1}$$

Let $z = a + bi$. We denote $|z| = \sqrt{a^2+b^2}$.

Example: **Conjugates and Absolute Values**

- $\left|\dfrac{a-b}{1-\overline{a}b}\right| = 1$.
- Lagrange's Identity:

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1\le i<j\le n} \left|a_i\overline{b_j} - a_j\overline{b_i}\right|^2 \tag{1.2}$$

- $|a_1 + \ldots + a_n| \le |a_1| + \ldots + |a_n|$.

1.2 Geometry Representation of Complex Numbers

The most usual way to visualize complex numbers is putting it real and imaginary part into a coordinate system.

1.2.1 Addition and Multiplication

The addition of complex numbers can be visualized as vector addition.

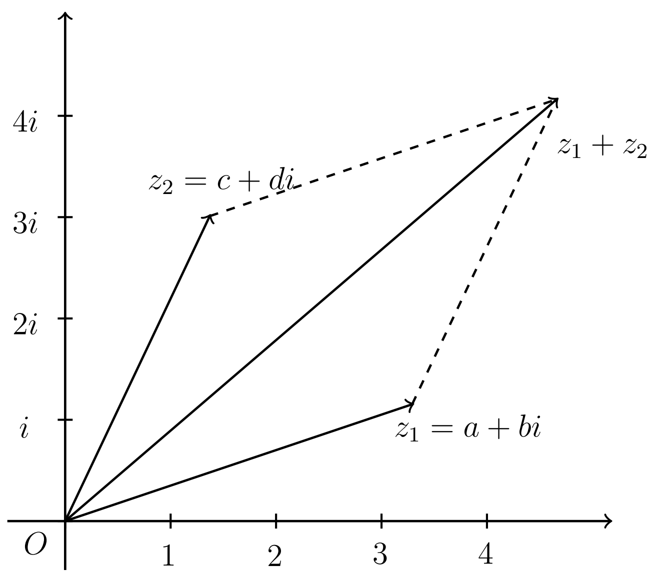

Figure 1.1: Complex Addition

In order to derive a geometric interpretation of multiplication of complex numbers we write it into polar coordinates. If the polar coordinates of point (α, β) is (r, φ), we know

$$\alpha = r\cos\varphi$$
$$\beta = r\sin\varphi$$

Hence we can write $z = \alpha + i\beta = r\left(\cos\varphi + \sin\varphi\right)$, which is the trigonometric form. In this form, we denote

$$r = |z|\,, \text{ and } \varphi = \arg z$$

Multiplying z_1 and z_2 we get

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left(\cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2 + \left(\cos\varphi_1 \sin\varphi_2 + \sin\varphi_1 \cos\varphi_2\right) i\right) \\ &= r_1 r_2 \left(\cos\left(\varphi_1 + \varphi_2\right) + i \sin\left(\varphi_1 + \varphi_2\right)\right) \end{aligned}$$

We see that the product has modulus $r_1 r_2$ and argument $\varphi_1 + \varphi_2$. We latter result is new, for which we can express it by

$$\arg\left(z_1 z_2\right) = \arg z_1 + \arg z_2 \tag{1.3}$$

By means of 1.3 we can produce a geometric construction of products of complex numbers by similar triangles.

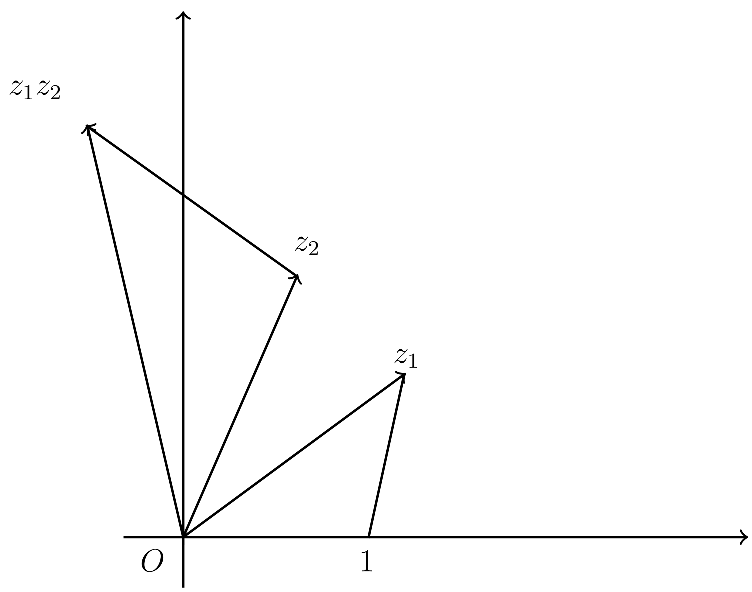

Figure 1.2: Complex Products

Remark:
We've noticed that this definition of argument is fairly intuitive but not very stern. Angle's are not defined yet and we use trigonometric functions. However, a perfect acceptable way is to express angles by the length of an arc, which is defined by definite integral.
However, in complex analysis we do not follow that rule, because we have a more direct route: the connection between exponential functions and trigonometric functions.

1.2.2 Binomial Equation

From the preceding result we derive that the powers of $a = r(\cos\varphi + i\sin\varphi)$ are given by (for $n > 0$)

$$a^n = r^n(\cos n\varphi + i \sin n\varphi) \tag{1.4}$$

This equation hold trivially for $n = 0$ and since

$$a^{-1} = r^{-1}(\cos\varphi - i\sin\varphi) = r^{-1}(\cos(-\varphi) + i\sin(-\varphi))$$

so it holds for all $n \in \mathbb{Z}$.

To solve the n^{th} root of a complex number a, we need to solve the equation

$$z^n = a$$

Suppose $a \neq 0$ and we write $a = r(\cos\varphi + i\sin\varphi)$ and

$$z = \rho(\cos\theta + i\sin\theta)$$

then we have

$$\rho^n(\cos n\theta + i\sin b\theta) = r(\cos\varphi + i\sin\varphi)$$

thus we have $\rho^n = r$ and $n\theta = \varphi + 2k\pi$.

$$z = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + k\frac{2\pi}{n}\right)\right)$$

where $k = 0, \ldots, n-1$. Therefore, we get

Theorem 1.2.1: Root of Complex Numbers

There are n n^{th} root of any complex number $\neq 0$, they have the same modulus and there arguments are equally spaced

Geometrically, the n n^{th} root forms a regular n-polygon.

Definition 1.2.1: Roots of Unity

The roots of $z^n = 1$ are called n^{th} root of unity. That is, if we set

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \tag{1.5}$$

then $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the roots of unity.

1.2.3 Analytic Geometry

In analytic geometry, points of a 2D plane can be represented by both x, y or $z, \overline{z}$.

For instance, the equation of a circle is $|z - a| = r$ or $(z - a)(\overline{z} - \overline{a}) = r^2$.

A straight line can be expressed as parametric equation $z = a + bt$ where $a, b \in \mathbb{C}$ and the parameter $t \in \mathbb{R}$. The angle between $z = a + bt$ and $z = a' + b't$ are just $\arg\frac{b}{b'}$.

1.2.4 The Spherical Representation

For many purposes it is useful to extent the system $\mathbb{C}$ to include the symbol ∞. We define $a + \infty = \infty + a = \infty$ for all $a \in \mathbb{C}$ and $b \cdot \infty = \infty \cdot b = \infty$ for all $b \neq 0$.

In the plane there is no room for the infinity, but we can introduce an ideal point called the point at infinity. The points in the plane and the point at infinity together constructed *The Extended Complex Plane*. To give a more intuitive geometric model, we consider S^2, which is the unit sphere in 3-dimensional space, $x_1^2 + x_2^2 + x_3^2 = 1$.

We have the 1-1 correspondence by stereographic projection.

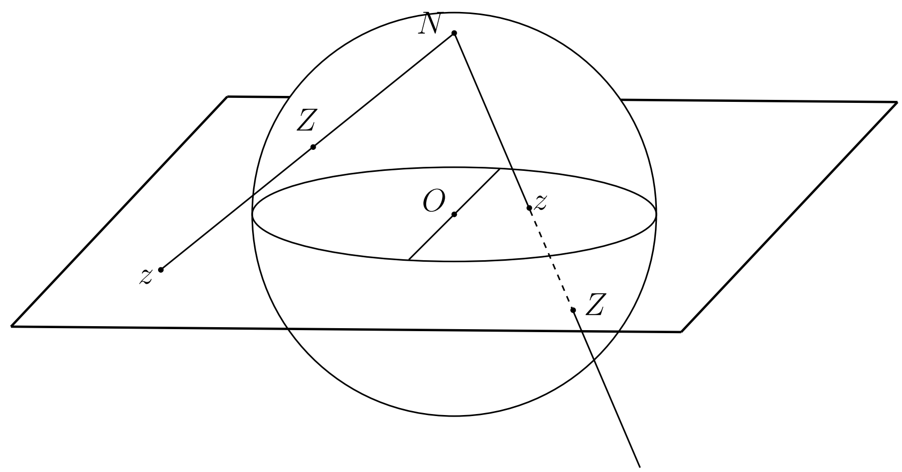

Figure 1.3: Stereographic Projection

With every point on S, except $(0, 0, 1)$, we associate a complex number

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

then we have

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}$$

hence

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

and further computation leads to

$$x_1 = \frac{z + \overline{z}}{1 + |z|^2}$$
$$x_2 = \frac{z - \overline{z}}{i(1 + |z|^2)}$$

An we let the point ∞ corresponds to $(0, 0, 1)$.

Remark:

The correspondence can be derived informally by similar triangles. We have $x_3 = \cos 2\theta$ and

$|z| = \cot\theta$. Thus $|z|^2 = \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1+x_3}{1-x_3}$

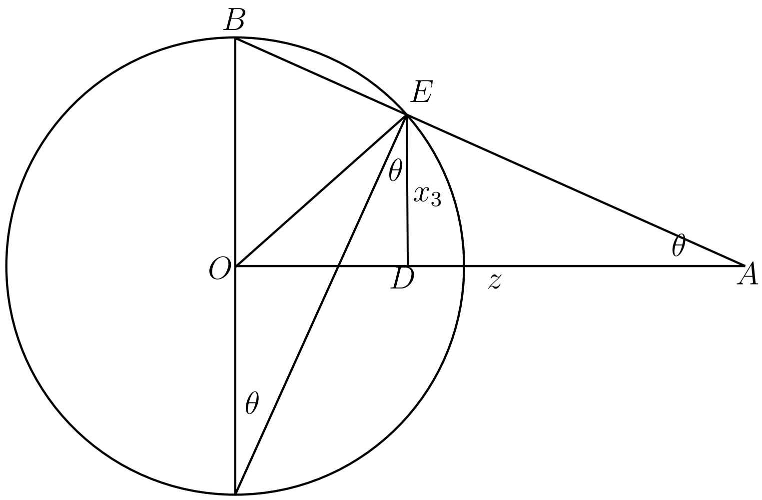

Figure 1.4: Derivation of Sphere

It is easy to calculate the distance $d(z, z')$ between the stereographic projections of points z and z'. Let the points are (x_1, x_2, x_3) and (x_1', x_2', x_3'), then the distance

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 = 2 - 2(x_1x_1' + x_2x_2' + x_3x_3')$$

And we simplify by

$$x_1x_1' + x_2x_2' + x_3x_3' = \frac{(1+|z|^2)(1+|z'|^2) - 2|z-z'|^2}{(1+|z|^2)(1+|z'|^2)}$$

Thus, we have

$$d(z, z') = \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}} \tag{1.6}$$

For $z' = \infty$ we have

$$d(z, \infty) = \frac{2}{\sqrt{1+|z|^2}} \tag{1.7}$$

Chapter 2

Complex Functions

2.1 Analytic Functions

We aim to extend the differentiation and integration of real analysis to the complex field. During this process, the range of capability is greatly restricted. Only analytic (or holomorphic) functions can be differentiated and integrated.

Practical notation:

We shall denote the symbols

- z and w always means variables in $\mathbb{C}$
- x and y can be in either $\mathbb{R}$ or $\mathbb{C}$
- t is always in $\mathbb{R}$
- $z = x + iy$ automatically means x and y are real.

We also denote $\mathbb{F}$ for $\mathbb{R}$ or $\mathbb{C}$.

We shall restricted our discussion to only functions defined on open sets.

2.1.1 Limits and Continuity

Definition 2.1.1: Limits

The function $f : \mathbb{F} \to \mathbb{F}$ is said to have a limit A at a, denoted

$$\lim_{x \to a} f(x) = A \tag{2.1}$$

iff the following is true.
For every $\epsilon > 0$ there exists $\delta > 0$ such that $\forall 0 < |x - a| < \delta$, we have $|f(x) - A| < \epsilon$. That is

$$\forall \epsilon > 0, \exists \delta > 0, (\forall 0 < |x - a| < \delta, |f(x) - A| < \epsilon) \tag{2.2}$$

There are also similar definitions for a or A to be ∞.

Theorem 2.1.1: Rational Operations on Limits

If $f, g : \mathbb{F} \to \mathbb{F}$ has

$$\lim_{x\to a} f(x) = A, \lim_{x\to a} g(x) = B$$

Then for any rational operation R, we have

$$\lim_{x\to a} R(f(x), g(x)) = R(A, B)$$

Proof. The proof is completely the same as those of real analysis. For we only need

$$|ab| = |a|\,|b| \text{ and } |a+b| \le |a| + |b|$$

□

Theorem 2.1.2: Equivalences of Limits

The following are equivalent.

- $\lim_{x\to a} f(x) = A$.
- $\lim_{x\to a} \overline{f(x)} = \overline{A}$.
- $\begin{cases} \lim_{x\to a} \operatorname{Re} f(x) = \operatorname{Re} A \\ \lim_{x\to a} \operatorname{Im} f(x) = \operatorname{Im} A \end{cases}$

Proof. We use $|\operatorname{Re} f(x) - \operatorname{Re} A| \le |f(x) - A| \le |\operatorname{Re} f(x) - \operatorname{Re} A| + |\operatorname{Im} f(x) - \operatorname{Im} A|$ would do. □

We defines continuous functions similarly

Definition 2.1.2: Continuity

$f : \mathbb{F} \to \mathbb{F}'$ is continuous at a iff

$$\lim_{x\to a} f(x) = f(a)$$

A continuous function is a function that is continuous at $\forall x \in$ domain.

Definition 2.1.3: Derivative

The derivative of a function $f : \mathbb{F} \to \mathbb{F}'$ is defined

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

we notice that we use a multiplication of the domain and range, which is not valid for two arbitrary fields $\mathbb{F}$ and $\mathbb{F}'$.

For a function $f : \mathbb{C} \to \mathbb{R}$, if it has a derivative at $x = a$, then it must be zero. For $h \in \mathbb{R}$, we have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a+ih) - f(a)}{ih}$$

The two equations are real and pure imaginary respectively, thus must be 0.

For a function $z : \mathbb{R} \to \mathbb{C}$, we have $z(t) = x(t) + iy(t)$, thus

$$z'(t) = x'(t) + iy'(t)$$

In short, the rational operations of derivatives and the chain rule also applied for the derivatives.

2.1.2 Analytic Functions

Definition 2.1.4: Analytic Functions

An analytic function is a function $f : \mathbb{C} \to \mathbb{C}$ that has a derivative for $\forall x \in$ domain.

The rational operations on analytic functions is also analytic (except of those points that have zero denominator, obviously). We begin by finding the necessary results of an analytic function.

- Analytic Functions are Continuous

  The definition of derivative can be rewritten as

  $$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

  Then

  $$\lim_{h \to 0} (f(z+h) - f(z)) = f'(z) \lim_{h \to 0} h = 0$$

  giving $\lim\limits_{h \to 0} f(z+h) = f(z)$ as expected.

  If we write $f(z) = u(z) + iv(z)$, then u, v are both continuous.

- The limit must be the same regardless of how h approaches 0.

  We write $z = x + iy$ then $u(z) = u(x, y)$ and $v(z) = v(x, y)$.

  If we let h approaches 0 in the real line, we have

  $$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

  Letting h approach 0 on the imaginary side, we have

  $$f'(z) = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Getting those together, we have

Theorem 2.1.3: The Cauchy-Riemann Equations

If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function, then let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.3}$$

or more shortly, we have

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \tag{2.4}$$

For the quantity $|f'(z)|^2$ we have

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial(u,v)}{\partial(x,y)}$$

We shall later prove that the derivative of an analytic function is analytic, that is, it has derivative of all orders. Using theorem 2.1.3, we have

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \Delta v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\end{aligned} \tag{2.5}$$

Definition 2.1.5: Harmonic Functions

A function u satisfying the Laplace equation $\Delta u = 0$ is said to be harmonic.
If two harmonic functions u and v satisfies the Cauchy-Riemann Equation, then v is the conjugate harmonic function of u. (u is the conjugate harmonic function of $-v$).

Thus the real and imaginary part of an analytic function are conjugate harmonic functions.

We wish to prove, if v is the conjugate harmonic function of u, then $u + iv$ is analytic. This is a sufficient and necessary condition.

Theorem 2.1.4: Analytic and Conjugate Harmonic

If $u(x, y)$ and $v(x, y)$ have continuous first-order partial derivatives which satisfies the Cauchy-Riemann Equations, then $f(z) = u(z) + iv(z)$ is an analytic function, and conversely.

Proof. The continuity of the first-order partial derivatives makes it possible to write

$$\begin{aligned}u(x+h, y+k) - u(x, y) &= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1 \\ v(x+h, y+k) - v(x, y) &= \frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2\end{aligned}$$

Thus we have

$$\begin{aligned}f(z+h+ik) - f(z) &= u(x+h, y+k) + iv(x+h, y+k) - u(x, y) - iv(x, y) \\ &= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1 + i\left(\frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2\right) \\ &= \frac{\partial u}{\partial x}h - \frac{\partial v}{\partial x}k + i\frac{\partial v}{\partial x}h + i\frac{\partial u}{\partial y}k + \epsilon_1 + i\epsilon_2 \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h + ik) + \epsilon_1 + i\epsilon_2\end{aligned}$$

Therefore we get

$$\lim_{h+ik\to 0}\frac{f(z+h+ik)-f(z)}{h+ik}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

We conclude that f is analytic. $\square$

The next result is pretty interesting. Consider a complex function with two real variables $f(x,y)$. We write $x=\frac{1}{2}(z+\overline{z})$ and $y=-\frac{1}{2}i(z-\overline{z})$, then we see $f(x(z,\overline{z}),y(z,\overline{z}))$ as a function of "independent" z and $\overline{z}$. We give a derivative *formally*, which is not means by limits.

$$\frac{\partial f}{\partial z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right),\qquad \frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right). \tag{2.6}$$

The derivative here is only formal, we can understand it as a definition. Then the Cauchy-Riemann Equations are equivalent to $\dfrac{\partial f}{\partial \overline{z}}=0$, thus we are tempted to say that analytic functions are "truly functions of z alone".

This gives us a fairly easy way to compute the conjugate harmonic function of $u(x,y)$. Just making $u+iv$ be function that has only $x+iy$ as variable.

If u is rational, then we have $f=u+iv$,

$$u(x,y)=\frac{1}{2}(f(z)+\overline{f(z)})=\frac{1}{2}(f(z)+f(\overline{z}))$$

thus we have

$$u(\frac{z}{2},\frac{z}{2i})=\frac{1}{2}\left(f(z)+\overline{f(0)}\right)$$

As adding a pure imaginary constant does not influence f, we can assume $f(0)$ to be real. Then we have $\overline{f(0)}=u(0,0)$. Take

$$f(z)=2u(\frac{z}{2},\frac{z}{2i})-u(0,0) \tag{2.7}$$

would do.

2.1.3 Polynomials

The simplest analytic function is the constant function and the identity function $z\mapsto z$. Since the sum and product of any two analytic functions are again analytic, we have

Theorem 2.1.5: Polynomials

Every Polynomial

$$P(z)=a_0+a_1z+\ldots+z_nz^n \tag{2.8}$$

is analytic.

For $\deg P>0$, $P(z)=0$ has at least one root in $\mathbb{C}$. Thus we have the complete factorization

$$P(z)=a_n(z-\alpha_1)\cdots(z-\alpha_n)$$

We are familiar with the order of zeros. A simple zero has order 1. If a zero α has order h, then we have

$$P(\alpha)=\ldots=P^{(h-1)}(\alpha)=0\text{ and }P^{(h)}(\alpha)\neq 0$$

As a result

Theorem 2.1.6: Lucas's Theorem

If all zeros of a polynomial $P(z)$ lies in a half plane, then all the zeros of $P'(z)$ lies in the same half plane.

Proof. We have

$$\frac{P'(z)}{P(z)} = \frac{1}{z-\alpha_1} + \ldots + \frac{1}{z-\alpha_n} \tag{2.9}$$

Suppose the half plane H is defined $\left\{z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} < 0\right\}$. If $z \notin H$, we have

$$\operatorname{Im} \frac{z-\alpha_k}{b} = \operatorname{Im} \frac{z-a}{b} - \operatorname{Im} \frac{\alpha_k - a}{b} > 0$$

then $\operatorname{Im} \dfrac{b}{z-\alpha_k} < 0$. Then we have

$$\operatorname{Im} \frac{bP'(z)}{P(z)} = \sum_{k=1}^{n} \operatorname{Im} \frac{b}{z-\alpha_k} < 0$$

Thus $P'(z) \neq 0$ □

A sharper formulation is that the smallest convex polygon that contains all zeros of $P(z)$ also contains all zeros of $P'(z)$. (as a convex polygon can be made up of half plains).

Theorem 2.1.7: Eneström-Kakeya Theorem

If $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ with $0 < a_0 \leq a_1 \leq \ldots \leq a_n$, then all zeros of $P(z)$ lies in $|z| \leq 1$.

Proof. For $|z| > 1$, we have

$$(z-1)P(z) = a_n z^{n+1} + (a_{n-1} - a_n) z^n + \ldots + (a_0 - a_1) z - a_0$$

Using the triangle inequality, we have

$$\begin{aligned}
|(z-1)P(z)| &\geq |a_n|\,|z|^{n+1} - |a_{n-1} - a_n|\,|z|^n - \ldots - |a_0 - a_1|\,|z| - |a_0| \\
&= a_n\,|z|^{n+1} - (a_n - a_{n-1})\,|z|^n - \ldots - (a_1 - a_0)\,|z| - a_0 \\
&= |z|^{n+1}\left(a_n - \frac{a_n - a_{n-1}}{|z|} - \ldots - \frac{a_1 - a_0}{|z|^n} - \frac{a_0}{|z|^{n+1}}\right) \\
&> |z|^{n+1}\left(a_n - (a_n - a_{n-1}) - \ldots - (a_1 - a_0) - a_0\right) \\
&= 0
\end{aligned}$$

□

2.1.4 Rational Functions

We turn to rational functions

$$R(z) = \frac{P(z)}{Q(z)} \tag{2.10}$$

Assuming that P and Q has no common roots. The roots of $Q(z)$ are called poles in $R(z)$. The order of each pole in defined respectively.

The derivative

$$R'(z) = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q^2(z)}$$

has the same poles as $R(z)$, with order increased by 1.

We extend the domain and range to the extended plain. **Then R is continuous**.

- If z_0 is a pole of $R(z)$, then define $R(z_0) = \infty$.
- For $R(\infty)$ we can use limit to define it but in that way it is hard to define the order of a zero or pole at ∞. Therefore we consider $R_1(z) = R(\frac{1}{z})$.

  Define $R(\infty) = R_1(0)$.

With the notation

$$R(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}$$

we have

$$R_1(z) = z^{m-n} \frac{a_0 z^n + \ldots + z_n}{b_0 z^m + \ldots + b_m}$$

- $m > n$ then ∞ is a zero or order $m - n$.
- $m < n$ the ∞ is a pole of order $n - m$.
- $m = n$ then $R(\infty) = \frac{a_n}{b_m}$

The total number of zeros or poles of $R(z)$ in the extended plane is both $\max\{m, n\}$.

Definition 2.1.6: Order of Rational Functions

The order of a rational function $R(z) = \dfrac{P(z)}{Q(z)}$ with $\deg P(z) = n$ and $\deg Q(z) = m$ is define as $\max\{n, m\}$.

Theorem 2.1.8: Zeros and Poles of the Extended Plane

A rational function that has degree p has p roots and p poles. Every equation $R(z)$ has exactly p roots.

A rational function of order 1 is linear

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \text{ with } \alpha\delta - \beta\gamma \neq 0 \tag{2.11}$$

The equation $\omega = S(z)$ has only one root, and

$$z = S^{-1}(\omega) = \frac{\delta\omega - \beta}{-\gamma\omega + \alpha}$$

Specifically, the linear transformation $z + a$ is called parallel translation, and $1/z$ is called inversion.

If $R(z)$ has a pole at ∞, divide $P(z)$ with $Q(z)$ till the degree of nominator and denominator are the same and we get

$$R(z) = G(z) + H(z)$$

where G is a polynomial without constant terms and $H(z)$ is finite at ∞. Then G is called the singular part of R at ∞.

This process can be done similarly to finite poles. Let $\beta_1, \ldots, \beta_q$ be distinct finite poles. Then the function $R(\beta_j + \frac{1}{\zeta})$ is a rational function that has a pole at ∞. We decompose it into

$$R(\beta_j + \frac{1}{\zeta}) = G_j(\zeta) + H_j(\zeta)$$

then

$$R(z) = G_j(\frac{1}{z - \beta_j}) + H_j(\frac{1}{z - \beta_j})$$

Then H_j is finite at β_j, and G_j is called the singular part of R at β_j.

Consider the expression

$$R(z) - G(z) - \sum_{j=1}^{q} G_j(\frac{1}{z - \beta_j})$$

It has no poles other than $\beta_1, \ldots, \beta_q, \infty$. But at each one of these points, the expression is finite. Thus it is a constant function according to theorem 2.1.8. We absorb it in $G(z)$ and get

$$R(z) = G(z) + \sum_{j=1}^{q} G_j(\frac{1}{z - \beta_j}) \tag{2.12}$$

where G, G_j are polynomials.

In analysis, this is how we do indefinite integration of rational functions.

2.2 Elementary Theory of Power Series

Polynomials and Rational functions are rare in general, and the easiest way to achieve greater variety is to form limits. Infinite sums of analytic functions have good chance to be analytic.

Definition 2.2.1: Limit of Sequence

Then sequence $\{a_n\}_{n \in \mathbb{Z}_+}$ has the limit A iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall n > N, |a_n - A| < \epsilon$$

If a sequence has finite limit, then it is convergent, otherwise it is divergent.

In analysis, we are familiar with the convergence criterion.

Theorem 2.2.1: Cauchy Criterion for Convergence

A sequence $\{a_n\}_{n \in \mathbb{Z}_+}$ is convergent iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall m, n > N, |a_n - a_m| < \epsilon$$

such sequence is called Cauchy sequence.

The real and complex part of a Cauchy sequence is also Cauchy sequence, thus convergent.

2.2.1 Series

A simple application of Cauchy's criterion allows us to identify convergence from existing sequences. If $|b_n - b_m| \leq |a_n - a_m|$ for all n, m, then $\{b_n\}$ is a contraction of $\{a_n\}$. Then the convergence of $\{a_n\}$ implies the convergence of $\{b_n\}$.

For an infinite series

$$a_1 + \cdots + a_n + \cdots \tag{2.13}$$

The partial sum is given

$$s_n = a_1 + \cdots + a_n$$

We say the series converges to A iff $\{s_n\}$ converges to A. We denote the series as $\sum_{n=1}^{\infty} a_n = A$.

If the series of absolute values

$$|a_1| + \cdots + |a_n| + \cdots \tag{2.14}$$

converges, then we say the series $\sum_{n=1}^{\infty} a_n$ to be absolutely convergent. From Cauchy's criterion, we know that absolutely convergent series is convergent.

2.2.2 Uniform Convergence

Definition 2.2.2: Uniform Convergence

A sequence of functions $\{f_n\}$ converges uniformly to f on a set D iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall n > N, \forall x \in D, |f_n(x) - f(x)| < \epsilon$$

A convergent series that do not converge uniformly is called pointwise convergent.

The Cauchy criterion for uniform convergence is similar to the one for convergence of sequences.

A sequence of functions $\{f_n\}$ converges uniformly to f on a set D iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall m, n > N, \forall x \in D, |f_n(x) - f_m(x)| < \epsilon$$

In practice, if $\{f_n(x)\}$ is a contraction of $\{a_n\}$, then if $\{a_n\}$ is convergent, $\{f_n(x)\}$ is uniformly convergent. This is called the **Weierstrass test**.

Theorem 2.2.2: Continuity of Uniform Convergence

The limit function of a uniformly convergent sequence of continuous functions is continuous.

2.2.3 Power Series

A power series is a series of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = \sum_{n=0}^{\infty} a_n z^n$$

where $a_n \in \mathbb{C}$. More generally we can translate the series to a power series centered at z_0 as

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Theorem 2.2.3: Abel Disk Theorem

For every power series $\sum_{i=1}^{\infty} a_n z^n$, there exists $0 \leq R \leq +\infty$, called the radius of convergence, that:

- The series converge absolutely for $|z| < R$. $\forall 0 \leq \rho < R$, the series converges uniformly on the disk $\{z \in \mathbb{C} : |z| \leq \rho\}$.
- For $|z| > R$, then the terms are not bounded, so the series is divergent.
- In $|z| < R$ the sum of the series is an analytic function. The derivative can be obtained by term-by-term differentiation, and the derivative series has the same radius of convergence.

This is rather similar to that of $\mathbb{R}$ in analysis. The radius R can be given by

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \tag{2.15}$$

called the Hadamard formula.

Proof.

- The first two statements we get by comparing the series with $\sum \rho^n$, using Weierstrass test.
- the derived series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same radius of convergence, for $\lim_{n\to\infty} \sqrt[n]{n} = 1$. To show that we can derive term-by-term, we write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = s_n(z) + R_n(z)$$

for $|z| < R$, also

$$f_1(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = \lim_{n\to\infty} s_n'(z)$$

Now we show $f'(z) = f_1(z)$. Consider the identity

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) = \left(\frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0)\right) + (s_n'(z_0) - f_1(z_0)) + \left(\frac{R_n(z) - R_n(z_0)}{z - z_0}\right)$$

Assume $z \neq z_0, |z|, |z_0| < \rho < R$, then as have

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}$$

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| \leq \sum_{k=n}^{\infty} k\,|a_k|\,\rho^{k-1}$$

For every $\epsilon > 0$, there exists sufficiently large n such that the last two terms are less than $\epsilon/2$. Then when z is close to z_0 we can make the first term as small as we want, thus

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0)\right| < \epsilon$$

□

We now have proved that a power series with positive radius of convergence is an analytic function, and has derivative of all orders. The derivative is also a power series with the same radius of convergence.

By continuously taking derivatives, we can get the n-th derivative of a power series $f(z)$:

$$f^{(n)}(z) = \sum_{k=n}^{\infty} a_k \frac{k!}{(k-n)!} z^{k-n} = \sum_{k=0}^{\infty} a_{n+k} \frac{(n+k)!}{k!} z^k$$

We see that $a_k = f^{(k)}(0)/k!$, so

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

This is the familiar Taylor series.

Remark:
We have shown that an analytic function that is formed by power series has arbitrary many derivatives, and the Taylor series converges to the function inside the convergence disk.
But it remains to be seen that every analytic function can be expressed as a power series. This is the content of the **Cauchy Integral Theorem**, which we shall prove later.

2.2.4 Abel limit Theorem

We cannot say much about convergence on the circle of radius R. However, if it converges at one point on the circle, then we can say some things about the “one side continuity” of the series. We loss no generality by assuming that $R = 1$.

Theorem 2.2.4: Abel Limit Theorem

If $\displaystyle\sum_{n=0}^{\infty} a_n$ converges, then $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ tends to $f(1)$ as z tends to 1 in a way that $\dfrac{|1-z|}{1-|z|} \leq K$ for some constant $K > 0$.

Geometrically it means that z stays in an angle $< \pi$ with vertex 1 and symmetric to the real axis. We say that the approach takes place in a Stolz angle.

Proof. Assume $\sum_{n=0}^{\infty} a_n = 0$ by adding a constant ti a_0. Then write $s_n = a_0 + a_1 + \cdots + a_n$ and use Abel tranform

$$f(z) = (1-z)(s_0 + s_1 z + \cdots + s_{n-1} z^{n-1}) + s_n z^n$$

As $s_n z^n \to 0$ we have

$$f(z) = (1-z)\sum_{i=1}^{\infty} s_n z^n$$

When $|1-z| \le K(1-|z|)$, choose large enough m such that $\forall n > m, |s_n| < \epsilon$, then we have

$$\begin{aligned}|f(z)| &\le |1-z|\left|\sum_{k=0}^{m-1} s_k z^k\right| + |1-z|\,\epsilon\sum_{k=m}^{\infty}|z|^k \\ &\le |1-z|\left|\sum_{k=0}^{m-1} s_k z^k\right| + \frac{|1-z|}{1-|z|}\epsilon\end{aligned}$$

The first term is arbitrarily small when $z \to 1$ and the second term is bounded by $K\epsilon$. Thus we have $|f(z) - f(1)| \to 0$. □

2.3 The Exponential and Trigonometric Functions

Back is analysis, we have noticed the similarity of Taylor series of exponential and trigonometric functions. We can define them in the complex plane as well, this time directly via power series, or a solution of a differential equation.

In this way, we can analytically define the logarithm and the argument of a complex number, and leading to a nongeometric definition of the angle.

2.3.1 the Exponential

Definition 2.3.1: Exponential Function

The exponential function is defined as the solution of the differential function

$$f'(z) = f(z) \qquad f(0) = 1 \tag{2.16}$$

Taking the series expansion it shows that if $f(z)$ can be expressed as a power series, then it must be of the form

$$f(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{2.17}$$

And $f(z)$ is denoted e^z.

Proposition: **Properties of the Exponential**

- The series converges for all $z \in \mathbb{C}$, for $\sqrt[n]{n!} \to \infty$.
- The addition property:

$$e^a e^b = e^{a+b} \tag{2.18}$$

Proof. We find that $D(e^z e^{c-a}) = e^z e^{c-z} - e^z e^{c-z} = 0$. Hence $e^z e^{c-z}$ is a constant. (Using Cauchy Riemann Equations and the real immediate value theorem). Setting $z = 0$ gives $e^z e^{c-z} = e^c$, letting $z = a, c = a + b$ would do. □

Specifically, $e^z e^{-z} = 1$, so $\forall z \in \mathbb{C}, e^z \neq 0$.

- For $x \in \mathbb{R}$ the series shows that $e^x > 0, x > 0$ and $0 < e^x < 1, x < 0$.
- The conjugate property:
$$e^{\bar{z}} = \overline{e^z} \tag{2.19}$$

Proof. We have $e^{\bar{z}} = \sum_{n=0}^{\infty} \frac{\bar{z}^n}{n!} = \sum_{n=0}^{\infty} \frac{\overline{z^n}}{n!} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \overline{e^z}$. □

2.3.2 The Trigonometric Functions

Definition 2.3.2: Trigonometric Functions

The trigonometric functions are defined by:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{2.20}$$

A series expansion shows that:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \qquad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \tag{2.21}$$

The other trigonometric functions are defined by cos and sin. All are rational functions of e^{iz}.

So we have the famous Euler's formula, directly from the definition.

$$e^{iz} = \cos z + i \sin z \tag{2.22}$$

Some familiar results:

$$\cos^2 z + \sin^2 z = 1$$

$$D \cos z = -\sin z, \qquad D \sin z = \cos z$$

The addition formulas

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

2.3.3 The Periodicity

We say that f is periodic with period $T \in \mathbb{C}$ if $f(z+T) = f(z)$ for all $z \in \mathbb{C}$.

Theorem 2.3.1: Periodicity of the Exponential

The exponential function e^z is periodic. The periods are of the form $c = ik\omega_0$, where $k \in \mathbb{Z}_+, \omega_0 \in \mathbb{R}_+$.

Proof. A period of e^z satisfies $e^{z+c} = e^z$, thus $e^c = 1$, writing $c = a + ib$, we have $e^{a+ib} = e^a e^{ib} = 1$, so taking norm we have $a = 0$ and $c = ib$.

For $y \in \mathbb{R}$, we use $D \sin y = \cos y \leq 1$ (by series form) and $\sin 0 = 0$, thus $\forall y > 0, \sin y < y$. In similar ways, we get

$$\cos y < 1 - \frac{y^2}{2} + \frac{y^4}{24}, y > 0$$

Thus we have $\cos \sqrt{3} < 0$, thus $\exists 0 < y_0 < \sqrt{3}, \cos y_0 = 0$. For

$$\cos^2 y_0 + \sin^2 y_0 = 1$$

we have $\sin y_0 + \pm 1$, thus $e^{iy_0} = \pm i$, thus $e^{4y_0} = 1$. We've shown that $4y_0$ is a period.

A more careful analysis shows that $4y_0$ is the smallest period. As $\forall 0 < y < y_0, \sin y > y(1 - y^2/6) > 0$ showing $\cos y$ is strictly decreasing. Also $\sin y$ is positive so $\sin y$ is strictly increasing. Therefore, e^{iy} cannot be $\pm 1, \pm i$ for $0 < y < 4y_0$. Thus the smallest period is $4y_0$. □

Definition 2.3.3: π

The period of the exponential function is denoted $c = 2k\pi i$.

We have

$$e^{\pi i/2} = i, \qquad e^{\pi i} = -1, \qquad e^{3\pi i/2} = -i, \qquad e^{2\pi i} = 1$$

Geometrically, the mapping $y \to e^{iy}, 0 \leq y < 2\pi$ is a 1-1 mapping from the interval $[0, 2\pi)$ to the unit circle S^1 in the complex plane. The image of y is the point on the unit circle with angle y with respect to the positive real axis.

Algebraically, the function $y \to e^{iy}$ defines a homomorphism from the additive group of $\mathbb{R}$ to the multiplicative group of S^1, the kernel is $2\pi\mathbb{Z}$.

2.4 The Logarithm

The logarithm is the inverse function of the exponential function.

If $e^z = e^{x+iy} = w$, we have

$$e^x = |w|, \qquad e^{iy} = \frac{w}{|w|} \tag{2.23}$$

First, $w \neq 0$, for $e^z \neq 0$. The first function has a unique solution $x = \log_{\mathbb{R}} |w|$, the real logarithm. The solution is a set $y + 2\pi\mathbb{Z}$ where $0 \leq y < 2\pi$. Therefore, the logarithm of a complex number has the form $z + 2\pi i\mathbb{Z}$.

Definition 2.4.1: Arguments and Logarithm

If $w \neq 0$, then let $e^z = w, 0 \leq \operatorname{Im} z < 2\pi$, we define the (principle) argument of w by $\arg w = \operatorname{Im} z$. And

$$\log w = \log_{\mathbb{R}} |w| + i \arg w \tag{2.24}$$

We can write an arbitrary complex number w as

$$w = re^{i\theta} \tag{2.25}$$

where $r = |w| \geq 0$ and $\theta = \arg w$ is the principal argument, i.e. $\theta \in [0, 2\pi)$.

The solution of $e^z = w$ can be written as $\log w + 2\pi i\mathbb{Z}$, we denote it by $\operatorname{Log} w$. The set $\operatorname{Arg} w = \arg w + 2\pi\mathbb{Z}$.

- $\operatorname{Log} w = \log w + 2\pi i\mathbb{Z}$.
- $\operatorname{Arg} w = \arg w + 2\pi\mathbb{Z}$.

Definition 2.4.2: Power

The power is defined as
$$a^b = e^{b\log a}e^{2\pi i k b}, k \in \mathbb{Z} \tag{2.26}$$

This is somewhat chaotic.

- If $a \in \mathbb{R}_+$, then usually we take $a^b = e^{b\log_{\mathbb{R}} a}$.
- If $b \in \mathbb{Z}$, then the above expression has only one value $e^{b\log a}$, which is just the repeated products of a for b times.

  Proof. We have $e^{b\log a} = e^{\log a}\cdots e^{\log a} = a^b$. □

- If $b = p/q$, then there are exactly q different values of a^b.

The addition theorem of exponential implies

$$\begin{aligned}\operatorname{Log}(ab) &= \operatorname{Log} a + \operatorname{Log} b\\ \operatorname{Arg}(ab) &= \operatorname{Arg} a + \operatorname{Arg} b\end{aligned} \tag{2.27}$$

Finally we consider the inverse trigonometric functions. The function

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = w$$

implies that e^{iz} satisfies $z + \frac{1}{z} = 2w$.

$$z = -i\operatorname{Log}\left(w + \sqrt{w^2 - 1}\right)$$

Note that $\sqrt{w^2-1}$ has usually two values, and Log has infinity.

The two values of $w + \sqrt{w^2-1}$ are reciprocal to each other, so we can write

$$\arccos w = \pm i\operatorname{Log}\left(w + \sqrt{w^2 - 1}\right), \text{ taking one of the two values of } \sqrt{} \tag{2.28}$$

We also have
$$\arcsin x = \frac{\pi}{2} - \arccos x \tag{2.29}$$

Chapter 3

Analytic Functions As Mappings

3.1 Conformality

Our goal here is to derive a geometrical intuition about what it means to be an analytic function.

3.1.1 Arcs and Closed Curves

Definition 3.1.1: Arcs

An arc γ on the complex plane is image of a continuous function $z : [\alpha, \beta] \to \mathbb{C}$, where $\alpha, \beta \in \mathbb{R}$. We denote the arc $z = z(t) = x(t) + iy(t)$.

As a continuous mapping of a closed interval, an arc is closed and thus compact and connected.

The choice of parameter t is arbitrary, and we can freely change it. If $\varphi : [\alpha', \beta'] \to [\alpha, \beta]$ is a strictly increasing function, then $z = z(\varphi(\tau))$ describes the same arc, but with a different parameter τ.

The following are some properties that describe an arc:

- **Differentiability**: The arc is differentiable (continuously differentiable) if the derivative $z'(t)$ exists and is continuous for all $t \in [\alpha, \beta]$.
- **Regularity**: The arc is regular if it is differentiable and the derivative $z'(t) \neq 0$ for all $t \in [\alpha, \beta]$.
- **Piecewise Differentiability**: The arc is piecewise differentiable iff:
  - It is continuous on $[\alpha, \beta]$.
  - It has continuous derivatives except for a finite number of points in $[\alpha, \beta]$.
  - At each breaking point, the left and right derivatives exist and are equal to the limit of $z'(t)$. (left and right continuous derivatives)
- **Piecewise Regularity**: The arc is piecewise regular if it is piecewise differentiable and $z'(t) \neq 0$ and the left and right derivatives at the breaking points are $\neq 0$.

Remark:
Differentiability and Regularity is invariant under change of parameters.

- **Jordan arc (Simple arc)**: An arc is simple if $z(t_1) = z(t_2) \to t_1 = t_2$. (it doesn't cross itself)
- **Closed Curve**: An arc is a closed curve if $z(\alpha) = z(\beta)$. For that, a shift of parameter can be done: taking $t_0 \in (\alpha, \beta)$ and defining a new one by taking t_0 as initial point.
- The **Opposite** of an arc γ is the arc defined by $\gamma^*(t) = \gamma(\beta + \alpha - t)$, where $t \in [\alpha, \beta]$.

3.1.2 Analytic Functions in Regions

The definition of an analytic function requires the approach to a point in arbitrary directions. So we have good reasons to restrict our discussion to open sets.

Definition 3.1.2: Analytic Functions in Open Sets

Let $\Omega \subseteq \mathbb{C}$ be open. A complex function $f : \Omega \to \mathbb{C}$ is analytic (holomorphic) if it has a derivative at every point in Ω.

It is obvious that the restriction of an analytic function to an open subset is also analytic. Also, for the components of open sets are open, we can assume that Ω is connected, that is, Ω is a region.

Definition 3.1.3: Regions

A region is an open connected subset of $\mathbb{C}$.

Definition 3.1.4: Analytic Functions on Arbitrary Sets

A function f is analytic on $A \subseteq \mathbb{C}$ if $\exists C \subseteq \mathbb{C}, A \subseteq C$ and C is open and f is analytic on C.

Remark:
For multivalued functions like the logarithm and inverse trigonometric functions, we can specify a well-behaved Ω and an analytic branch to make it single-valued and analytic on Ω. However, for certain functions like log, for some Ω it is impossible to find a single-valued analytic branch. For example, log is not analytic on $\mathbb{C} \setminus \{0\}$, which we shall prove later.

Theorem 3.1.1: The Inverse Function Theorem

Let $f : \Omega \to \mathbb{C}$ be an analytic function on a region Ω and let $z_0 \in \Omega$ be a point such that $f'(z_0) \neq 0$. Then there exists a neighborhood U of z_0 such that f is bijective from U onto its image $f(U)$, and the inverse function f^{-1} is also analytic on $f(U)$. The derivative of the inverse function at the point $w_0 = f(z_0)$ is given by

$$Df^{-1}(w_0) = \frac{1}{f'(z_0)}$$

Proof. The inverse function theorem for $\mathbb{R}^2$ states that

> Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function and let $Df(z_0)$ be the Jacobian matrix of f at z_0. If $\det Df(z_0) \neq 0$, then there exists a neighborhood U of

z_0 such that f is a bijection from U onto its image $f(U)$, and the inverse function f^{-1} is also continuously differentiable on $f(U)$. The derivative of the inverse function at $w_0 = f(z_0)$ is given by

$$Df^{-1}(w_0) = (Df(z_0))^{-1}$$

Now, if f is analytic, we see it as a function $\Omega \to \mathbb{R}^2$, then it is continuously differentiable, and we have $\det f(z_0) = |f'(z_0)|^2 \neq 0$. The inverse function f is also continuously differentiable, and we verify the Cauchy-Riemann equations for $g = f^{-1}$: Let $f = u + iv, g = \alpha + i\beta$, using $Df \cdot Dg = I_2$, we have

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Cauchy-Riemann equations for f imply that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Therefore, the Cauchy-Riemann equations hold for g, and thus g is analytic on $f(U)$. The derivative of the inverse function is given by chain rule. □

Proposition: **The Analytic Branch of Multivalue Functions**

- The root function is the inverse of the power function z^n for $n \in \mathbb{N}$, which is analytic on $\mathbb{C} \setminus \{0\}$. For each slice that has the form

  $$\Omega = \left\{ z \in \mathbb{C} : \arg z \in (\frac{2\pi k}{n}, \frac{2\pi(k+1)}{n}), z \neq 0 \right\}$$

  The function $z \mapsto z^n$ is bijective, and the image is $\mathbb{C} - [0, +\infty)$. The inverse function $z \mapsto z^{1/n}$ is analytic.

- The logarithm function is the inverse of the exponential function, which is analytic on $\mathbb{C} \setminus \{0\}$. For each slice that has the form

  $$\Omega = \{ z \in \mathbb{C} : \operatorname{Im} z \in (2\pi k, 2\pi(k+1)), z \neq 0 \}$$

  The function $z \mapsto e^z$ is bijective, and the image is $\mathbb{C} - [0, +\infty)$. The inverse function $z \mapsto \log z$ is analytic.

- The inverse trigonometric functions are the inverses of the trigonometric functions, which are analytic on $\mathbb{C}$. For each slice that has the form

  $$\Omega = \{ z \in \mathbb{C} : \operatorname{Re} z \in (2\pi k, 2\pi(k+1)), \operatorname{Im} z > 0 (\text{ or } < 0) \}$$

The function $z \mapsto \cos z$ is bijective, and the image is $\mathbb{C} - [-1, +\infty)$. The inverse function $z \mapsto \arcsin z$ is analytic.

We can also see this using the composite function

$$\arccos z = \pm i \log\left(z + \sqrt{z^2 - 1}\right)$$

Proposition: **The derivatives of inverse functions**

- The logarithm function:
$$\frac{d}{dz} \log z = \frac{1}{z}$$
- The root function:
$$\frac{d}{dz} z^{1/n} = \frac{1}{n} z^{\frac{1}{n}-1}$$
- The inverse trigonometric functions:
  - $\arcsin z$:
$$\frac{d}{dz} \arcsin z = \frac{1}{\sqrt{1-z^2}}$$
  - $\arccos z$:
$$\frac{d}{dz} \arccos z = -\frac{1}{\sqrt{1-z^2}}$$
  - $\arctan z$:
$$\frac{d}{dz} \arctan z = \frac{1}{1+z^2}$$

For a complicated multivalue function it takes some effort to fine the branch that is analytic on a given open set, usually a cut of $\mathbb{C}$. The key is to find a suitable slice of the complex plane that avoids the branch cuts of the function.

Example: **Branch Cut of Multivalue Functions**

- $f(z) = \sqrt{1-z} + \sqrt{1+z}$.

  For $\sqrt{z}$ a cut through arbitrary rays from 0 would do. So we can cut $(-\infty, -1] \cup [1, +\infty)$.
- $\log \log z$.

  Cutting $(-\infty, 0]$ would have an image $|\operatorname{Im} z| < \pi$. If the image should also cut the negative real axis, we can cut $(-\infty, 1]$, for $e^{(-\infty,0]} = (0, 1]$.

A more general way will be introduced via Cauchy integral later.

In $\mathbb{C}$ and open connected set is path-connected, so we can have the constant theorem and the mean value theorem for analytic functions.

Theorem 3.1.2: The Constant Theorem

An analytic function f on a region Ω is constant if and only if its derivative is zero everywhere in Ω.
Also, f is constant iff either $\operatorname{Re} f, \operatorname{Im} f, \arg f, |f|$ is constant on Ω.

Proof. Let $f = u + iv$, then the derivative being zero means that all partial derivatives of u and v are zero, which means that u and v are constant functions (joining a path would do). Therefore, f is constant.

If u is constant, then

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0$$

so do v.

If $u^2 + v^2$ is constant, then

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$$

The equations has non-zero solutions if the coefficient determinant $u^2 + v^2 = 0$. If $u^2 + v^2 = 0$ at a point, then it is constant due to our assumption that modulus is constant. Then $f = 0$ everywhere. If not, then the partial derivatives are zero, and thus f is constant.

Finally, if $\arg z$ is constant, then $u = kv$ for some constant k, but $u - kv$ is the real part of $(1 + ik)f$. So f is constant. ☐

3.2 Conformal Mappings

Suppose an arc $\gamma : z = z(t), \alpha \leq t \leq \beta$, contained in a reigion Ω. A continuous function $f : \Omega \to \mathbb{C}$ maps γ to the arc $\gamma' : w = w(t) = f(z(t))$ in the w-plane.

If f is analytic in Ω, and z' exists, we say

$$w'(t) = f'(z(t))z'(t) \tag{3.1}$$

For a point $z_0 = z(t_0)$, if $z'(t_0) \neq 0, f'(z_0) \neq 0$, then γ' has a tangent at t_0, the direction of the tangent is given by

$$\operatorname{Arg} w'(t_0) = \operatorname{Arg} f'(z_0) + \operatorname{Arg} z'(t_0)$$

From z-space to w-space, the angle of the tangent is rotated by $\operatorname{Arg} f'(z_0)$, the rotation does not depend on the curve. For each curve passing through z_0, the rotation would be the same. Curves passing z_0 having the same tangent are mapped to curves passing $w_0 = f(z_0)$ having the same tangent. The angle of two curves is invariant under the mapping, so we say that f is a **conformal mapping** at z_0.

Definition 3.2.1: Conformal Mapping

A function f is conformal at a point $z_0 \in \Omega$ if it is analytic in a neighborhood of z_0 and $f'(z_0) \neq 0$.

About the modulus of $|f'(z_0)|$, we have

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$$

Which means a segment at z_0 would stretch by a factor of $|f'(z_0)|$ in the w-plane. Together the conformality of f means in an infinite small neighborhood of z_0, the transformation f :

- Rotates an angle by $\operatorname{Arg} f'(z_0)$.
- Stretches a segment by a factor of $|f'(z_0)|$.

We shall see that the converse also holds for either case.

Theorem 3.2.1: Criterions for Conformal Mapping

Let $f : \Omega \to \mathbb{C}$ (thought of $\Omega_{\mathbb{R}} \to \mathbb{R}^2$) has continuous partial derivatives whose Jacobian determinant is nonzero. If one of the following conditions holds, then f or $\overline{f}$ is conformal at $z_0 \in \Omega$:

- The Jacobian matrix J preserves angles at z_0.
- The Jacobian matrix J multiples the length of segments at z_0 by a constant.

Proof. These are two sufficient conditions for J to be a multiple of an orthonormal matrix. □

If $\overline{f}$ is conformal, then f is called indirectly conformal.

Definition 3.2.2: Topological Mapping

If f is bijective and both f and f^{-1} are continuous, then f is called a **topological mapping** or a **homeomorphism**.

The inverse function theorem states that if f is analytic and $f'(z_0) \neq 0$, then f is a homeomorphism in a neighborhood of z_0. The inverse function f^{-1} is also analytic, and the derivative is given by the inverse of $f'(z_0)$.

However, if $f'(z) \neq 0$ for all $z \in \Omega$, we cannot say that f is topological at Ω, it maybe not a bijection. This may lead us to the concept of Riemann surfaces.

3.2.1 Length and Area

It is easy to notice that the length under a conformal mapping is multiplied by $|f'(z_0)|$, and area is multiplied by $|f'(z_0)|^2$. More rigorously:

f is conformal.

- Let γ be a continuously differentiable arc, with $z = z(t)$, then

$$L(\gamma) = \int_\alpha^\beta |z'(t)|\, dt$$

The length of the image arc would be

$$L(\gamma') = \int_\alpha^\beta |f'(z(t))z'(t)|\, dt = \int_\alpha^\beta |f'(z(t))|\, |z'(t)|\, dt$$

We can write it as

$$L(\gamma) = \int_\gamma |\mathrm{d}z| \qquad \text{and} \qquad L(\gamma') = \int_{\gamma'} |\mathrm{d}w| = \int_\gamma |f'(z)|\, |\mathrm{d}z|$$

the $|\mathrm{d}z|$ here is the same as $\mathrm{d}s$ is analysis.

- For a point set E whose area is given

$$A(E) = \iint_E \mathrm{d}x\mathrm{d}y$$

The area of the image set $f(E)$ is given by

$$A(f(E)) = \iint_E |u_x v_y - u_y v_x| \,\mathrm{d}x\mathrm{d}y = \iint_E |f'(z)|^2 \,\mathrm{d}x\mathrm{d}y$$

3.3 Linear Transformations

The first-order rational functions are conformal and topological on the extended complex plane. They have remarkable geometrical properties, and we can use them to do transformations that may simplify matters.

3.3.1 The Linear Group

A linear fractional transformation has the form

$$w = S(z) = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}, ad-bc \neq 0 \tag{3.2}$$

We assume that $S(\infty) = a/c$ and $S(-d/c) = \infty$. It has an inverse

$$z = S^{-1}(w) = \frac{dw-b}{-cw+a}, \qquad a,b,c,d \in \mathbb{C}, ad-bc \neq 0 \tag{3.3}$$

It is obvious that S is topological on $\mathbb{C}*$, and the inverse is also topological.

Remark:
We shall see that the extended complex plane is metrizable, with distances on the Riemann sphere.

We say that the linear fractional transformation S is normalized if $ad - bc = 1$. (Every linear fractional transformation can be normalized by dividing a, b, c, d by $ad - bc$, changing the sign of the coefficients would make no change.)

We can write the linear fractional transformation in matrix form: Letting $z = \frac{z_1}{z_2}$ and $w = \frac{w_1}{w_2}$, we have $w = Sz$ iff

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

In this way we can see that the linear fractional transformation forms a group. And the normalized linear fractional transformations form a subgroup of the group, denoted by $\mathrm{SL}(2, \mathbb{C})$.

Three special linear fractional transformations are of particular interest:

- The parallel translation: $w = z + \alpha$.
- Stretching and Rotation: $w = kz$.
- The inversion: $w = \frac{1}{z}$.

Every linear fractional transformation can be expressed as a composition of these three transformations. We say that the linear transformation has 3 degrees of freedom.

3.3.2 The Cross Ratio

Given 3 distinct points $z_2, z_3, z_4 \in \mathbb{C}^*$, there is a linear transformation S such that $S(z_2) = 0$, $S(z_3) = 1$, and $S(z_4) = \infty$. The transformation is given by

$$Sz = \frac{z - z_3}{z - z_4} / \frac{z_2 - z_3}{z_2 - z_4} \tag{3.4}$$

If one of z_2, z_3, z_4 is ∞, then it reduces to

$$Sz = \frac{z - z_3}{z - z_4}, \qquad \frac{z_2 - z_4}{z - z_4}, \qquad \frac{z - z_2}{z_2 - z_3} \tag{3.5}$$

respectively.

Definition 3.3.1: The Cross Ratio

The cross ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the linear transformation S that maps z_2, z_3, z_4 to $0, 1, \infty$, respectively. It is given by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \tag{3.6}$$

Remark:
The linear transformation above is uniquely determined by z_2, z_3, z_4.

Proof. If S, T satisfies the condition, then ST^{-1} would leave $0, 1, \infty$ invariant. Letting $ST^{-1}z = \dfrac{az + b}{cz + d}$, we have

$$\frac{b}{d} = 0 \qquad \frac{a + b}{c + d} = 1 \qquad \frac{a}{c} = \infty$$

which means that $b = c = 0, a = d$. So $ST^{-1}(z) = z$. $S = T$. □

We can see the cross ratio as some properties relate to the four points. It characterizes some properties of the four points, such as collinearity and concyclicity. The cross ratio is invariant under linear fractional transformations.
We can also say that

$$(z_1, z_2, z_3, z_4) = ((z_1, z_2, z_3, z_4), 0, 1, \infty)$$

Theorem 3.3.1: The Invariance of Cross Ratio

If $z_1, z_2, z_3, z_4 \in \mathbb{C}^*$ are distinct, and T is any linear fractional transformation, then the cross ratio is invariant under T:

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4) \tag{3.7}$$

Proof. Let $Sz = (z, z_2, z_3, z_4)$, then ST^{-1} maps Tz_2, Tz_3, Tz_4 to $0, 1, \infty$, respectively. So we have

$$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1, z_2, z_3, z_4)$$

□

Using this property it is easy to get the linear transformation if given $z_1, z_2, z_3 \mapsto w_1, w_2, w_3$. It must obey

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3) \tag{3.8}$$

Solving the equation we shall get $z \mapsto w(z)$.

Theorem 3.3.2: Concyclicity and Collinearity

The cross ratio $(z_1, z_2, z_3, z_4) \in \mathbb{R}$ iff the four points lies on a straight line or on a circle.

Proof. From elementary geometry we know that the opposite angles of a cyclic quadrilateral are supplementary, so using

$$\operatorname{Arg}(z_1, z_2, z_3, z_4) = \operatorname{Arg}\frac{z_1 - z_3}{z_1 - z_4} - \operatorname{Arg}\frac{z_2 - z_3}{z_2 - z_4}$$

would do.

For an analytical proof, we need only the fact that the real axis under arbitrary linear fractional transformation is mapped to a circle or a line. Let T^{-1} be the linear transformation, then $Tw \in \mathbb{R}$, that $Tw = \overline{Tw}$. So we have

$$\frac{aw + b}{cw + d} = \frac{\overline{a}\,\overline{w} + \overline{b}}{\overline{c}\,\overline{w} + \overline{d}}$$

$$(a\overline{c} - \overline{a}c)\, w\overline{w} + \left(a\overline{d} - \overline{b}c\right) w + (b\overline{c} - \overline{a}d)\, \overline{w} + \left(b\overline{d} - \overline{b}d\right) = 0$$

If $a\overline{c} - c\overline{a} = 0$, then this is the equation of a straight line, otherwise, we have

$$\left| w + \frac{\overline{a}d - \overline{c}b}{\overline{a}c - \overline{c}a} \right| = \left| \frac{ad - bc}{\overline{a}c - \overline{c}a} \right|$$

This is a circle. □

Corollary 3.3.1: Geometry of Linear Transformations

A linear transformation takes circles or lines to circles or lines.
For any two circle or line, there exists a linear transformation that maps one to the other.

Proof. Let C_1 and C_2 be two circles or lines, then we can find three points $z_1, z_2, z_3 \in C_1$ and three points $w_1, w_2, w_3 \in C_2$. We can find a linear transformation T such that $Tz_i = w_i$, for $i = 1, 2, 3$.

Then we have $\forall z \in C_1, (z, z_1, z_2, z_3) \in \mathbb{R}$ so $(Tz, Tz_1, Tz_2, Tz_3) \in \mathbb{R}$ so $Tz \in C_2$. Also, $\forall z \in C_2, T^{-1}(z) \in C_1$. Therefore, T is surjective, thus $T(C_1) = C_2$. □

3.3.3 Symmetry

For linear transformations with real coefficients, that is, $a, b, c, d \in \mathbb{R}$:

- $T(\mathbb{R}) = \mathbb{R}$.
- $T\overline{z} = \overline{Tz}$.

Proposition: **Real Transformations**

If $T(\mathbb{R}) = \mathbb{R}$, then $\forall z \in \mathbb{C}$, $T\overline{z} = \overline{Tz}$ and T can have real coefficients. (We say "can" for the coefficients maybe multiplied by a common factor.)

Proof. Assume that T takes $\mathbb{R}$ to $\mathbb{R}$. Letting $Tz = \dfrac{az+b}{cz+d}$.

Taking $z = 0$, we have $d = 0$ or $b/d \in \mathbb{R}$.

- If $d = 0$, then $c \neq 0$, taking $z = \infty$ would give $a/c \in \mathbb{R}$. Taking $z = 1$ we get $b/c \in \mathbb{R}$. So $a' = a/c, b' = b/c, c' = 1, d' = 0$ would do.
- If $d \neq 0$, then it takes similar discussion.

$\square$

In the general case, we have $T(\mathbb{R})$ being a circle or a line C, and we say $w = Tz$ and $w* = T\overline{z}$ are symmetric with respect to C.

Definition 3.3.2: Symmetry

If $C \subseteq \mathbb{C}$ is a line or circle, and $w, w^* \in \mathbb{C}$. Let T be any linear transformation that $T(C) = \mathbb{R}$. If $\overline{Tw} = Tw^*$, then we say that w and w^* are symmetric with respect to C.

We shall prove that the symmetry does not depend on the choice of T.

Proof. If S, T both takes C to $\mathbb{R}$, then ST^{-1} takes $\mathbb{R}$ to $\mathbb{R}$. If $\overline{Tw} = Tw^*$, then $Sw^* = ST^{-1}Tw^* = ST^{-1}\overline{Tw} = \overline{ST^{-1}Tw} = \overline{Sw}$. $\square$

Therefore, we rewrite the symmetry as

Theorem 3.3.3: Criterion for Symmetry

The points z, z^* are symmetric with respect to a line or circle C iff $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$ for some distinct points $z_1, z_2, z_3 \in C$.

The symmetric function $z \mapsto z^*$ is a bijection of $\mathbb{C} \to \mathbb{C}$, called a reflection. The only points that are symmetric to themselves are those on C.

Remark:
Note that reflection is NOT a linear transformation, just like the conjugation transform. However, two reflection would be a linear transformation.
Any reflection would have the form TST^{-1}, where T is a linear transformation that takes $\mathbb{R}$ to C and S is the conjugate operation.

Now we consider the geometrical implications of symmetry.

- Suppose C is a straight line, we take $z_3 = \infty$. Then we have

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\overline{z} - \overline{z_2}}{\overline{z_1} - \overline{z_2}}$$

Taking absolute values we get $|z^* - z_2| = |z - z_2|$, as z_2 can take any point on the line, and z, z^* are on the different half planes (taking Im), so C is the bisecting normal of the segment zz^*.

- If C is a finite circle with center a and radius R, then using the invariance of cross ratio we have

$$\overline{(z, z_1, z_2, z_3)} = \overline{(z-a, z_1-a, z_2-a, z_3-a)}$$

As $|z_1 - a| = |z_2 - a| = |z_3 - a| = R$, so we have

$$\begin{aligned}\overline{(z, z_1, z_2, z_3)} &= \left(\overline{z} - \overline{a}, \frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}\right) \\ &= \left(\frac{R^2}{\overline{z} - \overline{a}} + a, z_1, z_2, z_3\right) \text{ Using the cross ratio invariance}\end{aligned}$$

Therefore, as have $z^* = \dfrac{R^2}{\overline{z} - \overline{a}} + a$, or

$$(z^* - a)\overline{(z-a)} = R^2 \tag{3.9}$$

Note that the distances $|z^* - a|\,|z - a| = R^2$ being a constant, and the ratio $(z^* - a)/(z - a) \in \mathbb{R}$. Therefore, the points z^* and z are on the same line with a.

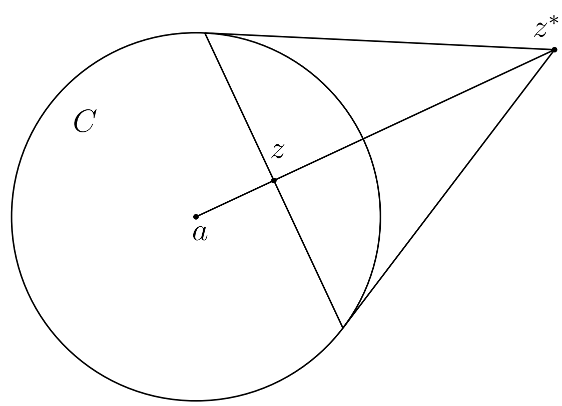

Figure 3.1: Symmetric Points to a Circle

Theorem 3.3.4: The Symmetric Principle

T is a linear transform, and C_1, C_2 are circles or lines. If T maps C_1 to C_2, then if z^*, z are symmetric with respect to C_1, then Tz^*, Tz are symmetric with respect to C_2.

Proof. Both taking the point to be conjugate to $\mathbb{R}$. □

We can use this principle to find transformations T. If $z_1 \in C_1$ maps to $z_2 \in C_2$, and z_2 mapping to w_2 are not on C_1, then we can solve

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

to find the transformation T.

3.3.4 Oriented Circles

The extended complex plane is homeomorphic to S_1 rather than R^2. So topologically, we need two charts to analyze the derivative of linear transformation. One is just $\mathbb{C}$, the other $\mathbb{C} \cup \{\infty\} - \{0\}$, by the inversion $z \mapsto 1/z$.

Because a linear transformation $S(z)$ is analytic, and

$$S'(z) = \frac{ad - bc}{(cz + d)^2} \tag{3.10}$$

is not zero for $z \neq -d/c, \infty$. We can see that the angle of two intersecting circle is preserved under linear transformations.

Definition 3.3.3: Orientation of Circles

Let z_1, z_2, z_3 be an ordered triple on C. A point z not on C is said to be on the right of C if $\operatorname{Im}(z, z_1, z_2, z_3) > 0$, and on the left of C if $\operatorname{Im}(z, z_1, z_2, z_3) < 0$.
This definition formulates what we mean by clockwise and counterclockwise orientation of circles.

We shall see that there are only two orientations. For a given circle, there are two regions, one is called left, and the other is called right, depending on which triple we use. The orientation of the circle is determined by the triple.

Proof. The invariance of cross ratio means that we only need to consider C being $\mathbb{R}$. Now

$$(z, z_1, z_2, z_3) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R}, ad - bc \neq 0$$

The coefficients depend on the triple. We have

$$\operatorname{Im}(z, z_1, z_2, z_3) = \frac{ad - bc}{|cz + d|^2} \operatorname{Im} z$$

Now we see that the distinction of right and left only depends on the upper and lower plane, which is right and which left depends on the triple, more specifically, the determinant $ad - bc$. □

Remark:
If we draw an arrow $z_1 \to z_2 \to z_3$, then the left and right regions are determined by the left and right direction of the arrow.
We define the points to the left of the circle are called **inside** the circle, and the points to the right of the circle are called **outside** the circle.

3.3.5 Families of Circles

We can use circles to visualize the linear transformations.

Consider

$$w = k \cdot \frac{z-a}{z-b}$$

Then $a \mapsto 0, b \mapsto \infty$.

- Circles that pass through a, b are mapped to circles that pass through $0, \infty$, which are straight lines passing through 0.
- Concentric circles centered at 0 in w-plane are given by $|w| = \rho$, the preimage is

$$\left|\frac{z-a}{z-b}\right| = \frac{\rho}{|k|}$$

These are Apollonius circles, with limit points a, b.

Definition 3.3.4: Circular Net

Let C_1 be the circles passing a, b, and C_2 be the Apollonius circles with limit points a, b. The family of circles $C_1 \cup C_2$ is called a **circular net** with respect to a, b.

Proposition: **Properties of Circular Nets**

- $\forall z \in C$, there is exactly one circle in C_1 that passes through z, and exactly one circle in C_2 that passes through z.
- Every circle in C_1 intersects every circle in C_2 at two points, with right angles.
- Reflection of one circle in C_1: Transfer every C_2 onto itself, and C_1 onto another C_1.
- Reflection of one circle in C_2: Transfer every C_1 onto itself, and C_2 onto another C_2.
- The limit points a, b are symmetric with respect to every circle in C_2, and not symmetric with respect to any other circle.

Proof. All these can be proven by mapping this to the w-plane. □

Remark:
Geometrically speaking,

- The circles in C_1 are defined by

$$\arg \frac{z-a}{z-b} = \text{constant}$$

- The circles in C_2 are defined by

$$\left|\frac{z-a}{z-b}\right| = \text{constant}$$

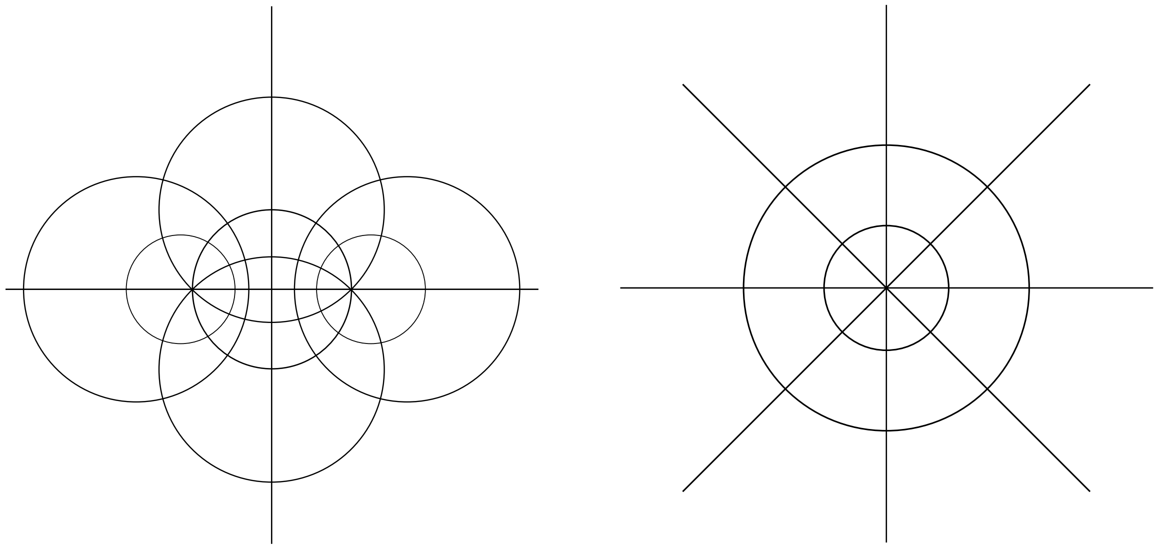

Figure 3.2: Circular Nets

If a linear transformation takes a, b to a', b', it can be written by

$$\frac{w-a'}{w-b'} = k \cdot \frac{z-a}{z-b} \tag{3.11}$$

It can be seen as a composition of first bringing a, b to $0, \infty$, then to a', b' by inverse transformation. It is clear that the transformation takes C_1 to C_1' and C_2 to C_2', where C_1' is the circles passing through a', b', and C_2' is the Apollonius circles with limit points a', b'.

The following are some special cases.

- If $a' = a, b' = b$, we say a, b are fixed points of T. In this case, the whole circular net is would map onto itself. The transformation can be visualized by studying k.
  - Taking arg we see that C_1 circles are changed to circles adding $\arg k$.
  - Taking modulus we see that C_2 circles are changed to circles multiplying $|k|$.
- Additionally, if $k \in \mathbb{R}$, then C_1 does not change, (if $k > 0$ then the orientation of C_1 is preserved, if $k < 0$ then the orientation is reversed). We call the transformation **Hyperbolic**.
- If $|k| = 1$, then C_2 does not change, and we call the transformation **Elliptic**.
- Every general transformation can be written as a composition of hyperbolic and elliptic transformations. (This is quite obvious, we can write $k = re^{i\theta}$. And the form of the two sides of equation 3.11 is preserved.)

Now we introduce another type of circular nets. Consider writing a linear transformation in the form

$$w = \frac{\omega}{z-a} + c \tag{3.12}$$

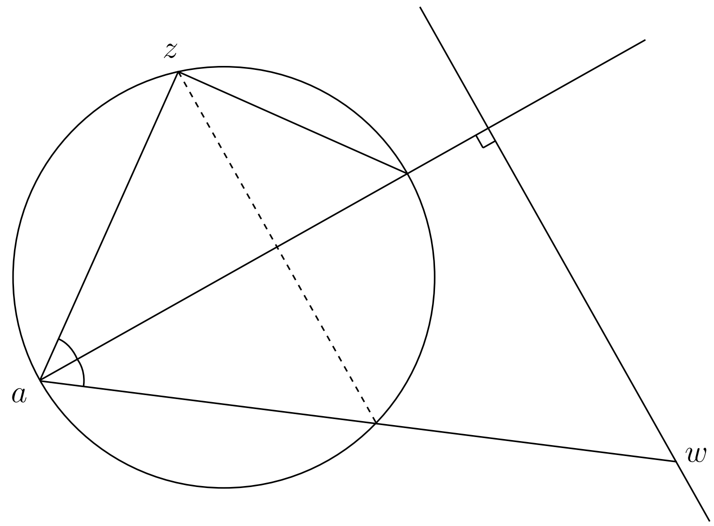

Figure 3.3: Parallel Lines and Cotangent Circles

As $a \mapsto \infty$, the circles passing through a becomes straight lines. The preimage of parallel lines are mutually tangent circles at a. (This can be seen by geometry, proof is easy)

The product of the diameter and the distance to the line is a constant w. To sets of cotangent circles may map to the $x - y$ grid in the w-plane.

Remark:
We can see the case as the limit case of the circular nets with two fixed points, where both Apollonius circle and the circles passing through the fixed points are degenerated cotangent ones. We also denote them C_1 and C_2.
The direction of the tangents is given by $\arg w$, as is seen.

Any linear transformation taking a to a' can be written as

$$\frac{\omega'}{w - a'} = \frac{\omega}{z - a} + c \tag{3.13}$$

If $a = a'$ is the only fixed point, then we have $\omega = \omega'$ and

$$\frac{\omega}{w - a} = \frac{\omega}{z - a} + c$$

A multiplicative parameter is arbitrary, assume $c \in \mathbb{R}$ and every C_1 is mapped onto itself, so we say the transformation is **Parabolic**, which are flows of C_2.

The fixed points of a linear transformation can be found by solving

$$z = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{3.14}$$

If it has two distinct roots, then the transformation is hyperbolic or ecliptic. However, if the two roots are the same, then the transformation is parabolic. The condition for a parabolic

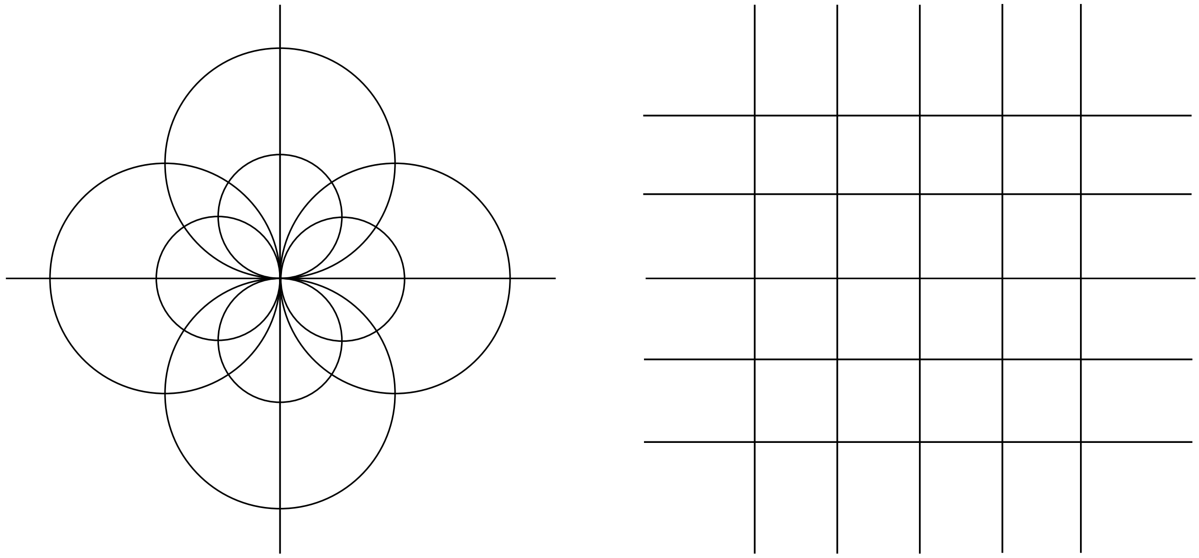

Figure 3.4: Circular Nets of One Fixed Point

transformation is:

$$(\alpha - \delta)^2 = 4\beta\gamma \tag{3.15}$$

- If the equation has two distinct roots a, b, we can write it in the form 3.11, and use the circular nets to visualize the transformation.
- To visualize parabolic transformations, we write it in the nets with one fixed point.

A linear function that is neither hyperbolic, elliptic, or parabolic is called **Loxodromic**.

3.4 Elementary Conformal Mappings

The visualization of conformal mappings is very important in complex analysis, as it gives us a direct intuition of analytic functions.

3.4.1 Level Curves

When a point-to-point visualization is hard to interpret, we can use curve families of known nature to gain a more direct intuition.

Definition 3.4.1: Level Curves

If $f : \Omega \to \mathbb{C}$ are determined by the real functions $f = u + iv$, then the curves defined by $u(x, y) = c$ or $v(x, y) = c$ for some constant c are called the **level curves** of f.

It is of conformal nature that the level curves of u and v are orthogonal to each other.

In a more general sense, any orthogonal net curves can be used to visualize the conformal mapping. As we did for linear transformations.

The Power $w = z^\alpha$ We use the polar coordinate chart here. Let $S(\varphi_1, \varphi_2)$ be the sector defined

$$S(\varphi_1, \varphi_2) = \left\{z = re^{i\varphi} : r > 0, \varphi_1 < \varphi < \varphi_2\right\} \tag{3.16}$$

This is indeed a region. The power function has the property

$$|w| = |z|^\alpha \qquad \text{and} \qquad \arg w = \alpha \arg z$$

(Well we take one branch of the multivalue function). So that $f(S(\varphi_1, \varphi_2)) = S(\alpha\varphi_1, \alpha\varphi_2)$. It is analytic for certain domain $S(\varphi_1, \varphi_2)$. The derivative

$$Dz^\alpha = De^{\alpha \log z} = \alpha \frac{w}{z}$$

The Exponential Function $w = e^z$ This one is well to understand. Some examples include

- Cartesian chart to polar chart.
- Straight lines to logarithmic spirals. (Circles and rays maybe)

3.4.2 A Brief Survey of Conformal Mappings

Our ultimate goal is to map a region Ω_1 to another Ω_2. It is advisable to do it in two parts:

- Mapping Ω_1 to a disk or half plane.
- Mapping the disk or half plane to Ω_2.

We shall later prove that this is possible for any region whose boundary is a simple closed curve.

Now our tools are: linear transformations, the exponential function, and the power function, the logarithm. They have properties that transform circles and lines to each other, so their use are limited to regions bounded by circles or lines.

Example: **Conformal Mappings to Half Plane or Circle**

We start by considering a region bounded by **two circular arcs.**

- If the two intersection points are a, b, we use $z_1 = (z-a)/(z-b)$ to make it to a sector.
- An appropriate power would make it into a half plane.
- If the two circles are tangent, then using $z_1 = 1/(z-a)$ will make it into a parallel strip. And a suitable exponential function would do the trick.

Now we map $\mathbb{C} - [-1, 1]$ to a circle.

- Use $z_1 = \dfrac{z+1}{z-1}$ to get $\mathbb{C} - (-\infty, 0]$.
- Use $z_2 = \sqrt{z_1}$ to get $\{z : \operatorname{Re} z > 0\}$.
- Use $w = \dfrac{z_2 - 1}{z_2 + 1}$ to get the unit disk.

The overall mapping is

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right) \text{ and } w = z - \sqrt{z^2 - 1} \tag{3.17}$$

We take a closer look at the mapping 3.17. Let $w = \rho e^{i\theta}$, where $\rho < 1$ or > 1, then we have

$$x = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\theta, \qquad y = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\theta$$

For circle and rays in w-plane, we eliminate ρ and θ respectively,

$$\frac{x^2}{\left(\frac{1}{2}(\rho + \frac{1}{\rho})\right)^2} + \frac{y^2}{\left(\frac{1}{2}(\rho - \frac{1}{\rho})\right)^2} = 1$$

$$\frac{x^2}{\cos^2\theta} + \frac{y^2}{\sin^2\theta} = 1$$

The circles maps to eclipses, and the rays to hyperbolas. The mapping is conformal, and the angles are preserved as perpendicular intersections.

Another less trivial example, we consider the cubic polynomial $w = a_0 z^3 + a_1 z^2 + a_2 z + a_3$. We can reduce it to $w = z^3 - 3z$ easily. (Setting $z = z_1 - a_1/3a_0$ to cancel out quadratic terms and normalize it)

Using the transformation 3.17, we let

$$z = \zeta + \frac{1}{\zeta}$$

take ζ to lay in either the inside or outside the disk. Then we have

$$w = \zeta^3 + \frac{1}{\zeta^3}.$$

The total effect is eclipse $\to$ circle $\to$ stretching rotation circle $\to$ eclipse. Or simply stretching rotation along the eclipse.

3.4.3 A Brief Survey on Riemann Surfaces

For non-surjective maps, the image of regions would overlap. We can see it as different layers of the same $\mathbb{C}$ plane. This is the original intuition of Riemann surfaces. However, we shall not formally define the concept, just to give some intuitive understanding.

Example: **Riemann Surfaces**

For the simple power z^n where $n > 1, n \in \mathbb{Z}$, the Riemann surfaces are like spirals around a vertical axis. After n turns we shall end up at the beginning.
We see it has fairly well topological properties, with locally Euclidean and all that.
The exponent e^z is the same, but with endless spirals.

A **fundamental region** is a region that maps to the whole space except for some cuts in a one-to-one manner. Identifying Riemann surfaces simply include Identifying fundamental regions and sticking different layers together according to the edge of the fundamental regions.

Chapter 4

Complex Integration

4.1 Fundamental Theorems

Similar to the real case, we have definite and indefinite integrals for complex field.

4.1.1 Line Integral

A direct generalization of the real integral is the integral of a function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$. If $f(t) = u(t) + iv(t)$, then

$$\int_a^b f(t)\mathrm{d}t = \int_a^b u(t)\mathrm{d}t + i\int_a^b v(t)\mathrm{d}t \tag{4.1}$$

It has similar properties as the real integral, such as linearity, additivity, and monotonicity. If $c \in \mathbb{C}$ we have

$$\int_a^b cf(t)\mathrm{d}t = c\int_a^b f(t)\mathrm{d}t$$

and if $a \leq b$ the fundamental inequality holds:

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\mathrm{d}t$$

This would naturally turn to our definition of a complex line integral.

Definition 4.1.1: Complex Line Integral

Let γ be a piecewise differentiable arc with equation $z = z(t)$, $a \leq t \leq b$. If $f : \gamma \to \mathbb{C}$ is continuous on γ, then the complex line integral of f along γ is defined as

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t \tag{4.2}$$

It is easy to show that the definition is invariant under reparametrization.

Remark:
The integral can be defined by the Riemann sum just like the real case. If γ is a polygonal

arc, then the integral can be computed as

$$\int_\gamma f(z)\mathrm{d}z = \lim \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) \tag{4.3}$$

where z_k are the points on γ with max distance tend to zero.

Proposition: **Some Properties of Line Integrals**

Similar of that of $\mathbb{R}^2$, we have

- The reverse direction of γ is $-\gamma$, then

$$\int_{-\gamma} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z \tag{4.4}$$

- Additive to the arcs:

$$\int_{\gamma_1+\gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z \tag{4.5}$$

- Expanding the integral, if $f = u + iv$ we have

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma u\mathrm{d}x - v\mathrm{d}y + i\int_\gamma v\mathrm{d}x + u\mathrm{d}y$$

- For respect to the arc length, we have

$$\int_\gamma f(z)\mathrm{d}s = \int_\gamma f(z)\,|\mathrm{d}z| = \int_a^b f(z(t))\,|z'(t)|\,\mathrm{d}t$$

  And the fundamental inequality holds:

$$\left|\int_\gamma f(z)\mathrm{d}s\right| \le \int_\gamma |f(z)|\mathrm{d}s$$

For integrals of the conjugate, we have

$$\int_\gamma f(z)\overline{\mathrm{d}z} = \overline{\int_\gamma \overline{f(z)}\mathrm{d}z} \tag{4.6}$$

So we can write:

$$\int_\gamma f(z)\mathrm{d}x = \frac{1}{2}\left(\int_\gamma f(z)\mathrm{d}z + \int_\gamma f(z)\overline{\mathrm{d}z}\right)$$
$$\int_\gamma f(z)\mathrm{d}y = \frac{1}{2i}\left(\int_\gamma f(z)\mathrm{d}z - \int_\gamma f(z)\overline{\mathrm{d}z}\right)$$

4.1.2 Rectifiable Arcs

Definition 4.1.2: Rectifiable Arcs

The length of an arc can be defined as the least upper bound of the sums of the lengths of the polygonal arcs that approximate it. That is,

$$|z(t_1) - z(t_0)| + \cdots + |z(t_n) - z(t_{n-1})| = \sum_{k=1}^{n} |z(t_k) - z(t_{k-1})|$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of the interval $[a, b]$. The arc is said to be rectifiable if the length is finite.

It is easy to show that piecewise differentiable arcs are rectifiable.

When γ is rectifiable, we can define the integral with respect to the arc length:

Definition 4.1.3: Integral with Respect to Arc Length

Let γ be a rectifiable arc, then the integral of f with respect to the arc length is defined as

$$\int_\gamma f(z)\mathrm{d}s = \lim \sum_{k=1}^{n} f(z_k)\,|z_k - z_{k-1}| \tag{4.7}$$

4.1.3 Line Integral As Functions of Arcs

If $p, q : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous in Ω, and γ is any piecewise differentiable arc in Ω, then we have a function

$$\gamma \mapsto \int_\gamma p\mathrm{d}x + q\mathrm{d}y \tag{4.8}$$

This is a functional on the space of arcs in Ω. In analysis, we've known that the integral is zero for every closed curve is called a **conservative field**.

Let p, q be continuously differentiable on a region $\Omega \subseteq \mathbb{R}^2$. The line integral $\int_\gamma p\mathrm{d}x + q\mathrm{d}y$ is independent of the path γ if and only if there exists a function $U : \Omega \to \mathbb{R}^2$ such that $p = \dfrac{\partial U}{\partial x}$ and $q = \dfrac{\partial U}{\partial y}$.

The function U is called a **potential function** of the field (p, q), and it is unique up to an additive constant.

Now, when do a continuous function $f : \mathbb{C} \to \mathbb{C}$ has a primitive function F such that $F' = f$?

Let $f = u + iv$, then we have

$$u = \frac{\partial F}{\partial x}, \quad v = \frac{\partial F}{\partial y}$$

Comparing to the last theorem, we have

Theorem 4.1.1: Indefinite Integralizable Functions

If f is continuous, and has continuous partial derivatives in a simply connected domain $\Omega \subseteq \mathbb{C}$, then there exists a function $F : \Omega \to \mathbb{C}$ such that $F' = f$ iff the integral $\displaystyle\int_\gamma f(z)\mathrm{d}z$ is only dependent on the endpoints of γ and not on the path taken.

NOTE that the region can have holes, but it must be simply connected.

An immediate example shows that

$$\int_\gamma (z-a)^n \mathrm{d}z = 0, n \in \mathbb{N} \tag{4.9}$$

As $(z-a)^n$ has a primitive function $F(z) = \dfrac{(z-a)^{n+1}}{n+1}$, which is analytic on $\mathbb{C}$. If $n < 0, n \neq -1$, then the result also holds for any closed curves that do not pass through a.

For $n = -1$, the result does not hold, for a circle $C : z = a + \rho e^{it}, 0 \leq t \leq 2\pi$, we have

$$\int_C \frac{\mathrm{d}z}{z-a} = \int_0^{2\pi} i\rho e^{it} \cdot \frac{1}{\rho e^{it}} \mathrm{d}t = 2\pi i \tag{4.10}$$

Remark:
This implies that it is impossible to define a single-valued branch of $\log(z-a)$ in an annulus $\rho_1 < |z-a| < \rho_2$.

4.1.4 Cauchy's Integral Theorem for Rectangles

Let R be a rectangle defined $a \leq x \leq b, c \leq y \leq d$. Let the boundary (counterclockwise) be ∂R. (Note that R is closed, so not a region)

Theorem 4.1.2: Cauchy's Integral Theorem on Rectangles

If f is analytic on R, (on an open region containing R), then

$$\int_{\partial R} f(z)\mathrm{d}z = 0 \tag{4.11}$$

Proof. Let

$$\eta(R) = \int_{\partial R} f(z)\mathrm{d}z$$

Divide R into four congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$ with the same orientation as R, then we have

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$$

We have one of the four rectangles $R^{(k)}$, denoted by R_1, having

$$|\eta(R_1)| \geq \frac{1}{4} |\eta(R)|$$

Repeat the process and get $R \supset R_1 \supset R_2 \supset \cdots$, we have

$$|\eta(R_n)| \geq \frac{1}{4^n} |\eta(R)|$$

From the nested chain of closed sets, we have a $z^* \in R_n$ for all n, so for given $\epsilon > 0$, there exists $\delta > 0$ such that $\forall z, |z - z^*| < \delta$, we have

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon$$

Choose large enough n that $R_n \subseteq \{z : |z - z^*| < \delta\}$. From theorem 4.1.1, we have

$$\int_{\partial R_n} \mathrm{d}z = 0, \qquad \int_{\partial R_n} z \mathrm{d}z = 0$$

So we write

$$\eta(R_n) = \int_{\partial R_n} (f(z) - f(z^*) - (z - z^*) f'(z^*)) \, \mathrm{d}z$$

Let d_n, L_n denote the diagonal length and perimeter of R_n, so we have $d_n = 2^{-n} d, L_n = 2^{-n} L$, so

$$|\eta(R_n)| \leq \epsilon \int_{\partial R_n} |z - z^*| \, |\mathrm{d}z| \leq \epsilon d_n L_n = \epsilon 4^{-n} dL$$

So we get

$$|\eta(R)| \leq \epsilon dL$$

As ϵ can be arbitrarily small, we have $\eta(R) = 0$. □

Now we weaken the conditions step by step.

Theorem 4.1.3: Cauchy's Theorem on Stained Rectangles

Let R' be a rectangle R omitting a finite number of interior points ζ_i, if f is analytic on R' and

$$\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0, i = 1, 2, \ldots, n$$

holds, then

$$\int_{\partial R} f(z) \mathrm{d}z = 0 \tag{4.12}$$

Proof. We shall only consider one point, as R can be splited into several rectangles R_i with at most one hole.

Let R_0 be a small rectangle surrounding ζ, splitting shows that

$$\int_{\partial R} f(z) \mathrm{d}z = \int_{\partial R_0} f(z) \mathrm{d}z$$

For any $\epsilon > 0$, choose small enough R_0 that

$$|f(z)| \leq \frac{\epsilon}{|z - \zeta|}$$

Thus,

$$\left| \int_{\partial R} f(z) \mathrm{d}z \right| \leq \epsilon \int_{\partial R_0} \frac{|\mathrm{d}z|}{|z - \zeta|} \leq 8\epsilon$$

□

Note the condition holds automatically if f is bounded on R'.

4.1.5 Cauchy's Theorem in Disk

Let Δ denote an open disk $|z - z_0| < \rho$.

Theorem 4.1.4: Cauchy's Theorem on Disk

If $f(z)$ is analytic in an open disk Δ. Then

$$\int_{\gamma} f(z)\mathrm{d}z = 0 \tag{4.13}$$

for every closed curve γ in Δ.

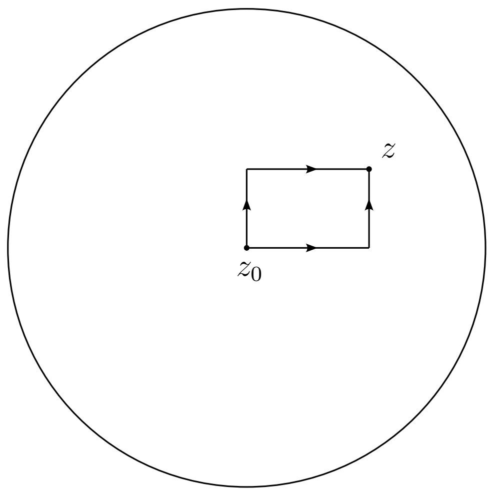

Figure 4.1: Diagram of F(z)

Proof. Let $F(z) = \displaystyle\int_{\sigma} f(z)\mathrm{d}z$, where σ is the rectangular route (either up or down is the same) from z_0 to z as shown in Figure 4.1. Then by the two routes respectively, we have

$$\frac{\partial F}{\partial x} = f(z), \quad \frac{\partial F}{\partial y} = if(z)$$

Therefore, F follows the Cauchy-Riemann equations, and $F'(z) = f(z)$. □

Similarly, we also have

Theorem 4.1.5: Cauchy's Theorem on Disk with Holes

Let Δ' be an open disk $|z - z_0| < \rho$ omitting a finite number of interior points ζ_i, if f is analytic on Δ' and

$$\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0, i = 1, 2, \ldots, n$$

then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

for every closed curve γ in Δ'.

4.2 Cauchy's Integral Formula

4.2.1 The Index of a Point

This formulate our notion of how many times a closed curve winds around a point not on the curve.

Lemma 4.2.1: The Index Lemma

If γ is a piecewise differentiable closed curve in $\mathbb{C}$, and a is a point not on γ, then there exists $n \in \mathbb{Z}$ that

$$\int_\gamma \frac{\mathrm{d}z}{z-a} = 2\pi i n \tag{4.14}$$

Proof. The simplest proof is computation. Let $\gamma : z = z(t)$, where $\alpha \le t \le \beta$, so we have

$$h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a}\mathrm{d}t$$

it is continuous and differentiable on $[\alpha, \beta]$.

$$h'(t) = \frac{z'(t)}{z(t)-a}$$

Multiplying to the left, we have the derivative of the function $e^{-h(t)}(z(t)-a)$ is zero. So we have

$$e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}$$

Since $z(\alpha) = z(\beta)$, we have $e^{h(\beta)} = 1$, so $h(\beta) = 2\pi i n$ for some integer n. $\square$

Definition 4.2.1: Index of a Point to a Curve

If γ is a piecewise differentiable closed curve in $\mathbb{C}$, and a is a point not on γ, then the index of a to γ is defined as

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-a} \tag{4.15}$$

n is also called the winding number of γ around a.

The following are some properties of the index:

- It is clear that $n(-\gamma, a) = -n(\gamma, a)$.
- If γ lies in an open disk, and a is outside of the open disk, then $n(\gamma, a) = 0$.

A closed curve γ is a closed point set, and its complement is open so can be divided into components as open regions. We say γ determines these regions.

Theorem 4.2.1: Regions by a Closed Curve

The function $a \mapsto n(\gamma, a)$ is constant for each region determined by γ.

Proof. As two points in a region can be joined by a polygonal path. We only need to prove when the segment ab does not meet γ and lies in the same region.

Outside the segment the function $\dfrac{z-a}{z-b}$ is never real and ≤ 0. Then $\log \dfrac{z-a}{z-b}$ can be analytically defined on $\mathbb{C} - (-\infty, 0]$. Its derivative is what we want:

$$\frac{\mathrm{d}}{\mathrm{d}z} \log \frac{z-a}{z-b} = \frac{1}{z-a} - \frac{1}{z-b}$$

Therefore, we have

$$\int_\gamma \frac{\mathrm{d}z}{z-a} - \int_\gamma \frac{\mathrm{d}z}{z-b} = 0$$

□

Specifically, if $|a|$ is sufficiently large, then $n(\gamma, a) = 0$. So the index is zero for the unbounded region determined by γ.

The case where $n(\gamma, a) = 1$ is worth noting. For convenience, we want $a = 0$.

Lemma 4.2.2: When Index is 1

Let z_1, z_2 be points on a closed curve γ which does not pass through 0. Following the direction of the curve, we denote γ_1 be the subarc from z_1 to z_2, and γ_2 be the subarc from z_2 to z_1. Also assume $\operatorname{Im} z_1 < 0, \operatorname{Im} z_2 > 0$. If γ_1 does not intersect the negative real axis, and γ_2 does not intersect the positive real axis, then $n(\gamma, 0) = 1$.

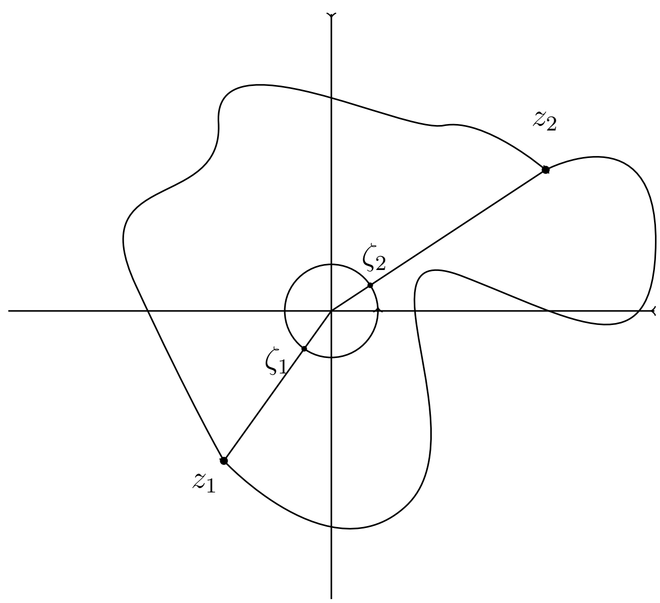

Figure 4.2: When Index is 1

Proof. Well this is a how we formulate our intuition of winding the origin once.

Take a small circle around the origin, and the origin belongs to the unbounded region of the two sides, so we have $n(\gamma, 0) = n(C, 0) = 1$. □

4.2.2 The Integral Formula

Let $f(z)$ be analytic on an open disk Δ. Let γ be a closed piecewise differentiable curve in Δ and a be a point not on γ.

We apply Cauchy's theorem on disks with holes 4.1.5 for the function

$$F(z) = \frac{f(z) - f(a)}{z - a} \tag{4.16}$$

We have

$$\int_\gamma F(z)\mathrm{d}z = 0 \tag{4.17}$$

$$\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z = f(a) \int_\gamma \frac{\mathrm{d}z}{z-a} \tag{4.18}$$

Theorem 4.2.2: Cauchy's Integral Theorem

Suppose f is analytic on an open disk Δ (there can be exceptional points ζ_j, such that $\lim_{z\to\zeta_j}(z-\zeta_j)f(z) = 0$), and γ be a closed piecewise differentiable curve in Δ and a is a point not on γ. Then

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z \tag{4.19}$$

NOTE here a may not be in Δ.

Remark:
The region which f is analytic on may also be not a disk, as long as theorem 4.1.5 holds, we can still apply the theorem.

An immediate usage is when $n(\gamma, a) = 1$, we have

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z, \quad a \notin \gamma, n(\gamma, a) = 1 \tag{4.20}$$

This is called the Cauchy's integral formula.

4.2.3 Higher Order Derivatives

The Cauchy's integral formula provided us with a powerful tool to study the local properties of analytic functions.

Lemma 4.2.3: Derivatives Under Integral

The function $\varphi(z,t)$ is defined for $z \in \Omega$ and $t \in [\alpha, \beta]$ and is continuous. If $\forall t, z \mapsto \varphi(z,t)$ is analytic, then

$$F(z) = \int_\alpha^\beta \varphi(z,t)\mathrm{d}t \tag{4.21}$$

is analytic in Ω and

$$F'(z) = \int_\alpha^\beta \frac{\partial \varphi(z,t)}{\partial z}\mathrm{d}t \tag{4.22}$$

Proof. Using the definition, we have

$$\frac{F(z) - F(z_0)}{z - z_0} = \int_\alpha^\beta \frac{\varphi(z,t) - \varphi(z_0,t)}{z - z_0}\mathrm{d}t$$

Just like the real case, get a small closed space around z_0, and continuity of φ in a closed set implies uniform continuity. Using the $\epsilon - \delta$ language it is easy. □

Let f be analytic on the region Ω. For a point $a \in \Omega$ consider its δ-neighborhood $\Delta = N_\delta(a) = \{z : |z - a| < \delta\}$ contained in Ω. In Δ we have a circle C around a, so we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta, \qquad \forall z \text{ inside } C$$

Provided that the derivative can be done inside the integral, we have

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta, \qquad \forall z \text{ inside } C \tag{4.23}$$

And higher order derivatives can be computed similarly:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta, \qquad \forall z \text{ inside } C \tag{4.24}$$

As the choice of a is arbitrary in Ω, we've proven that f has arbitrary order derivatives in Ω.

The following are some classical results of Cauchy's integral formula.

Theorem 4.2.3: The Mean Value Theorem of Analytic Functions

Let f be analytic on a disk Δ with radius r centered at a. Then

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\mathrm{d}\theta \tag{4.25}$$

Proof. Directly from Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - a}\mathrm{d}z$$

Where C is the circle $|z - a| = r$. Parametrize $z = a + re^{i\theta}$, we have

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\mathrm{d}\theta$$

□

Theorem 4.2.4: Morera's Theorem

Let f be continuous on a region $\Omega \subseteq \mathbb{C}$. If for every closed curve γ in Ω, we have

$$\int_\gamma f(z)\mathrm{d}z = 0 \tag{4.26}$$

then f is analytic on Ω.

Proof. We know that f is the derivative of some analytic function F on Ω. So f is analytic. □

Theorem 4.2.5: Cauchy's Estimation

Let f be analytic on a disk Δ with radius r centered at a. Then for any $z \in \Delta$, we have

$$\left|f^{(n)}(a)\right| \leq \frac{M}{r^n}n!, \quad M = \max_{|\zeta - a| = r} |f(\zeta)| \tag{4.27}$$

Proof. We use Cauchy's integral formula 4.24. If we have $|f(z)| \leq M$ for all $z \in C$, then let the radius of the circle C be r, we have

$$\left|f^{(n)}(a)\right| = \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta\right| \leq \frac{Mn!}{r^n} \tag{4.28}$$

□

Theorem 4.2.6: Liouville's Theorem

Let f be a bounded analytic function on $\mathbb{C}$. Then f is constant.

Proof. For $n = 1$ we have

$$|f'(z)| \leq \frac{M}{r}, \forall r > 0$$

So we have $f'(z) = 0$. □

This leads to a trivial proof of the fundamental theorem of algebra: Every non-constant complex polynomial $p(z)$ has at least one root in $\mathbb{C}$.

Proof. Suppose $P(z)$ is a polynomial with degree $n \geq 1$. Then $f(z) = 1/P(z)$ is bounded and analytic if there are no roots. By Liouville's theorem, $f(z)$ is constant, which contradicts. □

4.3 Local Properties of Analytic Functions

4.3.1 Removable Singularities and Taylor's Formula

We've shown that the Cauchy's Theorem holds for regions that have finite exceptional points ζ_j. However, we will see that these are not actual singularities. They can be filled and thus producing a whole analytic function on the entire region.

Theorem 4.3.1: Removable Singularities

If Ω is an open region, and Ω' is contracted by Ω omitting a point a. Suppose f is analytic on Ω'. Then there is an analytic function g on Ω which $\forall z \in \Omega', g(z) = f(z)$ iff we have

$$\lim_{z \to a}(z-a)f(z) = 0$$

The extended function g is unique.

Proof. The necessity and uniqueness is trivial to notice. To prove sufficiency we let D be an open disk that $\overline{D} \subseteq \Omega$ with center a. Define $g(z) = f(z)$ for $z \notin D$ and

$$g(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z}, \qquad z \in D$$

Then g is the desired function. □

If f is analytic on Ω and $a \in \Omega$, we apply the theorem to the function

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

Then $F(z)$ has an analytic extension on Ω, denoted $f_1(z)$ with $f_1(z) = f'(a)$. Resume the construction, we have

$$\begin{aligned} f(z) &= f(a) + (z-a)f_1(z) \\ f_1(z) &= f(a) + (z-a)f_2(z) \\ &\cdots\cdots \\ f_{n-1}(z) &= f(a) + (z-a)f_n(z) \end{aligned} \tag{4.29}$$

Getting together, we have

$$f(z) = f(a) + (z-a)f_1(a) + (z-a)^2 f_2(a) + \cdots + (z-a)^{n-1}f_{n-1}(a) + (z-a)^n f_n(z)$$

Taking n^{th} derivative, we have

$$f^{(n)}(a) = n! f_n(a)$$

Theorem 4.3.2: Taylor's Theorem

If f is analytic on Ω and $a \in \Omega$, then

$$f(z) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(a)(z-a)^k + f_n(z)(z-a)^n$$

Where f_n is analytic in Ω.

For $z \in D$, writing

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\zeta)}{\zeta - z}\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta-a)^n(\zeta - z)}\mathrm{d}\zeta - R(z)$$

The integrals in $R(z)$ has the form

$$F_\nu = \int_{\partial D} \frac{\mathrm{d}\zeta}{(\zeta - a)^\nu(\zeta - z)}, \qquad \nu \geq 1$$

Taking it as a function of a. We have

$$F_1 = \frac{1}{z-a}\int_{\partial D}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - a}\right)\mathrm{d}\zeta = 0$$

Then we have

$$F_{\nu+1} = \frac{1}{\nu!}\frac{\mathrm{d}^\nu F_1}{\mathrm{d}a^\nu} = 0$$

Therefore, $R = 0$ and we have

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - a)^n(\zeta - z)}, \qquad z \in D \tag{4.30}$$

4.3.2 Zeros and Poles

We'll see that local properties of an analytic function has surprising influence on the global properties of the function.

Theorem 4.3.3: All Derivatives Zero

Let f be analytic on Ω and $a \in \Omega$. If $f^{(n)}(a) = 0$ for all $n \geq 0$, then $f(z) = 0$ for all $z \in \Omega$.

Proof. We write

$$f(z) = f_n(z)(z-a)^n$$

according to Taylor's theorem. An estimation of f_n can be obtained by equation 4.30: If $M = \max\{f(z) : z \in \partial D\}$ and the radius of ∂D is R, then we have

$$|f_n(z)| \leq \frac{1}{2\pi}\int_{\partial D}\frac{M\,|\mathrm{d}\zeta|}{R^n(R - |z-a|)} = \frac{M}{R^n}\cdot\frac{R}{R - |z-a|}$$

Thus we have

$$|f(z)| \leq \left(\frac{|z-a|}{R}\right)^n \frac{MR}{R - |z-a|}, \qquad \forall z \in D, \forall n \in \mathbb{N}$$

As $|z-a|/R < 1$, we have $f(z) = 0$ for all $z \in D$.

To show that $f(z) = 0$ for all $z \in \Omega$, we denote

$$E_1 = \left\{z \in \Omega : \forall n \in \mathbb{N}, f^{(n)}(z) = 0\right\}, \qquad E_2 = \Omega - E_1$$

We have E_1 being open, as for any $z \in E_1$, we can find a disk D such that $f(z) = 0$ for all $n \geq 0$. E_2 is also open for the derivatives of f is continuous. Therefore, the connectedness of Ω implies that $E_1 = \Omega$ as $a \in E_1$. □

Local Properties of Zeros Now, if f is not identically zero on Ω, then if $f(a) = 0$, there exists a smallest $n \in \mathbb{N}$ that $f^{(n)}(a) \neq 0$. We call a a **zero of f of order n**. Now it is possible to write

$$f(z) = (z-a)^n f_n(z), \qquad \text{where } f_n \text{ is analytic on } \Omega \text{ and } f_n(a) \neq 0 \tag{4.31}$$

Remark:
We can see that the zeros of a nontrivial analytic function has the same local properties as the polynomials, which can also be written in this form.

As f_n is continuous, there is a neighborhood D of a such that $f_n(z) \neq 0$ for all $z \in D$. In that neighborhood, a is the only zero of f. Thus **All zeros of a nontrivial analytic function are isolated**.

An equivalent formulation is as follows:

> f, g are analytic on Ω. If there is a set $S \subseteq \Omega$ containing a limit point in S, such that $f(z) = g(z)$ for all $z \in S$, then $f(z) = g(z)$ for all $z \in \Omega$.

Particularly, if f is analytic on a subreigion $\Omega' \subseteq \Omega$, and $f(z) = 0$ holds in Ω', then $f(z) = 0$ for all $z \in \Omega$. This is true for all subspaces Ω' that do not reduce to points, like lines, curves, etc.

Poles If a function f is analytic on $\Omega - \{a\}$, then we say a is an **isolated singularity** of f on Ω. We've already considered removable singularities. Since we can extend the function to a whole analytic function on Ω, there is no need for further discussion.

Let a be an isolated singularity of f. If $\lim\limits_{z\to a} f(z) = \infty$, then we say a is a **pole** of f. In a near neighborhood Ω' of a, we have $f(z) \neq 0$, and the function $g(z) = 1/f(z)$ is defined and analytic on $\Omega' - \{a\}$. As $\lim_{z\to a} g(z) = 0$, then a is a removable singularity of g, with $g(a) = 0$. The zero a has finite order h, with $g(z) = (z-a)^h g_h(z), g_h(a) \neq 0$. So we write

$$f(z) = (z-a)^{-h} f_h(z), \qquad \text{where } f_h \text{ is analytic on } \Omega' \text{ and } f_h(a) \neq 0 \tag{4.32}$$

The number h is called the **order of the pole** of f at a. Similar to the zeros, we can see that the poles of a nontrivial analytic function is also isolated. (This is partly because our definition of poles require a limit process, which is not possible for a non-isolated singularity.)

Remark:
There are singularities that are not removable and not poles. The limit process near such singularities does not exist.

Definition 4.3.1: Meromorphic

A function f is called **meromorphic** on Ω if it is analytic on Ω except for some isolated singularities, which are all poles.

Theorem 4.3.4: Countable Poles for Meromorphic Functions

If f is meromorphic on Ω, then the poles of f are at most countable and have no limit points in Ω.

Proof. As poles are isolated, we can find a neighborhood U_z for each pole z that contains no other poles. As Ω is the subset of $\mathbb{C}$, which is second countable, we can find a countable basis for Ω. Take $z \in B_z \subseteq U_z$, where B_z is in the countable basis. Then each pole corresponds to a distinct element in the countable basis, so the poles are at most countable. $\square$

Proposition: **The Operations of Meromorphics**

The sum, difference, product, and quotient of two non-trivial(not zero) meromorphic functions is also meromorphic. The composition of a meromorphic function with an analytic function is also meromorphic.

Proof. For any point z, we can write $f(z) = (z - z_0)^n f_n(z)$, where $f_n(z_0) \neq 0$, for a non-zero, non-pole point, $n = 0$ would do. Substituting this for f, g it is easy to verify the result. $\square$

To make a clear classification of isolated singularities, we consider the expressions (where $\alpha \in \mathbb{R}$)

1. $\lim_{z \to a} |z - a|^\alpha |f(z)| = 0$.

2. $\lim_{z \to a} |z - a|^\alpha |f(z)| = \infty$.

If $f = 0$, then 1 always holds, and 2 never holds.

If f is not trivial, and 1 holds for certain α_0, then it holds for $\forall \alpha \geq \alpha_0$. Let $m \geq \alpha_0$ be an integer, then $(z - a)^m f(z)$ has a removable singularity that is a zero of finite order $k \in \mathbb{Z}_+$, so we have $(z - a)^m f(z) = (z - a)^k f_h(z)$, letting $h = m - k$, we have $f(z) = (z - a)^{-h} f_h(z)$, where f_h is analytic on Ω' and $f_h(a) \neq 0$. Therefore, 1 holds for all $\alpha > h$, and 2 holds for all $\alpha \leq h$.

If 2 holds for certain α_0, then it holds for $\forall \alpha \leq \alpha_0$. We have a similar discussion with the same result as above.

We conclude that there are three possibilities for an isolated singularity a of a nontrivial function f:

(i) 1 holds for all $\alpha \in \mathbb{R}$, then $f = 0$.

(ii) There is an integer h such that 1 holds for all $\alpha > h$ and 2 holds for all $\alpha \leq h$. The integer h is called the **algebraic order** of the singularity. It is positive for a pole, negative for a zero, and zero for a non-zero, non-pole analytic point.

(iii) Neither 1 nor 2 holds for any $\alpha \in \mathbb{R}$. Such singularities are called **essential isolated singularities**.

If a is a pole of order h, apply Taylor's theorem to the analytic function $(z - a)^h f(z)$, we have

$$(z - a)^h f(z) = B_h + B_{h-1}(z - a) + \cdots + B_1 (z - a)^{h-1} + \varphi(z)(z - a)^h$$

where $\varphi(z)$ is analytic at $z = a$. Thus, we have

$$f(z) = \frac{B_h}{(z - a)^h} + \frac{B_{h-1}}{(z - a)^{h-1}} + \cdots + \frac{B_1}{z - a} + \varphi(z) \tag{4.33}$$

The part $\dfrac{B_h}{(z - a)^h} + \dfrac{B_{h-1}}{(z - a)^{h-1}} + \cdots + \dfrac{B_1}{z - a}$ is called the **singular part** of f at a.

The essential isolated singularity is more complicated. We can not write it in a similar form as above. The neighborhood of an essential singularity can come close to any value in $\mathbb{C}$ and ∞, as we shall see.

Theorem 4.3.5: Weierstrass Theorem for Essential Isolated Singularities

Let f be analytic on $\Omega - \{a\}$, where a is an essential isolated singularity of f. Let $c \in \mathbb{C}$ be any number. Then for any neighborhood Ω' and any $r > 0$, there exists a point $z_0 \in \Omega', z_0 \neq a$ such that

$$|f(z_0) - c| < r$$

and there exists a point $z_1 \in \Omega', z_1 \neq a$ such that

$$|f(z_1)| > \frac{1}{r}$$

Proof. If it is not true, there exists $A \in \mathbb{C}, r > 0$ and a neighborhood Ω' of a such that $\forall z \in \Omega', z \neq a$, we have $|f(z) - A| \geq r$. Then $\forall \alpha < 0$, we have

$$\lim_{z \to a} |z - a|^{\alpha} |f(z) - A| = \infty$$

Well, a is not a essential isolated singularity of $f - A$, so there exists $\beta \in \mathbb{R}_+$ that

$$\lim_{z \to a} |z - a|^{\beta} |f(z) - A| = 0$$

As $\lim_{z \to a} |z - a|^{\beta} |A| = 0$, we have

$$\lim_{z \to a} |z - a|^{\beta} |f(z)| = 0$$

Contradicting that a is an essential isolated singularity of f.

The infinity part is similar, if $|f(z)| \leq 1/r$, then $\forall \alpha > 0$, we have $\lim_{z \to a} |z - a|^{\alpha} |f(z)| = 0$, contradicting again. □

Treating Infinity as a Point The notion of isolated singularity can also apply to ∞. The neighborhood of ∞ is seen as an open region containing some $\{z : |z| > R\}$. We can transfer the neighborhood of ∞ to the neighborhood of 0 by letting $g(z) = f(\frac{1}{z})$.

If we have $g(z) = z^h g_h(z)$, then $f(z) = g(\frac{1}{z}) = z^{-h} f_h(z)$, where $f_h(z) = g_h(\frac{1}{z})$. The classification of removable singularities, poles, and essential isolated singularities can be done similarly by consulting equations

$$\lim_{z \to \infty} \frac{|f(z)|}{|z|^{\alpha}} = 0, \quad \lim_{z \to \infty} \frac{|f(z)|}{|z|^{\alpha}} = \infty$$

If the singularity is non-trivial and non-essential, then there is a interger h such that the first limit holds for all $\alpha > h$ and the second limit holds for all $\alpha \leq h$. The integer h is called the **algebraic order** of f at ∞. It is positive for a pole, negative for a zero, and zero for a non-zero, non-pole analytic point.

The singular part expansion and Weierstrass theorem can also be done similarly.

4.3.3 The Local Mapping

We want to determine the number of zeros of an analytic function. Consider f is an analytic function (not identically zero) on an open disk Δ. Let γ be a closed curve in Δ that $\forall z \in \gamma, f(z) \neq$

0. For simplicity we only consider f has finite zeros, denoted $z_1, z_2, \ldots, z_n$ with multiplicity equals order. So we can write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n) g(z) \tag{4.34}$$

where g is analytic on Ω, and $\forall z \in \Omega, g(z) \neq 0$. Taking the logarithm (analytic for a small neighborhood of each point), we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)} \tag{4.35}$$

for any $z \neq z_j$ in Δ, particularly for $z \in \gamma$. We can integrate this on γ. Since $g(z) \neq 0$ on Ω, from Cauchy's Theorem we have

$$\int_\gamma \frac{g'(z)}{g(z)} \mathrm{d}z = 0$$

The definition of index yields

$$n(\gamma, z_1) + \cdots + n(\gamma, z_n) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z \tag{4.36}$$

If f has infinite zeros, then we choose a concentric smaller open disk Δ' that $\gamma \subseteq \Delta', \overline{\Delta'} \subseteq \Delta$ (This is possible due to normality). Then the compactness of $\overline{\Delta'}$ and the isolation of zeros implies that f has only finitely many zeros in $\overline{\Delta'}$, and the above equation still holds, as the zeros outside do not contribute to the sum.

Theorem 4.3.6: Number of Zeros

Let f be analytic on an open disk Δ (not identically zero), and z_j are distinct zeros of f that has order m_j. Then for every closed curve γ in Δ that does not pass through any z_j, we have

$$\sum_j m_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z \tag{4.37}$$

The sum has only finite terms that are not zero.

The function $w = f(z)$ maps γ into a closed curve Γ in the w-plane. And we have

$$\int_\Gamma \frac{\mathrm{d}w}{w} = \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z$$

Therefore, we have

$$n(\Gamma, 0) = \sum_j m_j n(\gamma, z_j) \tag{4.38}$$

Remark:
When γ is a circle, then all $n(\gamma, z_j)$ are either 0 or 1, and the number of zeros are calculated by theorem 4.3.6.

To find the roots of $f(z) = a$, apply theorem 4.3.6 to the function $g(z) = f(z) - a$. The zeros of g are the roots of $f(z) = a$, denoted as $z_j(a)$, so we have

$$\sum_j m_j(a) n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} \mathrm{d}z \tag{4.39}$$

$$n(\Gamma, a) = \sum_j m_j(a) n(\gamma, z_j(a)) \tag{4.40}$$

If a, b are in the same region determined by Γ, then $n(\Gamma, a) = n(\Gamma, b)$, so we have

$$\sum_j m_j(a) n(\gamma, z_j(a)) = \sum_j m_j(b) n(\gamma, z_j(b)) \tag{4.41}$$

Theorem 4.3.7: Local Variation of Roots

If f is analytic at a neighborhood of z_0 and $f(z_0) = w_0$, and $f(z) - w_0$ has a zero of order m at z_0. Then $\forall \epsilon > 0$ that z_0 is the only zero of $f(z) - w_0$ in $\overline{N_\epsilon(z_0)}$, there exists $\delta > 0$ such that $\forall w$ that $|w - w_0| < \delta$, $f(z) - w$ has exactly m zeros counting multiplicity in $N_\epsilon(z_0)$.

Proof. Letting γ be the circle $\partial N_\epsilon(z_0)$. Taking δ such that $N_\delta(w_0) \cap f(\gamma) = \emptyset$ would do. □

Remark:
If we take small enough ϵ, then $f'(z) \neq 0$ for all $z \in N_\epsilon(z_0) - \{z_0\}$. In this case, $\forall a \in w_0$, the m roots of $f(z) = a$ are all simple (of order 1).

Corollary 4.3.1: Image of Analytic Function on Open Sets

A nonconstant analytic function maps open sets onto open sets.

Proof. Let f be analytic on Ω and nonconstant, and $a \in \Omega$. Let $w_0 = f(a)$, then $f(z) - w_0$ has a zero of order $n \geq 1$ at a. Choose $\epsilon > 0$ such that $N_\epsilon(a) \subseteq \Omega$ and $f(z) - w_0$ has no other zeros in $\overline{N_\epsilon(a)}$. By theorem 4.3.7, there exists $\delta > 0$ such that $\forall w \in N_\delta(w_0)$, $f(z) - w$ has exactly n zeros in $N_\epsilon(a)$. Therefore, $N_\delta(w_0) \subseteq f(N_\epsilon(a))$, so $f(\Omega)$ is open. □

If the order of the zero is $m = 1$, then f is 1-1 on $f^{-1}(N_\delta(w_0)) \subseteq N_\epsilon(z_0)$. This is an open set due to continuity of f. We can take a smaller ϵ' such that $N_{\epsilon'}(z_0) \subseteq f^{-1}(N_\delta(w_0))$. Then f is injective on $N_{\epsilon'}(z_0)$, and thus a homeomorphism onto its image.

Corollary 4.3.2: Local Homeomorphism

If $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$, then there exists a neighborhood $N_\epsilon(z_0)$ such that f is conformal and is a homeomorphism from $N_\epsilon(z_0)$ onto its image.

Remark:
Note that f is analytic at z_0 means that it is analytic on a neighborhood of z_0. The condition $f'(z_0) \neq 0$ is equivalent to the condition that $f(z) - f(z_0)$ has a zero of order 1 at z_0.
This corollary can also be easily seen by the inverse function theorem, which states that if f is analytic at z_0 and $f'(z_0) \neq 0$, then there exists a neighborhood $N_\epsilon(z_0)$ such that f is a homeomorphism from $N_\epsilon(z_0)$ onto its image. We also note that the inverse function theorem cannot apply to the case that $f'(z_0) = 0$, as in corollary 4.3.1.

The inverse mapping is also conformal due to the inverse function theorem.
For order $n > 1$, we write

$$f(z) - w_0 = (z - z_0)^n g(z)$$

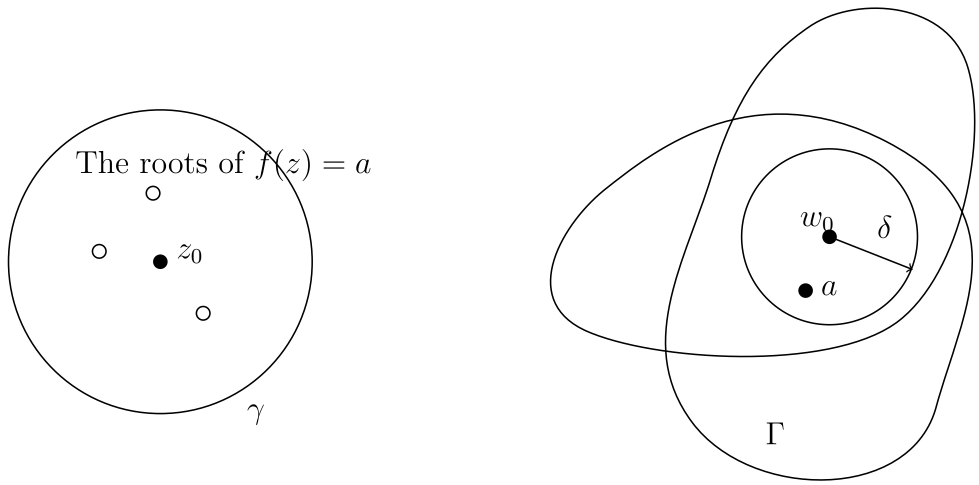

Figure 4.3: Local Mapping of an Analytic Function

where g is analytic at z_0 and $g(z_0) \neq 0$. Choose $\epsilon > 0$ such that $g(z) \in N_\delta(g(z_0))$ for all $z \in \Omega = N_\epsilon(z_0)$, where $\delta < |g(z_0)|$. In this neighborhood we define an analytic branch of $h(z) = \sqrt[n]{g(z)}$. As $g(\Omega)$ lies in a disk that do not contain zero, we do not need a branch cut.

$$f(z) - w_0 = \zeta^n(z), \qquad \zeta(z) = (z - z_0)h(z)$$

As $\zeta'(z_0) = h(z_0) \neq 0$, then ζ is a homeomorphism on a neighborhood of z_0. The function $w = w_0 + \zeta^n$ is familiar.

4.3.4 The Maximum Principle

Theorem 4.3.8: The Maximum Principle

If f is analytic and not constant in a region Ω, then $|f|$ has no maximum value in Ω.

Proof. Assume $z_0 \in \Omega$ and $|f(z_0)| = M = \max_{z\in\Omega} |f(z)|$. Then let D be a neighborhood of z_0 such that $D \subseteq \Omega$. Then $f(D)$ is a neighborhood of $f(z_0)$, and it contains some w_0 such that $|w_0| > M$. □

A restatement of the maximum principle is as follows:

> If f is analytic on a region Ω and E is a closed bounded subset of Ω, then $|f|$ has a maximum value on E, which is taken on the boundary of E.

The existence is due to the compactness of E, and the continuity of f. The maximum principle is then used to show that the maximum value is not taken in the interior of E.

Another computational proof is as follows:

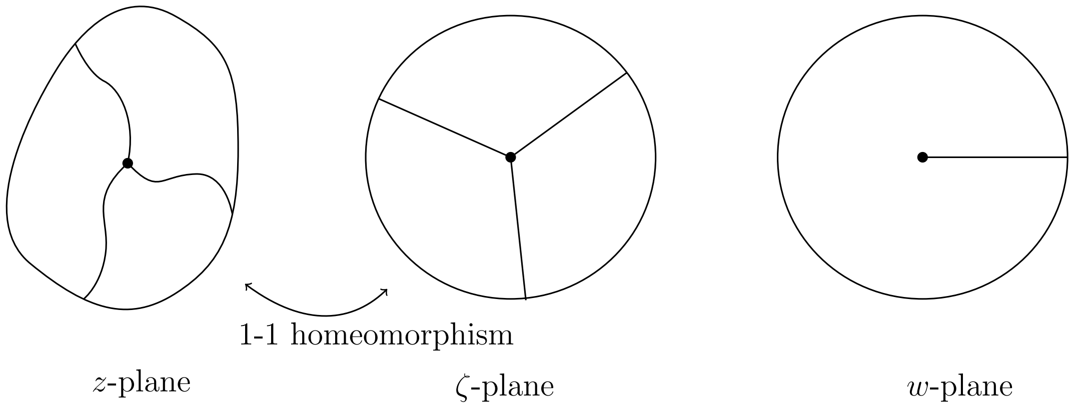

Figure 4.4: The Local Properties of Analytic Functions ($n = 3$)

Proof. Let D be a small disk centered at z_0 such that $\overline{D} \subseteq \Omega$. Then we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} \mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) \mathrm{d}\theta$$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + Re^{i\theta}) \right| \mathrm{d}\theta$$

Suppose $|f(z_0)|$ is a maximum, then f is constant on D. Which is a contradiction to the assumption that f is not constant in Ω. □

Now consider that f is analytic on an open set $D = \{z : |z| < R\}$ and continuous on $\overline{D}$. If $|f| \leq M$ for $z \in \partial D$, then we have $|f| \leq M$ for $z \in D$. There is a point $z_0 \in D$ that $|f(z_0)| = M$ iff f is a constant with norm M. So if we know that there is $f(z_1) < M$, we may want a better approximation.

Theorem 4.3.9: Schwarz Lemma

Let f be analytic on $D = \{z : |z| < 1\}$ and satisfies

- $f(0) = 0$.
- $\forall z \in D, |f(z)| \leq 1$.

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.
If $(\exists z_0 \in D, z_0 \neq 0, |f(z_0)| = |z_0|) \vee |f'(0)| = 1$, then $\exists c \in \mathbb{C}, |c| = 1$ that $f(z) = cz$.

Proof. Letting

$$f_1(z) = \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

This can be thought of an extended function of $f(z)/z$ thus is analytic on D. On ∂D, we have $|f_1(z)| = |f(z)| \le 1$. By the maximum principle, we have $|f_1(z)| \le 1$ for all $z \in D$. Therefore, $|f(z)| \le |z|$ and $|f'(0)| = |f_1(0)| \le 1$.

If either of the conditions hold, then $f_1(z) = c$. □

Generally speaking, if f is analytic on $D = \{z : |z| < R\}$ and $f(z) \le M$, and for some $|z_0| < R$ we have $f(z_0) = w_0, |w_0| < M$. We can find a linear fractional transformation $\zeta = Tz$ such that $Tz_0 = 0$ and $T(\partial D) = \{z : |z| = 1\}$ the unit circle. (This can be done by transforming z_0, z_0^* and some point on ∂D to $0, \infty, 1$ respectively, where z_0^* is the symmetric point of z_0 about the circle $|z| = R$.) Such a transformation can be written

$$Tz = \frac{R(z - z_0)}{R^2 - \overline{z_0}z}$$

Similarly, we take a linear fractional transformation $\omega = Sw$ such that $Sw_0 = 0$ and $S(\{w : |w| = M\}) = \{w : |w| = 1\}$. Then the function $g(z) = Sf(T^{-1}z)$ satisfies the conditions of Schwarz lemma, so we have $|g(z)| \le |z|$. Therefore, we have $|Sf(T^{-1}z)| \le |z|$ or $|Sf(z)| \le |Tz|$, explicitly,

$$\left| \frac{M(f(z) - w_0)}{M^2 - \overline{w_0}f(z)} \right| \le \left| \frac{R(z - z_0)}{R^2 - \overline{z_0}z} \right| \tag{4.42}$$

4.4 The General Cauchy Theorem

In previous discussions of Cauchy's theorem we've dealt with regions that are open disks. We now want to extend the theorem to more general regions, and open sets also (which may not be connected)

4.4.1 Chains and Cycles

Definition 4.4.1: Chains

Chains are the smallest Abelian group containing all directed curves in an open set $\Omega \subseteq \mathbb{C}$.

In the formulation, a chain can be written as

$$\gamma = a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n \tag{4.43}$$

where $a_j \in \mathbb{Z}, \gamma_j$ are directed curves. The zero and negative of the group is easy to find. An integral of a function f on a chain γ is defined as

$$\int_\gamma f(z)\mathrm{d}z = a_1 \int_{\gamma_1} f(z)\mathrm{d}z + a_2 \int_{\gamma_2} f(z)\mathrm{d}z + \cdots + a_n \int_{\gamma_n} f(z)\mathrm{d}z \tag{4.44}$$

it is clear that this definition is well-defined: different representations of the same chain would yield the same integral.

Definition 4.4.2: Cycles

A **cycle** is a chain that can be represented as a sum of closed curves.

A chain is a cycle if in its representation, the initial and terminal points of each directed curve appear the same number of times. A simple corollary is that *the integral of an exact differential on a cycle is zero.*

The index of a point with respect to a cycle is defined similarly to closed curves. We have the obvious additivity property:

$$n(\gamma, z) = a_1 n(\gamma_1, z) + a_2 n(\gamma_2, z) + \cdots + a_n n(\gamma_n, z) \tag{4.45}$$

4.4.2 Simple Connectedness

This notion formulates our intuition that there are no holes.

Definition 4.4.3: Simple Connectedness

A region $\Omega \subseteq \mathbb{C}$ is called **simply connected** if $\mathbb{C}_\infty - \Omega$ is connected in $\mathbb{C}_\infty$.

In this case, a parallel strip is simply connected, but the outside of a circle is not.

Remark:
This is not a common definition of simple connectedness. The common definition is that any closed curve in Ω can be contracted to a point in Ω. The two definitions are equivalent in $\mathbb{C}$, but not in general topological spaces.

Theorem 4.4.1: Criterion for Simply Connectedness

A region Ω is simply connected iff $\forall a \notin \Omega$ and $\forall$ cycles $\gamma \subseteq \Omega$ we have $n(\gamma, a) = 0$.

Proof. The $\Rightarrow$ direction is obvious, as $\mathbb{C}_{\pm\infty} - \Omega$ is connected, and ∞ belongs to it, then $\mathbb{C}_{\pm\infty} - \Omega$ is in the unbounded region determined by γ, thus $n(\gamma, a) = 0$.

To prove the $\Leftarrow$ direction, we need an explicit construction. We assume $\mathbb{C}_{\pm\infty} - \Omega = A \sqcup B$ for two disjoint closed sets, and $\infty \in B$ while A is bounded. Let $\delta > 0$ be the distance of A and B. Let $a \in A$, and Q be a net of squares covering the whole $\mathbb{C}$ with edge length $< \delta/\sqrt{2}$, and a is a center of a square $Q_0 \in Q$.

Consider the cycle

$$\gamma = \sum_{j \in J} \partial Q_j, \qquad J = \{j : Q_j \in Q, Q_j \cap A \neq \emptyset\} \tag{4.46}$$

consisting of all squares that intersects A (the sum is finite due to boundedness of A.) As a is contained in just one square in γ, we have $n(\gamma, a) = 1$. Also, we have γ does not intersects B, as the distance between A and B is greater than the diagonal of each square. γ does not meet A either, as every edge of square that intersects A are counted twice with opposite directions. Therefore, $\gamma \subseteq \Omega$, contradicting the assumption. $\square$

Cauchy's theorem is not generally true for non-simply connected regions. For example, consider the function $f(z) = 1/z$ on the region $\Omega = \mathbb{C} - \{0\}$. The function is analytic on Ω, but for the unit circle γ, we have

$$\int_\gamma f(z) \mathrm{d}z = \int_\gamma \frac{1}{z} \mathrm{d}z = 2\pi i$$

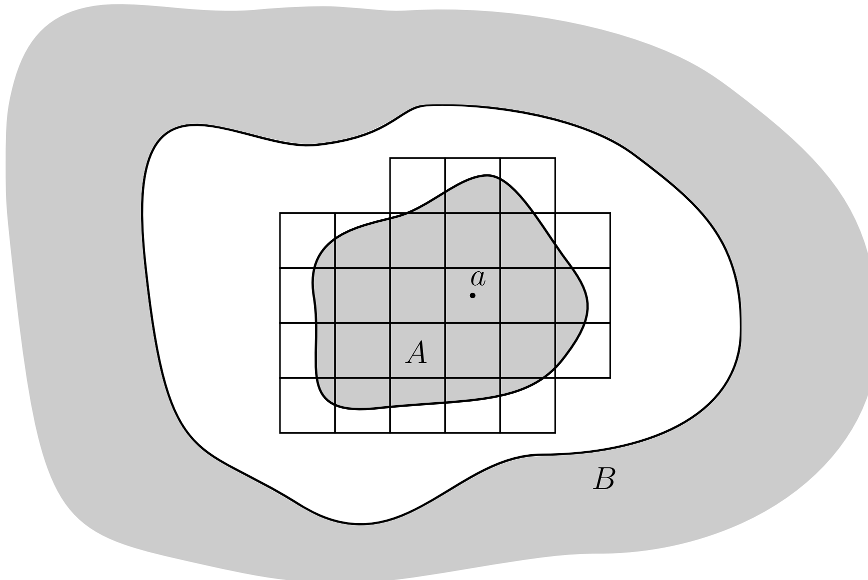

Figure 4.5: The Nets of Squares

4.4.3 Homology

Definition 4.4.4: Homology

A cycle γ in an open set Ω is said to be **homologous to zero** in Ω if $\forall a \notin \Omega$, we have $n(\gamma, a) = 0$. Denoted as $\gamma \sim 0(\text{mod } \Omega)$.
Two cycles γ_1, γ_2 in Ω are said to be **homologous** in Ω if $\gamma_1 - \gamma_2 \sim 0(\text{mod } \Omega)$. Denoted as $\gamma_1 \sim \gamma_2(\text{mod } \Omega)$.

For $\Omega' \subseteq \Omega$, it is obvious that $\gamma \sim 0(\text{mod } \Omega')$ implies $\gamma \sim 0(\text{mod } \Omega)$. If Ω is simply connected, then every cycle in Ω is homologous to zero in Ω.

4.4.4 The General Cauchy's Theorem

Theorem 4.4.2: The General Cauchy's Theorem

If f is analytic in an open set Ω and γ is a cycle in Ω that is homologous to zero in Ω, then

$$\int_\gamma f(z)\mathrm{d}z = 0 \tag{4.47}$$

Here are some frequently used corollaries:

- If f is analytic in a simply connected open set Ω, then for all cycles γ in Ω, we have $\int_\gamma f(z)\mathrm{d}z = 0$.
- Every analytic function in a simple connected region is an exact differential. That is, $\exists$ an analytic function F in Ω such that $F' = f$.

- If f is analytic and $\neq 0$ in a simply connected region Ω, then it is possible to define a single-value branch of $\log f(z)$ and $\sqrt[n]{f(z)}$ in Ω.

Proof. The first two are obvious, for the third statement, we have $f'(z)/f(z)$ is analytic, so there is an analytic F in Ω that $F'(z) = f'(z)/f(z)$. Thus we have

$$f(z)e^{-F(z)} \text{ has zero derivative so is constant.}$$

Let $z_0 \in \Omega$ and choose one of the values of $\log f(z_0)$. We have

$$f(z_0)e^{-F(z_0)} = e^{\log f(z_0) - F(z_0)} = f(z)e^{-F(z)}$$

So we let

$$\log f(z) = F(z) - F(z_0) + \log f(z_0), \qquad \sqrt[n]{f(z)} = \exp \frac{1}{n} \log f(z)$$

□

4.4.5 Proof of Cauchy's Theorem

We present a proof that is similar to the one of theorem 4.4.1.

First assume Ω is a bounded open set. Given $\delta > 0$, we cover the whole $\mathbb{C}$ with nets of squares. Denote $Q_j, j \in J$ by those closed square regions $Q_j \subseteq \Omega$. Then the set J is finite. We denote

$$\Gamma_\delta = \sum_{j \in J} \partial Q_j, \qquad \Omega_\delta = \operatorname{Int} \bigcup_{j \in J} Q_j \tag{4.48}$$

Let γ be a circle homologous to zero in Ω. Choose sufficiently small δ that $\gamma \subseteq \Omega_\delta$. (This is possible for γ and Ω^c are closed sets and have distance > 0.) Let $\zeta \in \Omega - \Omega_\delta$. (This is possible for $\overline{\Omega_\delta} \subseteq \Omega$.) Then $\zeta \in Q$ for some $Q \neq Q_j$. The let $\zeta_0 \in Q, \zeta_0 \notin \Omega$. Draw a straight line $\overline{\zeta\zeta_0}$ and it does not intersect Ω_δ. So we have $n(\gamma, \zeta) = n(\gamma, \zeta_0) = 0$. In particular, we have

$$\forall \zeta \in \Gamma_\delta, n(\gamma, \zeta) = 0$$

Suppose f is analytic in Ω. Let $z \in \operatorname{Int} Q_{j_0}$ for some $j_0 \in J$. Then we have

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} = \begin{cases} f(z), & \text{if } j_0 = j \\ 0, & \text{otherwise} \end{cases}$$

Thus we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z}$$

for every z in the interior of some square Q_j. As both side are continuous, we have this equation holds for all $z \in \Omega_\delta$. As a consequence, we have

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma \left(\frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} \right) \mathrm{d}z = \int_{\Gamma_\delta} \left(\frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{\zeta - z} \right) f(\zeta)\mathrm{d}\zeta = 0$$

This interchange of integral is due to the continuity of both z and ζ (Taking parametric form we can prove this using the real analysis theorem).

If Ω is unbounded, replace it with a bounded subregion that contains γ, and we still have our result.

4.4.6 Local Exact Differential

Definition 4.4.5: Local Exact Differential

A differential form $p\mathrm{d}x + q\mathrm{d}y$ is local exact in Ω if $\forall z \in \Omega$, there is a neighborhood Ω_z of z that $p\mathrm{d}x + q\mathrm{d}y$ is an exact differential in Ω_z.

A sufficient and necessary condition for local exactness is that $\displaystyle\int_\gamma p\mathrm{d}x + q\mathrm{d}y = 0$ holds for all $\gamma = \partial R$, where $R \subseteq \Omega$ is an open rectangle with edge parallel to the axes.

Proof. The sufficiency is obvious, taking a disk as the neighborhood. And define F similarly to the proof of theorem 4.1.4.

The necessity is also obvious, cover $\overline{R}$ with each neighborhood U_x of $x \in R$ that $p\mathrm{d}x + q\mathrm{d}y$ is exact. As $\overline{R}$ is compact, we can take a finite subcover $U_1, \ldots, U_n$. Therefore, divide the rectangle R into enough small rectangles that each is fully in one of the U_j. Then we have our result. ☐

Theorem 4.4.3: Locally Exact Implies Zero Cycle

If $p\mathrm{d}x + q\mathrm{d}y$ is locally exact in Ω, then for any cycle $\gamma \sim 0$ in Ω, we have

$$\int_\gamma p\mathrm{d}x + q\mathrm{d}y = 0 \tag{4.49}$$

As we can see, the general Cauchy's theorem is a special case of this theorem. However, there is no obvious way to modify the proof of the general Cauchy's theorem to prove this theorem. So we would follow another way.

Proof. Without the loss of generality, we shall assume Ω is bounded.

We first show that γ can be replaced by a polygon σ. Let the distance of γ to Ω^c be ρ. Let $\gamma : z = z(t), t \in [a, b]$. Then $z(t)$ is uniformly continuous on $[a, b]$. Take $\delta > 0$ that

$$\forall t, t' \in [a, b], |t - t'| < \delta \to |z(t) - z(t')| < \rho.$$

Divide $[a, b]$ into subintervals of length $< \delta$. Then each subarc γ_i of γ lies in a small disk Δ_i of radius ρ that lies entirely in Ω. Join the endpoints of each subarc γ_i to form a polygon σ_i with a vertical and a horizontal edge. Then $\sigma_i \subseteq \Delta_i$ as well.

It is easy to see that $p\mathrm{d}x + q\mathrm{d}y$ is exact in Δ_i for each i. (This is a result by multivariable calculus). Thus

$$\int_{\sigma_i} p\mathrm{d}x + q\mathrm{d}y = \int_{\gamma_i} p\mathrm{d}x + q\mathrm{d}y$$

Taking the sum of all σ_i, we have

$$\int_\sigma p\mathrm{d}x + q\mathrm{d}y = \int_\gamma p\mathrm{d}x + q\mathrm{d}y$$

Now we extend each segment that form σ to a rectangular grid. There are finite rectangles R_i and unbounded rectangles(strips) R'_j. Choose ANY $a_i \in R_i$ for all i, and we shall prove

$$\sigma = \sum_i n(\sigma, a_i)\partial R_i. \tag{4.50}$$

(Well this is quite obvious in an intuitive way, each rectangle inside contributes $n(\sigma, a_i)$ times to the total cycle).

To prove this, we let $\sigma_0 = \sum_i n(\sigma, a_i)\partial R_i$. And let $a'_j \in R'_j$. It is clear that

$$n(\partial R_i, a_k) = \delta_{ik}, \qquad n(\partial R_i, a'_j) = 0$$

Therefore, we have $n(\sigma_0, a'_j) = 0$. We also have $n(\sigma, a_j) = 0$ for a_j belongs to the unbounded region determined by σ. So we have

$$n(\sigma - \sigma_0, a) = 0, \qquad \forall a \text{ is in any } R_i \text{ or } R'_j$$

We shall conclude that $\sigma = \sigma_0$ from here: Let σ_{ik} be the common side of R_i and R_k. Choose orientation so that R_i lies on the left of σ_{ik}. Suppose $\sigma - \sigma_0$ contains $c\sigma_{ik}$. Then $\sigma' = \sigma - \sigma_0 - c\partial R_i$ does not contain σ_{ik}, so we have $n(\sigma', a_i) = n(\sigma', a_k)$. Thus we have $-c = 0$. The same holds true for all R_i, R'_j.

Now we prove that all $\overline{R_i}$ such that $n(\sigma, a_i) \neq 0$ are contained in Ω. If not so, take some $a_i \in \overline{R_i} \cap \Omega^c$, (then a_i do not lie on σ) then rake a very small neighborhood that $b_i \in R_i$ and b_i, a_i are in the same region determined by σ. So we have $n(\sigma, b_i) = n(\sigma, a_i) = 0$ for $\sigma \sim 0$ in Ω.

It is easy to show that $p\mathrm{d}x + q\mathrm{d}y$ is exact in $\overline{R_i}$, so we have

$$\int_\sigma p\mathrm{d}x + q\mathrm{d}y = \sum_i n(\sigma, a_i) \int_{\partial R_i} p\mathrm{d}x + q\mathrm{d}y = 0$$

□

4.4.7 Multiply Connected Regions

Definition 4.4.6: Multiply Connectedness

A region which is not simply connected is called **multiply connected**.
Let Ω be an open set. If $\mathbb{C}_\infty - \Omega$ has n components, then Ω is said to be n-**connected**. If $n = 1$, then Ω is simply connected. If $\mathbb{C}_\infty - \Omega$ has infinitely many components, then Ω is said to be **infinitely connected**.

In the case if finite connectedness, let $A_1, \ldots, A_n$ be the components of $\mathbb{C}_\infty - \Omega$ and assume $\infty \in A_n$. If γ is any cycle in Ω, then in each A_i we have $n(\gamma, a_i)$ is constant for all $a_i \in A_i$, and $n(\gamma, a) = 0$ for all $a \in A_n$.

Furthermore, duplication the construction method used in the proof of theorem 4.4.1, we can find $n-1$ cycles $\gamma_1, \ldots, \gamma_{n-1}$ in Ω such that $n(\gamma_i, a_j) = \delta_{ij}$ for all $a_j \in A_j$. For a given cycle $\gamma \in \Omega$, let $c_i = n(\gamma, a_i)$. Now every point outside Ω has zero index with respect to the cycle $\gamma - \sum_{i=1}^{n-1} c_i\gamma_i$. So we have

$$\gamma \sim \sum_{i=1}^{n-1} c_i\gamma_i \qquad (\mathrm{mod}\ \Omega) \tag{4.51}$$

As $c_1\gamma_1 + \cdots + c_{n-1}\gamma_{n-1} \sim 0$ would imply $c_1 = c_2 = \cdots = c_{n-1} = 0$, we have the uniqueness of this representation:

Theorem 4.4.4: The Representation of Cycle

Let Ω be an n-connected region, and let $\gamma_1, \ldots, \gamma_{n-1}$ be cycles in Ω such that $n(\gamma_i, a_j) = \delta_{ij}$ for all $a_j \in A_j$. Then every cycle γ in Ω can be uniquely represented as

$$\gamma \sim \sum_{i=1}^{n-1} c_i \gamma_i \qquad (\text{mod } \Omega)$$

where $c_i = n(\gamma, a_i)$ for some (and hence all) $a_i \in A_i$.
The γ_i are called the homology basis for the region Ω.

Therefore, the general Cauchy's theorem would implie: for an analytic function f in an n-connected region Ω, we have

$$\int_\gamma f(z)\mathrm{d}z = \sum_{i=1}^{n-1} c_i \int_{\gamma_i} f(z)\mathrm{d}z \tag{4.52}$$

We define

$$P_i = \int_{\gamma_i} f(z)\mathrm{d}z \tag{4.53}$$

as long as γ_i satisfies theorem 4.4.4, then P_i does not depends on the choice of γ_i. We call P_i the **modulus of periodicity** of the differential $f(z)\mathrm{d}z$ with respect to the cycle γ_i, or the **periods** of the indefinite integral $\int f(z)\mathrm{d}z$.

4.5 The Calculus of Residues

As it turns out, periods are likely easier to calculate.

4.5.1 The Residue Theorem

All theorems we derived from the Cauchy's Theorem on a disk is now valid in arbitrary regions for $\gamma \sim 0$.

Theorem 4.5.1: General Cauchy's Integral Formula

If f is analytic in an open set Ω and γ is a cycle in Ω that is homologous to zero in Ω, then for all $a \in \mathbb{C}, a \notin \gamma$, we have

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z \tag{4.54}$$

Remark:
There is no need for a separate talk on exceptional points, for they are removable singularities which could be ignored.

Now we come a function f which is analytic in a region Ω with finite isolated singularities $a_1, \ldots, a_n$. Denote $\Omega' = \Omega - \{a_1, \ldots, a_n\}$. For each a_i, there is a $\delta_j > 0$ such that the doubly connected region $0 < |z - a_j| < \delta_j$ is contained in Ω'. Draw a circle C_j around each a_j with radius $< \delta_j$. Let

$$P_j = \int_{C_j} f(z)\mathrm{d}z, \qquad R_j = \frac{P_j}{2\pi i}$$

Thus the function

$$f(z) - \frac{R_j}{z - a_j}$$

has a vanishing period around C_j.

Definition 4.5.1: Residue

Let f be analytic in a region Ω with isolated singularity a. The **residue** of f at a is the unique complex number R such that the function differential

$$\left(f(z) - \frac{R}{z-a}\right) \mathrm{d}z$$

is exact in an annulus $0 < |z - a| < \delta$ for some $\delta > 0$. Denoted $R = \operatorname{Res}_{z=a} f(z)$.

If there are infinite many isolated singularities, there are only infinite pf them in a closed disk covering γ, so it doesn't matter.

Theorem 4.5.2: The Residue Theorem

Let f be analytic in a region Ω with isolated singularities $a_j, j \in J$, and γ be a cycle in Ω that does not pass a_j. Then

$$\frac{1}{2\pi i} \int_\gamma f(z) \mathrm{d}z = \sum_{j \in J} n(\gamma, a_j) \operatorname{Res}_{z=a_j} f(z) \tag{4.55}$$

And only a finite number of a_j contributes to the sum.

Remark:
Indeed, this result holds for every point set a_j as long as it contains the singularities, the non-singularity points would contribute zero of the sum.

There is no direct and universal way to find residues for essential singularities. However, for poles we can tell another story. If a is a pole, then

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + B_0 + \varphi(z).$$

where $\varphi(z)$ is analytic in a neighborhood of a and $B_h \neq 0$. Well we can see that

$$\operatorname{Res}_{z=a} f(z) = B_1 \tag{4.56}$$

for the remainder is certianly exact in a no-center neighborhood of a. For simple poles, we have

$$\operatorname{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z) \tag{4.57}$$

Remark:
In presentations of Cauchy's theorem, the integral formula and the residue theorem which follow more classical lines, there is no mention of homology, nor is the notion of index used

explicitly. Instead, the curve γ to which the theorems are applied is supposed to form the complete boundary of a subregion of Ω, and the orientation is chosen so that the subregion lies to the left of Ω. With the general point of view it is still possible to isolate the classical case. All that is needed is to accept the following definition:

Definition 4.5.2: A Curve that Bounds a Region

A cycle γ is said to bounded the region Ω iff $n(\gamma, a) = 1$ for all $a \in \Omega$ and $n(\gamma, a) = 0$ for all $a \notin \Omega \wedge a \notin \gamma$.

4.5.2 The Argument Principle

Cauchy's Integral Formula is a special case of the residue theorem, taking that the function $f(z)/(z-a)$ has a simple pole at $z = a$ with residue $f(a)$.

Another application is to determine the number of zeros of an analytic function in a region, as theorem 4.3.6 states, which we now generalize to meromorphic functions.

Theorem 4.5.3: The Argument Principle

Let f be a meromorphic function in a region Ω with zeros a_j and poles b_k, with multiplicities m_j, n_k, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_j m_j n(\gamma, a_j) - \sum_k n_k n(\gamma, b_k) \tag{4.58}$$

for any cycle $\gamma \in \Omega, \gamma \sim 0$ which does not pass any zeros or poles.

Proof. The singularities of $f'(z)/f(z)$ would lie in the singularities and zeros of f. For a zero a of order h we can write $f(z) = (z-a)^h f_h(z)$, and thus $f'(z) = h(z-a)^{h-1}f_h(z) + (z-a)^h f_h'(z)$. And we have

$$\frac{f'(z)}{f(z)} = \frac{h}{z-a} + \frac{f_h'(z)}{f_h(z)}$$

As $f_h'(z)/f_h(z)$ is analytic around a, we have $\mathrm{Res}_{z=a} f'/f = h$.

Taking $h \to -h$ the result holds for poles. □

The left of equation (4.58) is $n(\Gamma, 0)$ where Γ is the image $f(\gamma)$. We would have $n(\Gamma, 0) = 0$ if Γ lies in a disk not containing the origin.

Theorem 4.5.4: Rouché's Theorem

Let $\gamma \sim 0$ in Ω and $n(\gamma, z) = 0$ or 1 for $\forall z \notin \gamma$. Suppose f, g are analytic in Ω and satisfy $|f(z) - g(z)| < |f(z)|$ on γ. Then f, g has the same number of zeros enclosed by γ. (Enclosure means the zeros of f and g that has index 1 with respect to γ, counting multiplicities)

Proof. Well the assumption shows that $f(z), g(z) \neq 0$ on γ. And we have

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1, \qquad \forall z \in \gamma$$

Let $F(z) = g(z)/f(z)$ then we have $n(F(\gamma), 0) = 0$, as the image curve lies on the disk radius 1 centered at 1. By the argument principle, we have our result. (That is, F has the same number of zeros and poles inside γ.) □

An application involves finding the number of zeros of f in a closed disk neighborhood of a point a. We approximate f by Taylor's Theorem:

$$f(z) = P_{n-1}(z-a) + (z-a)^n f_n(z)$$

where P_{n-1} is a polynomial of degree $n-1$ and $f_n(z)$ is analytic in the disk radius R. If we have $R^n |f_n(z)| < |P_{n-1}(z)|$ on $|z-a| = R$, then f and P_{n-1} has the same number of zeros inside the disk.

We can also generalize the argument principle. Take g be an analytic function in Ω and we have

$$g(z)\frac{f'(z)}{f(z)} = \frac{hg(a)}{z-a} + g_1(z) + \frac{f_n'(z)}{f_n(z)}g(z)$$

Thus we have $\operatorname{Res}_{z=a} g(z)f'(z)/f(z) = hg(a)$ for a zero a of order h. For a pole b of order k, we have similar results.

Theorem 4.5.5: Generalized Argument Principle

Let f be a meromorphic function in a region Ω with zeros a_j and poles b_k, with multiplicities m_j, n_k. Let g be an analytic function in Ω, then

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\mathrm{d}z = \sum_j m_j g(a_j) n(\gamma, a_j) - \sum_k n_k g(b_k) n(\gamma, b_k) \tag{4.59}$$

The Representation of Inverse Functions For an analytic function f in Ω and $f(z_0) = w_0$, we have $f(z) - w_0$ has a zero of order n at z_0. We know that for a small $\epsilon, \delta > 0$, the equation $f(z) = w$ has n roots $z_j(w)$ in the disk $|z - z_0| < \epsilon$ for $|w - w_0| < \delta$. We apply $g(z) = z^m$, we have

$$\sum_{j=1}^n z_j(w)^m = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-w} z^m \mathrm{d}z, \qquad \forall\, |w - w_0| < \delta \tag{4.60}$$

where $\gamma : |z - z_0| = \epsilon$.

For $j = 1, m = 1$ we have

$$f^{-1}(w) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-w} z \mathrm{d}z, \qquad \forall\, |w - w_0| < \delta \tag{4.61}$$

Thus the power sums (elementary symmetric functions) of $z_j(w)$ are analytic functions of w in a small neighborhood of w_0, thus we have $z_j(w)$ is the roots of a polynomial

$$z^n + a_1(w)z^{n-1} + \cdots + a_{n-1}(w)z + a_n(w) = 0 \tag{4.62}$$

whose coefficients are analytic functions of w in a small neighborhood of w_0. The coefficients are given by elementary symmetric functions of $z_j(w)$.

4.5.3 The Definite Integrals

The rational functions of $\sin$ **and** $\cos$

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\mathrm{d}\theta \tag{4.63}$$

where $R(x, y)$ is a rational function of x, y. We can write $z = e^{i\theta}$ and thus

$$I = -i\int_\gamma R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{\mathrm{d}z}{z}, \qquad \gamma : |z| = 1$$

The rational infinite integrals

$$I = \int_{-\infty}^{\infty} R(x)\mathrm{d}x \tag{4.64}$$

where the degree of denominator of $R(x)$ is at least 2 units greater than the degree of numerator, and we assume there is no poles on the real axis. We can take a segment $(-\rho, \rho)$ and the semicircle on the upper half plane. As $\rho \to \infty$ the integral on the semicircle vanishes, and we have

$$\int_{-\infty}^{\infty} R(x)\mathrm{d}x = 2\pi i \sum_{y>0} \operatorname{Res} R(z). \tag{4.65}$$

The same method also applies to

$$I = \int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x = 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz} \tag{4.66}$$

It is surprising to see that the same result holds for a simple zero at ∞.

Theorem 4.5.6: Rational Infinite Integral with Exponential

Let $R(x)$ be a rational function of x with $R(\infty) = 0$, and there are no poles on the real axis. Then

$$\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x = 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz} \tag{4.67}$$

Proof. We first prove the convergence of the integral. Consider

$$I(X_1, X_2) = \int_{-X_1}^{X_2} R(x)e^{ix}\mathrm{d}x, \qquad X_1, X_2 \in \mathbb{R}, -X_1 < X_2$$

for $Y > 0$, consider the rectangle

$$R = \{z = x + iy : -X_1 < x < X_2, 0 < y < Y\}$$

when X_1, X_2, Y are sufficiently large, this rectangle will cover all poles in the upper half plane. Also, we have $|zR(z)|$ is bounded, so the integral over the right side would be

$$\left|\int_{X_2}^{X_2+iY} R(z)e^{iz}\mathrm{d}z\right| \le C\int_0^Y \frac{e^{-y}}{|X_2+iy|}\mathrm{d}y < \frac{C}{X_2}\int_0^Y e^{-y}\mathrm{d}y < \frac{C}{X_2}$$

The same holds for the left side, and for the upper side, we have

$$\left|\int_{-X_1+iY}^{X_2+iY} R(z)e^{iz}\mathrm{d}z\right| \le C\int_{-X_1}^{X_2} \frac{e^{-Y}}{|x+iY|}\mathrm{d}x < C\frac{X_1+X_2}{Y}e^{-Y}$$

We also have the residue theorem

$$\int_{\partial R} R(z)e^{iz}\mathrm{d}z = 2\pi i\sum_{y>0}\operatorname{Res} R(z)e^{iz}$$

As $Y\to\infty$, the integral over the upper side would vanish, and we have

$$\left|\int_{-X_1}^{X_2} R(x)e^{ix}\mathrm{d}x - 2\pi i\sum_{y>0}\operatorname{Res} R(z)e^{iz}\right| < C\left(\frac{1}{X_1}+\frac{1}{X_2}\right)$$

This would imply that

$$\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x = 2\pi i\sum_{y>0}\operatorname{Res} R(z)e^{iz}$$

□

Now, if $R(z)$ has a simple pole at 0, we can adjust the rectangle a little to form a small semicircle of radius δ at 0.

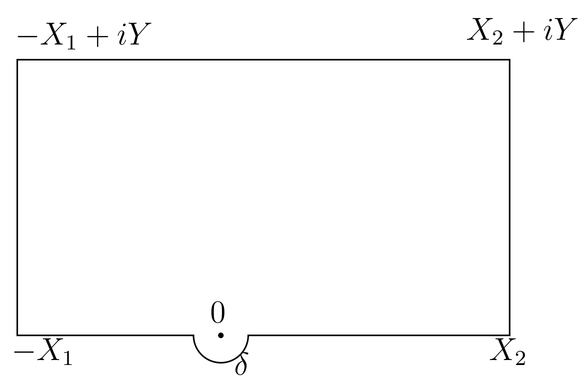

Figure 4.6: Rational Infinite Integral

If we write $R(z)e^{iz} = \frac{B}{z} + R_0(z)$, where $R_0(z)$ is analytic in the neighborhood of 0, then the integration of the first term would be πiB, and the integration of the second term would tend to 0 as $\delta\to 0$. Thus we have

$$\lim_{\delta\to 0}\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} R(x)e^{ix}\mathrm{d}x = 2\pi i\left(\sum_{y>0}\operatorname{Res}_{y>0} R(z)e^{iz} + \frac{1}{2}B\right) \tag{4.68}$$

The limit on the left side is called the **Cauchy Principal Value** of the integral, denoted by $\mathrm{P.V.}\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x$. So we have

$$\mathrm{P.V.}\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x = 2\pi i\sum_{y>0}\mathrm{Res}_{y>0} R(z)e^{iz} + \pi i\sum_{y=0}\mathrm{Res}_{y=0} R(z)e^{iz} \tag{4.69}$$

Example: **Cauchy Principal Value**

A direct application is

$$\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\mathrm{d}x = \pi i$$

Taking the imaginary part, we have

$$\int_{0}^{\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$$

We can also deduce this from Jordan's lemma:

Lemma 4.5.1: Jordan's Lemma

Let g be a complex function such that for sufficiently large R, g is continuous on the part-circle $C_R = \{|z| = R, \Im z \geq -a\}$, for some $a > 0$, and that

$$\lim_{z\to\infty} g(z) = 0$$

Then for any $\lambda > 0$, we have

$$\lim_{R\to\infty}\int_{C_R} g(z)e^{i\lambda z}\mathrm{d}z = 0 \tag{4.70}$$

Proof. Define $M(R) = \max_{z\in C_R}|g(z)|$, then we have $\lim_{R\to\infty} M(R) = 0$. First we have

$$|e^{i\lambda z}| = e^{-\lambda\,\mathrm{Im}\,z} \leq e^{\lambda a}$$

Divide the part-circle into three parts: two lower parts with $\mathrm{Im}\,z \leq 0$ and the upper part with $\mathrm{Im}\,z \geq 0$. For the lower part, we have

$$\left|\int_{\text{lower part}} g(z)e^{i\lambda z}\mathrm{d}z\right| \leq M(R)e^{\lambda a}\alpha R, \qquad \alpha = \arcsin\frac{a}{R}$$

tending to 0 as $R\to\infty$. For the upper part, we have

$$\left|\int_{\text{upper part}} g(z)e^{i\lambda z}\mathrm{d}z\right| \leq M(R)\int_0^{\pi} Re^{-\lambda R\sin\theta}\mathrm{d}\theta$$

Dealing with the integral, we have $\sin\theta \geq \frac{2\theta}{\pi}$ for $\theta\in[0,\pi/2]$, thus

$$\int_0^{\pi} Re^{-\lambda R\sin\theta}\mathrm{d}\theta \leq 2\int_0^{\pi/2} Re^{-\lambda R\frac{2\theta}{\pi}}\mathrm{d}\theta = \frac{\pi}{\lambda}(1-e^{-\lambda R}) < \frac{\pi}{\lambda}$$

So the integral on the upper part also tends to 0 as $R\to\infty$. $\square$

Another type of interest is

$$I = \int_0^\infty x^\alpha R(x)\mathrm{d}x, \qquad 0 < \alpha < 1. \tag{4.71}$$

where $R(z)$ has a zero of order ≥ 2 at ∞, and at most a simple pole at 0. (There are no poles on $(0, \infty)$).

Take $z \to t^2$, we have

$$I = 2\int_0^\infty t^{2\alpha+1} R(t^2)\mathrm{d}t$$

As $z^{2\alpha} = e^{2\alpha \operatorname{Log} z}$. We take the analytic branch, cutting the negative imaginary axis, and the imaginary part of $\operatorname{Log} z$ are taken to be in $(-\pi/2, 3\pi/2)$. Then we take the following cycle, with the small semicircle tends to 0 and the big semicircle tends to ∞. For a semicircle γ of radius r, we have $|z^4 R(z^2)|$ is bounded as $r \to \infty$, and $|z^2 R(z^2)|$ is bounded as $r \to 0$. Thus we have

$$\left|\int_\gamma z^{2\alpha+1} R(z^2)\mathrm{d}z\right| \leq r^{2\alpha+1} \int_0^\pi \frac{C_1}{r^2} r\mathrm{d}\theta = r^{2\alpha}\pi C_1 \to 0, \qquad r \to 0$$

$$\left|\int_\gamma z^{2\alpha+1} R(z^2)\mathrm{d}z\right| \leq r^{2\alpha+1} \int_0^\pi \frac{C_2}{r^4} r\mathrm{d}\theta = r^{2\alpha-2}\pi C_2 \to 0, \qquad r \to \infty$$

So we have

$$\int_{-\infty}^\infty z^{2\alpha+1} R(z^2)\mathrm{d}z = 2\pi i \sum_{y>0} \operatorname{Res} z^{2\alpha+1} R(z^2) \tag{4.72}$$

Also we have

$$\int_{-\infty}^\infty z^{2\alpha+1} R(z^2)\mathrm{d}z = \int_0^\infty (z^{2\alpha+1} + (-z)^{2\alpha+1}) R(z^2)\mathrm{d}z = \frac{1}{2}(1 - e^{2\pi i\alpha})I \tag{4.73}$$

Remark:
We can also use the second cycle in figure 4.7 to the original integral with slight change in adjustment and justification, and the result would be the same.

$$I = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum \operatorname{Res} z^\alpha R(z) \tag{4.74}$$

The logarithmic integrals

$$I = \int_0^\pi \log(\sin\theta)\mathrm{d}\theta \tag{4.75}$$

Consider the expression

$$1 - e^{2iz} = -2ie^{iz}\sin z = 1 - e^{-2y}(\cos 2x + i\sin 2x)$$

it falls on $(-\infty, 0]$ when $x = n\pi, y \leq 0$. Omitting this lines, we have an analytic branch (the principal branch) for the function $\log(1 - e^{2iz})$. Let

$$R = \{z = x + iy : 0 < x < \pi, 0 < y < Y\}$$

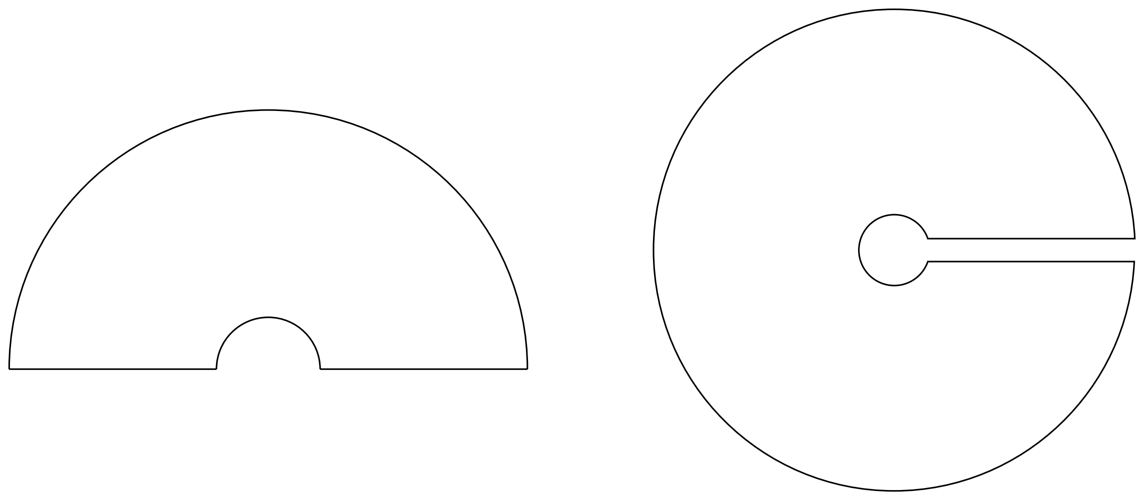

Figure 4.7: Infinite Rational Integral with Power

And let $\gamma = \partial R$, with the vertices at $0, \pi$ changed to small quarter circles of radius δ at 0 and π. As $Y \to \infty$, the integral on the upper side tends to 0, and that

$$\lim_{z\to 0}\left|\frac{1-e^{2iz}}{z}\right| = 2$$

which means that the integral $\log(1-e^{2iz})$ of quarter circles near $0, \pi$ would behave like $\delta \log \delta$ which tends to 0. Also, the side edge integral vanishes due to periodicity. Thus we have

$$\int_0^\pi \log(-2ie^{iz}\sin z)\mathrm{d}z = 2\pi i \sum_{y>0, 0<x<\pi} \operatorname{Res}\log(1-e^{2iz}) = 0 \tag{4.76}$$

We have $\log(-2ie^{ix}) = \log 2 + ix - \pi i/2$ (lying in the principal branch) so we have

$$\int_0^\pi \log(\sin x)\mathrm{d}x = -\int_0^\pi \log(-2ie^{ix})\mathrm{d}x = -\pi\log 2 \tag{4.77}$$

Now consider the type

$$I = \int_0^\infty R(x)\log x\mathrm{d}x \tag{4.78}$$

where $R(x)$ is a **even** rational function with degree of denominator at least 2 units greater than that of numerator. It does not have poles on the positive real axis.

We take the branch cut along the negative imaginary axis. Consider the cycle same in figure 4.7.

- In the left side edge $[-R, -r]$, we have $z = xe^{i\pi}$, thus $\log z = \log x + i\pi$.

$$\int_{-R}^{-r} R(z)\log z\mathrm{d}z = \int_r^R R(x)(\log x + i\pi)\mathrm{d}x$$

- In the outer semicircle, $z = Re^{i\theta}$, we have

$$|\log z| = \sqrt{(\log R)^2 + \theta^2} \leq \log R + \pi$$

$$\left| \int_{\text{outer semicircle}} R(z) \log z \mathrm{d}z \right| \leq \int_0^\pi \frac{C(\log R + \pi)}{R^2} R\mathrm{d}\theta = \frac{\pi C(\log R + \pi)}{R} \to 0, \qquad R \to \infty$$

- In the inner semicircle, $z = re^{i\theta}$, we have

$$\left| \int_{\text{inner semicircle}} R(z) \log z \mathrm{d}z \right| \leq R(0)\pi r \left(\log r + \pi\right) \to 0, \qquad r \to 0$$

Adding all parts together, we have

$$2I + i\pi \int_0^\infty R(x)\mathrm{d}x = 2\pi i \sum_{y>0} \operatorname{Res} R(z) \log z$$

$$I = \pi i \sum_{y>0} \operatorname{Res} R(z) \log z - \frac{i\pi}{2} \int_0^\infty R(x)\mathrm{d}x \tag{4.79}$$

The Bergman Kernel Sometimes complex integration can be used to evaluate area integrals.

Theorem 4.5.7: Bergman's Kernel Formula

If f is analytic and bounded for the unit disk $|z| < 1$, and let $|\zeta| < 1$, then we have

$$f(\zeta) = \frac{1}{\pi} \int_{|z|<1} \frac{f(z)}{(1 - \bar{z}\zeta)^2} \mathrm{d}x\mathrm{d}y \tag{4.80}$$

Proof. Write it in polar coordinates, we have

$$\text{RHS} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{f(re^{i\theta})}{(1 - re^{-i\theta}\zeta)^2} r\mathrm{d}r\mathrm{d}\theta = 2\int_0^1 \left(\frac{1}{2\pi i} \int_{|z|=r} \frac{zf(z)}{(z - r^2\zeta)^2} \mathrm{d}z \right) r\mathrm{d}r$$

As $|r^2\zeta| < |r|$, we have

$$\text{RHS} = 2\int_0^1 (f(r^2\zeta) + r^2\zeta f'(r^2\zeta)) r\mathrm{d}r = r^2 f(r^2\zeta) \Big|_0^1 = \text{LHS}$$

□

4.6 Harmonic Functions

4.6.1 Definition and Basic Properties

Definition 4.6.1: Harmonic Function

A function $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ define on a region Ω is called **harmonic** if it is twice continuously differentiable and satisfies the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.81}$$

In polar coordinates, the Laplace equation becomes

$$r^2 \Delta u = r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{4.82}$$

This would imply that any analytic function depending only on r has the form $u(r) = A + B\log r$.

If u is harmonic in Ω, then

$$f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \tag{4.83}$$

is analytic in Ω, according to theorem 2.1.4.

We now pass to the differential form

$$f(z)\mathrm{d}z = \left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + i\left(-\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y\right)$$

The first part is just $\mathrm{d}u$, and the second part is local exact, which potential is the conjugate harmonic function v of u.

Remark:

In a simple connected region Ω, we can always find a harmonic conjugate v of u such that $f(z)\mathrm{d}z = \mathrm{d}u + i\mathrm{d}v$. In this case, we have $F = u + iv$ is analytic in Ω, and $f(z)\mathrm{d}z = \mathrm{d}F$.

However in general, the second part is not exact (if Ω is not single valued especially), we denote

$$^*\mathrm{d}u = -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$$

called the conjugate differential of u. Then we have

$$f\mathrm{d}z = \mathrm{d}u + i\,^*\mathrm{d}u \tag{4.84}$$

According to theorem 4.4.3, for any cycle $\gamma \sim 0$ in Ω, we have

$$\int_\gamma f\mathrm{d}z = \int_\gamma \mathrm{d}u + i\int_\gamma {}^*\mathrm{d}u = 0$$

which is consistent with Cauchy's theorem. Also, this would imply

$$\int_\gamma \frac{\partial u}{\partial n}\,|\mathrm{d}z| = 0 \tag{4.85}$$

which is easily proved by the divergence theorem, where $\partial u/\partial n$ is the normal derivative of u on γ.

In a simply connected region Ω, we can define the harmonic conjugate v of u without difficulty. In multiple connected regions, the conjugate functions has periods:

$$\int_{\gamma_i} {}^*\mathrm{d}u = \int_{\gamma_i} \frac{\partial u}{\partial n} |\mathrm{d}z|$$

where γ_i are cycles in a homology basis.

Theorem 4.6.1: Pair Harmonic Functions

If u_1, u_2 are harmonic in Ω, and $\gamma \sim 0$ in Ω, then

$$\int_\gamma u_1 {}^*\mathrm{d}u_2 - u_2 {}^*\mathrm{d}u_1 = 0 \tag{4.86}$$

Proof. The simplest proof is by calculation, we have

$$\begin{aligned} u_1 {}^*\mathrm{d}u_2 - u_2 {}^*\mathrm{d}u_1 &= u_1 \left(-\frac{\partial u_2}{\partial y}\mathrm{d}x + \frac{\partial u_2}{\partial x}\mathrm{d}y \right) - u_2 \left(-\frac{\partial u_1}{\partial y}\mathrm{d}x + \frac{\partial u_1}{\partial x}\mathrm{d}y \right) \\ &= \left(-u_1 \frac{\partial u_2}{\partial y} + u_2 \frac{\partial u_1}{\partial y} \right) \mathrm{d}x + \left(u_1 \frac{\partial u_2}{\partial x} - u_2 \frac{\partial u_1}{\partial x} \right) \mathrm{d}y \end{aligned}$$

This is certainly a local exact differential, as we have

$$\frac{\partial}{\partial y}\left(-u_1 \frac{\partial u_2}{\partial y} + u_2 \frac{\partial u_1}{\partial y} \right) - \frac{\partial}{\partial x}\left(u_1 \frac{\partial u_2}{\partial x} - u_2 \frac{\partial u_1}{\partial x} \right) = -u_1 \frac{\partial^2 u_2}{\partial y^2} + u_2 \frac{\partial^2 u_1}{\partial y^2} - u_1 \frac{\partial^2 u_2}{\partial x^2} + u_2 \frac{\partial^2 u_1}{\partial x^2} = 0$$

□

4.6.2 The Mean Value Theorem

Apply theorem 4.6.1 to the pair $u_1(z) = \log r$ and $u_2(z) = u(z)$, where u is harmonic in $|z| < \rho$. We choose $\Omega = \{z : 0 < |z| < \rho\}$ and $\gamma = C_1 - C_2$ where $C_i : |z| = r_i < \rho$. Thus

$$\log r_1 \int_{C_1} r_1 \frac{\partial u}{\partial r}\mathrm{d}\theta - \int_{C_1} u\mathrm{d}\theta = \log r_2 \int_{C_2} r_2 \frac{\partial u}{\partial r}\mathrm{d}\theta - \int_{C_2} u\mathrm{d}\theta$$

Therefore, we have

$$\log r \int_{|z|=r} r \frac{\partial u}{\partial r}\mathrm{d}\theta - \int_{|z|=r} u\mathrm{d}\theta = \text{const} \qquad \forall r < \rho \tag{4.87}$$

If we take $u_1 = 1, u_2 = u$, we have

$$\int_{|z|=r} r \frac{\partial u}{\partial r}\mathrm{d}\theta = \text{const}, \qquad \forall r < \rho \tag{4.88}$$

Theorem 4.6.2: Mean Value of a Harmonic Function

Let u be a harmonic function in an annulus $r_1 < |z| < r_2$, then the arithmetic mean of u is

$$\frac{1}{2\pi} \int_{|z|=r} u\mathrm{d}\theta = \alpha \log r + \beta. \tag{4.89}$$

where $\alpha, \beta \in \mathbb{R}$ are constants. If u is harmonic in the whole disk $r < \rho$, then $\alpha = 0, \beta = u(0)$. If u is harmonic in the disk $B(z_0, r_0)$, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\mathrm{d}\theta \tag{4.90}$$

Proof. This can also be directly proved by the proof of theorem 4.3.8. ☐

Remark:
For a slightly different case, if u is harmonic in $r < \rho$, and continuous in $r \leq \rho$, then the theorem also holds:

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})\mathrm{d}\theta$$

The proof uses the uniform continuity due to the close disk.

Theorem 4.6.3: The Maximum Principle for Harmonic Functions

A nonconstant harmonic function u is defined on a region Ω, then u does not have a maximum or minimum value on Ω.

Proof. Using the similar method proving maximum value theorem of analytic functions 4.3.8 would do. Let $A = \{z : u(z) = \max u\}$, and $B = \Omega - A$. Then $\forall z \in A$, take a small disk and the mean value theorem says that u is constant in the disk, so A is open. As u is continuous, $B = u^{-1}((-\infty, \max u))$ is also open, which contradicts to the connectedness of Ω. ☐

4.6.3 Poisson's Formula

Theorem 4.6.4: The Uniqueness Theorem of Dirichlet Boundary

Let u, v be harmonic functions in a region Ω, and continuous on $\overline{\Omega}$. Then if $u = v$ on the boundary $\partial\Omega$, then $u = v$ in Ω.

Proof. We have $u - v$ is harmonic in Ω and continuous on $\overline{\Omega}$. The maximum and minimum of $u - v$ takes place on $\partial\Omega$, which is constant in Ω. Thus $u = v$ in Ω. ☐

We now want to solve u if its boundary value are given. For simplicity we consider disks first.

- First, we can use the mean value theorem to evaluate the center.
- Use a linear transform to carry any point to the center. Let

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \overline{a}\zeta}. \tag{4.91}$$

mapping $|\zeta| \leq 1 \mapsto |z| \leq R, 0 \mapsto a$. Now $u(S(\zeta))$ is harmonic in $|\zeta| < 1$, and continuous on $|\zeta| \leq 1$. Thus

$$u(a) = \frac{1}{2\pi}\int_{|\zeta|=1} u(S(\zeta))\mathrm{d}\theta_\zeta$$

From

$$\zeta = \frac{R(z-a)}{R^2 - \bar{a}z}$$

we have

$$\mathrm{d}\theta_\zeta = -i\frac{\mathrm{d}\zeta}{\zeta} = -i\left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right)\mathrm{d}z = \left(\frac{z}{z-a} + \frac{\bar{a}z}{R^2 - \bar{a}z}\right)\mathrm{d}\theta_z$$

From $R^2 = z\bar{z}$ we have

$$\mathrm{d}\theta_\zeta = \frac{R^2 - |a|^2}{|z-a|^2}\mathrm{d}\theta_z$$

Theorem 4.6.5: Poisson's Formula

Suppose u is harmonic for $|z| < R$ and continuous on $|z| \leq R$, then for any $a \in \mathbb{C}$ with $|a| < R$, we have

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} u(z)\frac{R^2 - |a|^2}{|z-a|^2}\mathrm{d}\theta_z = \frac{1}{2\pi}\int_{|z|=R} \operatorname{Re}\frac{z+a}{z-a}u(z)\mathrm{d}\theta_z \tag{4.92}$$

Poisson's Formula also gives us a explicit representation of the conjugate harmonic function v of u in the disk $|z| < R$: as

$$u(z) = \operatorname{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\zeta+z}{\zeta-z}u(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\right)$$

We have

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{\zeta+z}{\zeta-z}u(\zeta)\frac{\mathrm{d}\zeta}{\zeta} + iC, \qquad C \in \mathbb{R} \tag{4.93}$$

is an analytic function, and $f = u + iv$. This is known as the Schwarz integral formula.

As a special case, take $u = 1$, we have

$$\int_{|z|=R}\frac{R^2 - |a|^2}{|z-a|^2}\mathrm{d}\theta_z = 2\pi \tag{4.94}$$

4.6.4 Schwarz's Theorem

We can relax the continuity condition on the boundary a little. As long as u has sufficient regularity (say piecewise continuity) on the boundary, the integral in Poisson's formula still makes sense. We then define $u = \operatorname{Re} f$ as in the Schwarz integral formula. Then u is harmonic in $|z| < R$. The question whether u matches the boundary condition or not is answered by the following theorem.

For simplicity, choose $R = 1$, define $U(\theta), \theta \in [0, 2\pi)$ to be a piecewise continuous real function, and let

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z}U(\theta)\mathrm{d}\theta, \qquad |z| < 1 \tag{4.95}$$

Then $U \mapsto P_U(z)$ is a linear positive functional given that $U > 0 \to P_U(z) > 0$.

As $P_c = c$, we have

$$U(\theta) \in [m, M] \to P_U(z) \in [m, M] \tag{4.96}$$

Theorem 4.6.6: Schwarz's Theorem

Let $U(\theta)$ be a piecewise continuous real function on $[0, 2\pi]$, and let $P_U(z)$ be defined as above. Then for any $\theta_0 \in [0, 2\pi]$ where U is continuous, we have $P_U(z)$ is harmonic in $|z| < 1$, and

$$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0) \tag{4.97}$$

Proof. We've already proved that $P_U(z)$ is harmonic in $|z| < 1$. Now we need to show the limit.

- Decompose the unit circle to open arcs $\overline{C_1 \sqcup C_2}$, and let $U_i = U|_{C_i}$. Then $P_U = P_{U_1} + P_{U_2}$.
- P_{U_1} is harmonic on $\mathbb{C} - C_1$, being the real part of an analytic function. And we also have
$$\operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}.$$
So $P_{U_1}(z) = 0$ for $z \in C_2$. Through continuity we conclude that $P_{U_1}(z) \to 0$ as $z \to e^{i\theta_0}$, where $e^{i\theta_0} \in C_2$.
- Suppose $U(\theta_0) = 0$, otherwise we can replace U by $U - U(\theta_0)$. Given $\epsilon > 0$, find C_1, C_2 that $e^{i\theta_0} \in C_2$ and $\forall e^{i\theta} \in C_2, |U(\theta)| < \epsilon/2$. Due to equation 4.96, we have $|P_{U_2}(z)| < \epsilon/2$ for $|z| < 1$. The continuity of P_{U_1} near $e^{i\theta_0}$ implies that $\exists \delta > 0$ such that $|P_{U_1}(z)| < \epsilon/2$ for $\left|z - e^{i\theta_0}\right| < \delta$. Thus we have
$$|P_U(z) - U(\theta_0)| = |P_{U_1}(z) + P_{U_2}(z) - U(\theta_0)| < \epsilon, \qquad \left|z - e^{i\theta_0}\right| < \delta$$
which implies the limit.

□

A Geometric Interpretation of Poisson's Formula Let θ^* be such that on the unit circle $e^{i\theta}, z, e^{i\theta^*}$ are collinear. Then

$$P_U(z) = \frac{1}{2\pi} \int_0^{2\pi} U(\theta) \mathrm{d}\theta^* = \frac{1}{2\pi} \int_0^{2\pi} U(\theta^*) \mathrm{d}\theta \tag{4.98}$$

4.6.5 The Reflection Principle

We notice that if $u(z)$ harmonic, then $u(\overline{z})$ is also harmonic, and if $f(z)$ is analytic, then $\overline{f(\overline{z})}$ is also analytic.

Proof. This is quite straightforward. Let $g(z) = \overline{f(\overline{z})}$, then

$$g(z + \Delta z) - g(z) = \overline{f(\overline{z} + \overline{\Delta z}) - f(\overline{z})} = \overline{f'(\overline{z})\overline{\Delta z} + o(|\Delta z|)} = \overline{f'(\overline{z})}\Delta z + o(|\Delta z|)$$

which implies that $g'(z) = \overline{f'(\overline{z})}$. If $u = \operatorname{Re} f$ then $u(\overline{z}) = \operatorname{Re} g(z)$ so $u(\overline{z})$ is harmonic. □

This can be simply interpreted as flipping both Ω and the range about the real axis. Let $\Omega^* = \{\overline{z} : z \in \Omega\}$.

Consider when $\Omega = \Omega^*$, as Ω is connected, it must contain some open interval $I \subseteq \mathbb{R}$. Assume f is real on I. As $f(z) - \overline{f(\overline{z})}$ is analytic on I, which implies that $f(z) = \overline{f(\overline{z})}$ on Ω. Setting $f = u + iv$, we have $u(z) = u(\overline{z})$ and $v(z) = -v(\overline{z})$.

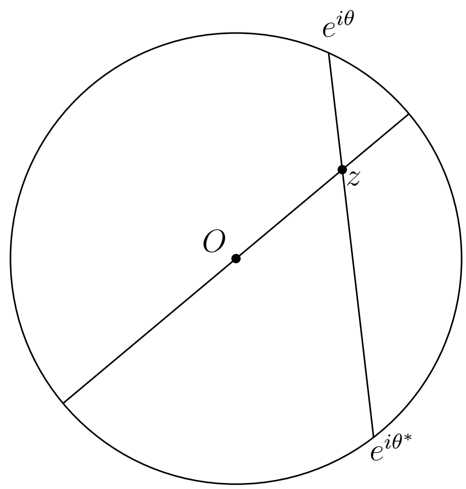

Figure 4.8: Geometry of Poisson's Formula

Theorem 4.6.7: The Reflection Principle

Let Ω be a symmetric region $\Omega = \Omega^*$, and $\Omega^+ = \Omega \cap \{z : \operatorname{Im} z > 0\}$ and $\sigma = \Omega \cap \mathbb{R}$. Suppose v is continuous in $\Omega^+ \cup \sigma$, harmonic in Ω^+, and $v = 0$ on σ. Then v can be extended to a harmonic function in Ω by setting $v(\overline{z}) = -v(z)$.
If v is the imaginary part of an analytic function f in Ω^+, then f has an analytic extension to Ω by setting $f(\overline{z}) = \overline{f(z)}$.

Proof. We construct $V(z)$ as the theorem states.

$$V(z) = \begin{cases} v(z), & z \in \Omega^+ \\ 0, & z \in \sigma \\ -v(\overline{z}), & z \in \Omega^- \end{cases}$$

and prove that V is harmonic in Ω. For $x_0 \in \sigma$ consider a disk $\Delta \subseteq \Omega$ centered at x_0. Then let P_V be the Poisson Integral with boundary values of V on $\partial\Delta$. We have $V - P_V$ being harmonic on on $\Delta \cap \Omega^+$, and $V(z) - P_V(z) = 0$ on σ due to obvious symmetry. Considering the uppse half disk, we have a boundary 0 Dirichlet problem, which shows that $V = P_V$ on the upper half disk, so $V = P_V$ on Δ, therefore, V is harmonic at x_0.

The next part of the theorem follows smoothly, as choosing a conjugate function on the disk that coincides with v on Ω^+ would do. $\square$

Taking linear fractional transformations (to the domain and image), we can generalize the theorem to symmetry with respect to circles and other lines.

Theorem 4.6.8: Generalized Reflection Principle

Let C, C' be circles or lines in $\mathbb{C}$. Let Ω be a region symmetric with respect to C, $\Omega = \Omega^*$, and Ω^+, Ω^- be two sides of Ω to C. If f is analytic in Ω^+, continuous on $\overline{\Omega^+}$, and $f(C) \subseteq C'$, then f can be extended to a function F in Ω by setting $f(z^*[C]) = f(z)^*[C']$. (where $z^*[C]$ is the reflection of z with respect to C).

Chapter 5

Series and Product Developments

A useful tool to explicitly represent an analytic function.

5.1 Power Series

5.1.1 Weierstrass' Theorem

In analysis, uniform convergence is crucial for studying the regularity properties of a limit function.

The Domain We now have a sequence of analytic functions $\{f_n\}$ defined on Ω_n. As we want to consider the limit function f in Ω, it would make sense that

$$\forall z \in \Omega, \exists n_0 \in \mathbb{N}, \forall n > n_0, z \in \Omega_n.$$

A typical case is that $\Omega_n \subsetneq \Omega_{n+1}$ for all $n \in \mathbb{Z}_+$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. In this way, however, the convergence is not uniform as no f_n is defined on Ω. So we consider a weaker assumption, but still strong enough: The inner compact uniform convergence.

Theorem 5.1.1: The Uniform Convergence Implies Analyticality

Let $\{f_n\}$ be a sequence of analytic functions defined on regions Ω_n with limit function $f(z) = \lim_{n\to\infty} f_n(z)$ defined on Ω. If for every compact set $K \subset \Omega$, f_n is uniformly convergent on K, then f is analytic on Ω.
Moreover, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact $K \subseteq \Omega$.

Proof. Let $\Delta : |z - a| \leq r$ be a closed disk in Ω, then for enough large n, we have $\Delta \subseteq \Omega_n$ (All Ω_n forms an open cover of Δ, thus has a finite subcover). For any closed curve γ in Δ, we have

$$\int_\gamma f(z)\mathrm{d}z = \lim_{n\to\infty} \int_\gamma f_n(z)\mathrm{d}z = 0.$$

due to uniform convergence. Thus, by Morera's theorem, f is analytic on Δ. Since Δ is arbitrary, we conclude that f is analytic on Ω.

Explicitly, we can write

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)}{\zeta - z}\mathrm{d}\zeta,$$

where $C : |\zeta - a| = r$ and $|z - a| < r$. Due to uniform convergence, we can exchange the limit and the integral:

$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i}\int_C \lim_{n\to\infty} \frac{f_n(\zeta)}{\zeta - z}\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta.$$

showing f is analytic in the disk. Following the same route, we have

$$f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta,$$

So we have

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta = f'(z). \tag{5.1}$$

In a compact set, convergence is uniform, thus we have finished the proof. □

Repeated applications suggest that $f_n^{(k)}(z)$ converges uniformly to $f^{(k)}(z)$ on every compact set $K \subseteq \Omega$.

Apply this to series, we naturally have the following theorem.

Theorem 5.1.2: Weierstrass Theorem

Let $\{f_n\}$ be a sequence of analytic functions with limit function $f(z) = \sum_{n=1}^{\infty} f_n(z)$ defined on Ω. If for every compact set $K \subseteq \Omega$, $\sum f_n$ is uniformly convergent on K, then f is analytic on Ω.
Moreover, $\sum f_n'(z)$ converges uniformly to $f'(z)$ on every compact $K \subseteq \Omega$.

Proposition: **Uniform Convergence on Compact Sets**

As K is compact so is closed, then the maximum and minimum of $|f_n(z) - f_m(z)|$ is achieved on ∂K. So uniform convergence on K is equivalent to uniform convergence on ∂K.

Theorem 5.1.3: Hurwitz Theorem

Let $\{f_n\}$ be a sequence of analytic functions defined on Ω, $f_n(z) \neq 0$, with limit function $f(z) = \lim_{n\to\infty} f_n(z)$ defined on Ω. If for every compact set $K \subset \Omega$, $\{f_n\}$ converges uniformly to f, then $f = 0$ or $\forall z \in \Omega, f(z) \neq 0$.

Proof. Suppose f is not identically zero, then the zeros are isolated. $\forall z_0 \in \Omega$, let $r > 0$ that $\forall 0 < |z - z_0| \leq r, f(z) \neq 0$. Denote

$$C = \min\{f(z) : |z - z_0| = r\} > 0$$

then $1/f_n$ converges uniformly to $1/f$ on $|z - z_0| = r$. Thus, there exists n_0 such that for all $n > n_0$, we have

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n'(z)}\mathrm{d}z = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\mathrm{d}z = 0.$$

From theorem 4.3.6, we know that the integral counts the number of zeros of f_n in $|z - z_0| < r$, which is zero. So $f(z_0) \neq 0$. □

5.1.2 The Taylor Series

According to theorem 4.3.2, let f be analytic in the region Ω. For ant $z, z_0 \in \Omega$ we have

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k + f_{n+1}(z)(z-z_0)^{n+1}.$$

Let $D \subseteq \Omega$ be a disk containing z with center at z_0 and radius ρ. Then we have

$$f_{n+1}(z) = \int_{\partial D} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)}.$$

Denote $M = \max\{|f(z)| : z \in \partial D\}$, then obtain

$$\left| f_{n+1}(z)(z-z_0)^{n+1} \right| \leq \frac{M\left|z-z_0\right|^{n+1}}{\rho^n(\rho - |z-z_0|)}. \tag{5.2}$$

which means that the remainder term inner compact uniform converges to zero in D as $n \to \infty$.

Theorem 5.1.4: Taylor's Series

Let f be analytic in the region Ω. For any $z_0 \in \Omega$ and any open disk $D \subseteq \Omega$ centered at z_0, the Taylor series

$$S(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \qquad z \in D$$

converges uniformly to $f(z)$ on every compact subset of D.

Therefore, the radius of convergence is at least the distance from z_0 to the boundary of Ω. (Well, it can be larger)

Remark:
The uniqueness of Taylor series is guaranteed by taking the derivative of the series term by term and evaluating at z_0.

Proposition: **The Derivatives and Integral of Taylor Series**

Let f be analytic in the region Ω and $z_0 \in \Omega$. The Taylor series of f at z_0 is

$$S(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \qquad z \in D$$

if D is an open disk centered at z_0 contained in Ω. Then $f(z) = S(z)$ for $z \in D$ and $S(z)$ converges uniformly to $f(z)$ on every compact subset of D.
Independently speaking, S is a power series, which is inner compact uniform convergent and analytic in an open disk D of radius R centered at z_0, and diverges for $|z-z_0| > R$. Moreover,

- The derivative of the series is

$$S'(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{(k-1)!}(z-z_0)^{k-1}, \qquad z \in D.$$

- The integral of the series is

$$\int_{z_0}^{z} S(\zeta)\mathrm{d}\zeta = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{(k+1)!}(z-z_0)^{k+1}, \qquad z \in D.$$

which have the same radius of convergence R as S.

Proof. This is the Abel Disk Theorem 2.2.3. □

Example: **Taylor Series of Elementary Functions**

- Exponential Function

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots, \qquad z \in \mathbb{C}$$

  This is of course the definition of e^z, but it is also the Taylor series at $z_0 = 0$ because of the uniqueness of Taylor series.

- Sine and Cosine Function

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots, \qquad z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots, \qquad z \in \mathbb{C}$$

- Power and logarithm: For a non-$\mathbb{Z}_+$ power or logarithm, we need first to choose an analytic branch near the origin. For $(1+z)^\mu$ or $\log(1+z)$, we can choose the principal branch with a cut along $(-\infty, -1]$. Then the Taylor series at $z_0 = 0$ is

$$(1+z)^\mu = 1 + \mu z + \binom{\mu}{2} z^2 + \cdots + \binom{\mu}{n} z^n + \cdots, \qquad z \in \mathbb{C}, |z| < 1$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^{n-1}\frac{z^n}{n} + \cdots, \qquad z \in \mathbb{C}, |z| < 1$$

  The radius of convergence is 1 for both series if $\mu \notin \mathbb{Z}_+$, for if $R > 1$, then all the derivative would be bounded for $|z| < 1$, which is not the case. For $\mu \in \mathbb{Z}_+$, then the series converges in $\mathbb{R}$.

- Arctangent and Arcsine: We determine the branch by

$$\arctan z = \int_0^z \frac{1}{1+\zeta^2}\mathrm{d}\zeta, \qquad \arcsin z = \int_0^z \frac{1}{\sqrt{1-\zeta^2}}\mathrm{d}\zeta, \qquad z \in \mathbb{C}, |z| < 1.$$

The path is taken to be any path from 0 to z that lies in the unit open disk. Through term-by-term integration, we have

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots + (-1)^{n-1}\frac{z^{2n-1}}{2n-1} + \cdots, \qquad z \in \mathbb{C}, |z| < 1$$

$$\arcsin z = z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots + \frac{(2n-1)!!}{(2n)!!}\frac{z^{2n+1}}{2n+1} + \cdots, \qquad z \in \mathbb{C}, |z| < 1$$

The radius of convergence is 1 for both series.

Notation 5.1.1: $[z^n]$

Denote $[z^n]$ to be any function which is analytic and has a zero of order at least n at $z = 0$. For any analytic function at the origin, we can say

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + [z^{n+1}], \tag{5.3}$$

Where the coefficients are uniquely determined by the Taylor series.

Therefore, finding the Taylor series coefficients of $f(z)$ is equivalent to finding the polynomial P_n that $f(z) - P_n(z) = [z^{n+1}]$ has a zero of order at least $n+1$ at $z = 0$. We can say that no matter the degree of P_n is larger than n, the coefficients of P_n up to z^n are uniquely determined.

Proof. Assume there are two such polynomials P_n and Q_n, then $P_n(z) - Q_n(z) = [z^{n+1}]$. Assume that $P_n(z) - Q_n(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots + a_m z^m$ with $a_k \neq 0$ and $k \leq n$, then $P_n(z) - Q_n(z)$ has a zero of order $k \leq n$ at $z = 0$, which is a contradiction. □

Therefore, we cay easily say that if

$$f(z) = P_n(z) + [z^{n+1}], \qquad g(z) = Q_n(z) + [z^{n+1}],$$
$$f(z)g(z) = P_n(z)Q_n(z) + [z^{n+1}],$$

So as to determine the Taylor series of fg at $z = 0$ up to z^n, we only need to determine the product of the polynomials P_n and Q_n up to z^n. Explicitly, we have:

Proposition: **Taylor Series of Operation of Functions**

Let f and g be analytic in the region Ω and $z_0 \in \Omega$. The Taylor series of f and g at z_0 are

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \qquad z \in D$$

$$g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!}(z - z_0)^k, \qquad z \in D$$

if D is an open disk centered at z_0 contained in Ω. Then $f(z)$ and $g(z)$ can be expressed as power series in D, which converge uniformly to $f(z)$ and $g(z)$ on every compact subset of D. Moreover,

- For any constant $c \in \mathbb{C}$, the Taylor series of cf at z_0 is

$$cf(z) = \sum_{k=0}^{\infty} c\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \qquad z \in D.$$

- The Taylor series of $f+g$ at z_0 is

$$f(z)+g(z) = \sum_{k=0}^{\infty} \left(\frac{f^{(k)}(z_0)}{k!} + \frac{g^{(k)}(z_0)}{k!}\right)(z-z_0)^k, \qquad z \in D.$$

- The Taylor series of fg at z_0 is

$$f(z)g(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{f^{(j)}(z_0)}{j!}\frac{g^{(k-j)}(z_0)}{(k-j)!}\right)(z-z_0)^k, \qquad z \in D.$$

- If $g(z_0) \neq 0$, then the Taylor series of f/g at z_0 can be determined by:

  First letting R_n be such that $P_n = Q_n R_n + [z^{n+1}]$, then taking the coefficients of R_n up to z^n to be the coefficients of the Taylor series of f/g at z_0 up to z^n. Explicitly, we have

$$\frac{f(z)}{g(z)} = \sum_{k=0}^{\infty} r_k (z-z_0)^k, \qquad z \in D,$$

  This is because $f - R_n g = [z^{n+1}]$. And as $g(z_0) \neq 0$, $f/g - R_n = [z^{n+1}]$.

- For the composite function $f(g(z))$, for simplicity let $z_0 = 0$ and $g(0) = 0$, then set

$$f(w) = P_n(w) + [w^{n+1}], \qquad g(z) = Q_n(z) + [z^{n+1}], \quad Q_n(0) = 0,$$

  Then we have

$$f(g(z)) = P_n(Q_n(z) + [z^{n+1}]) + [z^{n+1}] = P_n(Q_n(z)) + [z^{n+1}].$$

- For the inverse function $f^{-1}(z)$, let $z_0 = 0$ and $f(0) = 0$, assume $f'(0) \neq 0$ so there is a local homeomorphism inverse function. Let $f(z) = P_n(z) + [z^{n+1}]$ and $f^{-1}(z) = Q_n(z) + [z^{n+1}]$ with $P_n(0) = Q_n(0) = 0$, then we have

$$z = f(f^{-1}(z)) = P_n(Q_n(z)) + [z^{n+1}].$$

  The proof of exiestence and uniqueness of Q_n is due to the Lagrange Inversion Theorem.

Theorem 5.1.5: Lagrange Inversion Theorem

Let f be analytic in the region Ω and $z_0 \in \Omega$. The Taylor series of f at z_0 is

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \qquad z \in D$$

if D is an open disk centered at z_0 contained in Ω. Then $f(z)$ can be expressed as a power series in D, which converges uniformly to $f(z)$ on every compact subset of D.
Moreover, if $f'(z_0) \neq 0$, then there exists a local inverse function f^{-1} such that $f^{-1}(f(z)) = z$ for all z in a neighborhood of z_0. The Taylor series of f^{-1} at $w_0 = f(z_0)$ is

$$f^{-1}(w) = z_0 + \sum_{n=1}^{\infty} \frac{(w-w_0)^n}{n!} \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left(\frac{z-z_0}{f(z)-w_0}\right)^n\right]_{z=z_0}, \qquad w \in D',$$

where D' is an open disk centered at w_0 contained in the range of the local inverse function.

5.1.3 The Laurent Series

A series of the form

$$b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} + \cdots$$

can be viewed as a power series in $1/z$. So it will converge in a region of the form $|z| > R$ for some $R > 0$ and diverge for $|z| < R$.

For a more general form is called a Laurent series:

Definition 5.1.1: Laurent Series

A Laurent series centered at z_0 is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \tag{5.4}$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}$.

A Laurent series can be viewed as the sum of two series: a power series in $z - z_0$ and a power series in $1/(z-z_0)$. It would converge iff both series converge. Suppose the power series converges for $|z - z_0| < R_2$ and the other converges for $|z - z_0| > R_1$, then the Laurent series converges in the annulus $R_1 < |z - z_0| < R_2$. Note that if $R_1 \geq R_2$, then there is no convergence region.

Conversely, we can start with an analytic function whose analyticity domain contains an annulus $R_1 < |z - a| < R_2$, then we can represent it as a Laurent series in that annulus.

Theorem 5.1.6: Laurent's Theorem

Let f be analytic in the annulus $R_1 < |z - a| < R_2$. Then f can be represented as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \qquad R_1 < |z-a| < R_2,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \mathrm{d}\zeta, \qquad R_1 < r < R_2.$$

The series converges uniformly to f on every compact subset of the annulus.

Proof. We try to represent $f = f_1 + f_2$, and DEFINE

$$f_1(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta, \qquad |z-a|<r<R_2,$$

$$f_2(z) = -\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta, \qquad R_1<r<|z-a|\,.$$

Both integral is irrelevant to the choice of r in the given range. By Cauchy's integral formula, we have $f(z) = f_1(z) + f_2(z)$ for $R_1 < |z-a| < R_2$. (Adding the two integrals means integrate via two circles with opposite orientation, which ~ 0).

Also, we can see that f_1 is an analytic function in $|z-a| < R_2$ and f_2 is an analytic function in $|z-a| > R_1$. So we can expand them as power series:

$$f_1(z) = \sum_{n=0}^{\infty} A_n(z-a)^n, \qquad |z-a|<R_2, A_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\mathrm{d}\zeta,$$

For f_2 we consider the transformation $z = 1/z' + a, \zeta = 1/\zeta' + a$, then

$$f_2(1/z'+a) = \frac{1}{2\pi i}\int_{|\zeta'|=1/r}\frac{f(1/\zeta'+a)}{\zeta'-z'}\frac{z'}{\zeta'}\mathrm{d}\zeta' = \sum_{n=0}^{\infty} B_n(z')^n, \qquad |z'|<1/R_1,$$

where

$$\begin{aligned} B_n &= \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}(z')^n}\int_{|\zeta'|=1/r}\frac{f(1/\zeta'+a)}{\zeta'-z'}\frac{z'}{\zeta'}\mathrm{d}\zeta'\right]_{z'=0} \\ &= \frac{1}{2\pi i}\int_{|\zeta'|=1/r}\frac{f(1/\zeta'+a)}{(\zeta')^{n+1}}\mathrm{d}\zeta' \\ &= \frac{1}{2\pi i}\int_{|\zeta-a|=r} f(\zeta)(\zeta-a)^{n-1}\mathrm{d}\zeta. \end{aligned}$$

$$f_2(z) = \sum_{n=0}^{\infty} B_n(1/(z-a))^n = \sum_{n=-\infty}^{-1} B_{-n}(z-a)^n, \qquad |z-a|>R_1.$$

In all, we can write:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n, \qquad R_1<|z-a|<R_2, \qquad a_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\mathrm{d}\zeta.$$

□

The situation can be easily extended to $R_1 = 0$.

5.2 Partial Fractions and Factorization

There are two ways to represent a rational function: partial fractions and factorization of both numerator and denominator. Both are based on the fundamental theorem of algebra. We shall do this in a more general setting, that is, meromorphic functions.

5.2.1 Partial Fractions

Assume f is meromorphic in a region Ω, for each pole b_ν there is a neighborhood that contains no other poles. So we can write the Laurent series of f at b_ν:

$$P_\nu(\frac{1}{z-b_\nu}) + \text{ positive power series of } (z-b_\nu),$$

where P_ν is a polynomial in $1/(z-b_\nu)$ without constant term. There are also finite terms in P_ν because b_ν is a pole and $\lim_{z\to b_\nu} f(z)(z-b_\nu)^m = c \neq \infty$ for some $m \in \mathbb{Z}_+$.

Then we can subtract all these principal parts from f:

$$f(z) = \sum_\nu P_\nu(\frac{1}{z-b_\nu}) + g(z). \tag{5.5}$$

For there are at most countable poles in Ω, the sum is well-defined. However, there are generally infinite poles and the sum may not converge. So we need to introduce some convergence factors to make the sum convergent. There is no harm done if we substract some analytic functions from P_ν, as long as we add them to $g(z)$.

Theorem 5.2.1: Mittag-Leffler Theorem

Let $\{b_\nu\}$ be a sequence of complex numbers with $\lim_{\nu\to\infty} b_\nu = \infty$, and P_ν be polynomials without constant term. Then there exists meromorphic function f in $\mathbb{C}$ whose poles are exactly at b_ν and the principal part of the Laurent series of f at b_ν is $P_\nu(1/(z-b_\nu))$. And all such functions can be expressed in the following way:

$$f(z) = \sum_{\nu=1}^{\infty} \left[P_\nu(\frac{1}{z-b_\nu}) - p_\nu(z) \right] + g(z), \tag{5.6}$$

where $p_\nu(z)$ are polynomials chosen such that the series absolutely converges, and $g(z)$ is an entire function (analytic in $\mathbb{C}$).

Proof. Suppose $b_\nu \neq 0$ for all ν, otherwise we can just perform a slight translation. The function $P_\nu(1/(z-b_\nu))$ is analytic in $|z| < |b_\nu|$. So we can expand it as a Taylor series at $z=0$. Choose $p_\nu(z)$ to be a partial sum of the Taylor series at $z=0$, ending at the degree n_ν.

From the proof of theorem 4.3.3, we know that if the maximum of $|P_\nu|$ for $|z| \le |b_\nu|/2$ is M_ν, then

$$\left| P_\nu(\frac{1}{z-b_\nu}) - p_\nu(z) \right| \le 2M_\nu \left(\frac{2|z|}{|b_\nu|} \right)^{n_\nu+1}, \qquad |z| \le \frac{|b_\nu|}{4}.$$

We choose n_ν such that

$$2^{n_\nu} \ge M_\nu 2^\nu$$

For any R, there are only finite ν such that $|b_\nu| \le 4R$. For sufficiently large ν, that all $|b_\nu| > 4R$, we have

$$\left| P_\nu(\frac{1}{z-b_\nu}) - p_\nu(z) \right| \le 2M_\nu \left(\frac{2R}{|b_\nu|} \right)^{n_\nu+1} \le 2M_\nu \left(\frac{1}{2} \right)^{n_\nu+1} \le 2^{-\nu}.$$

So if we omit the first finite terms, the series converges uniformly and absolutely on $|z| \le R$. As R is arbitrary, the series converges uniformly and absolutely on every compact subset of $\mathbb{C}$. So the sum is meromorphic in $\mathbb{C}$ with poles exactly at b_ν and the principal part of the Laurent series of f at b_ν is $P_\nu(1/(z-b_\nu))$. ∎

Remark:
Conversely, we can say that given a meromorphic function f in $\mathbb{C}$ with poles at b_ν, and the principal part of the Laurent series of f at b_ν is $P_\nu(1/(z-b_\nu))$. Then we can find a suitable g that has the same poles and principal parts, so $f-g$ is entire. If we merge the $f-g$ into the g in the theorem, then we have the same form as in the theorem expressing f.
For a slight clarification, if f is meromorphic in $\mathbb{C}$ with infinitely many poles, then the poles must have a limit point at ∞. This is because any closed subset of $\mathbb{C}$ is compact, so there are only finite poles in that.

Example: **The Mittag-Leffler Theorem**

- Consider the function $\dfrac{\pi^2}{\sin^2 \pi z}$. It has poles at all integers, and the principal part of the Laurent series at $n \in \mathbb{Z}$ is $\dfrac{1}{(z-n)^2}$. So we can express it as

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z),$$

where $g(z)$ is an entire function. We prove that $g(z) = 0$. First notice that both sides have period 1, so does $g(z)$. Also if we let $z = x + iy$ then

$$|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x$$

which means that $\pi^2/\sin^2 \pi z$ tends uniformly to 0 as $y \to \pm\infty$. And the series also tends uniformly to 0 as $y \to \pm\infty$. So $g(z)$ tends to 0 uniformly as $y \to \pm\infty$. As $g(z)$ is periodic, it is bounded in $\mathbb{C}$. By Liouville's theorem, $g(z)$ is a constant. So $g(z) = 0$.

So we can infer that g is bounded in $0 \le x \le 1$ and from periodicity, it is bounded in $\mathbb{C}$. By Liouville's theorem, g is a constant. As $g(iy) \to 0$ as $y \to \pm\infty$, we have $g(z) = 0$. Therefore,

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

- Consider taking the integral of both sides, and the integral of the left side is $-\pi \cot \pi z$. The integral of the right side is inner compact uniform convergent in every domain that does not contain any integer. However, the series of $1/(z-n)$ does not converge. So we need to introduce some convergence factors. We can take

$$\sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) + \frac{1}{z} = \sum_{n \neq 0} \frac{z}{n(z-n)} + \frac{1}{z}.$$

Indeed, this is just given by termwise integration from 0. So we have

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \frac{z}{n(z-n)} + C = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} + C.$$

For both sides are odd functions, $C = 0$. Therefore,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

- Conversely, we now study the sum

$$\lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2}.$$

Which obviously is a meromorphic function with poles at all integers. Separate the odd and even terms, we have

$$\sum_{n=-(2k+1)}^{2k+1} \frac{(-1)^n}{z-n} = \sum_{n=-k}^{k} \frac{1}{z-2n} - \sum_{n=-k-1}^{k} \frac{1}{z-(2n+1)}.$$

which has limit

$$\frac{\pi}{2} \cot \frac{\pi z}{2} - \frac{\pi}{2} \cot \frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z}.$$

So we have

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n}. \tag{5.7}$$

5.2.2 Infinite Products

An infinite product is an expression of the form

$$\prod_{n=1}^{\infty} (1 + a_n), \qquad a_n \in \mathbb{C}, a_n \neq -1. \tag{5.8}$$

whose limit definition is given by

$$\lim_{N\to\infty} \prod_{n=1}^{N} (1 + a_n).$$

Taking the logarithm, we have

$$\sum_{n=1}^{\infty} \log(1 + a_n).$$

where the branch of the logarithm is chosen to be the principal branch. Denote

$$P_n = \prod_{k=1}^{n} (1 + a_k), \qquad S_n = \sum_{k=1}^{n} \log(1 + a_k).$$

Then we have $P_n = e^{S_n}$. If $S_n \to S$, then $P_n \to e^S$. So the infinite product converges to e^S. Conversely, if $P_n \to P$, then it is not necessary that $S_n \to \log P$ in the principle branch. But we shall prove that if $P \neq 0$, then there exists an integer m such that $S_n \to \log P + 2m\pi i$.

Proof. SORRY □

Theorem 5.2.2: Infinite Products

The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges to a nonzero limit iff the series $\sum_{n=1}^{\infty}\log(1+a_n)$ converges to a limit (the logarithm is taken in the principal branch).

Furthermore, as

$$\lim_{z\to 0}\frac{\log(1+z)}{z}=1,$$

We say that the infinite product converges absolutely if $\sum_{n=1}^{\infty}\log(1+a_n)$ converges absolutely. And we have the following theorem:

Theorem 5.2.3: Absolute Convergence of Infinite Products

The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely iff the series $\sum_{n=1}^{\infty}|a_n|$ converges.

It is quite clear to understand the uniform convergence of infinite products of functions.

Definition 5.2.1: Uniform Convergence of Infinite Products

An infinite product of functions

$$\prod_{n=1}^{\infty}(1+f_n(z))$$

converges uniformly in a set E if the sequence of partial products

$$P_N(z)=\prod_{n=1}^{N}(1+f_n(z))$$

converges uniformly in E as $N\to\infty$.

We can also verify uniform convergence via logarithms.

Proposition: **Uniform Convergence by logarithm**

The infinite product of functions

$$\prod_{n=1}^{\infty}(1+f_n(z))$$

with $1+f_n(z)\neq 0$ for all n and $z\in E$, converges uniformly in E iff the series

$$\sum_{n=1}^{\infty}\log\left(1+f_n(z)\right)$$

converges uniformly in E (the logarithm is taken in the principal branch).

Proof. SORRY □

Theorem 5.2.4: Uniform Convergence by Series

A sufficient and necessary condition for the infinite product of functions

$$\prod_{n=1}^{\infty}(1+f_n(z))$$

with $1+f_n(z)\neq 0$ for all n and $z\in E$, to converge absolutely and uniformly in E is that the series

$$\sum_{n=1}^{\infty}|f_n(z)|$$

converges uniformly in E.

Remark:
Here we see that the presence of zeros in the factors do give some difficulties. But we can restrict our attention to sets that only allow finite number of factors to be zero. If we omit these factors, then the remaining product is what we discussed above. Therefore, we are saying that the above theorems still hold in this case.

Proposition: **Compact Convergence Validity**

If the infinite sequence of analytic functions

$$f_1(z), f_2(z), \ldots$$

converges uniformly on every compact subset of a region Ω, then it defines an analytic function in Ω.

5.2.3 Canonical Products

We have developed the series representation of meromorphic functions. Now we turn to the product representation. We start with entire functions.

Definition 5.2.2: Entire Functions

A function f is entire if it is analytic in $\mathbb{C}$.

If g is an entire function so is $f=e^g$ and $\forall z, f(z)\neq 0$. Conversely, if f is entire and $\forall z, f(z)\neq 0$, then we can say that there exists an entire function g such that $f=e^g$.

Proof. Consider f'/f, which is analytic in $\mathbb{C}$. So there exists an entire function g such that $g'=f'/f$. We have $f(z)e^{-g(z)}$ has zero derivative, so $f(z)=e^{g(z)+C}$. □

We can generalize this method to construct entire functions with a finite number of zeros. Assume f has m-order zero for 0, and all other zeros being $a_1,\ldots,a_N$, counting multiplicities.

Then we can write

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right), \tag{5.9}$$

simply because

$$\frac{f(z)}{z^m \prod_{n=1}^{N} (z - a_n)}$$

has no zeros and entire.

As there are at most countable zeros for an entire function, we can try to generalize this to infinite zeros. We are tempted to write

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right),$$

If the product converges uniformly on every compact subset of $\mathbb{C}$, it defines an analytic function in $\mathbb{C}$. Also, as each factor has zeros exactly at a_n, the product has zeros exactly at a_n. Them this form is valid. However, the product converges absolutely iff $\sum_{n=1}^{\infty} |1/a_n|$ converges, and when this happens, it also converges uniformly on every closed disk $|z| \leq R$. But in many cases, the zeros do not satisfy this condition.

This is quite similar to the situation of Mittag-Leffler theorem, where we need to introduce some convergence factors to make the product convergent. We can do this by proving the existence of polynomials $p_n(z)$ such that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)} \tag{5.10}$$

converges to an entire function.

Theorem 5.2.5: Weierstrass Factorization Theorem

Let $\{a_n\}$ be a sequence of complex numbers with $\lim_{n\to\infty} |a_n| = \infty$. Then there exists an entire function f whose zeros are exactly at a_n, counting multiplicities. And all such functions can be expressed in the following way:

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)}, \tag{5.11}$$

where m is the order of zero at 0, $g(z)$ is an entire function, and $p_n(z)$ are polynomials chosen such that the product converges uniformly on every compact subset of $\mathbb{C}$. Here we can choose

$$p_n(z) = \sum_{k=1}^{m_n} \frac{1}{k} \left(\frac{z}{a_n}\right)^k, \tag{5.12}$$

for a suitable sequence of non-negative integers m_n. Specifically, we can take $m_n = n$.

Proof. Consider that the product converges with the series with general term

$$r_n = \log\left(1 - \frac{z}{a_n}\right) + p_n(z).$$

where the branch of the logarithm is chosen so that $\operatorname{Im} r_n \in (-\pi, \pi]$.

Take $R > 0$, we consider the $|a_n| > R$ part. In the disk $|z| \le R$, the principal branch of $\log(1 - z/a_n)$ can be expanded as

$$\log\left(1 - \frac{z}{a_n}\right) = -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{z}{a_n}\right)^k.$$

We choose $p_n(z)$ to be the partial sum of the above series, by

$$p_n(z) = \sum_{k=1}^{m_n} \frac{1}{k}\left(\frac{z}{a_n}\right)^k. \tag{5.13}$$

$$r_n = -\sum_{k=m_n+1}^{\infty} \frac{1}{k}\left(\frac{z}{a_n}\right)^k, \qquad |r_n| \le \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1} \frac{1}{1 - R/|a_n|}.$$

We shall take m_n that the series

$$\sum_{n=1}^{\infty} \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1} \tag{5.14}$$

converges. If this happens, then we have $|r_n| \to 0$ as $n \to \infty$. So for sufficiently large n, r_n has imaginary part in $(-\pi, \pi)$ as required. Also, this estimation implies uniform and absolute convergence of $\sum r_n$ in $|z| \le R$. So for large n, the product is an analytic function in $|z| \le R$. Add the finite number of terms omitted, the product is still analytic.

Now, if we take $m_n = n$, then the series converges for any $R > 0$ because $\lim_{n\to\infty} |a_n| = \infty$. So the product converges uniformly on every compact subset of $\mathbb{C}$, defining an entire function. □

Corollary 5.2.1: Structure if Meromorphic Functions

Every meromorphic function in $\mathbb{C}$ can be expressed as the quotient of two entire functions.

Proof. Let F be a meromorphic function in $\mathbb{C}$, then take an entire function g whose zeros are exactly at the poles of F, counting multiplicities. (poles are countable tending to ∞, by the compactness of closed disks). Then $f = Fg$ is entire. □

The representation of theorem 5.2.5 is interesting when we can take all $m_n = h$ to be the same. From the proof we know that this is possible if

$$\sum_{n=1}^{\infty} \frac{1}{h+1}\left(\frac{R}{|a_n|}\right)^{h+1} < \infty, \forall R \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < \infty. \tag{5.15}$$

Definition 5.2.3: Canonical Product

Define h to be the smallest non-negative integer such that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} < \infty.$$

Then h is called the genus of the sequence $\{a_n\}$. The expression

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left[\left(\frac{z}{a_n}\right)+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h\right] \tag{5.16}$$

is called the canonical product of genus h associated with the sequence $\{a_n\}$.

If we use the canonical product, then the Weierstrass factorization theorem can be uniquely expressed as

$$f(z)=z^m e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left[\left(\frac{z}{a_n}\right)+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h\right],$$

If g reduces to a polynomial, then we say that f is of finite genus. The genus of f is defined to be $\max\{\deg g, h\}$.

Remark:
The genus of an entire function gives a measure of the growth of the function as $|z|\to\infty$. We shall study this in detail later.

As an example, consider that

$$\sin\pi z = z e^{g(z)}\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}.$$

Here we take $h=1$ as $\sum 1/n^2$ converges and $\sum 1/n$ diverges. To find $g(z)$, we take logarithm and differentiate both sides: (Due to uniform convergence on every compact subset that does not contain any integer).

$$\pi\cot\pi z=\frac{1}{z}+g'(z)+\sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right).$$

From previous example, we know that $g'(z)=0$. Also, as $\sin\pi z/z\to\pi$ as $z\to 0$, we have $e^{g(z)}=\pi$, so

$$\sin\pi z=\pi z\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}=\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right). \tag{5.17}$$

The last equality is due to pairing the n and $-n$ terms. So sin is of genus 1.

Theorem 5.2.6: The Interpolation Theorem

Let $\{a_n\}$ be a sequence of different complex numbers with $\lim_{n\to\infty}|a_n|=\infty$, and let $\{A_n\}$ be a sequence of complex numbers. Then there exists an entire function f such that

$$f(a_n)=A_n,\qquad \forall n.$$

Proof. Let $g(z)$ be an entire function whose zeros are exactly at a_n, all simple zeros (by Weierstrass factorization theorem). Then consider

$$f(z)=\sum_{n=1}^{\infty}A_n\frac{g(z)}{g'(a_n)(z-a_n)}e^{\gamma_n(z-a_n)},$$

SORRY, to be continued □

5.2.4 The Gamma Function

The Sine function example shows a way to define functions via their zeros. It has both side zeros. The simplest function for negative side zeros is the canonical product

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}. \tag{5.18}$$

where easily

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi}. \tag{5.19}$$

We also notice that $G(z-1)$ has simple poles at negative integers and 0. So is $zG(z)$. So we can write

$$G(z-1) = ze^{\gamma(z)}G(z), \tag{5.20}$$

where γ is entire. By taking logarithm and differentiating both sides, we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+z-1} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n}\right).$$

Subtracting both sides, we have $\gamma'(z) = 0$. So $\gamma(z) = \gamma$ is a constant. Evaluating at $z = 1$, we have

$$G(0) = e^{\gamma}G(1) \Rightarrow e^{-\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}$$

The right side restates as

$$\prod_{n=1}^{N} \frac{n+1}{n} e^{-1/n} = (N+1)e^{-\sum_{n=1}^{N} 1/n}$$

Thus we have

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right), \tag{5.21}$$

called the Euler-Mascheroni constant, with approximate value $0.5772\ldots$.

Take $H(z) = G(z)e^{\gamma z}$, then we have $H(z-1) = zH(z)$, then $\Gamma(z) = 1/(zH(z))$ satisfies

$$\Gamma(z+1) = z\Gamma(z).$$

Our definition leads to the explicit formula

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}. \tag{5.22}$$

And

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{5.23}$$

It has poles at $0, -1, -2, \ldots$ all simple poles, without zeros. Also, we have $\Gamma(1) = 1$, so by induction, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_+$. Also from the functional equation, we have $\Gamma(1/2) = \sqrt{\pi}$.

5.3 Entire Functions

In this part we study the growth of entire functions related to their zeros and their genus.

5.3.1 Jensen's Formula

If f is analytic, then $\log|f(z)|$ is a harmonic function except at the zeros of f. So if f is analytic in a disk $|z| \le \rho$ and has no zeros in the disk, then by the mean value property of harmonic functions, we have

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})| \mathrm{d}\theta.$$

The equation still holds if f has zeros on the boundary $|z| = \rho$. For each zero $\rho e^{i\theta_i}$ on the boundary (only finite many), we can factor it out as

$$g(z) = \frac{f(z)}{\prod(z - \rho e^{i\theta_i})},$$

So we have

$$\log\left|\frac{f(0)}{\prod(-\rho e^{i\theta_i})}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(\rho e^{i\theta})}{\prod(\rho e^{i\theta} - \rho e^{i\theta_i})}\right| \mathrm{d}\theta.$$

It is sufficient to prove that

$$\log\rho = \frac{1}{2\pi}\int_0^{2\pi} \log|\rho e^{i\theta} - \rho e^{i\theta_i}| \mathrm{d}\theta \Leftrightarrow \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - 1| \mathrm{d}\theta = 0.$$

which is obvious by $\displaystyle\int_0^{\pi} \log\sin\theta \mathrm{d}\theta = -\pi\log 2$. Adding the equations for all zeros on the boundary, we have the desired result.

Remark:
Here the integral can be understood as an improper integral if there are zeros on the boundary.

Now we consider the case that f has zeros in the disk. Let the zeros of f in $|z| < \rho$ be $a_1, a_2, \ldots, a_n$, counting multiplicities. Assume that none of them is 0. We can factor them out as

$$F(z) = f(z)\prod_{i=1}^{n} \frac{\rho^2 - \overline{a_i}z}{\rho(z - a_i)}. \tag{5.24}$$

Notice the right side is a linear fractional transformation that maps the disk onto itself, so $|F(z)| = |f(z)|$ on $|z| = \rho$. Also, F has no zeros in $|z| < \rho$. So by the previous result, we have

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(\rho e^{i\theta})| \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})| \mathrm{d}\theta.$$

Substituting back, we have the Jensen's formula:

Theorem 5.3.1: Jensen's Formula

Let f be analytic in $|z| \leq \rho$ with zeros $a_1, a_2, \ldots, a_n$ in $|z| < \rho$, counting multiplicities. Then if $f(0) \neq 0$, we have

$$\log|f(0)| = \sum_{i=1}^{n} \log \frac{|a_i|}{\rho} + \frac{1}{2\pi} \int_0^{2\pi} \log|f(\rho e^{i\theta})| \mathrm{d}\theta. \tag{5.25}$$

If $f(0) = 0$ with order h, we can factor out the zero at 0 by

$$F_1(z) = F(z) \left(\frac{\rho}{z}\right)^h$$

and apply the previous result to $F_1(z)$, we have

$$\log|F_1(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(\rho e^{i\theta})| \mathrm{d}\theta. \tag{5.26}$$

We can also generalize Jensen's formula just as Poisson formula, if $f(z) \neq 0$, we obtain

$$\log|f(z)| = -\sum_{i=1}^{n} \log \left| \frac{\rho^2 - \overline{a_i} z}{\rho(z - a_i)} \right| + \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re} \left\{ \frac{\rho e^{i\theta} + z}{\rho e^{i\theta} - z} \right\} \log|f(\rho e^{i\theta})| \mathrm{d}\theta, \tag{5.27}$$

called the Poisson-Jensen formula.

5.3.2 Hadamard's Factorization Theorem

Let f be an entire function with zeros $a_1, a_2, \ldots$, and $f(0) \neq 0$. From the Weierstrass factorization theorem, we have

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)}, \qquad p_n(z) = \sum_{k=1}^{m_n} \frac{1}{k} \left(\frac{z}{a_n}\right)^k.$$

Definition 5.3.1: Order of Entire Function

Denote $M(r) = \max_{|z|=r} |f(z)|$. The order of f is defined as

$$\lambda = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}.$$

Equivalently, λ is the smallest number that for every $\epsilon > 0$, we have

$$M(r) \leq \exp(r^{\lambda + \epsilon})$$

for sufficiently large r.

As we can see, both genus and order measure the growth of entire functions. We have the following important theorem relating them.

Theorem 5.3.2: Genus and Order

Let f be an entire function. Then the genus h and the order λ of f satisfy

$$h \leq \lambda \leq h+1. \tag{5.28}$$

Proof. Assume f has genus h. Then e^g has order $\leq h$. The convergence of the canonical product implies $\sum 1/|a_n|^{h+1}$ converges. Denote the canonical product by $P(z)$, and write

$$E_h(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^h}{h}\right). \quad E_0(u) = 1-u.$$

We shall prove that $\log|E_h(u)| \leq (2h+1)|u|^{h+1}$ for all u.

If $|u| < 1$ we have

$$\log|E_h(u)| \leq \frac{|u|^{h+1}}{h+1} + \cdots \leq \frac{1}{h+1}\frac{|u|^{h+1}}{1-|u|}$$

□

Chapter 6

Laplace Transform

6.1 Definition and Basic Properties

If a function $f : \mathbb{R} \to \mathbb{R}$ is piecewise continuous on every finite interval and absolutely integrable on $(-\infty, +\infty)$, then the **Fourier Transform** of f is defined as

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt, \qquad \omega \in \mathbb{R}. \tag{6.1}$$

The inverse Fourier Transform is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}d\omega. \tag{6.2}$$

To deal with functions that does not have so good convergence properties, we introduce the **Laplace Transform**. The Laplace Transform only considers the function on the positive real axis and adds an exponential decay factor to ensure convergence. Therefore, we shall always assume that $f(x) = 0$ for $x < 0$ when dealing with Laplace Transforms.

Denote the Heaviside step function by

$$h(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases} \tag{6.3}$$

Definition 6.1.1: Laplace Transform

et $f : [0, +\infty) \to \mathbb{C}$ be a piecewise continuous function on every finite interval. And suppose there exists a constant $c \in \mathbb{R}$ such that $|f(t)| \leq Me^{ct}$ for some $M > 0$ and all t. The **Laplace Transform** of f is defined as

$$\mathcal{L}\{f(t)\} = F(p) = \int_{0}^{+\infty} f(t)e^{-pt}dt, \qquad p \in \mathbb{C}, \mathrm{Re}(p) > c. \tag{6.4}$$

The inner compact convergence is quite easy to verify. For $\mathrm{Re}(p) = \sigma > c$, we have

$$|f(t)e^{-pt}| \leq Me^{-(\sigma-c)t},$$

By the Weierstrass M-test, the integral converges uniformly on any compact subset of the half-plane $\mathrm{Re}(p) > c$.

Also, the derivative integral

$$\int_0^\infty \frac{\partial}{\partial p}\left(f(t)e^{-pt}\right)dt = \int_0^\infty -tf(t)e^{-pt}dt$$

Also converges uniformly on any compact subset of the half-plane $\mathrm{Re}(p) > c+\epsilon$ for any $\epsilon > 0$. Thus, by the theorem of differentiation under the integral sign, $F(p)$ is holomorphic on the half-plane $\mathrm{Re}(p) > c$ and

$$F'(p) = \int_0^\infty \frac{\partial}{\partial p}\left(f(t)e^{-pt}\right)dt = \int_0^\infty -tf(t)e^{-pt}dt. \tag{6.5}$$

So we have

$$\lim_{\sigma\to\infty} F(p) = 0.$$

Example: **Laplace Transform**

- $\mathcal{L}\{e^{at}\} = \dfrac{1}{p-a}, \quad \mathrm{Re}(p) > \mathrm{Re}(a).$
- $\mathcal{L}\{t^\alpha\} = \dfrac{\Gamma(\alpha+1)}{p^{\alpha+1}}, \quad \mathrm{Re}(p) > 0, \alpha > -1.$

6.2 Basic Properties

Proposition: **Basic Properties of Laplace Transform**

Let $f(t)$ and $g(t)$ be piecewise continuous functions on every finite interval and of exponential order c_f and c_g respectively. Then we have

- Linearity: $\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$, for any $a, b \in \mathbb{C}$. This also holds for inverse Laplace Transform.
- The Similarity Theorem: $\mathcal{L}\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{p}{a}\right)$, for any $a > 0, \mathrm{Re}(p) > ac_f$.
- The Differentiation Theorem: $\mathcal{L}\{f'(t)\} = pF(p) - f(0)$, for $\mathrm{Re}(p) > c_f$. More generally, $\mathcal{L}\{f^{(n)}(t)\} = p^nF(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$.

  Conversely, $\mathcal{L}\{(-t)^nf(t)\} = F^{(n)}(p)$, for $\mathrm{Re}(p) > c_f$.
- The Integration Theorem: $\mathcal{L}\left\{\displaystyle\int_0^t f(\tau)d\tau\right\} = \dfrac{1}{p}F(p)$, for $\mathrm{Re}(p) > c_f$.

  Conversely, $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_p^\infty F(q)dq$, for $\mathrm{Re}(p) > c_f$. Taking $p \to 0$, we have

  $$\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty F(p)dp.$$
- The Delay Theorem: $\mathcal{L}\{f(t-a)h(t-a)\} = e^{-ap}F(p)$, for any $a \geq 0, \mathrm{Re}(p) > c_f$.

- The Shift Theorem: $\mathcal{L}\{e^{at}f(t)\} = F(p-a)$, for any $a \in \mathbb{C}, \mathrm{Re}(p) > c_f + \mathrm{Re}(a)$.
- The Periodicity Theorem: If $f(t)$ is periodic with period $T > 0$, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T f(t)e^{-pt}dt, \quad \mathrm{Re}(p) > c_f.$$

- The Convolution Theorem: If we define the convolution of f and g as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau,$$

then we have

$$\mathcal{L}\{(f * g)(t)\} = F(p)G(p), \quad \mathrm{Re}(p) > \max\{c_f, c_g\}.$$

Some useful Laplace Transforms are listed below:

- $\mathcal{L}\{\cos(\omega t)\} = \dfrac{p}{p^2+\omega^2}, \quad \mathrm{Re}(p) > 0.$
- $\mathcal{L}\{\sin(\omega t)\} = \dfrac{\omega}{p^2+\omega^2}, \quad \mathrm{Re}(p) > 0.$
- $\mathcal{L}\{\sinh(\omega t)\} = \dfrac{\omega}{p^2-\omega^2}, \quad \mathrm{Re}(p) > |\mathrm{Re}(\omega)|.$
- $\mathcal{L}\{\cosh(\omega t)\} = \dfrac{p}{p^2-\omega^2}, \quad \mathrm{Re}(p) > |\mathrm{Re}(\omega)|.$

6.3 The Inverse Transform Methods

The Rational Function Method If $F(p)$ is a rational function, i.e., $F(p) = \frac{A(p)}{B(p)}$ where $A(p)$ and $B(p)$ are polynomials, then we can use partial fraction decomposition to write $F(p)$ as a sum of simpler fractions whose inverse Laplace Transforms are known.

$$F(p) = \sum_k \sum_{s=1}^{m_k} \frac{A_{k,s}}{(p-a_k)^s} \tag{6.6}$$

where a_k are the poles of $F(p)$ and m_k are their respective multiplicities. The inverse Laplace Transform can then be found using known transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{(p-a)^s}\right\} = \frac{t^{s-1}e^{at}}{(s-1)!}. \tag{6.7}$$

6.3.1 The Bromwich Integral Method

The inverse Laplace Transform can be computed using the Bromwich integral (also known as the inverse Laplace integral):

If f is continuous at $t \geq 0$ then

$$f(t) = \mathcal{L}^{-1}\{F(p)\} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(p)e^{pt}dp, \tag{6.8}$$

where σ is a real number such that $\sigma > c$, with c being the exponential order of $f(t)$.

Proof. Take $p = \sigma + is$, then from the Fourier formula, we have

$$\mathscr{F}(s) = \int_{-\infty}^{+\infty} f(t)h(t)e^{-\sigma t}e^{-ist}dt = F(\sigma + is).$$

$$f(t)h(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\sigma + is)e^{ist}ds.$$

which is exactly the Bromwich integral. □

As it turns out, the Bromwich integral can be evaluated using the residue theorem.

Theorem 6.3.1: Bromwich Integral and Residue Theorem

Let $F(p)$ be the Laplace Transform of a piecewise continuous function $f(t)$ of exponential order c. Suppose that $F(p)$ has finite many poles $p_1, p_2, \ldots, p_n$ all in the half-plane $\mathrm{Re}(p) < \sigma$, where $\sigma > c$ is sufficiently large, and the integral

$$\int_{\sigma - i\infty}^{\sigma + i\infty} F(p)dp$$

converges absolutely. Then for $t > 0$, we have

$$f(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(p)e^{pt}dp = \sum_{k=1}^{n} \mathrm{Res}\left(F(p)e^{pt}, p_k\right). \tag{6.9}$$

Proof. Taking the line and large semicircle contour in the left half-plane, by the Jordan's lemma, the integral on the semicircle vanishes as its radius goes to infinity. Thus, by the residue theorem, we have the above result. □

6.3.2 The Series Method

Theorem 6.3.2: The Series Method of Laplace Transform

Suppose the Laplace Transform $F(p)$ is analytic at ∞ and have the Laurent series expansion

$$F(p) = \sum_{n=1}^{\infty} \frac{c_n}{p^n}, \quad |p| > R.$$

Then the inverse Laplace Transform of $F(p)$ is given by

$$f(t) = \sum_{n=1}^{\infty} \frac{c_n}{(n-1)!}t^{n-1}.$$

The other way around also holds.