Mathematical Analysis

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Contents

Chapter 1

Continuous Mappings (General)

1.1 Metric Spaces

1.1.1 Definition and Examples

Definition 1.1.1: Metric Spaces

A set X is a metric space if it has a function

$$d : X \times X \to \mathbb{R} \tag{1.1}$$

such that

- $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$.
- $d(x_1, x_2) = d(x_2, x_1)$.
- $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$.

Note that setting $x_3 = x_1$ in triangle inequality we have $d(x_1, x_2) \geq 0$.

Example: **Metrics on $\mathbb{R}^n$**

In $\mathbb{R}^n$ we have the traditional Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \tag{1.2}$$

Or we can have a more general

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \text{ where } p \geq 1. \tag{1.3}$$

The validity comes from the Minkowski inequality.
Generalizing by $p \to \infty$ we have clearly

$$d(x, y) = \max_{a \leq x \leq b} |x_i - y_i| \tag{1.4}$$

Example: **Metrics on $C[a,b]$**

Similarly in $C[a,b]$, that is the continuous functions on $[a,b]$, we can define

$$d_p(f,g) = \left(\int_a^b |f-g|^p(x)\mathrm{d}x\right)^{\frac{1}{p}}, \text{ where } p \geq 1 \tag{1.5}$$

and limiting to infinity we have

$$d(f,g) = \sup |f(x) - g(x)| \tag{1.6}$$

1.1.2 Open and Closed Sets of a Metric Space

Definition 1.1.2: Open Balls

For $\delta > 0$ and $a \in X$, we define the set

$$B(a,\delta) = \{x \in X \mid d(a,x) < \delta\} \tag{1.7}$$

to be the open ball with center $a \in X$ and radius δ or the δ-neighborhood of a.

Definition 1.1.3: Open Sets and Closed Sets

A set $G \subseteq X$ is open in (X,d) if

$$\forall x \in G, \exists \delta > 0, B(x,\delta) \subseteq G$$

A set $F \subseteq X$ is closed iff $X - F$ is open.
An open set containing x is said to be a neighborhood of x.

We now denote the closed ball

$$\tilde{B}(a,r) = \{x \in X \mid d(a,x) \leq r\} \tag{1.8}$$

Definition 1.1.4: Interior, Exterior and Boundary points

Let $E \subseteq X$

- An interior point of E iff some neighborhood of it $\subseteq E$.
- An exterior point of E iff some neighborhood of it $\subseteq X - E$.
- A boundary point of E is neither an interior point nor an exterior point of E.

Definition 1.1.5: Limit Points

A point $a \in X$ is a limit point of $E \subseteq X$ iff $\forall$ neighborhood $O(a)$ we have $E \cap O(a)$ is infinite.
We denote $\overline{E} = E \cup$ the limit points of E.

Proposition: **Condition to be Closed**

A set $F \subseteq X$ is closed iff it contains all its limit points. That is $F = \overline{F}$.

1.1.3 Subspaces of a Metric Space

Definition 1.1.6: Subspace of a Metric Space

A metric space (X_1, d_1) is a subspace of (X, d) iff

- $X_1 \subseteq X$.
- $\forall a, b \in X_1, d_1(a, b) = d(a, b)$.

Proposition: **Open sets in Subspaces**

If (X_1, d_1) is a subspace of (X, d), then the open sets in X_1 is exactly $X_1 \cap E$ where E is an open set of X.

1.1.4 Direct Product of Metric Spaces

If (X_1, d_1) and (X_2, d_2) are two metric spaces, one can introduce a metric on the set $X_1 \times X_2$. Like

$$d((x_1, x_2), (x_1', x_2')) = \sqrt{d_1^2(x_1, x_1') + d_2^2(x_2, x_2')}$$

$$d((x_1, x_2), (x_1', x_2')) = d_1(x_1, x_1') + d_2(x_2, x_2')$$

$$d((x_1, x_2), (x_1', x_2')) = \max\{d_1(x_1, x_1'), d_2(x_2, x_2')\}$$

1.2 Topological Spaces

Definition 1.2.1: Topological Spaces

A set X has a topology $\mathcal{T}$, where $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X that are called open sets, with the restriction

- $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
- $\forall \alpha \in A, \mathcal{T}_\alpha \in \mathcal{T}$, we have $\bigcup_{\alpha \in A} \mathcal{T}_\alpha \in \mathcal{T}$.
- $\forall \mathcal{T}_i \in \mathcal{T}, \bigcap_{i=1}^{n} \mathcal{T}_i \in \mathcal{T}$.

A topology can be generated by a metric as above. We now introduce base of a topology.

Definition 1.2.2: Base of a Topology

A base of a topological space $(X, \mathcal{T})$ is a set $\mathcal{B} \subseteq \mathcal{T}$ such that

$$\forall G \in \mathcal{T}, G = \bigcup_{\alpha \in A} B_\alpha, \text{ for some } B_\alpha \in \mathcal{B}$$

The minimal cardinality among all the bases of a topological space is called its weight.

Thus all the open balls is a base of the topology given by a metric.

Example: **The germs of Continuous Functions**

Consider the set $C(\mathbb{R}, \mathbb{R})$ of real-valued continuous functions defined on the entire $\mathbb{R}$ line. For an $a \in \mathbb{R}$, we define an equivalence relation $\sim$:

$$f \sim g \Leftrightarrow \exists \text{ a neighborhood } U(a), \forall x \in U(a), f(x) = g(x) \tag{1.9}$$

We denote the equivalent class (called germs) f_a.
We now define a neighborhood of f_a. Let f be a function that generates f_a, the set $\{f_x \mid x \in \mathbb{R}\}$ is a neighborhood of f_a. Taking all the neighborhoods as a base we get a topology.

Definition 1.2.3: Hausdorff Space

A topological space is Hausdorff if *any two distinct points have non-intersecting neighborhoods.*

Definition 1.2.4: Dense

A set $E \subseteq X$ is (everywhere) dense in X if

$$\forall x \in X, \forall U(x), E \cap U(x) \neq \emptyset$$

It is easy to show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 1.2.5: Separable Spaces

A metric space having a countable dense set is called separable.

1.3 Compact Sets

1.3.1 Definition

Definition 1.3.1: Compact Sets

A set K in topological space $(X, \mathcal{T})$ is compact if every open cover of K has a finite subcover.

Proposition: **Compact Conditions**

A subset $K \subseteq X$ is compact in (X, d) iff K is compact in (K, d).
Which mean that compactness has some sense of locality.

Proof. Using proposition 1.1.3 would do. □

Lemma 1.3.1: Compact Sets are Closed

If K is a compact set in a Hausdorff space $(X, \mathcal{T})$, then K is closed in X

Proof. We shall show that every limit point of K belongs to K. Suppose $x_0 \notin K$ is a limit point of K, then $\forall x \in K$,we construct an open neighborhood $G(x)$ such that $\exists O_x(x_0) \cap G(x) = \emptyset$, then all of $G(x)$ forms an open cover of K. Select a finite subcover $G(x_1), \ldots, G(x_n)$, then $O = \bigcap_{i=1}^{n} O_{x_i}(x_0)$ is a neighborhood of x_0 but $K \cap O = \emptyset$, so x_0 cannot be a limit point of K. □

Lemma 1.3.2: Nested Compact Sets

If $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ is a nested sequence of nonempty compact sets, then $\bigcap_{i=1}^{\infty} K_i$ is nonempty.

Proof. By lemma 1.3.1 the sets $G_i = K_1 - K_i$ are open in K_1. If the intersection $\bigcap_{i=1}^{\infty} G_i$ is empty, then the sequence $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots$ forms a covering of K_1. Extracting a finite subcover gives a contradiction. □

Lemma 1.3.3: Closed subsets of Compact Sets

A closed subset F of a compact set K is itself compact.

Proof. Let $\{G_\alpha\}$ be an open covering of F. Adjoining $\{G_\alpha\} \cup K \backslash F$ we obtain an open covering of K. □

1.3.2 Metric Compact Sets

Definition 1.3.2: ϵ-grid

The set $E \subseteq X$ is called an ϵ-grid in the metric space (X, d) if for $\forall x \in X, \exists e \in E, d(e, x) < \epsilon$.

Lemma 1.3.4: Finite ϵ-grid

If a metric space (X, d) is compact, then for $\forall \epsilon > 0$ there exists a finite ϵ-grid in X.

Proof. $\forall x \in K$ we choose an open ball $B(x, \epsilon)$. From the open covering of K by these balls we select a finite subcover $B(x_i, \epsilon)$, and the x_i forms a finite ϵ-grid. □

Theorem 1.3.1: Criterion for Compactness in a metric space

A metric space (X, d) is compact iff from each sequence there is a subsequence that converges to a point in K.

1.4 Connected Topological Spaces

Definition 1.4.1: Connected Topological Spaces

A topological space (X, τ) is connected if the only clopen subsets are X and $\emptyset$. (It cannot be represented as the union of two disjoint nonempty open/closed subsets).

1.5 Complete Metric Spaces

Definition 1.5.1: Complete Metric Spaces

A metric space (X, d) is complete if every Cauchy sequence of its points is convergent.

1.5.1 Completion of a Metric Space

Definition 1.5.2: Completion of a Metric Space

The smallest complete metric space containing a given metric space (X, d) is the completion of (X, d).

1.6 Continuous Mapping of Topological Space

1.6.1 The limit of a Mapping

Definition 1.6.1: Limit of a Mapping

Let $f : X \to Y$. Let $\mathcal{B}$ be a basis of X. Then the point $A \in Y$ is the limit of the mapping f over basis $\mathcal{B}$ if $\forall$ neighborhood $V(A)$ of $A \in Y$ there exists $B \in \mathcal{B}$ such that $f(B) \subseteq V(A)$. Denoted $\lim_{\mathcal{B}} f(x) = A$.

If (X, d_X) and (Y, d_Y) are metric spaces, we can rephrase the definition as follows by $\epsilon - \delta$ language:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in X (0 < d_X(a, x) < \delta \to d_Y(f(x), A) < \epsilon)$$

Then we denote

$$\lim_{x \to a} f(x) = A.$$

Definition 1.6.2: Continuity on Topological Spaces

A mapping $f : X \to Y$ of a topological space (X, τ_X) and (Y, τ_Y) is continuous at a point $a \in X$ if for every neighborhood $V(f(a)) \subseteq Y$, there exists a neighborhood $U(a) \subseteq X$ such that $f(U(a)) \subseteq V(f(a))$.

$$\forall V(f(a)), \exists U(a), f(U(a)) \subseteq V(f(a)).$$

The mapping $f : X \to Y$ is continuous iff it is continuous at every point $x \in X$. The set of continuous mappings $f : X \to Y$ will be denoted $C(X, Y)$.

In the sense of metric space, we can rephrase that:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in X(0 < d_X(a, x) < \delta \to d_Y(f(x), f(a)) < \epsilon)$$

or

$$\lim_{x \to a} f(x) = f(a).$$

1.6.2 Local Properties of a Continuous Mapping

Theorem 1.6.1: Continuity of Composition

Let X, Y, Z be topological spaces, if $f : X \to Y$ is continuous at $a \in X$ and $g : Y \to Z$ is continuous at $f(a) \in Y$, then $g \circ f$ is continuous at a.

Chapter 2

General Differential Calculus

2.1 Normed Vector Spaces

Differentiation is the process of finding the best local linear approximation of a function.

2.2 Linear and Multilinear Transformations

2.3 The Differential of a Mapping

Definition 2.3.1: Differentiable

Let X and Y be normed vector spaces. A mapping $f : E \to Y$ of a set $E \subseteq X$ is differentiable at an interior point $x \in E$ if there extsts a continuous linear transform $L(x) : X \to Y$ such that

$$f(x+h) - f(x) = L(x)h + \alpha(x;h)$$

where

$$\lim_{h \to 0, x+h \in E} |\alpha(x;h)|_Y \cdot |h|_X^{-1} = 0.$$

The function $L(x) \in \mathscr{L}(X,Y)$ is called the differential, the tangent mapping or the derivative of f at x. We denote $L(x)$ by $\mathrm{d}f(x), Df(x)$ or $f'(x)$.

Theorem 2.3.1: Uniqueness of Differential

If $f : X \to Y$ is differentiable at $x \in X$, its differential $L(x)$ is unique.

Proof. Let $L_1(x)$ and $L_2(x)$ satisfy the condition. Then

$$f(x+h) - f(x) - L_1(x)h = \alpha_1(x;h)$$
$$f(x+h) - f(x) - L_2(x)h = \alpha_2(x;h)$$

Setting $L(x) = L_1(x) - L_2(x)$ and $\alpha(x;h) = \alpha_1(x;h) - \alpha_2(x;h)$, so $\alpha(x;h) = o(h)$ as $h \to 0$. And we have

$$L(x)h = \alpha(x;h)$$

We have

$$|L(x)h| = \frac{|L(x)(\lambda h)|}{|\lambda|} = \frac{|\alpha(x;\lambda h)|}{|\lambda h|}|h| \to 0, \text{ as } \lambda \to 0 \ .$$

Thus $\forall h \neq 0, L(x)h = 0$, thus $L(x) = 0$. □

If E is an open subset of X and $f : E \to Y$ is a mapping that is differential at $\forall x \in E$, then the function $f' : E \to \mathscr{L}(X;Y)$ is called the derivative of f. Keep in mind that $f'(x) \in \mathscr{L}(X;Y)$ is a linear transform.

2.3.1 The General Rules for Differentiation

Proposition: **Rules for Differential**

Let X, Y, Z be normed spaces and U, V open sets in X, Y respectively.

- **Linearity:** If f_1, f_2 are differentiable at x, then $f_1 + f_2$ is differentiable at x, and

$$(\lambda_1 f_1 + \lambda_2 f_2)'(x) = \lambda_1 f_1'(x) + \lambda_2 f_2'(x).$$

- **Composition Chain Rule:** $f : U \to V$ is differentiable at $x \in U \subseteq X$, and $g : V \to Z$ if differentiable at $f(x) = y \in V \subseteq Y$, then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x)$$

- **Inverse Mapping:** If $f : U \to Y$ is continuous at $x \in U \subseteq X$ and has a continuous inverse $f^{-1} : V \to X$ in the neighborhood at $f(x)$. Then if f is differential at x and $f'(x)$ has a continuous inverse, then the mapping f^{-1} is differentiable at $f(x)$ with

$$(f^{-1})'(f(x)) = (f'(x))^{-1}$$

2.3.2 The Partial Derivatives of a Mapping

Let $U = U(a)$ be a neighborhood of $a \in X = X_1 \times \cdots \times X_m$, and $f : U \to Y$ be a mapping. In this case

$$y = f(x) = f(x_1, \ldots, x_m)$$

Fixing all variables other then x_i, we have a mapping

$$f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_m) = \varphi_i(x_i)$$

defined in some neighborhood U_i of $a_i \in X$.

The mapping φ_i is called the partial mapping with respect to x_i at $a \in X$.

Definition 2.3.2: Partial Derivative

If φ_i is differentiable at $x_i = a_i$, then its derivative is called the partial derivative of f at a with respect to x_i. Denoted

$$\partial_i f(a) \qquad D_i f(a) \qquad \frac{\partial f}{\partial x_i}(a) \qquad f'_{x_i}(a)$$

Note that $\partial_i f(a) \in \mathscr{L}(X_i; Y)$.

Proposition: **Total Derivative and Partial Derivative**

If the mapping $f : X \to Y$ is differentiable at $a = (a_1, \ldots, a_m) \in X$, then it has partial derivative of each variable and the derivative of f is:

$$df(a)h = \partial_1 f(a)h_1 + \cdots + \partial_m f(a)h_m.$$

where $h = (h_1, \ldots, h_m) \in TX(a)$.

2.4 The Finite-Increasement Theorem