

# *Complex Analysis*

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# Contents

<b>1</b>	<b>Complex Numbers</b>	<b>1</b>
1.1	The Algebra of Complex Numbers . . . . .	1
1.1.1	Arithmetic Operations . . . . .	1
1.1.2	Square Roots . . . . .	2
1.1.3	Conjugation and Absolute Value . . . . .	2
1.2	Geometry Representation of Complex Numbers . . . . .	3
1.2.1	Addition and Multiplication . . . . .	3
1.2.2	Binomial Equation . . . . .	5
1.2.3	Analytic Geometry . . . . .	5
1.2.4	The Spherical Representation . . . . .	6
<b>2</b>	<b>Complex Functions</b>	<b>9</b>
2.1	Analytic Functions . . . . .	9
2.1.1	Limits and Continuity . . . . .	9
2.1.2	Analytic Functions . . . . .	11
2.1.3	Polynomials . . . . .	13
2.1.4	Rational Functions . . . . .	14
2.2	Elementary Theory of Power Series . . . . .	16
2.2.1	Series . . . . .	17
2.2.2	Uniform Convergence . . . . .	17
2.2.3	Power Series . . . . .	17
2.2.4	Abel limit Theorem . . . . .	19
2.3	The Exponential and Trigonometric Functions . . . . .	20
2.3.1	the Exponential . . . . .	20
2.3.2	The Trigonometric Functions . . . . .	21
2.3.3	The Periodicity . . . . .	21
2.4	The Logarithm . . . . .	22
<b>3</b>	<b>Analytic Functions As Mappings</b>	<b>25</b>
3.1	Conformality . . . . .	25
3.1.1	Arcs and Closed Curves . . . . .	25
3.1.2	Analytic Functions in Regions . . . . .	26
3.2	Conformal Mappings . . . . .	29
3.2.1	Length and Area . . . . .	30
3.3	Linear Transformations . . . . .	31
3.3.1	The Linear Group . . . . .	31
3.3.2	The Cross Ratio . . . . .	32

3.3.3	Symmetry . . . . .	33
3.3.4	Oriented Circles . . . . .	36
3.3.5	Families of Circles . . . . .	37
3.4	Elementary Conformal Mappings . . . . .	40
3.4.1	Level Curves . . . . .	40
3.4.2	A Brief Survey of Conformal Mappings . . . . .	41
3.4.3	A Brief Survey on Riemann Surfaces . . . . .	42
<b>4</b>	<b>Complex Integration</b>	<b>45</b>
4.1	Fundamental Theorems . . . . .	45
4.1.1	Line Integral . . . . .	45
4.1.2	Rectifiable Arcs . . . . .	47
4.1.3	Line Integral As Functions of Arcs . . . . .	47
4.1.4	Cauchy's Integral Theorem for Rectangles . . . . .	48
4.1.5	Cauchy's Theorem in Disk . . . . .	50
4.2	Cauchy's Integral Formula . . . . .	51
4.2.1	The Index of a Point . . . . .	51
4.2.2	The Integral Formula . . . . .	52
4.2.3	Higher Order Derivatives . . . . .	53
4.3	Local Properties of Analytic Functions . . . . .	55
4.3.1	Removable Singularities and Taylor's Formula . . . . .	55
4.3.2	Zeros and Poles . . . . .	56
4.3.3	The Local Mapping . . . . .	60
4.3.4	The Maximum Principle . . . . .	63
4.4	The General Cauchy Theorem . . . . .	65
4.4.1	Chains and Cycles . . . . .	65
4.4.2	Simple Connectedness . . . . .	65
4.4.3	Homology . . . . .	67
4.4.4	The General Cauchy's Theorem . . . . .	67
4.4.5	Proof of Cauchy's Theorem . . . . .	68
4.4.6	Local Exact Differential . . . . .	68
4.4.7	Multiply Connected Regions . . . . .	70
4.5	The Calculus of Residues . . . . .	71
4.5.1	The Residue Theorem . . . . .	71
4.5.2	The Argument Principle . . . . .	73
4.5.3	The Definite Integrals . . . . .	74
4.6	Harmonic Functions . . . . .	79
4.6.1	Definition and Basic Properties . . . . .	79
4.6.2	The Mean Value Theorem . . . . .	81
4.6.3	Poisson's Formula . . . . .	82
4.6.4	Schwarz's Theorem . . . . .	83
4.6.5	The Reflection Principle . . . . .	84
<b>5</b>	<b>Series and Product Developments</b>	<b>87</b>
5.1	Power Series . . . . .	87
5.1.1	Weierstrass' Theorem . . . . .	87
5.1.2	The Taylor Series . . . . .	89

5.1.3	The Laurent Series . . . . .	93
5.2	Partial Fractions and Factorization . . . . .	95
5.2.1	Partial Fractions . . . . .	95
5.2.2	Infinite Products . . . . .	97
5.2.3	Canonical Products . . . . .	98



# Chapter 1

## Complex Numbers

### 1.1 The Algebra of Complex Numbers

#### 1.1.1 Arithmetic Operations

The complex field  $\mathbb{C}$  is defined to be  $\mathbb{R} \times \mathbb{R}$ , with  $a + ib$  stands for  $(a, b)$ . The addition and multiplication is defined as follows.

##### Definition 1.1.1: Complex Field

The complex field  $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$ , with addition and multiplication:

- **Addition:**  $(a + ib) + (c + id) = (a + c) + (b + d)i$ .
- **Multiplication:**  $(a + ib)(c + id) = (ac - bd) + (ad + bc)i$ .

We can verify that  $(\mathbb{C}, +, \cdot)$  indeed is a field. Its additive identity being 0 and multiplicative identity 1.

---

*Remark:*

We can also define the complex field in a more abstract way. Showing only properties of the field that has to be matched.

Let  $\mathbb{F}$  be a field. Then  $\mathbb{C}$  is a subfield of  $\mathbb{F}$  that matches the following properties.

- $\mathbb{R}$  is a subfield of  $\mathbb{F}$ . (isomorphic)
- $\exists \alpha \in \mathbb{F}, \alpha^2 + 1 = 0$ . Let one of them be  $i$ .
- $\mathbb{C}$  is the subfield generated by  $(\mathbb{R}, i)$ .

We can prove that  $\mathbb{C}$  does not depend on the choice of  $\mathbb{F}$  and  $i$ . The existence of such a field is constructed by 1.1.1 above.

---

There are also isomorphic representations of the complex field. For example,

- $\left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \cong \mathbb{C}$ .

- $\mathbb{R}[x]/(x^2 + 1)\mathbb{R}[x] \cong \mathbb{C}$ .

### 1.1.2 Square Roots

Now let's consider some properties that are special about the complex field. The original intuition of constructing  $\mathbb{C}$  is that not all reals have square roots. We denote the square root of  $-1$  to  $i$  so as to fill the blank. We now state that square root is closed under  $\mathbb{C}$ .

#### Theorem 1.1.1: Square Roots of Complex Numbers

$$\forall a \in \mathbb{C}, \exists b \in \mathbb{C}, a = b^2$$

*Proof.* We can explicitly find the square roots using the result that every non-negative reals have square roots.

Let  $\alpha + i\beta \in \mathbb{C}$ , we find

$$(x + iy)^2 = \alpha + i\beta$$

that is,

$$\begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta \end{aligned}$$

Form there we get

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2$$

giving

$$x^2 + y^2 = \sqrt{\alpha^2 + \beta^2}$$

that is,

$$\begin{cases} x^2 = \frac{1}{2} \left( \alpha + \sqrt{\alpha^2 + \beta^2} \right) \\ y^2 = \frac{1}{2} \left( -\alpha + \sqrt{\alpha^2 + \beta^2} \right) \end{cases}$$

Only two result matches, as we can see.

□

### 1.1.3 Conjugation and Absolute Value

Define  $a - bi$  to be conjugation of  $a + bi$ . And define

$$\operatorname{Re} a = \frac{a + \bar{a}}{2}, \operatorname{Im} a = \frac{a - \bar{a}}{2i}$$

#### Theorem 1.1.2: Conjugations

Let  $R(a_1, \dots, a_n)$  be any rational operation applied to  $a_1, \dots, a_n \in \mathbb{C}$ , then

$$\overline{R(a_1, \dots, a_n)} = R(\bar{a}_1, \dots, \bar{a}_n) \quad (1.1)$$

Let  $z = a + bi$ . We denote  $|z| = \sqrt{a^2 + b^2}$ .



---

*Example:* **Conjugates and Absolute Values**

---

- $\left| \frac{a-b}{1-\bar{a}b} \right| = 1.$

- Lagrange's Identity:

$$\left| \sum_{i=1}^n a_i b_i \right|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 - \sum_{1 \leq i < j \leq n} |a_i \bar{b}_j - a_j \bar{b}_i|^2 \quad (1.2)$$

- $|a_1 + \dots + a_n| \leq |a_1| + \dots + |a_n|.$

---

## 1.2 Geometry Representation of Complex Numbers

The most usual way to visualize complex numbers is putting it real and imaginary part into a coordinate system.

### 1.2.1 Addition and Multiplication

The addition of complex numbers can be visualized as vector addition.

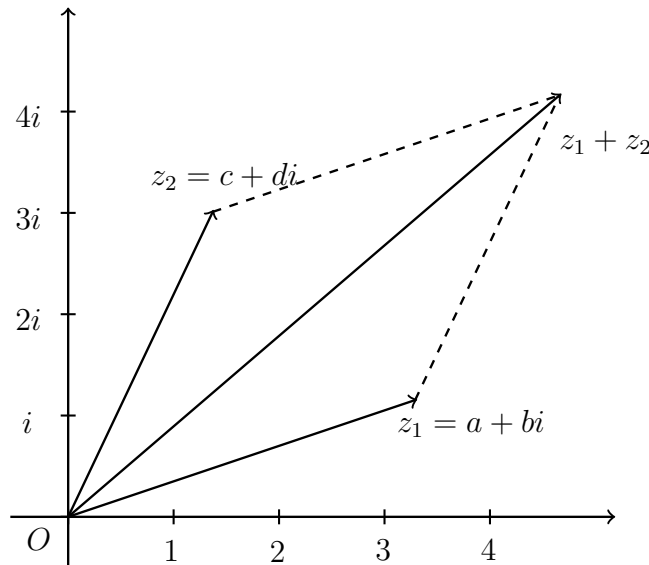


Figure 1.1: Complex Addition

In order to derive a geometric interpretation of multiplication of complex numbers we write it

into polar coordinates. If the polar coordinates of point  $(\alpha, \beta)$  is  $(r, \varphi)$ , we know

$$\begin{aligned}\alpha &= r \cos \varphi \\ \beta &= r \sin \varphi\end{aligned}$$

Hence we can write  $z = \alpha + i\beta = r(\cos \varphi + i \sin \varphi)$ , which is the trigonometric form. In this form, we denote

$$r = |z|, \text{ and } \varphi = \arg z$$

Multiplying  $z_1$  and  $z_2$  we get

$$\begin{aligned}z_1 z_2 &= r_1 r_2 (\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2 + (\cos \varphi_1 \sin \varphi_2 + \sin \varphi_1 \cos \varphi_2) i) \\ &= r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))\end{aligned}$$

We see that the product has modulus  $r_1 r_2$  and argument  $\varphi_1 + \varphi_2$ . We latter result is new, for which we can express it by

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad (1.3)$$

By means of 1.3 we can produce a geometric construction of products of complex numbers by similar triangles.

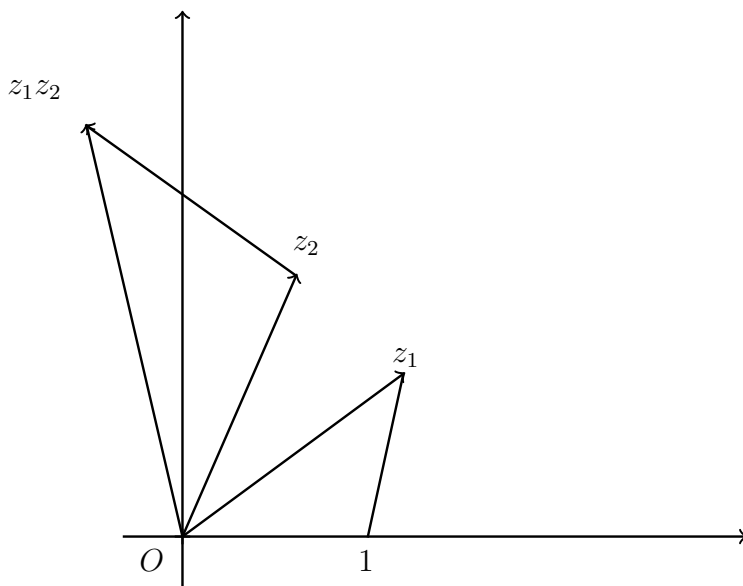


Figure 1.2: Complex Products

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*Remark:*

We've noticed that this definition of argument is fairly intuitive but not very stern. Angle's are not defined yet and we use trigonometric functions. However, a perfect acceptable way is to express angles by the length of an arc, which is defined by definite integral.

However, in complex analysis we do not follow that rule, because we have a more direct route: the connection between exponential functions and trigonometric functions.

---

### 1.2.2 Binomial Equation

From the preceding result we derive that the powers of  $a = r(\cos \varphi + i \sin \varphi)$  are given by (for  $n > 0$ )

$$a^n = r^n(\cos n\varphi + i \sin n\varphi) \quad (1.4)$$

This equation hold trivially for  $n = 0$  and since

$$a^{-1} = r^{-1}(\cos \varphi - i \sin \varphi) = r^{-1}(\cos(-\varphi) + i \sin(-\varphi))$$

so it holds for all  $n \in \mathbb{Z}$ .

To solve the  $n^{\text{th}}$  root of a complex number  $a$ , we need to solve the equation

$$z^n = a$$

Suppose  $a \neq 0$  and we write  $a = r(\cos \varphi + i \sin \varphi)$  and

$$z = \rho(\cos \theta + i \sin \theta)$$

then we have

$$\rho^n(\cos n\theta + i \sin n\theta) = r(\cos \varphi + i \sin \varphi)$$

thus we have  $\rho^n = r$  and  $n\theta = \varphi + 2k\pi$ .

$$z = \sqrt[n]{r} \left( \cos \left( \frac{\varphi}{n} + k \frac{2\pi}{n} \right) + i \sin \left( \frac{\varphi}{n} + k \frac{2\pi}{n} \right) \right)$$

where  $k = 0, \dots, n-1$ . Therefore, we get

#### Theorem 1.2.1: Root of Complex Numbers

There are  $n$   $n^{\text{th}}$  root of any complex number  $\neq 0$ , they have the same modulus and there arguments are equally spaced

Geometrically, the  $n$   $n^{\text{th}}$  root forms a regular  $n$ -polygon.

#### Definition 1.2.1: Roots of Unity

The roots of  $z^n = 1$  are called  $n^{\text{th}}$  root of unity. That is, if we set

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \quad (1.5)$$

then  $1, \omega, \omega^2, \dots, \omega^{n-1}$  are the roots of unity.

### 1.2.3 Analytic Geometry

In analytic geometry, points of a 2D plane can be represented by both  $x, y$  or  $z, \bar{z}$ .

For instance, the equation of a circle is  $|z - a| = r$  or  $(z - a)(\bar{z} - \bar{a}) = r^2$ .

A straight line can be expressed as parametric equation  $z = a + bt$  where  $a, b \in \mathbb{C}$  and the parameter  $t \in \mathbb{R}$ . The angle between  $z = a + bt$  and  $z = a' + b't$  are just  $\arg \frac{b}{b'}$ .

### 1.2.4 The Spherical Representation

For many purposes it is useful to extend the system  $\mathbb{C}$  to include the symbol  $\infty$ . We define  $a + \infty = \infty + a = \infty$  for all  $a \in \mathbb{C}$  and  $b \cdot \infty = \infty \cdot b = \infty$  for all  $b \neq 0$ .

In the plane there is no room for the infinity, but we can introduce an ideal point called the point at infinity. The points in the plane and the point at infinity together constructed *The Extended Complex Plane*. To give a more intuitive geometric model, we consider  $S^2$ , which is the unit sphere in 3-dimensional space,  $x_1^2 + x_2^2 + x_3^2 = 1$ .

We have the 1-1 correspondence by stereographic projection.

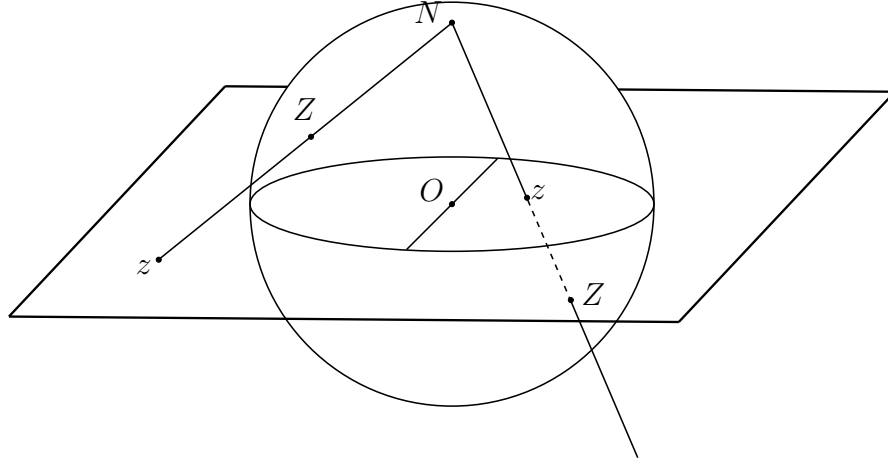


Figure 1.3: Stereographic Projection

With every point on  $S$ , except  $(0, 0, 1)$ , we associate a complex number

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

then we have

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}$$

hence

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

and further computation leads to

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2}$$

$$x_2 = \frac{z - \bar{z}}{i(1 + |z|^2)}$$

And we let the point  $\infty$  corresponds to  $(0, 0, 1)$ .

*Remark:*

The correspondence can be derived informally by similar triangles. We have  $x_3 = \cos 2\theta$  and  $|z| = \cot \theta$ . Thus  $|z|^2 = \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1 + x_3}{1 - x_3}$

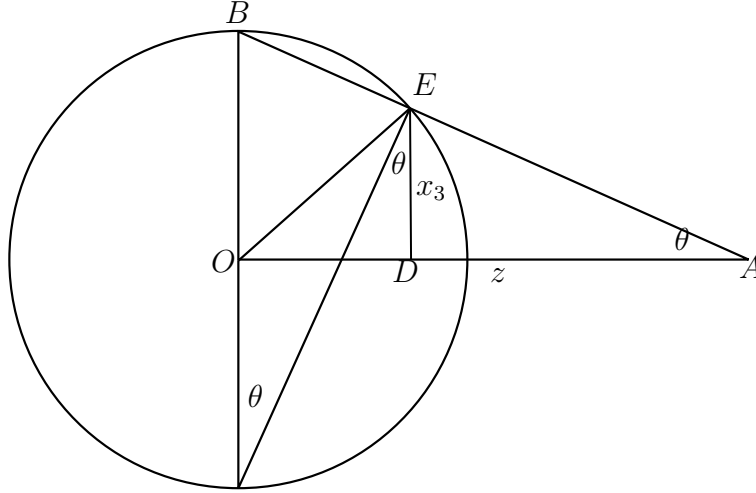


Figure 1.4: Derivation of Sphere

It is easy to calculate the distance  $d(z, z')$  between the stereographic projections of points  $z$  and  $z'$ . Let the points be  $(x_1, x_2, x_3)$  and  $(x'_1, x'_2, x'_3)$ , then the distance

$$(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2 = 2 - 2(x_1x'_1 + x_2x'_2 + x_3x'_3)$$

And we simplify by

$$x_1x'_1 + x_2x'_2 + x_3x'_3 = \frac{(1 + |z|^2)(1 + |z'|^2) - 2|z - z'|^2}{(1 + |z|^2)(1 + |z'|^2)}$$

Thus, we have

$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}} \quad (1.6)$$

For  $z' = \infty$  we have

$$d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}} \quad (1.7)$$



# Chapter 2

## Complex Functions

### 2.1 Analytic Functions

We aim to extend the differentiation and integration of real analysis to the complex field. During this process, the range of capability is greatly restricted. Only analytic (or holomorphic) functions can be differentiated and integrated.

---

Practical notation:

---

We shall denote the symbols

- $z$  and  $w$  always means variables in  $\mathbb{C}$
- $x$  and  $y$  can be in either  $\mathbb{R}$  or  $\mathbb{C}$
- $t$  is always in  $\mathbb{R}$
- $z = x + iy$  automatically means  $x$  and  $y$  are real.

We also denote  $\mathbb{F}$  for  $\mathbb{R}$  or  $\mathbb{C}$ .

---

We shall restricted our discussion to only functions defined on open sets.

#### 2.1.1 Limits and Continuity

##### Definition 2.1.1: Limits

The function  $f : \mathbb{F} \rightarrow \mathbb{F}$  is said to have a limit  $A$  at  $a$ , denoted

$$\lim_{x \rightarrow a} f(x) = A \quad (2.1)$$

iff the following is true.

For every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\forall 0 < |x - a| < \delta$ , we have  $|f(x) - A| < \epsilon$ . That is

$$\forall \epsilon > 0, \exists \delta > 0, (\forall 0 < |x - a| < \delta, |f(x) - A| < \epsilon) \quad (2.2)$$

There are also similar definitions for  $a$  or  $A$  to be  $\infty$ .

**Theorem 2.1.1: Rational Operations on Limits**

If  $f, g : \mathbb{F} \rightarrow \mathbb{F}$  has

$$\lim_{x \rightarrow a} f(x) = A, \lim_{x \rightarrow a} g(x) = B$$

Then for any rational operation  $R$ , we have

$$\lim_{x \rightarrow a} R(f(x), g(x)) = R(A, B)$$

*Proof.* The proof is completely the same as those of real analysis. For we only need

$$|ab| = |a| |b| \quad \text{and} \quad |a + b| \leq |a| + |b|$$

□

**Theorem 2.1.2: Equivalences of Limits**

The following are equivalent.

- $\lim_{x \rightarrow a} f(x) = A$ .
- $\lim_{x \rightarrow a} \overline{f(x)} = \overline{A}$ .
- $\begin{cases} \lim_{x \rightarrow a} \operatorname{Re} f(x) = \operatorname{Re} A \\ \lim_{x \rightarrow a} \operatorname{Im} f(x) = \operatorname{Im} A \end{cases}$

*Proof.* We use  $|\operatorname{Re} f(x) - \operatorname{Re} A| \leq |f(x) - A| \leq |\operatorname{Re} f(x) - \operatorname{Re} A| + |\operatorname{Im} f(x) - \operatorname{Im} A|$  would do. □

We defines continuous functions similarly

**Definition 2.1.2: Continuity**

$f : \mathbb{F} \rightarrow \mathbb{F}'$  is continuous at  $a$  iff

$$\lim_{x \rightarrow a} f(x) = f(a)$$

A continuous function is a function that is continuous at  $\forall x \in \text{domain}$ .

**Definition 2.1.3: Derivative**

The derivative of a function  $f : \mathbb{F} \rightarrow \mathbb{F}'$  is defined

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

we notice that we use a multiplication of the domain and range, which is not valid for two arbitrary fields  $\mathbb{F}$  and  $\mathbb{F}'$ .

For a function  $f : \mathbb{C} \rightarrow \mathbb{R}$ , if it has a derivative at  $x = a$ , then it must be zero. For  $h \in \mathbb{R}$ , we have

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(a + ih) - f(a)}{ih}$$



The two equations are real and pure imaginary respectively, thus must be 0.

For a function  $z : \mathbb{R} \rightarrow \mathbb{C}$ , we have  $z(t) = x(t) + iy(t)$ , thus

$$z'(t) = x'(t) + iy'(t)$$

In short, the rational operations of derivatives and the chain rule also applied for the derivatives.

### 2.1.2 Analytic Functions

#### Definition 2.1.4: Analytic Functions

An analytic function is a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  that has a derivative for  $\forall x \in \text{domain}$ .

The rational operations on analytic functions is also analytic (except of those points that have zero denominator, obviously). We begin by finding the necessary results of an analytic function.

- Analytic Functions are Continuous

The definition of derivative can be rewritten as

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

Then

$$\lim_{h \rightarrow 0} (f(z+h) - f(z)) = f'(z) \lim_{h \rightarrow 0} h = 0$$

giving  $\lim_{h \rightarrow 0} f(z+h) = f(z)$  as expected.

If we write  $f(z) = u(z) + iv(z)$ , then  $u, v$  are both continuous.

- The limit must be the same regardless of how  $h$  approaches 0.

We write  $z = x + iy$  then  $u(z) = u(x, y)$  and  $v(z) = v(x, y)$ .

If we let  $h$  approaches 0 in the real line, we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Letting  $h$  approach 0 on the imaginary side, we have

$$f'(z) = \lim_{k \rightarrow 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Getting those together, we have

#### Theorem 2.1.3: The Cauchy-Riemann Equations

If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is an analytic function, then let  $f(z) = u(x, y) + iv(x, y)$  where  $z = x + iy$ , then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (2.3)$$

or more shortly, we have

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0 \quad (2.4)$$

For the quantity  $|f'(z)|^2$  we have

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \frac{\partial(u, v)}{\partial(x, y)}$$

We shall later prove that the derivative of an analytic function is analytic, that is, it has derivative of all orders. Using theorem 2.1.3, we have

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \Delta v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\end{aligned}\tag{2.5}$$

#### Definition 2.1.5: Harmonic Functions

A function  $u$  satisfying the Laplace equation  $\Delta u = 0$  is said to be harmonic.

If two harmonic functions  $u$  and  $v$  satisfies the Cauchy-Riemann Equation, then  $v$  is the conjugate harmonic function of  $u$ . ( $u$  is the conjugate harmonic function of  $-v$ ).

Thus the real and imaginary part of an analytic function are conjugate harmonic functions.

We wish to prove, if  $v$  is the conjugate harmonic function of  $u$ , then  $u + iv$  is analytic. This is a sufficient and necessary condition.

#### Theorem 2.1.4: Analytic and Conjugate Harmonic

If  $u(x, y)$  and  $v(x, y)$  have continuous first-order partial derivatives which satisfies the Cauchy-Riemann Equations, then  $f(z) = u(z) + iv(z)$  is an analytic function, and conversely.

*Proof.* The continuity of the first-order partial derivatives makes it possible to write

$$\begin{aligned}u(x + h, y + k) - u(x, y) &= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1 \\ v(x + h, y + k) - v(x, y) &= \frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2\end{aligned}$$

Thus we have

$$\begin{aligned}f(z + h + ik) - f(z) &= u(x + h, y + k) + iv(x + h, y + k) - u(x, y) - iv(x, y) \\ &= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1 + i\left(\frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2\right) \\ &= \frac{\partial u}{\partial x}h - \frac{\partial v}{\partial x}k + i\frac{\partial v}{\partial x}h + i\frac{\partial u}{\partial y}k + \epsilon_1 + i\epsilon_2 \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h + ik) + \epsilon_1 + i\epsilon_2\end{aligned}$$

Therefore we get

$$\lim_{h+ik \rightarrow 0} \frac{f(z + h + ik) - f(z)}{h + ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

We conclude that  $f$  is analytic. □

The next result is pretty interesting. Consider a complex function with two real variables  $f(x, y)$ . We write  $x = \frac{1}{2}(z + \bar{z})$  and  $y = -\frac{1}{2}i(z - \bar{z})$ , then we see  $f(x(z, \bar{z}), y(z, \bar{z}))$  as a function of “independent”  $z$  and  $\bar{z}$ . We give a derivative *formally*, which is not means by limits.

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right). \quad (2.6)$$

The derivative here is only formal, we can understand it as a definition. Then the Cauchy-Riemann Equations are equivalent to  $\frac{\partial f}{\partial \bar{z}} = 0$ , thus we are tempted to say that analytic functions are “truly functions of  $z$  alone”.

This gives us a fairly easy way to compute the conjugate harmonic function of  $u(x, y)$ . Just making  $u + iv$  be function that has only  $x + iy$  as variable.

If  $u$  is rational, then we have  $f = u + iv$ ,

$$u(x, y) = \frac{1}{2}(f(z) + \overline{f(z)}) = \frac{1}{2}(f(z) + f(\bar{z}))$$

thus we have

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2} \left( f(z) + \overline{f(0)} \right)$$

As adding a pure imaginary constant does not influence  $f$ , we can assume  $f(0)$  to be real. Then we have  $\overline{f(0)} = u(0, 0)$ . Take

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0, 0) \quad (2.7)$$

would do.

### 2.1.3 Polynomials

The simplest analytic function is the constant function and the identity function  $z \mapsto z$ . Since the sum and product of any two analytic functions are again analytic, we have

#### Theorem 2.1.5: Polynomials

Every Polynomial

$$P(z) = a_0 + a_1 z + \dots + z_n z^n \quad (2.8)$$

is analytic.

For  $\deg P > 0$ ,  $P(z) = 0$  has at least one root in  $\mathbb{C}$ . Thus we have the complete factorization

$$P(z) = a_n(z - \alpha_1) \cdots (z - \alpha_n)$$

We are familiar with the order of zeros. A simple zero has order 1. If a zero  $\alpha$  has order  $h$ , then we have

$$P(\alpha) = \dots = P^{(h-1)}(\alpha) = 0 \text{ and } P^{(h)}(\alpha) \neq 0$$

As a result

#### Theorem 2.1.6: Lucas's Theorem

If all zeros of a polynomial  $P(z)$  lies in a half plane, then all the zeros of  $P'(z)$  lies in the same half plane.

*Proof.* We have

$$\frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + \dots + \frac{1}{z - \alpha_n} \quad (2.9)$$

Suppose the half plane  $H$  is defined  $\left\{ z \in \mathbb{C} : \operatorname{Im} \frac{z - a}{b} < 0 \right\}$ . If  $z \notin H$ , we have

$$\operatorname{Im} \frac{z - \alpha_k}{b} = \operatorname{Im} \frac{z - a}{b} - \operatorname{Im} \frac{\alpha_k - a}{b} > 0$$

then  $\operatorname{Im} \frac{b}{z - \alpha_k} < 0$ . Then we have

$$\operatorname{Im} \frac{bP'(z)}{P(z)} = \sum_{k=1}^n \operatorname{Im} \frac{b}{z - \alpha_k} < 0$$

Thus  $P'(z) \neq 0$  □

A sharper formulation is that the smallest convex polygon that contains all zeros of  $P(z)$  also contains all zeros of  $P'(z)$ . (as a convex polygon can be made up of half plains).

### Theorem 2.1.7: Eneström-Kakeya Theorem

If  $P(z) = a_0 + a_1z + \dots + a_nz^n$  with  $0 < a_0 \leq a_1 \leq \dots \leq a_n$ , then all zeros of  $P(z)$  lies in  $|z| \leq 1$ .

*Proof.* For  $|z| > 1$ , we have

$$(z - 1)P(z) = a_nz^{n+1} + (a_{n-1} - a_n)z^n + \dots + (a_0 - a_1)z - a_0$$

Using the triangle inequality, we have

$$\begin{aligned} |(z - 1)P(z)| &\geq |a_n| |z|^{n+1} - |a_{n-1} - a_n| |z|^n - \dots - |a_0 - a_1| |z| - |a_0| \\ &= a_n |z|^{n+1} - (a_n - a_{n-1}) |z|^n - \dots - (a_1 - a_0) |z| - a_0 \\ &= |z|^{n+1} \left( a_n - \frac{a_n - a_{n-1}}{|z|} - \dots - \frac{a_1 - a_0}{|z|^n} - \frac{a_0}{|z|^{n+1}} \right) \\ &> |z|^{n+1} (a_n - (a_n - a_{n-1}) - \dots - (a_1 - a_0) - a_0) \\ &= 0 \end{aligned}$$

□

## 2.1.4 Rational Functions

We turn to rational functions

$$R(z) = \frac{P(z)}{Q(z)} \quad (2.10)$$

Assuming that  $P$  and  $Q$  has no common roots. The roots of  $Q(z)$  are called poles in  $R(z)$ . The order of each pole is defined respectively.

The derivative

$$R'(z) = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q^2(z)}$$

has the same poles as  $R(z)$ , with order increased by 1.

We extend the domain and range to the extended plain. **Then  $R$  is continuous.**

- If  $z_0$  is a pole of  $R(z)$ , then define  $R(z_0) = \infty$ .
- For  $R(\infty)$  we can use limit to define it but in that way it is hard to define the order of a zero or pole at  $\infty$ . Therefore we consider  $R_1(z) = R(\frac{1}{z})$ .

Define  $R(\infty) = R_1(0)$ .

With the notation

$$R(z) = \frac{a_0 + a_1z + \dots + a_nz^n}{b_0 + b_1z + \dots + b_mz^m}$$

we have

$$R_1(z) = z^{m-n} \frac{a_0z^n + \dots + z_n}{b_0z^m + \dots + b_m}$$

- $m > n$  then  $\infty$  is a zero of order  $m - n$ .
- $m < n$  then  $\infty$  is a pole of order  $n - m$ .
- $m = n$  then  $R(\infty) = \frac{a_n}{b_m}$

The total number of zeros or poles of  $R(z)$  in the extended plane is both  $\max\{m, n\}$ .

#### Definition 2.1.6: Order of Rational Functions

The order of a rational function  $R(z) = \frac{P(z)}{Q(z)}$  with  $\deg P(z) = n$  and  $\deg Q(z) = m$  is defined as  $\max\{n, m\}$ .

#### Theorem 2.1.8: Zeros and Poles of the Extended Plane

A rational function that has degree  $p$  has  $p$  roots and  $p$  poles. Every equation  $R(z)$  has exactly  $p$  roots.

A rational function of order 1 is linear

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \text{ with } \alpha\delta - \beta\gamma \neq 0 \quad (2.11)$$

The equation  $\omega = S(z)$  has only one root, and

$$z = S^{-1}(\omega) = \frac{\delta\omega - \beta}{-\gamma\omega + \alpha}$$

Specifically, the linear transformation  $z + a$  is called parallel translation, and  $1/z$  is called inversion.

If  $R(z)$  has a pole at  $\infty$ , divide  $P(z)$  with  $Q(z)$  till the degree of nominator and denominator are the same and we get

$$R(z) = G(z) + H(z)$$

where  $G$  is a polynomial without constant terms and  $H(z)$  is finite at  $\infty$ . Then  $G$  is called the singular part of  $R$  at  $\infty$ .

This process can be done similarly to finite poles. Let  $\beta_1, \dots, \beta_q$  be distinct finite poles. Then the function  $R(\beta_j + \frac{1}{\zeta})$  is a rational function that has a pole at  $\infty$ . We decompose it into

$$R(\beta_j + \frac{1}{\zeta}) = G_j(\zeta) + H_j(\zeta)$$

then

$$R(z) = G_j(\frac{1}{z - \beta_j}) + H_j(\frac{1}{z - \beta_j})$$

Then  $H_j$  is finite at  $\beta_j$ , and  $G_j$  is called the singular part of  $R$  at  $\beta_j$ .

Consider the expression

$$R(z) - G(z) - \sum_{j=1}^q G_j(\frac{1}{z - \beta_j})$$

It has no poles other than  $\beta_1, \dots, \beta_q, \infty$ . But at each one of these points, the expression is finite. Thus it is a constant function according to theorem 2.1.8. We absorb it in  $G(z)$  and get

$$R(z) = G(z) + \sum_{j=1}^q G_j(\frac{1}{z - \beta_j}) \quad (2.12)$$

where  $G, G_j$  are polynomials.

In analysis, this is how we do indefinite integration of rational functions.

## 2.2 Elementary Theory of Power Series

Polynomials and Rational functions are rare in general, and the easiest way to achieve greater variety is to form limits. Infinite sums of analytic functions have good chance to be analytic.

### Definition 2.2.1: Limit of Sequence

Then sequence  $\{a_n\}_{n \in \mathbb{Z}_+}$  has the limit  $A$  iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall n > N, |a_n - A| < \epsilon$$

If a sequence has finite limit, then it is convergent, otherwise it is divergent.

In analysis, we are familiar with the convergence criterion.

### Theorem 2.2.1: Cauchy Criterion for Convergence

A sequence  $\{a_n\}_{n \in \mathbb{Z}_+}$  is convergent iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall m, n > N, |a_n - a_m| < \epsilon$$

such sequence is called Cauchy sequence.

The real and complex part of a Cauchy sequence is also Cauchy sequence, thus convergent.

### 2.2.1 Series

A simple application of Cauchy's criterion allows us to identify convergence from existing sequences. If  $|b_n - b_m| \leq |a_n - a_m|$  for all  $n, m$ , then  $\{b_n\}$  is a contraction of  $\{a_n\}$ . Then the convergence of  $\{a_n\}$  implies the convergence of  $\{b_n\}$ .

For an infinite series

$$a_1 + \cdots + a_n + \cdots \quad (2.13)$$

The partial sum is given

$$s_n = a_1 + \cdots + a_n$$

We say the series converges to  $A$  iff  $\{s_n\}$  converges to  $A$ . We denote the series as  $\sum_{n=1}^{\infty} a_n = A$ .

If the series of absolute values

$$|a_1| + \cdots + |a_n| + \cdots \quad (2.14)$$

converges, then we say the series  $\sum_{n=1}^{\infty} a_n$  to be absolutely convergent. From Cauchy's criterion, we know that absolutely convergent series is convergent.

### 2.2.2 Uniform Convergence

#### Definition 2.2.2: Uniform Convergence

A sequence of functions  $\{f_n\}$  converges uniformly to  $f$  on a set  $D$  iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall n > N, \forall x \in D, |f_n(x) - f(x)| < \epsilon$$

A convergent series that do not converge uniformly is called pointwise convergent.

The Cauchy criterion for uniform convergence is similar to the one for convergence of sequences.

A sequence of functions  $\{f_n\}$  converges uniformly to  $f$  on a set  $D$  iff

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \text{ such that } \forall m, n > N, \forall x \in D, |f_n(x) - f_m(x)| < \epsilon$$

In practice, if  $\{f_n(x)\}$  is a contraction of  $\{a_n\}$ , then if  $\{a_n\}$  is convergent,  $\{f_n(x)\}$  is uniformly convergent. This is called the **Weierstrass test**.

#### Theorem 2.2.2: Continuity of Uniform Convergence

The limit function of a uniformly convergent sequence of continuous functions is continuous.

### 2.2.3 Power Series

A power series is a series of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = \sum_{n=0}^{\infty} a_n z^n$$

where  $a_n \in \mathbb{C}$ . More generally we can translate the series to a power series centered at  $z_0$  as

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

**Theorem 2.2.3: Abel Disk Theorem**

For every power series  $\sum_{n=1}^{\infty} a_n z^n$ , there exists  $0 \leq R \leq +\infty$ , called the radius of convergence, that:

- The series converge absolutely for  $|z| < R$ .  $\forall 0 \leq \rho < R$ , the series converges uniformly on the disk  $\{z \in \mathbb{C} : |z| \leq \rho\}$ .
- For  $|z| > R$ , then the terms are not bounded, so the series is divergent.
- In  $|z| < R$  the sum of the series is an analytic function. The derivative can be obtained by term-by-term differentiation, and the derivative series has the same radius of convergence.

This is rather similar to that of  $\mathbb{R}$  in analysis. The radius  $R$  can be given by

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \quad (2.15)$$

called the Hadamard formula.

*Proof.* • The first two statements we get by comparing the series with  $\sum \rho^n$ , using Weierstrass test.

- the derived series  $\sum_{n=1}^{\infty} n a_n z^{n-1}$  has the same radius of convergence, for  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ . To show that we can derive term-by-term, we write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = s_n(z) + R_n(z)$$

for  $|z| < R$ , also

$$f_1(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = \lim_{n \rightarrow \infty} s'_n(z)$$

Now we show  $f'(z) = f_1(z)$ . Consider the identity

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) = \left( \frac{s_n(z) - s_n(z_0)}{z - z_0} - s'_n(z_0) \right) + (s'_n(z_0) - f_1(z_0)) + \left( \frac{R_n(z) - R_n(z_0)}{z - z_0} \right)$$

Assume  $z \neq z_0$ ,  $|z|, |z_0| < \rho < R$ , then as have

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}$$

$$\left| \frac{R_n(z) - R_n(z_0)}{z - z_0} \right| \leq \sum_{k=n}^{\infty} k |a_k| \rho^{k-1}$$

For every  $\epsilon > 0$ , there exists sufficiently large  $n$  such that the last two terms are less than  $\epsilon/2$ . Then when  $z$  is close to  $z_0$  we can make the first term as small as we want, thus

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \right| < \epsilon$$

□



We now have proved that a power series with positive radius of convergence is an analytic function, and has derivative of all orders. The derivative is also a power series with the same radius of convergence.

By continuously taking derivatives, we can get the  $n$ -th derivative of a power series  $f(z)$  :

$$f^{(n)}(z) = \sum_{k=n}^{\infty} a_k \frac{k!}{(k-n)!} z^{k-n} = \sum_{k=0}^{\infty} a_{n+k} \frac{(n+k)!}{k!} z^k$$

We see that  $a_k = f^{(k)}(0)/k!$ , so

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

This is the familiar Taylor series.

---

*Remark:*

We have shown that an analytic function that is formed by power series has arbitrary many derivatives, and the Taylor series converges to the function inside the convergence disk.

But it remains to be seen that every analytic function can be expressed as a power series.

This is the content of the **Cauchy Integral Theorem**, which we shall prove later.

---

### 2.2.4 Abel limit Theorem

We cannot say much about convergence on the circle of radius  $R$ . However, if it converges at one point on the circle, then we can say some things about the “one side continuity” of the series. We lose no generality by assuming that  $R = 1$ .

#### Theorem 2.2.4: Abel Limit Theorem

If  $\sum_{n=0}^{\infty} a_n$  converges, then  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  tends to  $f(1)$  as  $z$  tends to 1 in a way that  $\frac{|1-z|}{1-|z|} \leq K$  for some constant  $K > 0$ .

Geometrically it means that  $z$  stays in an angle  $< \pi$  with vertex 1 and symmetric to the real axis. We say that the approach takes place in a Stolz angle.

*Proof.* Assume  $\sum_{n=0}^{\infty} a_n = 0$  by adding a constant to  $a_0$ . Then write  $s_n = a_0 + a_1 + \cdots + a_n$  and use Abel transform

$$s_n(z) = (1-z)(s_0 + s_1 z + \cdots + s_{n-1} z^{n-1}) + s_n z^n$$

As  $s_n z^n \rightarrow 0$  we have

$$f(z) = (1-z) \sum_{i=1}^{\infty} s_i z^i$$

When  $|1 - z| \leq K(1 - |z|)$ , choose large enough  $m$  such that  $\forall n > m, |s_n| < \epsilon$ , then we have

$$\begin{aligned} |f(z)| &\leq |1 - z| \left| \sum_{k=0}^{m-1} s_k z^k \right| + |1 - z| \epsilon \sum_{k=m}^{\infty} |z|^k \\ &\leq |1 - z| \left| \sum_{k=0}^{m-1} s_k z^k \right| + \frac{|1 - z|}{1 - |z|} \epsilon \end{aligned}$$

The first term is arbitrarily small when  $z \rightarrow 1$  and the second term is bounded by  $K\epsilon$ . Thus we have  $|f(z) - f(1)| \rightarrow 0$ .  $\square$

## 2.3 The Exponential and Trigonometric Functions

Back in analysis, we have noticed the similarity of Taylor series of exponential and trigonometric functions. We can define them in the complex plane as well, this time directly via power series, or a solution of a differential equation.

In this way, we can analytically define the logarithm and the argument of a complex number, and leading to a nongeometric definition of the angle.

### 2.3.1 the Exponential

#### Definition 2.3.1: Exponential Function

The exponential function is defined as the solution of the differential function

$$f'(z) = f(z) \quad f(0) = 1 \quad (2.16)$$

Taking the series expansion it shows that if  $f(z)$  can be expressed as a power series, then it must be of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (2.17)$$

And  $f(z)$  is denoted  $e^z$ .

---

#### Proposition: Properties of the Exponential

---

- The series converges for all  $z \in \mathbb{C}$ , for  $\sqrt[n]{n!} \rightarrow \infty$ .
- The addition property:

$$e^a e^b = e^{a+b} \quad (2.18)$$

*Proof.* We find that  $D(e^z e^{c-z}) = e^z e^{c-z} - e^z e^{c-z} = 0$ . Hence  $e^z e^{c-z}$  is a constant. (Using Cauchy Riemann Equations and the real immediate value theorem). Setting  $z = 0$  gives  $e^z e^{c-z} = e^c$ , letting  $z = a, c = a + b$  would do.  $\square$

Specifically,  $e^z e^{-z} = 1$ , so  $\forall z \in \mathbb{C}, e^z \neq 0$ .

- For  $x \in \mathbb{R}$  the series shows that  $e^x > 0, x > 0$  and  $0 < e^x < 1, x < 0$ .
-

- The conjugate property:

$$e^{\bar{z}} = \overline{e^z} \quad (2.19)$$

*Proof.* We have  $e^{\bar{z}} = \sum_{n=0}^{\infty} \frac{\bar{z}^n}{n!} = \sum_{n=0}^{\infty} \frac{\overline{z^n}}{n!} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \overline{e^z}$ . □

### 2.3.2 The Trigonometric Functions

#### Definition 2.3.2: Trigonometric Functions

The trigonometric functions are defined by:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad (2.20)$$

A series expansion shows that:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad (2.21)$$

The other trigonometric functions are defined by  $\cos$  and  $\sin$ . All are rational functions of  $e^{iz}$ .

So we have the famous Euler's formula, directly from the definition.

$$e^{iz} = \cos z + i \sin z \quad (2.22)$$

Some familiar results:

$$\cos^2 z + \sin^2 z = 1$$

$$D \cos z = -\sin z, \quad D \sin z = \cos z$$

The addition formulas

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

### 2.3.3 The Periodicity

We say that  $f$  is periodic with period  $T \in \mathbb{C}$  if  $f(z + T) = f(z)$  for all  $z \in \mathbb{C}$ .

#### Theorem 2.3.1: Periodicity of the Exponential

The exponential function  $e^z$  is periodic. The periods are of the form  $c = ik\omega_0$ , where  $k \in \mathbb{Z}_+$ ,  $\omega_0 \in \mathbb{R}_+$ .

*Proof.* A period of  $e^z$  satisfies  $e^{z+c} = e^z$ , thus  $e^c = 1$ , writing  $c = a + ib$ , we have  $e^{a+ib} = e^a e^{ib} = 1$ , so taking norm we have  $a = 0$  and  $c = ib$ .

For  $y \in \mathbb{R}$ , we use  $D \sin y = \cos y \leq 1$  (by series form) and  $\sin 0 = 0$ , thus  $\forall y > 0, \sin y < y$ . In similar ways, we get

$$\cos y < 1 - \frac{y^2}{2} + \frac{y^4}{24}, y > 0$$

Thus we have  $\cos \sqrt{3} < 0$ , thus  $\exists 0 < y_0 < \sqrt{3}$ ,  $\cos y_0 = 0$ . For

$$\cos^2 y_0 + \sin^2 y_0 = 1$$

we have  $\sin y_0 = \pm 1$ , thus  $e^{iy_0} = \pm i$ , thus  $e^{4y_0} = 1$ . We've shown that  $4y_0$  is a period.

A more careful analysis shows that  $4y_0$  is the smallest period. As  $\forall 0 < y < y_0$ ,  $\sin y > y(1 - y^2/6) > 0$  showing  $\cos y$  is strictly decreasing. Also  $\sin y$  is positive so  $\sin y$  is strictly increasing. Therefore,  $e^{iy}$  cannot be  $\pm 1, \pm i$  for  $0 < y < 4y_0$ . Thus the smallest period is  $4y_0$ .  $\square$

### Definition 2.3.3: $\pi$

The period of the exponential function is denoted  $c = 2k\pi i$ .

We have

$$e^{\pi i/2} = i, \quad e^{\pi i} = -1, \quad e^{3\pi i/2} = -i, \quad e^{2\pi i} = 1$$

Geometrically, the mapping  $y \rightarrow e^{iy}$ ,  $0 \leq y < 2\pi$  is a 1-1 mapping from the interval  $[0, 2\pi)$  to the unit circle  $S^1$  in the complex plane. The image of  $y$  is the point on the unit circle with angle  $y$  with respect to the positive real axis.

Algebraically, the function  $y \rightarrow e^{iy}$  defines a homomorphism from the additive group of  $\mathbb{R}$  to the multiplicative group of  $S^1$ , the kernel is  $2\pi\mathbb{Z}$ .

## 2.4 The Logarithm

The logarithm is the inverse function of the exponential function.

If  $e^z = e^{x+iy} = w$ , we have

$$e^x = |w|, \quad e^{iy} = \frac{w}{|w|} \quad (2.23)$$

First,  $w \neq 0$ , for  $e^z \neq 0$ . The first function has a unique solution  $x = \log_{\mathbb{R}} |w|$ , the real logarithm. The solution is a set  $y + 2\pi\mathbb{Z}$  where  $0 \leq y < 2\pi$ . Therefore, the logarithm of a complex number has the form  $z + 2\pi i\mathbb{Z}$ .

### Definition 2.4.1: Arguments and Logarithm

If  $w \neq 0$ , then let  $e^z = w$ ,  $0 \leq \text{Im } z < 2\pi$ , we define the (principle) argument of  $w$  by  $\arg w = \text{Im } z$ . And

$$\log w = \log_{\mathbb{R}} |w| + i \arg w \quad (2.24)$$

We can write an arbitrary complex number  $w$  as

$$w = re^{i\theta} \quad (2.25)$$

where  $r = |w| \geq 0$  and  $\theta = \arg w$  is the principal argument, i.e.  $\theta \in [0, 2\pi)$ .

The solution of  $e^z = w$  can be written as  $\log w + 2\pi i\mathbb{Z}$ , we denote it by  $\text{Log } w$ . The set  $\text{Arg } w = \arg w + 2\pi\mathbb{Z}$ .

- $\text{Log } w = \log w + 2\pi i\mathbb{Z}$ .
- $\text{Arg } w = \arg w + 2\pi\mathbb{Z}$ .

**Definition 2.4.2: Power**

The power is defined as

$$a^b = e^{b \log a} e^{2\pi i k b}, k \in \mathbb{Z} \quad (2.26)$$

This is somewhat chaotic.

- If  $a \in \mathbb{R}_+$ , then usually we take  $a^b = e^{b \log_{\mathbb{R}} a}$ .
- If  $b \in \mathbb{Z}$ , then the above expression has only one value  $e^{b \log a}$ , which is just the repeated products of  $a$  for  $b$  times.

*Proof.* We have  $e^{b \log a} = e^{\log a} \dots e^{\log a} = a^b$ . □

- If  $b = p/q$ , then there are exactly  $q$  different values of  $a^b$ .

The addition theorem of exponential implies

$$\begin{aligned} \text{Log}(ab) &= \text{Log } a + \text{Log } b \\ \text{Arg}(ab) &= \text{Arg } a + \text{Arg } b \end{aligned} \quad (2.27)$$

Finally we consider the inverse trigonometric functions. The function

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = w$$

implies that  $e^{iz}$  satisfies  $z + \frac{1}{z} = 2w$ .

$$z = -i \text{Log} \left( w + \sqrt{w^2 - 1} \right)$$

Note that  $\sqrt{w^2 - 1}$  has usually two values, and  $\text{Log}$  has infinity.

The two values of  $w + \sqrt{w^2 - 1}$  are reciprocal to each other, so we can write

$$\arccos w = \pm i \text{Log} \left( w + \sqrt{w^2 - 1} \right), \text{ taking one of the two values of } \sqrt{\phantom{x}} \quad (2.28)$$

We also have

$$\arcsin x = \frac{\pi}{2} - \arccos x \quad (2.29)$$



# Chapter 3

## Analytic Functions As Mappings

### 3.1 Conformality

Our goal here is to derive a geometrical intuition about what it means to be an analytic function.

#### 3.1.1 Arcs and Closed Curves

##### Definition 3.1.1: Arcs

An arc  $\gamma$  on the complex plane is image of a continuous function  $z : [\alpha, \beta] \rightarrow \mathbb{C}$ , where  $\alpha, \beta \in \mathbb{R}$ . We denote the arc  $z = z(t) = x(t) + iy(t)$ .

As a continuous mapping of a closed interval, an arc is closed and thus compact and connected.

The choice of parameter  $t$  is arbitrary, and we can freely change it. If  $\varphi : [\alpha', \beta'] \rightarrow [\alpha, \beta]$  is a strictly increasing function, then  $z = z(\varphi(\tau))$  describes the same arc, but with a different parameter  $\tau$ .

The following are some properties that describe an arc:

- **Differentiability:** The arc is differentiable (continuously differentiable) if the derivative  $z'(t)$  exists and is continuous for all  $t \in [\alpha, \beta]$ .
- **Regularity:** The arc is regular if it is differentiable and the derivative  $z'(t) \neq 0$  for all  $t \in [\alpha, \beta]$ .
- **Piecewise Differentiability:** The arc is piecewise differentiable iff:
  - It is continuous on  $[\alpha, \beta]$ .
  - It has continuous derivatives except for a finite number of points in  $[\alpha, \beta]$ .
  - At each breaking point, the left and right derivatives exist and are equal to the limit of  $z'(t)$ . (left and right continuous derivatives)
- **Piecewise Regularity:** The arc is piecewise regular if it is piecewise differentiable and  $z'(t) \neq 0$  and the left and right derivatives at the breaking points are  $\neq 0$ .

---

*Remark:*

Differentiability and Regularity is invariant under change of parameters.

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- **Jordan arc (Simple arc):** An arc is simple if  $z(t_1) = z(t_2) \rightarrow t_1 = t_2$ . (it doesn't cross itself)
- **Closed Curve:** An arc is a closed curve if  $z(\alpha) = z(\beta)$ . For that, a shift of parameter can be done: taking  $t_0 \in (\alpha, \beta)$  and defining a new one by taking  $t_0$  as initial point.
- The **Opposite** of an arc  $\gamma$  is the arc defined by  $\gamma^*(t) = \gamma(\beta + \alpha - t)$ , where  $t \in [\alpha, \beta]$ .

### 3.1.2 Analytic Functions in Regions

The definition of an analytic function requires the approach to a point in arbitrary directions. So we have good reasons to restrict our discussion to open sets.

#### Definition 3.1.2: Analytic Functions in Open Sets

Let  $\Omega \subseteq \mathbb{C}$  be open. A complex function  $f : \Omega \rightarrow \mathbb{C}$  is analytic (holomorphic) if it has a derivative at every point in  $\Omega$ .

It is obvious that the restriction of an analytic function to an open subset is also analytic. Also, for the components of open sets are open, we can assume that  $\Omega$  is connected, that is,  $\Omega$  is a region.

#### Definition 3.1.3: Regions

A region is an open connected subset of  $\mathbb{C}$ .

#### Definition 3.1.4: Analytic Functions on Arbitrary Sets

A function  $f$  is analytic on  $A \subseteq \mathbb{C}$  if  $\exists C \subseteq \mathbb{C}, A \subseteq C$  and  $C$  is open and  $f$  is analytic on  $C$ .

#### Remark:

For multivalued functions like the logarithm and inverse trigonometric functions, we can specify a well-behaved  $\Omega$  and an analytic branch to make it single-valued and analytic on  $\Omega$ . However, for certain functions like  $\log$ , for some  $\Omega$  it is impossible to find a single-valued analytic branch. For example,  $\log$  is not analytic on  $\mathbb{C} \setminus \{0\}$ , which we shall prove later.

#### Theorem 3.1.1: The Inverse Function Theorem

Let  $f : \Omega \rightarrow \mathbb{C}$  be an analytic function on a region  $\Omega$  and let  $z_0 \in \Omega$  be a point such that  $f'(z_0) \neq 0$ . Then there exists a neighborhood  $U$  of  $z_0$  such that  $f$  is bijective from  $U$  onto its image  $f(U)$ , and the inverse function  $f^{-1}$  is also analytic on  $f(U)$ . The derivative of the inverse function at the point  $w_0 = f(z_0)$  is given by

$$Df^{-1}(w_0) = \frac{1}{f'(z_0)}$$

*Proof.* The inverse function theorem for  $\mathbb{R}^2$  states that

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a continuously differentiable function and let  $Df(z_0)$  be the Jacobian matrix of  $f$  at  $z_0$ . If  $\det Df(z_0) \neq 0$ , then there exists a neighborhood  $U$  of



$z_0$  such that  $f$  is a bijection from  $U$  onto its image  $f(U)$ , and the inverse function  $f^{-1}$  is also continuously differentiable on  $f(U)$ . The derivative of the inverse function at  $w_0 = f(z_0)$  is given by

$$Df^{-1}(w_0) = (Df(z_0))^{-1}$$

Now, if  $f$  is analytic, we see it as a function  $\Omega \rightarrow \mathbb{R}^2$ , then it is continuously differentiable, and we have  $\det f'(z_0) = |f'(z_0)|^2 \neq 0$ . The inverse function  $f$  is also continuously differentiable, and we verify the Cauchy-Riemann equations for  $g = f^{-1}$ : Let  $f = u + iv, g = \alpha + i\beta$ , using  $Df \cdot Dg = I_2$ , we have

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Cauchy-Riemann equations for  $f$  imply that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Therefore, the Cauchy-Riemann equations hold for  $g$ , and thus  $g$  is analytic on  $f(U)$ . The derivative of the inverse function is given by chain rule.  $\square$

**Proposition: The Analytic Branch of Multivalued Functions**

- The root function is the inverse of the power function  $z^n$  for  $n \in \mathbb{N}$ , which is analytic on  $\mathbb{C} \setminus \{0\}$ . For each slice that has the form

$$\Omega = \left\{ z \in \mathbb{C} : \arg z \in \left( \frac{2\pi k}{n}, \frac{2\pi(k+1)}{n} \right), z \neq 0 \right\}$$

The function  $z \mapsto z^n$  is bijective, and the image is  $\mathbb{C} - [0, +\infty)$ . The inverse function  $z \mapsto z^{1/n}$  is analytic.

- The logarithm function is the inverse of the exponential function, which is analytic on  $\mathbb{C} \setminus \{0\}$ . For each slice that has the form

$$\Omega = \{z \in \mathbb{C} : \operatorname{Im} z \in (2\pi k, 2\pi(k+1)), z \neq 0\}$$

The function  $z \mapsto e^z$  is bijective, and the image is  $\mathbb{C} - [0, +\infty)$ . The inverse function  $z \mapsto \log z$  is analytic.

- The inverse trigonometric functions are the inverses of the trigonometric functions, which are analytic on  $\mathbb{C}$ . For each slice that has the form

$$\Omega = \{z \in \mathbb{C} : \operatorname{Re} z \in (2\pi k, 2\pi(k+1)), \operatorname{Im} z > 0 \text{ (or } < 0)\}$$

The function  $z \mapsto \cos z$  is bijective, and the image is  $\mathbb{C} - [-1, +\infty)$ . The inverse function  $z \mapsto \arcsin z$  is analytic.

We can also see this using the composite function

$$\arccos z = \pm i \log \left( z + \sqrt{z^2 - 1} \right)$$

---

*Proposition:* **The derivatives of inverse functions**

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- The logarithm function:

$$\frac{d}{dz} \log z = \frac{1}{z}$$

- The root function:

$$\frac{d}{dz} z^{1/n} = \frac{1}{n} z^{\frac{1}{n}-1}$$

- The inverse trigonometric functions:

–  $\arcsin z$ :

$$\frac{d}{dz} \arcsin z = \frac{1}{\sqrt{1-z^2}}$$

–  $\arccos z$ :

$$\frac{d}{dz} \arccos z = -\frac{1}{\sqrt{1-z^2}}$$

–  $\arctan z$ :

$$\frac{d}{dz} \arctan z = \frac{1}{1+z^2}$$


---

For a complicated multivalued function it takes some effort to find the branch that is analytic on a given open set, usually a cut of  $\mathbb{C}$ . The key is to find a suitable slice of the complex plane that avoids the branch cuts of the function.

---

*Example:* **Branch Cut of Multivalued Functions**

---

- $f(z) = \sqrt{1-z} + \sqrt{1+z}$ .

For  $\sqrt{z}$  a cut through arbitrary rays from 0 would do. So we can cut  $(-\infty, -1] \cup [1, +\infty)$ .

- $\log \log z$ .

Cutting  $(-\infty, 0]$  would have an image  $|\operatorname{Im} z| < \pi$ . If the image should also cut the negative real axis, we can cut  $(-\infty, 1]$ , for  $e^{(-\infty, 0]} = (0, 1]$ .

A more general way will be introduced via Cauchy integral later.

---

In  $\mathbb{C}$  and open connected set is path-connected, so we can have the constant theorem and the mean value theorem for analytic functions.

**Theorem 3.1.2: The Constant Theorem**

An analytic function  $f$  on a region  $\Omega$  is constant if and only if its derivative is zero everywhere in  $\Omega$ .

Also,  $f$  is constant iff either  $\operatorname{Re} f, \operatorname{Im} f, \arg f, |f|$  is constant on  $\Omega$ .

*Proof.* Let  $f = u + iv$ , then the derivative being zero means that all partial derivatives of  $u$  and  $v$  are zero, which means that  $u$  and  $v$  are constant functions (joining a path would do). Therefore,  $f$  is constant.

If  $u$  is constant, then

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = 0$$

so do  $v$ .

If  $u^2 + v^2$  is constant, then

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = -u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x} = 0$$

The equations has non-zero solutions if the coefficient determinant  $u^2 + v^2 = 0$ . If  $u^2 + v^2 = 0$  at a point, then it is constant due to our assumption that modulus is constant. Then  $f = 0$  everywhere. If not, then the partial derivatives are zero, and thus  $f$  is constant.

Finally, if  $\arg z$  is constant, then  $u = kv$  for some constant  $k$ , but  $u - kv$  is the real part of  $(1 + ik)f$ . So  $f$  is constant.  $\square$

## 3.2 Conformal Mappings

Suppose an arc  $\gamma : z = z(t), \alpha \leq t \leq \beta$ , contained in a region  $\Omega$ . A continuous function  $f : \Omega \rightarrow \mathbb{C}$  maps  $\gamma$  to the arc  $\gamma' : w = w(t) = f(z(t))$  in the  $w$ -plane.

If  $f$  is analytic in  $\Omega$ , and  $z'$  exists, we say

$$w'(t) = f'(z(t))z'(t) \tag{3.1}$$

For a point  $z_0 = z(t_0)$ , if  $z'(t_0) \neq 0, f'(z_0) \neq 0$ , then  $\gamma'$  has a tangent at  $t_0$ , the direction of the tangent is given by

$$\operatorname{Arg} w'(t_0) = \operatorname{Arg} f'(z_0) + \operatorname{Arg} z'(t_0)$$

From  $z$ -space to  $w$ -space, the angle of the tangent is rotated by  $\operatorname{Arg} f'(z_0)$ , the rotation does not depend on the curve. For each curve passing through  $z_0$ , the rotation would be the same. Curves passing  $z_0$  having the same tangent are mapped to curves passing  $w_0 = f(z_0)$  having the same tangent. The angle of two curves is invariant under the mapping, so we say that  $f$  is a **conformal mapping** at  $z_0$ .

**Definition 3.2.1: Conformal Mapping**

A function  $f$  is conformal at a point  $z_0 \in \Omega$  if it is analytic in a neighborhood of  $z_0$  and  $f'(z_0) \neq 0$ .

About the modulus of  $|f'(z_0)|$ , we have

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$$

Which means a segment at  $z_0$  would stretch by a factor of  $|f'(z_0)|$  in the  $w$ -plane. Together the conformality of  $f$  means in an infinite small neighborhood of  $z_0$ , the transformation  $f$  :

- Rotates an angle by  $\text{Arg } f'(z_0)$ .
- Stretches a segment by a factor of  $|f'(z_0)|$ .

We shall see that the converse also holds for either case.

### Theorem 3.2.1: Criteria for Conformal Mapping

Let  $f : \Omega \rightarrow \mathbb{C}$  (thought of  $\Omega_{\mathbb{R}} \rightarrow \mathbb{R}^2$ ) has continuous partial derivatives whose Jacobian determinant is nonzero. If one of the following conditions holds, then  $f$  or  $\bar{f}$  is conformal at  $z_0 \in \Omega$ :

- The Jacobian matrix  $J$  preserves angles at  $z_0$ .
- The Jacobian matrix  $J$  multiplies the length of segments at  $z_0$  by a constant.

*Proof.* These are two sufficient conditions for  $J$  to be a multiple of an orthonormal matrix.  $\square$

If  $\bar{f}$  is conformal, then  $f$  is called indirectly conformal.

### Definition 3.2.2: Topological Mapping

If  $f$  is bijective and both  $f$  and  $f^{-1}$  are continuous, then  $f$  is called a **topological mapping** or a **homeomorphism**.

The inverse function theorem states that if  $f$  is analytic and  $f'(z_0) \neq 0$ , then  $f$  is a homeomorphism in a neighborhood of  $z_0$ . The inverse function  $f^{-1}$  is also analytic, and the derivative is given by the inverse of  $f'(z_0)$ .

However, if  $f'(z) \neq 0$  for all  $z \in \Omega$ , we cannot say that  $f$  is topological at  $\Omega$ , it maybe not a bijection. This may lead us to the concept of Riemann surfaces.

## 3.2.1 Length and Area

It is easy to notice that the length under a conformal mapping is multiplied by  $|f'(z_0)|$ , and area is multiplied by  $|f'(z_0)|^2$ . More rigorously:

$f$  is conformal.

- Let  $\gamma$  be a continuously differentiable arc, with  $z = z(t)$ , then

$$L(\gamma) = \int_{\alpha}^{\beta} |z'(t)| dt$$

The length of the image arc would be

$$L(\gamma') = \int_{\alpha}^{\beta} |f'(z(t))z'(t)| dt = \int_{\alpha}^{\beta} |f'(z(t))| |z'(t)| dt$$

We can write it as

$$L(\gamma) = \int_{\gamma} |dz| \quad \text{and} \quad L(\gamma') = \int_{\gamma'} |dw| = \int_{\gamma} |f'(z)| |dz|$$

the  $|dz|$  here is the same as  $ds$  is analysis.

- For a point set  $E$  whose area is given

$$A(E) = \iint_E dx dy$$

The area of the image set  $f(E)$  is given by

$$A(f(E)) = \iint_E |u_x v_y - u_y v_x| dx dy = \iint_E |f'(z)|^2 dx dy$$

### 3.3 Linear Transformations

The first-order rational functions are conformal and topological on the extended complex plane. They have remarkable geometrical properties, and we can use them to do transformations that may simplify matters.

#### 3.3.1 The Linear Group

A linear fractional transformation has the form

$$w = S(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \quad (3.2)$$

We assume that  $S(\infty) = a/c$  and  $S(-d/c) = \infty$ . It has an inverse

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \quad (3.3)$$

It is obvious that  $S$  is topological on  $\mathbb{C}^*$ , and the inverse is also topological.

*Remark:*

We shall see that the extended complex plane is metrizable, with distances on the Riemann sphere.

We say that the linear fractional transformation  $S$  is normalized if  $ad - bc = 1$ . (Every linear fractional transformation can be normalized by dividing  $a, b, c, d$  by  $ad - bc$ , changing the sign of the coefficients would make no change.)

We can write the linear fractional transformation in matrix form: Letting  $z = \frac{z_1}{z_2}$  and  $w = \frac{w_1}{w_2}$ , we have  $w = Sz$  iff

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

In this way we can see that the linear fractional transformation forms a group. And the normalized linear fractional transformations form a subgroup of the group, denoted by  $SL(2, \mathbb{C})$ .

Three special linear fractional transformations are of particular interest:

- The parallel translation:  $w = z + \alpha$ .
- Stretching and Rotation:  $w = kz$ .
- The inversion:  $w = \frac{1}{z}$ .

Every linear fractional transformation can be expressed as a composition of these three transformations. We say that the linear transformation has 3 degrees of freedom.

### 3.3.2 The Cross Ratio

Given 3 distinct points  $z_2, z_3, z_4 \in \mathbb{C}^*$ , there is a linear transformation  $S$  such that  $S(z_2) = 0$ ,  $S(z_3) = 1$ , and  $S(z_4) = \infty$ . The transformation is given by

$$Sz = \frac{z - z_3}{z - z_4} \bigg/ \frac{z_2 - z_3}{z_2 - z_4} \quad (3.4)$$

If one of  $z_2, z_3, z_4$  is  $\infty$ , then it reduces to

$$Sz = \frac{z - z_3}{z - z_4}, \quad \frac{z_2 - z_4}{z - z_4}, \quad \frac{z - z_2}{z_2 - z_3} \quad (3.5)$$

respectively.

#### Definition 3.3.1: The Cross Ratio

The cross ratio  $(z_1, z_2, z_3, z_4)$  is the image of  $z_1$  under the linear transformation  $S$  that maps  $z_2, z_3, z_4$  to  $0, 1, \infty$ , respectively. It is given by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \quad (3.6)$$

*Remark:*

The linear transformation above is uniquely determined by  $z_2, z_3, z_4$ .

*Proof.* If  $S, T$  satisfies the condition, then  $ST^{-1}$  would leave  $0, 1, \infty$  invariant. Letting  $ST^{-1}z = \frac{az + b}{cz + d}$ , we have

$$\frac{b}{d} = 0 \quad \frac{a+b}{c+d} = 1 \quad \frac{a}{c} = \infty$$

which means that  $b = c = 0, a = d$ . So  $ST^{-1}(z) = z$ .  $S = T$ . □

We can see the cross ratio as some properties relate to the four points. It characterizes some properties of the four points, such as collinearity and concyclicity. The cross ratio is invariant under linear fractional transformations.

We can also say that

$$(z_1, z_2, z_3, z_4) = ((z_1, z_2, z_3, z_4), 0, 1, \infty)$$

#### Theorem 3.3.1: The Invariance of Cross Ratio

If  $z_1, z_2, z_3, z_4 \in \mathbb{C}^*$  are distinct, and  $T$  is any linear fractional transformation, then the cross ratio is invariant under  $T$ :

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4) \quad (3.7)$$

*Proof.* Let  $Sz = (z, z_2, z_3, z_4)$ , then  $ST^{-1}$  maps  $Tz_2, Tz_3, Tz_4$  to  $0, 1, \infty$ , respectively. So we have

$$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1, z_2, z_3, z_4)$$

□

Using this property it is easy to get the linear transformation if given  $z_1, z_2, z_3 \mapsto w_1, w_2, w_3$ . It must obey

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3) \quad (3.8)$$

Solving the equation we shall get  $z \mapsto w(z)$ .

### Theorem 3.3.2: Concyclicity and Collinearity

The cross ratio  $(z_1, z_2, z_3, z_4) \in \mathbb{R}$  iff the four points lies on a straight line or on a circle.

*Proof.* From elementary geometry we know that the opposite angles of a cyclic quadrilateral are supplementary, so using

$$\text{Arg}(z_1, z_2, z_3, z_4) = \text{Arg} \frac{z_1 - z_3}{z_1 - z_4} - \text{Arg} \frac{z_2 - z_3}{z_2 - z_4}$$

would do.

For an analytical proof, we need only the fact that the real axis under arbitrary linear fractional transformation is mapped to a circle or a line. Let  $T^{-1}$  be the linear transformation, then  $Tw \in \mathbb{R}$ , that  $Tw = \overline{Tw}$ . So we have

$$\frac{aw + b}{cw + d} = \frac{\bar{a} \bar{w} + \bar{b}}{\bar{c} \bar{w} + \bar{d}}$$

$$(a\bar{c} - \bar{a}c) w\bar{w} + (a\bar{d} - \bar{a}d) w + (b\bar{c} - \bar{b}c) \bar{w} + (b\bar{d} - \bar{b}d) = 0$$

If  $a\bar{c} - \bar{a}c = 0$ , then this is the equation of a straight line, otherwise, we have

$$\left| w + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - \bar{c}a} \right| = \left| \frac{ad - bc}{\bar{a}c - \bar{c}a} \right|$$

This is a circle. □

### Corollary 3.3.1: Geometry of Linear Transformations

A linear transformation takes circles or lines to circles or lines.

For any two circle or line, there exists a linear transformation that maps one to the other.

*Proof.* Let  $C_1$  and  $C_2$  be two circles or lines, then we can find three points  $z_1, z_2, z_3 \in C_1$  and three points  $w_1, w_2, w_3 \in C_2$ . We can find a linear transformation  $T$  such that  $Tz_i = w_i$ , for  $i = 1, 2, 3$ .

Then we have  $\forall z \in C_1, (z, z_1, z_2, z_3) \in \mathbb{R}$  so  $(Tz, Tz_1, Tz_2, Tz_3) \in \mathbb{R}$  so  $Tz \in C_2$ . Also,  $\forall z \in C_2, T^{-1}(z) \in C_1$ . Therefore,  $T$  is surjective, thus  $T(C_1) = C_2$ . □

### 3.3.3 Symmetry

For linear transformations with real coefficients, that is,  $a, b, c, d \in \mathbb{R}$ :

- $T(\mathbb{R}) = \mathbb{R}$ .
- $T\bar{z} = \overline{Tz}$ .

---

*Proposition:* **Real Transformations**

---

If  $T(\mathbb{R}) = \mathbb{R}$ , then  $\forall z \in \mathbb{C}$ ,  $T\bar{z} = \overline{Tz}$  and  $T$  can have real coefficients. (We say “can” for the coefficients maybe multiplied by a common factor.)

*Proof.* Assume that  $T$  takes  $\mathbb{R}$  to  $\mathbb{R}$ . Letting  $Tz = \frac{az + b}{cz + d}$ .

Taking  $z = 0$ , we have  $d = 0$  or  $b/d \in \mathbb{R}$ .

- If  $d = 0$ , then  $c \neq 0$ , taking  $z = \infty$  would give  $a/c \in \mathbb{R}$ . Taking  $z = 1$  we get  $b/c \in \mathbb{R}$ . So  $a' = a/c, b' = b/c, c' = 1, d' = 0$  would do.
- If  $d \neq 0$ , then it takes similar discussion.

□

In the general case, we have  $T(\mathbb{R})$  being a circle or a line  $C$ , and we say  $w = Tz$  and  $w^* = T\bar{z}$  are symmetric with respect to  $C$ .

### Definition 3.3.2: Symmetry

If  $C \subseteq \mathbb{C}$  is a line or circle, and  $w, w^* \in \mathbb{C}$ . Let  $T$  be any linear transformation that  $T(C) = \mathbb{R}$ . If  $\overline{Tz} = Tw^*$ , then we say that  $w$  and  $w^*$  are symmetric with respect to  $C$ .

We shall prove that the symmetry does not depend on the choice of  $T$ .

*Proof.* If  $S, T$  both takes  $C$  to  $\mathbb{R}$ , then  $ST^{-1}$  takes  $\mathbb{R}$  to  $\mathbb{R}$ . If  $\overline{Tz} = Tw^*$ , then  $Sw^* = ST^{-1}Tw^* = ST^{-1}\overline{Tz} = \overline{ST^{-1}Tz} = \overline{Sw}$ . □

Therefore, we rewrite the symmetry as

### Theorem 3.3.3: Criterion for Symmetry

The points  $z, z^*$  are symmetric with respect to a line or circle  $C$  iff  $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$  for some distinct points  $z_1, z_2, z_3 \in C$ .

The symmetric function  $z \mapsto z^*$  is a bijection of  $\mathbb{C} \rightarrow \mathbb{C}$ , called a reflection. The only points that are symmetric to themselves are those on  $C$ .

*Remark:*

Note that reflection is NOT a linear transformation, just like the conjugation transform. However, two reflection would be a linear transformation.

Any reflection would have the form  $TST^{-1}$ , where  $T$  is a linear transformation that takes  $\mathbb{R}$  to  $C$  and  $S$  is the conjugate operation.

Now we consider the geometrical implications of symmetry.

- Suppose  $C$  is a straight line, we take  $z_3 = \infty$ . Then we have

$$\frac{z^* - z_2}{z_1 - z_2} = \frac{\bar{z} - \bar{z}_2}{\bar{z}_1 - \bar{z}_2}$$

Taking absolute values we get  $|z^* - z_2| = |z - z_2|$ , as  $z_2$  can take any point on the line, and  $z, z^*$  are on the different half planes (taking  $\text{Im}$ ), so  $C$  is the bisecting normal of the segment  $zz^*$ .



- If  $C$  is a finite circle with center  $a$  and radius  $R$ , then using the invariance of cross ratio we have

$$\overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)}$$

As  $|z_1 - a| = |z_2 - a| = |z_3 - a| = R$ , so we have

$$\begin{aligned} \overline{(z, z_1, z_2, z_3)} &= \left( \bar{z} - \bar{a}, \frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a} \right) \\ &= \left( \frac{R^2}{\bar{z} - \bar{a}} + a, z_1, z_2, z_3 \right) \text{ Using the cross ratio invariance} \end{aligned}$$

Therefore, as have  $z^* = \frac{R^2}{\bar{z} - \bar{a}} + a$ , or

$$(z^* - a)\overline{(z - a)} = R^2 \quad (3.9)$$

Note that the distances  $|z^* - a| |z - a| = R^2$  being a constant, and the ratio  $(z^* - a)/(z - a) \in \mathbb{R}$ . Therefore, the points  $z^*$  and  $z$  are on the same line with  $a$ .

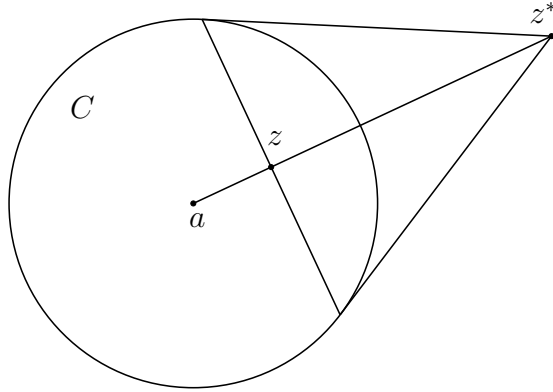


Figure 3.1: Symmetric Points to a Circle

#### Theorem 3.3.4: The Symmetric Principle

$T$  is a linear transform, and  $C_1, C_2$  are circles or lines. If  $T$  maps  $C_1$  to  $C_2$ , then if  $z^*, z$  are symmetric with respect to  $C_1$ , then  $Tz^*, Tz$  are symmetric with respect to  $C_2$ .

*Proof.* Both taking the point to be conjugate to  $\mathbb{R}$ . □

We can use this principle to find transformations  $T$ . If  $z_1 \in C_1$  maps to  $z_2 \in C_2$ , and  $z_2$  mapping to  $w_2$  are not on  $C_1$ , then we can solve

$$(w, w_1, w_2, w_2^*) = (z, z_1, z_2, z_2^*)$$

to find the transformation  $T$ .

### 3.3.4 Oriented Circles

The extended complex plane is homeomorphic to  $S_1$  rather than  $R^2$ . So topologically, we need two charts to analyze the derivative of linear transformation. One is just  $\mathbb{C}$ , the other  $\mathbb{C} \cup \{\infty\} - \{0\}$ , by the inversion  $z \mapsto 1/z$ .

Because a linear transformation  $S(z)$  is analytic, and

$$S'(z) = \frac{ad - bc}{(cz + d)^2} \quad (3.10)$$

is not zero for  $z \neq -d/c, \infty$ . We can see that the angle of two intersecting circle is preserved under linear transformations.

#### Definition 3.3.3: Orientation of Circles

Let  $z_1, z_2, z_3$  be an ordered triple on  $C$ . A point  $z$  not on  $C$  is said to be on the right of  $C$  if  $\text{Im}(z, z_1, z_2, z_3) > 0$ , and on the left of  $C$  if  $\text{Im}(z, z_1, z_2, z_3) < 0$ . This definition formulates what we mean by clockwise and counterclockwise orientation of circles.

We shall see that there are only two orientations. For a given circle, there are two regions, one is called left, and the other is called right, depending on which triple we use. The orientation of the circle is determined by the triple.

*Proof.* The invariance of cross ratio means that we only need to consider  $C$  being  $\mathbb{R}$ . Now

$$(z, z_1, z_2, z_3) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, ad - bc \neq 0$$

The coefficients depend on the triple. We have

$$\text{Im}(z, z_1, z_2, z_3) = \frac{ad - bc}{|cz + d|^2} \text{Im } z$$

Now we see that the distinction of right and left only depends on the upper and lower plane, which is right and which left depends on the triple, more specifically, the determinant  $ad - bc$ .  $\square$

---

#### Remark:

If we draw an arrow  $z_1 \rightarrow z_2 \rightarrow z_3$ , then the left and right regions are determined by the left and right direction of the arrow.

We define the points to the left of the circle are called **inside** the circle, and the points to the right of the circle are called **outside** the circle.

---

### 3.3.5 Families of Circles

We can use circles to visualize the linear transformations.

Consider

$$w = k \cdot \frac{z - a}{z - b}$$

Then  $a \mapsto 0, b \mapsto \infty$ .

- Circles that pass through  $a, b$  are mapped to circles that pass through  $0, \infty$ , which are straight lines passing through  $0$ .
- Concentric circles centered at  $0$  in  $w$ -plane are given by  $|w| = \rho$ , the preimage is

$$\left| \frac{z - a}{z - b} \right| = \frac{\rho}{|k|}$$

These are Apollonius circles, with limit points  $a, b$ .

#### Definition 3.3.4: Circular Net

Let  $C_1$  be the circles passing  $a, b$ , and  $C_2$  be the Apollonius circles with limit points  $a, b$ . The family of circles  $C_1 \cup C_2$  is called a **circular net** with respect to  $a, b$ .

#### Proposition: Properties of Circular Nets

- $\forall z \in C$ , there is exactly one circle in  $C_1$  that passes through  $z$ , and exactly one circle in  $C_2$  that passes through  $z$ .
- Every circle in  $C_1$  intersects every circle in  $C_2$  at two points, with right angles.
- Reflection of one circle in  $C_1$ : Transfer every  $C_2$  onto itself, and  $C_1$  onto another  $C_1$ .
- Reflection of one circle in  $C_2$ : Transfer every  $C_1$  onto itself, and  $C_2$  onto another  $C_2$ .
- The limit points  $a, b$  are symmetric with respect to every circle in  $C_2$ , and not symmetric with respect to any other circle.

*Proof.* All these can be proven by mapping this to the  $w$ -plane. □

*Remark:*

Geometrically speaking,

- The circles in  $C_1$  are defined by

$$\arg \frac{z - a}{z - b} = \text{constant}$$

- The circles in  $C_2$  are defined by

$$\left| \frac{z - a}{z - b} \right| = \text{constant}$$

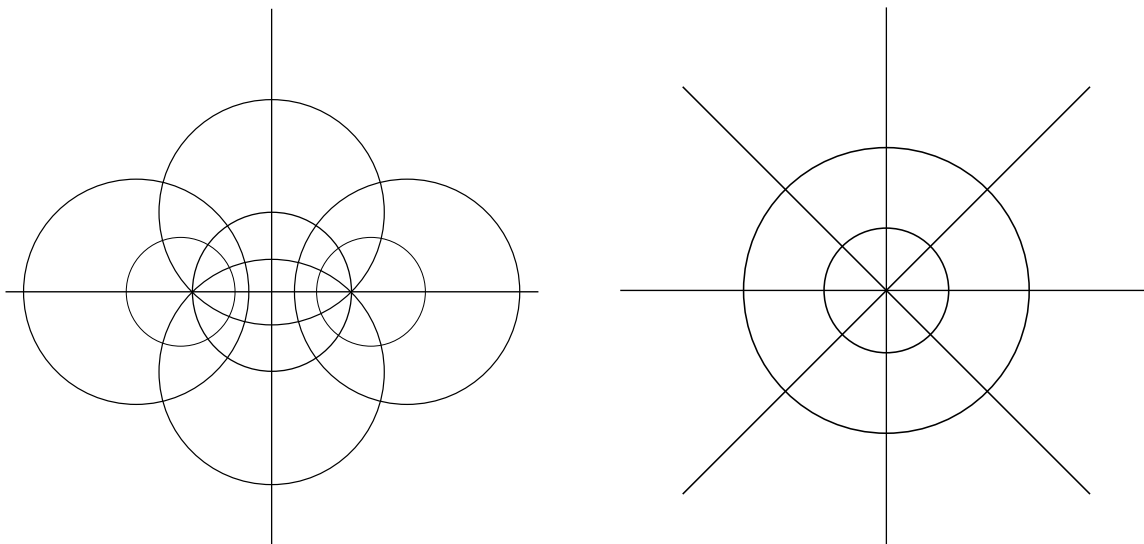


Figure 3.2: Circular Nets

If a linear transformation takes  $a, b$  to  $a', b'$ , it can be written by

$$\frac{w - a'}{w - b'} = k \cdot \frac{z - a}{z - b} \quad (3.11)$$

It can be seen as a composition of first bringing  $a, b$  to  $0, \infty$ , then to  $a', b'$  by inverse transformation. It is clear that the transformation takes  $C_1$  to  $C'_1$  and  $C_2$  to  $C'_2$ , where  $C'_1$  is the circles passing through  $a', b'$ , and  $C'_2$  is the Apollonius circles with limit points  $a', b'$ .

The following are some special cases.

- If  $a' = a, b' = b$ , we say  $a, b$  are fixed points of  $T$ . In this case, the whole circular net is would map onto itself. The transformation can be visualized by studying  $k$ .
  - Taking  $\arg$  we see that  $C_1$  circles are changed to circles adding  $\arg k$ .
  - Taking modulus we see that  $C_2$  circles are changed to circles multiplying  $|k|$ .
- Additionally, if  $k \in \mathbb{R}$ , then  $C_1$  does not change, (if  $k > 0$  then the orientation of  $C_1$  is preserved, if  $k < 0$  then the orientation is reversed). We call the transformation **Hyperbolic**.
- If  $|k| = 1$ , then  $C_2$  does not change, and we call the transformation **Elliptic**.
- Every general transformation can be written as a composition of hyperbolic and elliptic transformations. (This is quite obvious, we can write  $k = re^{i\theta}$ . And the form of the two sides of equation 3.11 is preserved.)

Now we introduce another type of circular nets. Consider writing a linear transformation in the form

$$w = \frac{\omega}{z - a} + c \quad (3.12)$$

As  $a \mapsto \infty$ , the circles passing through  $a$  becomes straight lines. The preimage of parallel lines are mutually tangent circles at  $a$ . (This can be seen by geometry, proof is easy)

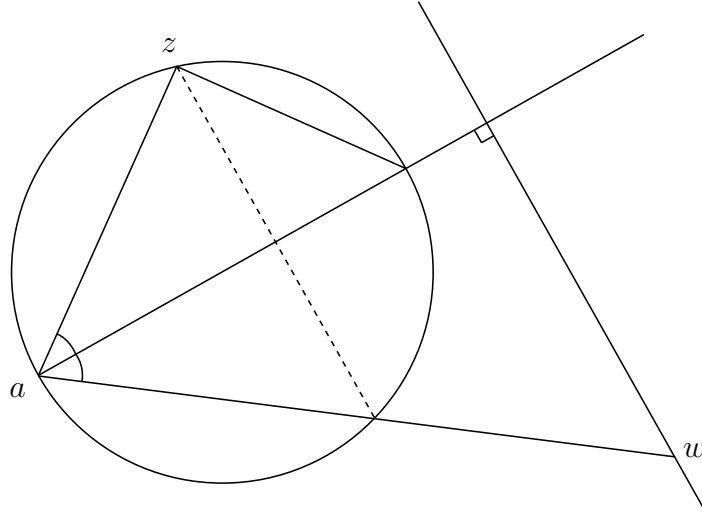


Figure 3.3: Parallel Lines and Cotangent Circles

The product of the diameter and the distance to the line is a constant  $w$ . To sets of cotangent circles may map to the  $x - y$  grid in the  $w$ -plane.

---

*Remark:*

We can see the case as the limit case of the circular nets with two fixed points, where both Apollonius circle and the circles passing through the fixed points are degenerated cotangent ones. We also denote them  $C_1$  and  $C_2$ .

The direction of the tangents is given by  $\arg w$ , as is seen.

---

Any linear transformation taking  $a$  to  $a'$  can be written as

$$\frac{\omega'}{w - a'} = \frac{\omega}{z - a} + c \quad (3.13)$$

If  $a = a'$  is the only fixed point, then we have  $\omega = \omega'$  and

$$\frac{\omega}{w - a} = \frac{\omega}{z - a} + c$$

A multiplicative parameter is arbitrary, assume  $c \in \mathbb{R}$  and every  $C_1$  is mapped onto itself, so we say the transformation is **Parabolic**, which are flows of  $C_2$ .

The fixed points of a linear transformation can be found by solving

$$z = \frac{\alpha z + \beta}{\gamma z + \delta} \quad (3.14)$$

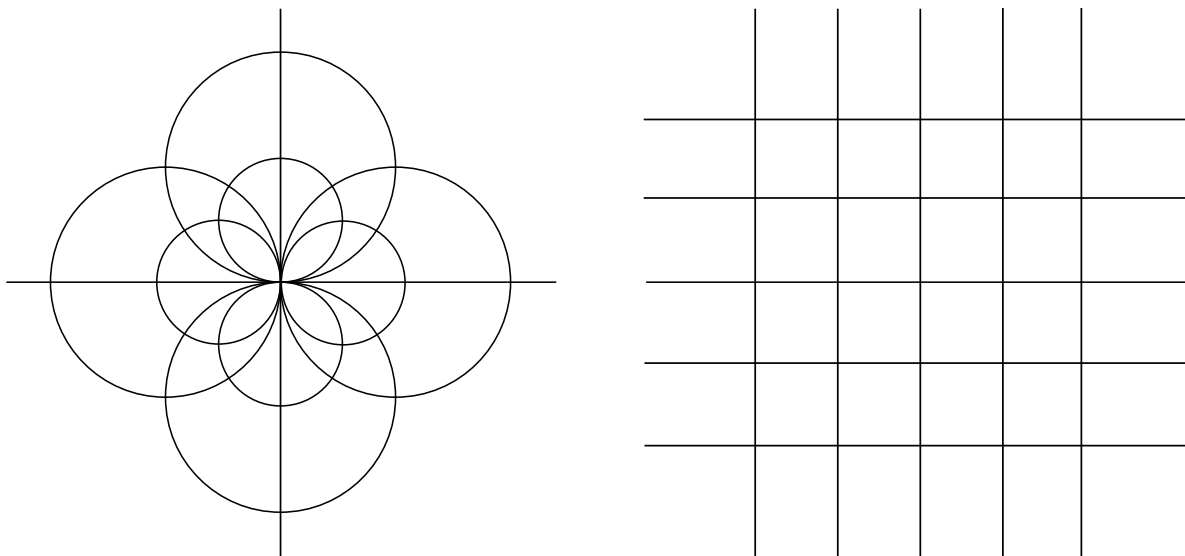


Figure 3.4: Circular Nets of One Fixed Point

If it has two distinct roots, then the transformation is hyperbolic or elliptic. However, if the two roots are the same, then the transformation is parabolic. The condition for a parabolic transformation is:

$$(\alpha - \delta)^2 = 4\beta\gamma \quad (3.15)$$

- If the equation has two distinct roots  $a, b$ , we can write it in the form 3.11, and use the circular nets to visualize the transformation.
- To visualize parabolic transformations, we write it in the nets with one fixed point.

A linear function that is neither hyperbolic, elliptic, or parabolic is called **Loxodromic**.

## 3.4 Elementary Conformal Mappings

The visualization of conformal mappings is very important in complex analysis, as it gives us a direct intuition of analytic functions.

### 3.4.1 Level Curves

When a point-to-point visualization is hard to interpret, we can use curve families of known nature to gain a more direct intuition.

#### Definition 3.4.1: Level Curves

If  $f : \Omega \rightarrow \mathbb{C}$  are determined by the real functions  $f = u + iv$ , then the curves defined by  $u(x, y) = c$  or  $v(x, y) = c$  for some constant  $c$  are called the **level curves** of  $f$ .

It is of conformal nature that the level curves of  $u$  and  $v$  are orthogonal to each other.

In a more general sense, any orthogonal net curves can be used to visualize the conformal mapping. As we did for linear transformations.

**The Power**  $w = z^\alpha$  We use the polar coordinate chart here. Let  $S(\varphi_1, \varphi_2)$  be the sector defined

$$S(\varphi_1, \varphi_2) = \{z = re^{i\varphi} : r > 0, \varphi_1 < \varphi < \varphi_2\} \quad (3.16)$$

This is indeed a region. The power function has the property

$$|w| = |z|^\alpha \quad \text{and} \quad \arg w = \alpha \arg z$$

(Well we take one branch of the multivalued function). So that  $f(S(\varphi_1, \varphi_2)) = S(\alpha\varphi_1, \alpha\varphi_2)$ . It is analytic for certain domain  $S(\varphi_1, \varphi_2)$ . The derivative

$$Dz^\alpha = De^{\alpha \log z} = \alpha \frac{w}{z}$$

**The Exponential Function**  $w = e^z$  This one is well to understand. Some examples include

- Cartesian chart to polar chart.
- Straight lines to logarithmic spirals. (Circles and rays maybe)

### 3.4.2 A Brief Survey of Conformal Mappings

Our ultimate goal is to map a region  $\Omega_1$  to another  $\Omega_2$ . It is advisable to do it in two parts:

- Mapping  $\Omega_1$  to a disk or half plane.
- Mapping the disk or half plane to  $\Omega_2$ .

We shall later prove that this is possible for any region whose boundary is a simple closed curve.

Now our tools are: linear transformations, the exponential function, and the power function, the logarithm. They have properties that transform circles and lines to each other, so their use are limited to regions bounded by circles or lines.

#### *Example:* Conformal Mappings to Half Plane or Circle

We start by considering a region bounded by **two circular arcs**.

- If the two intersection points are  $a, b$ , we use  $z_1 = (z - a)/(z - b)$  to make it to a sector.
- An appropriate power would make it into a half plane.
- If the two circles are tangent, then using  $z_1 = 1/(z - a)$  will make it into a parallel strip. And a suitable exponential function would do the trick.

Now we map  $\mathbb{C} - [-1, 1]$  to a circle.

- Use  $z_1 = \frac{z + 1}{z - 1}$  to get  $\mathbb{C} - (-\infty, 0]$ .

- Use  $z_2 = \sqrt{z_1}$  to get  $\{z : \operatorname{Re} z > 0\}$ .
- Use  $w = \frac{z_2 - 1}{z_2 + 1}$  to get the unit disk.

The overall mapping is

$$z = \frac{1}{2} \left( w + \frac{1}{w} \right) \text{ and } w = z - \sqrt{z^2 - 1} \quad (3.17)$$

We take a closer look at the mapping 3.17. Let  $w = \rho e^{i\theta}$ , where  $\rho < 1$  or  $> 1$ , then we have

$$x = \frac{1}{2} \left( \rho + \frac{1}{\rho} \right) \cos \theta, \quad y = \frac{1}{2} \left( \rho - \frac{1}{\rho} \right) \sin \theta$$

For circle and rays in  $w$ -plane, we eliminate  $\rho$  and  $\theta$  respectively,

$$\frac{x^2}{\left( \frac{1}{2} \left( \rho + \frac{1}{\rho} \right) \right)^2} + \frac{y^2}{\left( \frac{1}{2} \left( \rho - \frac{1}{\rho} \right) \right)^2} = 1$$

$$\frac{x^2}{\cos^2 \theta} + \frac{y^2}{\sin^2 \theta} = 1$$

The circles maps to ellipses, and the rays to hyperbolas. The mapping is conformal, and the angles are preserved as perpendicular intersections.

Another less trivial example, we consider the cubic polynomial  $w = a_0 z^3 + a_1 z^2 + a_2 z + a_3$ . We can reduce it to  $w = z^3 - 3z$  easily. (Setting  $z = z_1 - a_1/3a_0$  to cancel out quadratic terms and normalize it)

Using the transformation 3.17, we let

$$z = \zeta + \frac{1}{\zeta}$$

take  $\zeta$  to lay in either the inside or outside the disk. Then we have

$$w = \zeta^3 + \frac{1}{\zeta^3}.$$

The total effect is ellipse  $\rightarrow$  circle  $\rightarrow$  stretching rotation circle  $\rightarrow$  ellipse. Or simply stretching rotation along the ellipse.

### 3.4.3 A Brief Survey on Riemann Surfaces

For non-surjective maps, the image of regions would overlap. We can see it as different layers of the same  $\mathbb{C}$  plane. This is the original intuition of Riemann surfaces. However, we shall not formally define the concept, just to give some intuitive understanding.

---

*Example:* **Riemann Surfaces**

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For the simple power  $z^n$  where  $n > 1, n \in \mathbb{Z}$ , the Riemann surfaces are like spirals around a vertical axis. After  $n$  turns we shall end up at the beginning.

We see it has fairly well topological properties, with locally Euclidean and all that.

The exponent  $e^z$  is the same, but with endless spirals.

---

A **fundamental region** is a region that maps to the whole space except for some cuts in a one-to-one manner. Identifying Riemann surfaces simply include Identifying fundamental regions and sticking different layers together according to the edge of the fundamental regions.



# Chapter 4

## Complex Integration

### 4.1 Fundamental Theorems

Similar to the real case, we have definite and indefinite integrals for complex field.

#### 4.1.1 Line Integral

A direct generalization of the real integral is the integral of a function  $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{C}$ . If  $f(t) = u(t) + iv(t)$ , then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt \quad (4.1)$$

It has similar properties as the real integral, such as linearity, additivity, and monotonicity. If  $c \in \mathbb{C}$  we have

$$\int_a^b cf(t)dt = c \int_a^b f(t)dt$$

and if  $a \leq b$  the fundamental inequality holds:

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt$$

This would naturally turn to our definition of a complex line integral.

#### Definition 4.1.1: Complex Line Integral

Let  $\gamma$  be a piecewise differentiable arc with equation  $z = z(t)$ ,  $a \leq t \leq b$ . If  $f : \gamma \rightarrow \mathbb{C}$  is continuous on  $\gamma$ , then the complex line integral of  $f$  along  $\gamma$  is defined as

$$\int_{\gamma} f(z)dz = \int_a^b f(z(t))z'(t)dt \quad (4.2)$$

It is easy to show that the definition is invariant under reparametrization.

---

*Remark:*

The integral can be defined by the Riemann sum just like the real case. If  $\gamma$  is a polygonal

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arc, then the integral can be computed as

$$\int_{\gamma} f(z)dz = \lim \sum_{k=1}^n f(z_k)(z_k - z_{k-1}) \quad (4.3)$$

where  $z_k$  are the points on  $\gamma$  with max distance tend to zero.

*Proposition:* **Some Properties of Line Integrals**

Similar of that of  $\mathbb{R}^2$ , we have

- The reverse direction of  $\gamma$  is  $-\gamma$ , then

$$\int_{-\gamma} f(z)dz = - \int_{\gamma} f(z)dz \quad (4.4)$$

- Additive to the arcs:

$$\int_{\gamma_1 + \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz \quad (4.5)$$

- Expanding the integral, if  $f = u + iv$  we have

$$\int_{\gamma} f(z)dz = \int_{\gamma} udx - vdy + i \int_{\gamma} vdx + udy$$

- For respect to the arc length, we have

$$\int_{\gamma} f(z)ds = \int_{\gamma} f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

And the fundamental inequality holds:

$$\left| \int_{\gamma} f(z)ds \right| \leq \int_{\gamma} |f(z)|ds$$

For integrals of the conjugate, we have

$$\int_{\gamma} f(z)\overline{dz} = \overline{\int_{\gamma} \overline{f(z)}dz} \quad (4.6)$$

So we can write:

$$\begin{aligned} \int_{\gamma} f(z)dx &= \frac{1}{2} \left( \int_{\gamma} f(z)dz + \int_{\gamma} f(z)\overline{dz} \right) \\ \int_{\gamma} f(z)dy &= \frac{1}{2i} \left( \int_{\gamma} f(z)dz - \int_{\gamma} f(z)\overline{dz} \right) \end{aligned}$$

### 4.1.2 Rectifiable Arcs

#### Definition 4.1.2: Rectifiable Arcs

The length of an arc can be defined as the least upper bound of the sums of the lengths of the polygonal arcs that approximate it. That is,

$$|z(t_1) - z(t_0)| + \cdots + |z(t_n) - z(t_{n-1})| = \sum_{k=1}^n |z(t_k) - z(t_{k-1})|$$

where  $a = t_0 < t_1 < \cdots < t_n = b$  is a partition of the interval  $[a, b]$ . The arc is said to be rectifiable if the length is finite.

It is easy to show that piecewise differentiable arcs are rectifiable.

When  $\gamma$  is rectifiable, we can define the integral with respect to the arc length:

#### Definition 4.1.3: Integral with Respect to Arc Length

Let  $\gamma$  be a rectifiable arc, then the integral of  $f$  with respect to the arc length is defined as

$$\int_{\gamma} f(z) ds = \lim \sum_{k=1}^n f(z_k) |z_k - z_{k-1}| \quad (4.7)$$

### 4.1.3 Line Integral As Functions of Arcs

If  $p, q : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are continuous in  $\Omega$ , and  $\gamma$  is any piecewise differentiable arc in  $\Omega$ , then we have a function

$$\gamma \mapsto \int_{\gamma} p dx + q dy \quad (4.8)$$

This is a functional on the space of arcs in  $\Omega$ . In analysis, we've known that the integral is zero for every closed curve is called a **conservative field**.

Let  $p, q$  be continuously differentiable on a region  $\Omega \subseteq \mathbb{R}^2$ . The line integral  $\int_{\gamma} p dx + q dy$  is independent of the path  $\gamma$  if and only if there exists a function  $U : \Omega \rightarrow \mathbb{R}^2$  such that  $p = \frac{\partial U}{\partial x}$  and  $q = \frac{\partial U}{\partial y}$ .

The function  $U$  is called a **potential function** of the field  $(p, q)$ , and it is unique up to an additive constant.

Now, when do a continuous function  $f : \mathbb{C} \rightarrow \mathbb{C}$  has a primitive function  $F$  such that  $F' = f$ ? Let  $f = u + iv$ , then we have

$$u = \frac{\partial F}{\partial x}, \quad v = \frac{\partial F}{\partial y}$$

Comparing to the last theorem, we have

**Theorem 4.1.1: Indefinite Integralizable Functions**

If  $f$  is continuous, and has continuous partial derivatives in a simply connected domain  $\Omega \subseteq \mathbb{C}$ , then there exists a function  $F : \Omega \rightarrow \mathbb{C}$  such that  $F' = f$  iff the integral  $\int_{\gamma} f(z)dz$  is only dependent on the endpoints of  $\gamma$  and not on the path taken.

NOTE that the region can have holes, but it must be simply connected.

An immediate example shows that

$$\int_{\gamma} (z - a)^n dz = 0, n \in \mathbb{N} \quad (4.9)$$

As  $(z - a)^n$  has a primitive function  $F(z) = \frac{(z - a)^{n+1}}{n+1}$ , which is analytic on  $\mathbb{C}$ . If  $n < 0, n \neq -1$ , then the result also holds for any closed curves that do not pass through  $a$ .

For  $n = -1$ , the result does not hold, for a circle  $C : z = a + \rho e^{it}, 0 \leq t \leq 2\pi$ , we have

$$\int_C \frac{dz}{z - a} = \int_0^{2\pi} i\rho e^{it} \cdot \frac{1}{\rho e^{it}} dt = 2\pi i \quad (4.10)$$

---

*Remark:*

This implies that it is impossible to define a single-valued branch of  $\log(z - a)$  in an annulus  $\rho_1 < |z - a| < \rho_2$ .

---

**4.1.4 Cauchy's Integral Theorem for Rectangles**

Let  $R$  be a rectangle defined  $a \leq x \leq b, c \leq y \leq d$ . Let the boundary (counterclockwise) be  $\partial R$ . (Note that  $R$  is closed, so not a region)

**Theorem 4.1.2: Cauchy's Integral Theorem on Rectangles**

If  $f$  is analytic on  $R$ , (on an open region containing  $R$ ), then

$$\int_{\partial R} f(z)dz = 0 \quad (4.11)$$

*Proof.* Let

$$\eta(R) = \int_{\partial R} f(z)dz$$

Divide  $R$  into four congruent rectangles  $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$  with the same orientation as  $R$ , then we have

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$$

We have one of the four rectangles  $R^{(k)}$ , denoted by  $R_1$ , having

$$|\eta(R_1)| \geq \frac{1}{4} |\eta(R)|$$

Repeat the process and get  $R \supset R_1 \supset R_2 \supset \dots$ , we have

$$|\eta(R_n)| \geq \frac{1}{4^n} |\eta(R)|$$

From the nested chain of closed sets, we have a  $z^* \in R_n$  for all  $n$ , so for given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\forall z, |z - z^*| < \delta$ , we have

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon$$

Choose large enough  $n$  that  $R_n \subseteq \{z : |z - z^*| < \delta\}$ . From theorem 4.1.1, we have

$$\int_{\partial R_n} dz = 0, \quad \int_{\partial R_n} z dz = 0$$

So we write

$$\eta(R_n) = \int_{\partial R_n} (f(z) - f(z^*) - (z - z^*)f'(z^*)) dz$$

Let  $d_n, L_n$  denote the diagonal length and perimeter of  $R_n$ , so we have  $d_n = 2^{-n}d, L_n = 2^{-n}L$ , so

$$|\eta(R_n)| \leq \epsilon \int_{\partial R_n} |z - z^*| |dz| \leq \epsilon d_n L_n = \epsilon 4^{-n} dL$$

So we get

$$|\eta(R)| \leq \epsilon dL$$

As  $\epsilon$  can be arbitrarily small, we have  $\eta(R) = 0$ . □

Now we weaken the conditions step by step.

### Theorem 4.1.3: Cauchy's Theorem on Stained Rectangles

Let  $R'$  be a rectangle  $R$  omitting a finite number of interior points  $\zeta_i$ , if  $f$  is analytic on  $R'$  and

$$\lim_{z \rightarrow \zeta_i} (z - \zeta_i) f(z) = 0, i = 1, 2, \dots, n$$

holds, then

$$\int_{\partial R} f(z) dz = 0 \tag{4.12}$$

*Proof.* We shall only consider one point, as  $R$  can be split into several rectangles  $R_i$  with at most one hole.

Let  $R_0$  be a small rectangle surrounding  $\zeta$ , splitting shows that

$$\int_{\partial R} f(z) dz = \int_{\partial R_0} f(z) dz$$

For any  $\epsilon > 0$ , choose small enough  $R_0$  that

$$|f(z)| \leq \frac{\epsilon}{|z - \zeta|}$$

Thus,

$$\left| \int_{\partial R} f(z) dz \right| \leq \epsilon \int_{\partial R_0} \frac{|dz|}{|z - \zeta|} \leq 8\epsilon$$

□

Note the condition holds automatically if  $f$  is bounded on  $R'$ .

### 4.1.5 Cauchy's Theorem in Disk

Let  $\Delta$  denote an open disk  $|z - z_0| < \rho$ .

#### Theorem 4.1.4: Cauchy's Theorem on Disk

If  $f(z)$  is analytic in an open disk  $\Delta$ . Then

$$\int_{\gamma} f(z) dz = 0 \quad (4.13)$$

for every closed curve  $\gamma$  in  $\Delta$ .

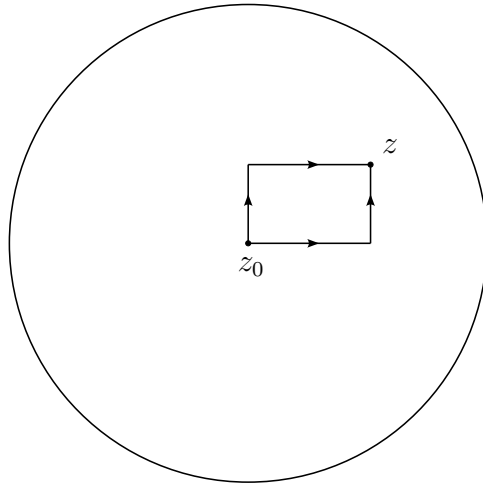


Figure 4.1: Diagram of  $F(z)$

*Proof.* Let  $F(z) = \int_{\sigma} f(z) dz$ , where  $\sigma$  is the rectangular route (either up or down is the same) from  $z_0$  to  $z$  as shown in Figure 4.1. Then by the two routes respectively, we have

$$\frac{\partial F}{\partial x} = f(z), \quad \frac{\partial F}{\partial y} = if(z)$$

Therefore,  $F$  follows the Cauchy-Riemann equations, and  $F'(z) = f(z)$ . □

Similarly, we also have



**Theorem 4.1.5: Cauchy's Theorem on Disk with Holes**

Let  $\Delta'$  be an open disk  $|z - z_0| < \rho$  omitting a finite number of interior points  $\zeta_i$ , if  $f$  is analytic on  $\Delta'$  and

$$\lim_{z \rightarrow \zeta_i} (z - \zeta_i)f(z) = 0, i = 1, 2, \dots, n$$

then

$$\int_{\gamma} f(z) dz = 0$$

for every closed curve  $\gamma$  in  $\Delta'$ .

## 4.2 Cauchy's Integral Formula

### 4.2.1 The Index of a Point

This formulate our notion of how many times a closed curve winds around a point not on the curve.

**Lemma 4.2.1: The Index Lemma**

If  $\gamma$  is a piecewise differentiable closed curve in  $\mathbb{C}$ , and  $a$  is a point not on  $\gamma$ , then there exists  $n \in \mathbb{Z}$  that

$$\int_{\gamma} \frac{dz}{z - a} = 2\pi i n \quad (4.14)$$

*Proof.* The simplest proof is computation. Let  $\gamma : z = z(t)$ , where  $\alpha \leq t \leq \beta$ , so we have

$$h(t) = \int_{\alpha}^t \frac{z'(t)}{z(t) - a} dt$$

it is continuous and differentiable on  $[\alpha, \beta]$ .

$$h'(t) = \frac{z'(t)}{z(t) - a}$$

Multiplying to the left, we have the derivative of the function  $e^{-h(t)}(z(t) - a)$  is zero. So we have

$$e^{h(t)} = \frac{z(t) - a}{z(\alpha) - a}$$

Since  $z(\alpha) = z(\beta)$ , we have  $e^{h(\beta)} = 1$ , so  $h(\beta) = 2\pi i n$  for some integer  $n$ . □

**Definition 4.2.1: Index of a Point to a Curve**

If  $\gamma$  is a piecewise differentiable closed curve in  $\mathbb{C}$ , and  $a$  is a point not on  $\gamma$ , then the index of  $a$  to  $\gamma$  is defined as

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a} \quad (4.15)$$

$n$  is also called the winding number of  $\gamma$  around  $a$ .

The following are some properties of the index:

- It is clear that  $n(-\gamma, a) = -n(\gamma, a)$ .
- If  $\gamma$  lies in an open disk, and  $a$  is outside of the open disk, then  $n(\gamma, a) = 0$ .

A closed curve  $\gamma$  is a closed point set, and its complement is open so can be divided into components as open regions. We say  $\gamma$  determines these regions.

### Theorem 4.2.1: Regions by a Closed Curve

The function  $a \mapsto n(\gamma, a)$  is constant for each region determined by  $\gamma$ .

*Proof.* As two points in a region can be joined by a polygonal path. We only need to prove when the segment  $ab$  does not meet  $\gamma$  and lies in the same region.

Outside the segment the function  $\frac{z-a}{z-b}$  is never real and  $\leq 0$ . Then  $\log \frac{z-a}{z-b}$  can be analytically defined on  $\mathbb{C} - (-\infty, 0]$ . Its derivative is what we want:

$$\frac{d}{dz} \log \frac{z-a}{z-b} = \frac{1}{z-a} - \frac{1}{z-b}$$

Therefore, we have

$$\int_{\gamma} \frac{dz}{z-a} - \int_{\gamma} \frac{dz}{z-b} = 0$$

□

Specifically, if  $|a|$  is sufficiently large, then  $n(\gamma, a) = 0$ . So the index is zero for the unbounded region determined by  $\gamma$ .

The case where  $n(\gamma, a) = 1$  is worth noting. For convenience, we want  $a = 0$ .

### Lemma 4.2.2: When Index is 1

Let  $z_1, z_2$  be points on a closed curve  $\gamma$  which does not pass through 0. Following the direction of the curve, we denote  $\gamma_1$  be the subarc from  $z_1$  to  $z_2$ , and  $\gamma_2$  be the subarc from  $z_2$  to  $z_1$ . Also assume  $\text{Im } z_1 < 0, \text{Im } z_2 > 0$ . If  $\gamma_1$  does not intersect the negative real axis, and  $\gamma_2$  does not intersect the positive real axis, then  $n(\gamma, 0) = 1$ .

*Proof.* Well this is how we formulate our intuition of winding the origin once.

Take a small circle around the origin, and the origin belongs to the unbounded region of the two sides, so we have  $n(\gamma, 0) = n(C, 0) = 1$ . □

## 4.2.2 The Integral Formula

Let  $f(z)$  be analytic on an open disk  $\Delta$ . Let  $\gamma$  be a closed piecewise differentiable curve in  $\Delta$  and  $a$  be a point not on  $\gamma$ .

We apply Cauchy's theorem on disks with holes 4.1.5 for the function

$$F(z) = \frac{f(z) - f(a)}{z - a} \tag{4.16}$$

We have

$$\int_{\gamma} F(z) dz = 0 \tag{4.17}$$

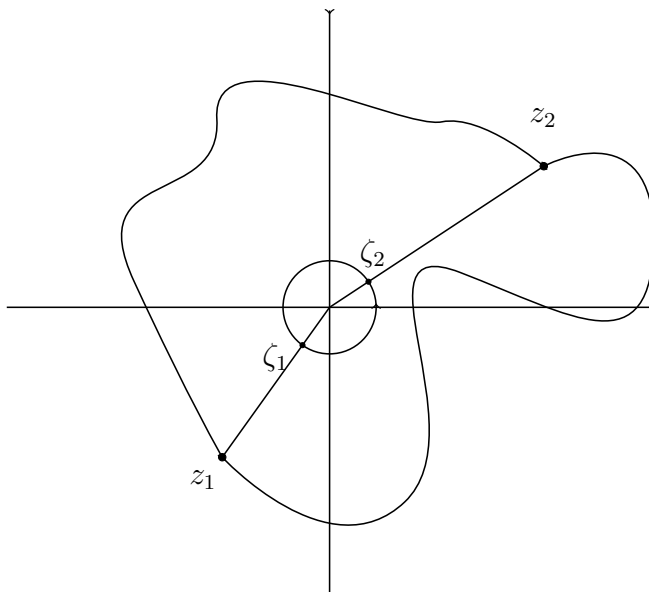


Figure 4.2: When Index is 1

$$\int_{\gamma} \frac{f(z)}{z-a} dz = f(a) \int_{\gamma} \frac{dz}{z-a} \quad (4.18)$$

### Theorem 4.2.2: Cauchy's Integral Theorem

Suppose  $f$  is analytic on an open disk  $\Delta$  (there can be exceptional points  $\zeta_j$ , such that  $\lim_{z \rightarrow \zeta_j} (z - \zeta_j)f(z) = 0$ ), and  $\gamma$  be a closed piecewise differentiable curve in  $\Delta$  and  $a$  is a point not on  $\gamma$ . Then

$$n(\gamma, a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz \quad (4.19)$$

NOTE here  $a$  may not be in  $\Delta$ .

*Remark:*

The region which  $f$  is analytic on may also be not a disk, as long as theorem 4.1.5 holds, we can still apply the theorem.

An immediate usage is when  $n(\gamma, a) = 1$ , we have

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz, \quad a \notin \gamma, n(\gamma, a) = 1 \quad (4.20)$$

This is called the Cauchy's integral formula.

### 4.2.3 Higher Order Derivatives

The Cauchy's integral formula provided us with a powerful tool to study the local properties of analytic functions.

**Lemma 4.2.3: Derivatives Under Integral**

The function  $\varphi(z, t)$  is defined for  $z \in \Omega$  and  $t \in [\alpha, \beta]$  and is continuous. If  $\forall t, z \mapsto \varphi(z, t)$  is analytic, then

$$F(z) = \int_{\alpha}^{\beta} \varphi(z, t) dt \quad (4.21)$$

is analytic in  $\Omega$  and

$$F'(z) = \int_{\alpha}^{\beta} \frac{\partial \varphi(z, t)}{\partial z} dt \quad (4.22)$$

*Proof.* Using the definition, we have

$$\frac{F(z) - F(z_0)}{z - z_0} = \int_{\alpha}^{\beta} \frac{\varphi(z, t) - \varphi(z_0, t)}{z - z_0} dt$$

Just like the real case, get a small closed space around  $z_0$ , and continuity of  $\varphi$  in a closed set implies uniform continuity. Using the  $\epsilon - \delta$  language it is easy.  $\square$

Let  $f$  be analytic on the region  $\Omega$ . For a point  $a \in \Omega$  consider its  $\delta$ -neighborhood  $\Delta = N_{\delta}(a) = \{z : |z - a| < \delta\}$  contained in  $\Omega$ . In  $\Delta$  we have a circle  $C$  around  $a$ , so we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta, \quad \forall z \text{ inside } C$$

Provided that the derivative can be done inside the integral, we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta, \quad \forall z \text{ inside } C \quad (4.23)$$

And higher order derivatives can be computed similarly:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad \forall z \text{ inside } C \quad (4.24)$$

As the choice of  $a$  is arbitrary in  $\Omega$ , we've proven that  $f$  has arbitrary order derivatives in  $\Omega$ .

The following are some classical results of Cauchy's integral formula.

**Theorem 4.2.3: Morera's Theorem**

Let  $f$  be continuous on a region  $\Omega \subseteq \mathbb{C}$ . If for every closed curve  $\gamma$  in  $\Omega$ , we have

$$\int_{\gamma} f(z) dz = 0 \quad (4.25)$$

then  $f$  is analytic on  $\Omega$ .

*Proof.* We know that  $f$  is the derivative of some analytic function  $F$  on  $\Omega$ . So  $f$  is analytic.  $\square$

**Theorem 4.2.4: Cauchy's Estimation**

Let  $f$  be analytic on a disk  $\Delta$  with radius  $r$  centered at  $a$ . Then for any  $z \in \Delta$ , we have

$$|f^{(n)}(z)| \leq \frac{M}{r^n} n!, \quad M = \max_{|\zeta - a| = r} |f(\zeta)| \quad (4.26)$$

*Proof.* We use Cauchy's integral formula 4.24. If we have  $|f(z)| \leq M$  for all  $z \in C$ , then let the radius of the circle  $C$  be  $r$ , we have

$$|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| \leq \frac{Mn!}{r^n} \quad (4.27)$$

□

### Theorem 4.2.5: Liouville's Theorem

Let  $f$  be a bounded analytic function on  $\mathbb{C}$ . Then  $f$  is constant.

*Proof.* For  $n = 1$  we have

$$|f'(z)| \leq \frac{M}{r}, \forall r > 0$$

So we have  $f'(z) = 0$ .

□

This leads to a trivial proof of the fundamental theorem of algebra: Every non-constant complex polynomial  $p(z)$  has at least one root in  $\mathbb{C}$ .

*Proof.* Suppose  $P(z)$  is a polynomial with degree  $n \geq 1$ . Then  $f(z) = 1/P(z)$  is bounded and analytic if there are no roots. By Liouville's theorem,  $f(z)$  is constant, which contradicts. □

## 4.3 Local Properties of Analytic Functions

### 4.3.1 Removable Singularities and Taylor's Formula

We've shown that the Cauchy's Theorem holds for regions that have finite exceptional points  $\zeta_j$ . However, we will see that these are not actual singularities. They can be filled and thus producing a whole analytic function on the entire region.

### Theorem 4.3.1: Removable Singularities

If  $\Omega$  is an open region, and  $\Omega'$  is contracted by  $\Omega$  omitting a point  $a$ . Suppose  $f$  is analytic on  $\Omega'$ . Then there is an analytic function  $g$  on  $\Omega$  which  $\forall z \in \Omega', g(z) = f(z)$  iff we have

$$\lim_{z \rightarrow a} (z - a)f(z) = 0$$

The extended function  $g$  is unique.

*Proof.* The necessity and uniqueness is trivial to notice. To prove sufficiency we let  $D$  be an open disk that  $\overline{D} \subseteq \Omega$  with center  $a$ . Define  $g(z) = f(z)$  for  $z \notin D$  and

$$g(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta) d\zeta}{\zeta - z}, \quad z \in D$$

Then  $g$  is the desired function.

□

If  $f$  is analytic on  $\Omega$  and  $a \in \Omega$ , we apply the theorem to the function

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

Then  $F(z)$  has an analytic extension on  $\Omega$ , denoted  $f_1(z)$  with  $f_1(z) = f'(a)$ . Resume the construction, we have

$$\begin{aligned} f(z) &= f(a) + (z-a)f_1(z) \\ f_1(z) &= f(a) + (z-a)f_2(z) \\ &\dots\dots\dots \\ f_{n-1}(z) &= f(a) + (z-a)f_n(z) \end{aligned} \tag{4.28}$$

Getting together, we have

$$f(z) = f(a) + (z-a)f_1(a) + (z-a)^2f_2(a) + \dots + (z-a)^{n-1}f_{n-1}(a) + (z-a)^nf_n(z)$$

Taking  $n^{\text{th}}$  derivative, we have

$$f^{(n)}(a) = n!f_n(a)$$

### Theorem 4.3.2: Taylor's Theorem

If  $f$  is analytic on  $\Omega$  and  $a \in \Omega$ , then

$$f(z) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(a)(z-a)^k + f_n(z)(z-a)^n$$

Where  $f_n$  is analytic in  $\Omega$ .

For  $z \in D$ , writing

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - a)^n(\zeta - z)} d\zeta - R(z)$$

The integrals in  $R(z)$  has the form

$$F_\nu = \int_{\partial D} \frac{d\zeta}{(\zeta - a)^\nu(\zeta - z)}, \quad \nu \geq 1$$

Taking it as a function of  $a$ . We have

$$F_1 = \frac{1}{z-a} \int_{\partial D} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - a} \right) d\zeta = 0$$

Then we have

$$F_{\nu+1} = \frac{1}{\nu!} \frac{d^\nu F_1}{da^\nu} = 0$$

Therefore,  $R = 0$  and we have

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - a)^n(\zeta - z)}, \quad z \in D \tag{4.29}$$

### 4.3.2 Zeros and Poles

We'll see that local properties of an analytic function has surprising influence on the global properties of the function.

### Theorem 4.3.3: All Derivatives Zero

Let  $f$  be analytic on  $\Omega$  and  $a \in \Omega$ . If  $f^{(n)}(a) = 0$  for all  $n \geq 0$ , then  $f(z) = 0$  for all  $z \in \Omega$ .

*Proof.* We write

$$f(z) = f_n(z)(z - a)^n$$

according to Taylor's theorem. An estimation of  $f_n$  can be obtained by equation 4.29: If  $M = \max \{f(z) : z \in \partial D\}$  and the radius of  $\partial D$  is  $R$ , then we have

$$|f_n(z)| \leq \frac{1}{2\pi} \int_{\partial D} \frac{M |d\zeta|}{R^n (R - |z - a|)} = \frac{M}{R^n} \cdot \frac{R}{R - |z - a|}$$

Thus we have

$$|f(z)| \leq \left( \frac{|z - a|}{R} \right)^n \frac{MR}{R - |z - a|}, \quad \forall z \in D, \forall n \in \mathbb{N}$$

As  $|z - a|/R < 1$ , we have  $f(z) = 0$  for all  $z \in D$ .

To show that  $f(z) = 0$  for all  $z \in \Omega$ , we denote

$$E_1 = \{z \in \Omega : \forall n \in \mathbb{N}, f^{(n)}(z) = 0\}, \quad E_2 = \Omega - E_1$$

We have  $E_1$  being open, as for any  $z \in E_1$ , we can find a disk  $D$  such that  $f(z) = 0$  for all  $n \geq 0$ .  $E_2$  is also open for the derivatives of  $f$  is continuous. Therefore, the connectedness of  $\Omega$  implies that  $E_1 = \Omega$  as  $a \in E_1$ .  $\square$

**Local Properties of Zeros** Now, if  $f$  is not identically zero on  $\Omega$ , then if  $f(a) = 0$ , there exists a smallest  $n \in \mathbb{N}$  that  $f^{(n)}(a) \neq 0$ . We call  $a$  a **zero of  $f$  of order  $n$** . Now it is possible to write

$$f(z) = (z - a)^n f_n(z), \quad \text{where } f_n \text{ is analytic on } \Omega \text{ and } f_n(a) \neq 0 \quad (4.30)$$

*Remark:*

We can see that the zeros of a nontrivial analytic function has the same local properties as the polynomials, which can also be written in this form.

As  $f_n$  is continuous, there is a neighborhood  $D$  of  $a$  such that  $f_n(z) \neq 0$  for all  $z \in D$ . In that neighborhood,  $a$  is the only zero of  $f$ . Thus **All zeros of a nontrivial analytic function are isolated.**

An equivalent formulation is as follows:

$f, g$  are analytic on  $\Omega$ . If there is a set  $S \subseteq \Omega$  containing a limit point in  $S$ , such that  $f(z) = g(z)$  for all  $z \in S$ , then  $f(z) = g(z)$  for all  $z \in \Omega$ .

Particularly, if  $f$  is analytic on a subregion  $\Omega' \subseteq \Omega$ , and  $f(z) = 0$  holds in  $\Omega'$ , then  $f(z) = 0$  for all  $z \in \Omega$ . This is true for all subspaces  $\Omega'$  that do not reduce to points, like lines, curves, etc.

**Poles** If a function  $f$  is analytic on  $\Omega - \{a\}$ , then we say  $a$  is an **isolated singularity** of  $f$  on  $\Omega$ . We've already considered removable singularities. Since we can extend the function to a whole analytic function on  $\Omega$ , there is no need for further discussion.

Let  $a$  be an isolated singularity of  $f$ . If  $\lim_{z \rightarrow a} f(z) = \infty$ , then we say  $a$  is a **pole** of  $f$ . In a near neighborhood  $\Omega'$  of  $a$ , we have  $f(z) \neq 0$ , and the function  $g(z) = 1/f(z)$  is defined and analytic on  $\Omega' - \{a\}$ . As  $\lim_{z \rightarrow a} g(z) = 0$ , then  $a$  is a removable singularity of  $g$ , with  $g(a) = 0$ . The zero  $a$  has finite order  $h$ , with  $g(z) = (z - a)^h g_h(z)$ ,  $g_h(a) \neq 0$ . So we write

$$f(z) = (z - a)^{-h} f_h(z), \quad \text{where } f_h \text{ is analytic on } \Omega' \text{ and } f_h(a) \neq 0 \quad (4.31)$$

The number  $h$  is called the **order of the pole** of  $f$  at  $a$ . Similar to the zeros, we can see that the poles of a nontrivial analytic function is also isolated. (This is partly because our definition of poles require a limit process, which is not possible for a non-isolated singularity.)

*Remark:*

There are singularities that are not removable and not poles. The limit process near such singularities does not exist.

#### Definition 4.3.1: Meromorphic

A function  $f$  is called **meromorphic** on  $\Omega$  if it is analytic on  $\Omega$  except for some isolated singularities, which are all poles.

#### Theorem 4.3.4: Countable Poles for Meromorphic Functions

If  $f$  is meromorphic on  $\Omega$ , then the poles of  $f$  are at most countable and have no limit points in  $\Omega$ .

*Proof.* As poles are isolated, we can find a neighborhood  $U_z$  for each pole  $z$  that contains no other poles. As  $\Omega$  is the subset of  $\mathbb{C}$ , which is second countable, we can find a countable basis for  $\Omega$ . Take  $z \in B_z \subseteq U_z$ , where  $B_z$  is in the countable basis. Then each pole corresponds to a distinct element in the countable basis, so the poles are at most countable.  $\square$

#### Proposition: The Operations of Meromorphics

The sum, difference, product, and quotient of two non-trivial(not zero) meromorphic functions is also meromorphic. The composition of a meromorphic function with an analytic function is also meromorphic.

*Proof.* For any point  $z$ , we can write  $f(z) = (z - z_0)^n f_n(z)$ , where  $f_n(z_0) \neq 0$ , for a non-zero, non-pole point,  $n = 0$  would do. Substituting this for  $f, g$  it is easy to verify the result.  $\square$

To make a clear classification of isolated singularities, we consider the expressions (where  $\alpha \in \mathbb{R}$ )

1.  $\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = 0.$
2.  $\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = \infty.$

If  $f = 0$ , then 1 always holds, and 2 never holds.

If  $f$  is not trivial, and 1 holds for certain  $\alpha_0$ , then it holds for  $\forall \alpha \geq \alpha_0$ . Let  $m \geq \alpha_0$  be an integer, then  $(z - a)^m f(z)$  has a removable singularity that is a zero of finite order  $k \in \mathbb{Z}_+$ , so we have  $(z - a)^m f(z) = (z - a)^k f_h(z)$ , letting  $h = m - k$ , we have  $f(z) = (z - a)^{-h} f_h(z)$ , where  $f_h$  is analytic on  $\Omega'$  and  $f_h(a) \neq 0$ . Therefore, 1 holds for all  $\alpha > h$ , and 2 holds for all  $\alpha \leq h$ .

If 2 holds for certain  $\alpha_0$ , then it holds for  $\forall \alpha \leq \alpha_0$ . We have a similar discussion with the same result as above.

We conclude that there are three possibilities for an isolated singularity  $a$  of a nontrivial function  $f$ :

- (i) 1 holds for all  $\alpha \in \mathbb{R}$ , then  $f = 0$ .



- (ii) There is an integer  $h$  such that 1 holds for all  $\alpha > h$  and 2 holds for all  $\alpha \leq h$ . The integer  $h$  is called the **algebraic order** of the singularity. It is positive for a pole, negative for a zero, and zero for a non-zero, non-pole analytic point.
- (iii) Neither 1 nor 2 holds for any  $\alpha \in \mathbb{R}$ . Such singularities are called **essential isolated singularities**.

If  $a$  is a pole of order  $h$ , apply Taylor's theorem to the analytic function  $(z - a)^h f(z)$ , we have

$$(z - a)^h f(z) = B_h + B_{h-1}(z - a) + \cdots + B_1(z - a)^{h-1} + \varphi(z)(z - a)^h$$

where  $\varphi(z)$  is analytic at  $z = a$ . Thus, we have

$$f(z) = \frac{B_h}{(z - a)^h} + \frac{B_{h-1}}{(z - a)^{h-1}} + \cdots + \frac{B_1}{z - a} + \varphi(z) \quad (4.32)$$

The part  $\frac{B_h}{(z - a)^h} + \frac{B_{h-1}}{(z - a)^{h-1}} + \cdots + \frac{B_1}{z - a}$  is called the **singular part** of  $f$  at  $a$ .

The essential isolated singularity is more complicated. We can not write it in a similar form as above. The neighborhood of an essential singularity can come close to any value in  $\mathbb{C}$  and  $\infty$ , as we shall see.

#### Theorem 4.3.5: Weierstrass Theorem for Essential Isolated Singularities

Let  $f$  be analytic on  $\Omega - \{a\}$ , where  $a$  is an essential isolated singularity of  $f$ . Let  $c \in \mathbb{C}$  be any number. Then for any neighborhood  $\Omega'$  and any  $r > 0$ , there exists a point  $z_0 \in \Omega'$ ,  $z_0 \neq a$  such that

$$|f(z_0) - c| < r$$

and there exists a point  $z_1 \in \Omega'$ ,  $z_1 \neq a$  such that

$$|f(z_1)| > \frac{1}{r}$$

*Proof.* If it is not true, there exists  $A \in \mathbb{C}$ ,  $r > 0$  and a neighborhood  $\Omega'$  of  $a$  such that  $\forall z \in \Omega'$ ,  $z \neq a$ , we have  $|f(z) - A| \geq r$ . Then  $\forall \alpha < 0$ , we have

$$\lim_{z \rightarrow a} |z - a|^\alpha |f(z) - A| = \infty$$

Well,  $a$  is not a essential isolated singularity of  $f - A$ , so there exists  $\beta \in \mathbb{R}_+$  that

$$\lim_{z \rightarrow a} |z - a|^\beta |f(z) - A| = 0$$

As  $\lim_{z \rightarrow a} |z - a|^\beta |A| = 0$ , we have

$$\lim_{z \rightarrow a} |z - a|^\beta |f(z)| = 0$$

Contradicting that  $a$  is an essential isolated singularity of  $f$ .

The infinity part is similar, if  $|f(z)| \leq 1/r$ , then  $\forall \alpha > 0$ , we have  $\lim_{z \rightarrow a} |z - a|^\alpha |f(z)| = 0$ , contradicting again.  $\square$

**Treating Infinity as a Point** The notion of isolated singularity can also apply to  $\infty$ . The neighborhood of  $\infty$  is seen as an open region containing some  $\{z : |z| > R\}$ . We can transfer the neighborhood of  $\infty$  to the neighborhood of 0 by letting  $g(z) = f(\frac{1}{z})$ .

If we have  $g(z) = z^h g_h(z)$ , then  $f(z) = g(\frac{1}{z}) = z^{-h} f_h(z)$ , where  $f_h(z) = g_h(\frac{1}{z})$ . The classification of removable singularities, poles, and essential isolated singularities can be done similarly by consulting equations

$$\lim_{z \rightarrow \infty} \frac{|f(z)|}{|z|^\alpha} = 0, \quad \lim_{z \rightarrow \infty} \frac{|f(z)|}{|z|^\alpha} = \infty$$

If the singularity is non-trivial and non-essential, then there is a interger  $h$  such that the first limit holds for all  $\alpha > h$  and the second limit holds for all  $\alpha \leq h$ . The integer  $h$  is called the **algebraic order** of  $f$  at  $\infty$ . It is positive for a pole, negative for a zero, and zero for a non-zero, non-pole analytic point.

The singular part expansion and Weierstrass theorem can also be done similarly.

### 4.3.3 The Local Mapping

We want to determine the number of zeros of an analytic function. Consider  $f$  is an analytic function (not identically zero) on an open disk  $\Delta$ . Let  $\gamma$  be a closed curve in  $\Delta$  that  $\forall z \in \gamma, f(z) \neq 0$ . For simplicity we only consider  $f$  has finite zeros, denoted  $z_1, z_2, \dots, z_n$  with multiplicity equals order. So we can write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)g(z) \quad (4.33)$$

where  $g$  is analytic on  $\Omega$ , and  $\forall z \in \Omega, g(z) \neq 0$ . Taking the logarithm (analytic for a small neighborhood of each point), we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)} \quad (4.34)$$

for any  $z \neq z_j$  in  $\Delta$ , particularly for  $z \in \gamma$ . We can integrate this on  $\gamma$ . Since  $g(z) \neq 0$  on  $\Omega$ , from Cauchy's Theorem we have

$$\int_{\gamma} \frac{g'(z)}{g(z)} dz = 0$$

The definition of index yields

$$n(\gamma, z_1) + \cdots + n(\gamma, z_n) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz \quad (4.35)$$

If  $f$  has infinite zeros, then we choose a concentric smaller open disk  $\Delta'$  that  $\gamma \subseteq \Delta', \overline{\Delta'} \subseteq \Delta$  (This is possible due to normality). Then the compactness of  $\overline{\Delta'}$  and the isolation of zeros implies that  $f$  has only finitely many zeros in  $\overline{\Delta'}$ , and the above equation still holds, as the zeros outside do not contribute to the sum.

**Theorem 4.3.6: Number of Zeros**

Let  $f$  be analytic on an open disk  $\Delta$  (not identically zero), and  $z_j$  are distinct zeros of  $f$  that has order  $m_j$ . Then for every closed curve  $\gamma$  in  $\Delta$  that does not pass through any  $z_j$ , we have

$$\sum_j m_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz \quad (4.36)$$

The sum has only finite terms that are not zero.

The function  $w = f(z)$  maps  $\gamma$  into a closed curve  $\Gamma$  in the  $w$ -plane. And we have

$$\int_{\Gamma} \frac{dw}{w} = \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

Therefore, we have

$$n(\Gamma, 0) = \sum_j m_j n(\gamma, z_j) \quad (4.37)$$

*Remark:*

When  $\gamma$  is a circle, then all  $n(\gamma, z_j)$  are either 0 or 1, and the number of zeros are calculated by theorem 4.3.6.

To find the roots of  $f(z) = a$ , apply theorem 4.3.6 to the function  $g(z) = f(z) - a$ . The zeros of  $g$  are the roots of  $f(z) = a$ , denoted as  $z_j(a)$ , so we have

$$\sum_j m_j(a) n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_{\gamma} \frac{g'(z)}{g(z)} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - a} dz \quad (4.38)$$

$$n(\Gamma, a) = \sum_j m_j(a) n(\gamma, z_j(a)) \quad (4.39)$$

If  $a, b$  are in the same region determined by  $\Gamma$ , then  $n(\Gamma, a) = n(\Gamma, b)$ , so we have

$$\sum_j m_j(a) n(\gamma, z_j(a)) = \sum_j m_j(b) n(\gamma, z_j(b)) \quad (4.40)$$

**Theorem 4.3.7: Local Variation of Roots**

If  $f$  is analytic at a neighborhood of  $z_0$  and  $f(z_0) = w_0$ , and  $\overline{f(z)} - w_0$  has a zero of order  $m$  at  $z_0$ . Then  $\forall \epsilon > 0$  that  $z_0$  is the only zero of  $f(z) - w_0$  in  $\overline{N_{\epsilon}(z_0)}$ , there exists  $\delta > 0$  such that  $\forall w$  that  $|w - w_0| < \delta$ ,  $f(z) - w$  has exactly  $m$  zeros counting multiplicity in  $N_{\epsilon}(z_0)$ .

*Proof.* Letting  $\gamma$  be the circle  $\partial N_{\epsilon}(z_0)$ . Taking  $\delta$  such that  $N_{\delta}(w_0) \cap f(\gamma) = \emptyset$  would do.  $\square$

*Remark:*

If we take small enough  $\epsilon$ , then  $f'(z) \neq 0$  for all  $z \in N_{\epsilon}(z_0) - \{z_0\}$ . In this case,  $\forall a \in w_0$ , the  $m$  roots of  $f(z) = a$  are all simple (of order 1).

### Corollary 4.3.1: Image of Analytic Function on Open Sets

A nonconstant analytic function maps open sets onto open sets.

*Proof.* Let  $f$  be analytic on  $\Omega$  and nonconstant, and  $a \in \Omega$ . Let  $w_0 = f(a)$ , then  $f(z) - w_0$  has a zero of order  $n \geq 1$  at  $a$ . Choose  $\epsilon > 0$  such that  $N_\epsilon(a) \subseteq \Omega$  and  $f(z) - w_0$  has no other zeros in  $\overline{N_\epsilon(a)}$ . By theorem 4.3.7, there exists  $\delta > 0$  such that  $\forall w \in N_\delta(w_0)$ ,  $f(z) - w$  has exactly  $n$  zeros in  $N_\epsilon(a)$ . Therefore,  $N_\delta(w_0) \subseteq f(N_\epsilon(a))$ , so  $f(\Omega)$  is open.  $\square$

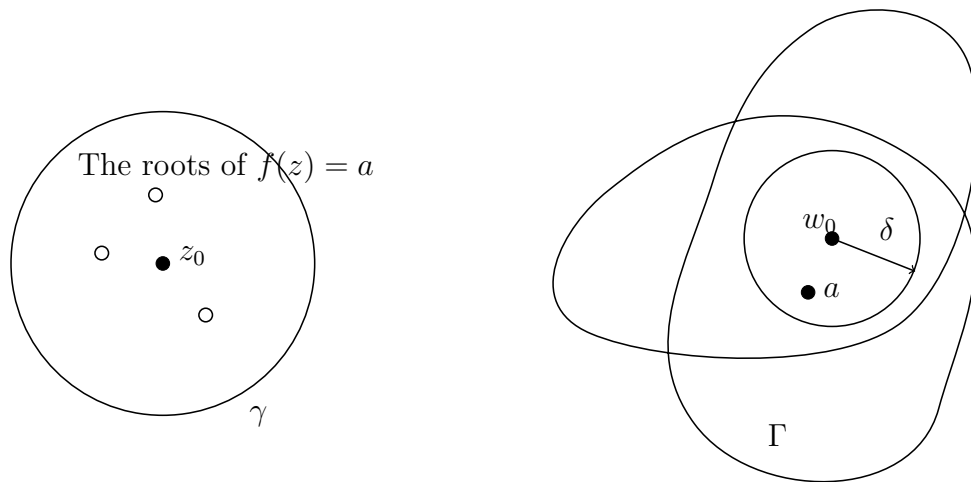


Figure 4.3: Local Mapping of an Analytic Function

If the order of the zero is  $m = 1$ , then  $f$  is 1-1 on  $f^{-1}(N_\delta(w_0)) \subseteq N_\epsilon(z_0)$ . This is an open set due to continuity of  $f$ . We can take a smaller  $\epsilon'$  such that  $N_{\epsilon'}(z_0) \subseteq f^{-1}(N_\delta(w_0))$ . Then  $f$  is injective on  $N_{\epsilon'}(z_0)$ , and thus a homeomorphism onto its image.

### Corollary 4.3.2: Local Homeomorphism

If  $f(z)$  is analytic at  $z_0$  with  $f'(z_0) \neq 0$ , then there exists a neighborhood  $N_\epsilon(z_0)$  such that  $f$  is conformal and is a homeomorphism from  $N_\epsilon(z_0)$  onto its image.

#### Remark:

Note that  $f$  is analytic at  $z_0$  means that it is analytic on a neighborhood of  $z_0$ . The condition  $f'(z_0) \neq 0$  is equivalent to the condition that  $f(z) - f(z_0)$  has a zero of order 1 at  $z_0$ .

This corollary can also be easily seen by the inverse function theorem, which states that if  $f$  is analytic at  $z_0$  and  $f'(z_0) \neq 0$ , then there exists a neighborhood  $N_\epsilon(z_0)$  such that  $f$  is a homeomorphism from  $N_\epsilon(z_0)$  onto its image. We also note that the inverse function theorem cannot apply to the case that  $f'(z_0) = 0$ , as in corollary 4.3.1.

The inverse mapping is also conformal due to the inverse function theorem.

For order  $n > 1$ , we write

$$f(z) - w_0 = (z - z_0)^n g(z)$$

where  $g$  is analytic at  $z_0$  and  $g(z_0) \neq 0$ . Choose  $\epsilon > 0$  such that  $g(z) \in N_\delta(g(z_0))$  for all  $z \in \Omega = N_\epsilon(z_0)$ , where  $\delta < |g(z_0)|$ . In this neighborhood we define an analytic branch of  $h(z) = \sqrt[n]{g(z)}$ . As  $g(\Omega)$  lies in a disk that do not contain zero, we do not need a branch cut.

$$f(z) - w_0 = \zeta^n(z), \quad \zeta(z) = (z - z_0)h(z)$$

As  $\zeta'(z_0) = h(z_0) \neq 0$ , then  $\zeta$  is a homeomorphism on a neighborhood of  $z_0$ . The function  $w = w_0 + \zeta^n$  is familiar.

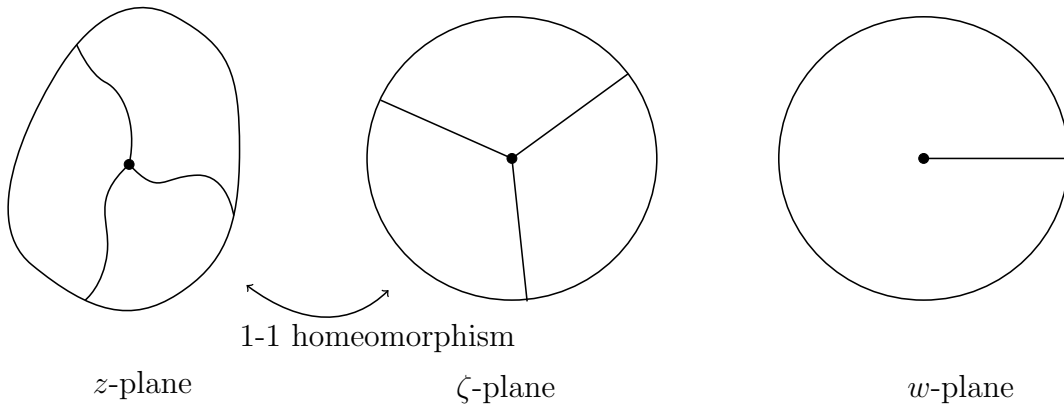


Figure 4.4: The Local Properties of Analytic Functions ( $n = 3$ )

#### 4.3.4 The Maximum Principle

##### Theorem 4.3.8: The Maximum Principle

If  $f$  is analytic and not constant in a region  $\Omega$ , then  $|f|$  has no maximum value in  $\Omega$ .

*Proof.* Assume  $z_0 \in \Omega$  and  $|f(z_0)| = M = \max_{z \in \Omega} |f(z)|$ . Then let  $D$  be a neighborhood of  $z_0$  such that  $D \subseteq \Omega$ . Then  $f(D)$  is a neighborhood of  $f(z_0)$ , and it contains some  $w_0$  such that  $|w_0| > M$ .  $\square$

A restatement of the maximum principle is as follows:

If  $f$  is analytic on a region  $\Omega$  and  $E$  is a closed bounded subset of  $\Omega$ , then  $|f|$  has a maximum value on  $E$ , which is taken on the boundary of  $E$ .

The existence is due to the compactness of  $E$ , and the continuity of  $f$ . The maximum principle is then used to show that the maximum value is not taken in the interior of  $E$ .

Another computational proof is as follows:

*Proof.* Let  $D$  be a small disk centered at  $z_0$  such that  $\overline{D} \subseteq \Omega$ . Then we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta$$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{i\theta})| d\theta$$

Suppose  $|f(z_0)|$  is a maximum, then  $f$  is constant on  $D$ . Which is a contradiction to the assumption that  $f$  is not constant in  $\Omega$ .  $\square$

Now consider that  $f$  is analytic on an open set  $D = \{z : |z| < R\}$  and continuous on  $\overline{D}$ . If  $|f| \leq M$  for  $z \in \partial D$ , then we have  $|f| \leq M$  for  $z \in D$ . There is a point  $z_0 \in D$  that  $|f(z_0)| = M$  iff  $f$  is a constant with norm  $M$ . So if we know that there is  $f(z_1) < M$ , we may want a better approximation.

### Theorem 4.3.9: Schwarz Lemma

Let  $f$  be analytic on  $D = \{z : |z| < 1\}$  and satisfies

- $f(0) = 0$ .
- $\forall z \in D, |f(z)| \leq 1$ .

Then  $|f(z)| \leq |z|$  and  $|f'(0)| \leq 1$ .

If  $(\exists z_0 \in D, z_0 \neq 0, |f(z_0)| = |z_0|) \vee |f'(0)| = 1$ , then  $\exists c \in \mathbb{C}, |c| = 1$  that  $f(z) = cz$ .

*Proof.* Letting

$$f_1(z) = \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

This can be thought of an extended function of  $f(z)/z$  thus is analytic on  $D$ . On  $\partial D$ , we have  $|f_1(z)| = |f(z)| \leq 1$ . By the maximum principle, we have  $|f_1(z)| \leq 1$  for all  $z \in D$ . Therefore,  $|f(z)| \leq |z|$  and  $|f'(0)| = |f_1(0)| \leq 1$ .

If either of the conditions hold, then  $f_1(z) = c$ .  $\square$

Generally speaking, if  $f$  is analytic on  $D = \{z : |z| < R\}$  and  $f(z) \leq M$ , and for some  $|z_0| < R$  we have  $f(z_0) = w_0, |w_0| < M$ . We can find a linear fractional transformation  $\zeta = Tz$  such that  $Tz_0 = 0$  and  $T(\partial D) = \{z : |z| = 1\}$  the unit circle. (This can be done by transforming  $z_0, z_0^*$  and some point on  $\partial D$  to  $0, \infty, 1$  respectively, where  $z_0^*$  is the symmetric point of  $z_0$  about the circle  $|z| = R$ .) Such a transformation can be written

$$Tz = \frac{R(z - z_0)}{R^2 - \overline{z_0}z}$$

Similarly, we take a linear fractional transformation  $\omega = Sw$  such that  $Sw_0 = 0$  and  $S(\{w : |w| = M\}) = \{w : |w| = 1\}$ . Then the function  $g(z) = Sf(T^{-1}z)$  satisfies the conditions of Schwarz lemma, so we have  $|g(z)| \leq |z|$ . Therefore, we have  $|Sf(T^{-1}z)| \leq |z|$  or  $|Sf(z)| \leq |Tz|$ , explicitly,

$$\left| \frac{M(f(z) - w_0)}{M^2 - \overline{w_0}f(z)} \right| \leq \left| \frac{R(z - z_0)}{R^2 - \overline{z_0}z} \right| \quad (4.41)$$

## 4.4 The General Cauchy Theorem

In previous discussions of Cauchy's theorem we've dealt with regions that are open disks. We now want to extend the theorem to more general regions, and open sets also (which may not be connected)

### 4.4.1 Chains and Cycles

#### Definition 4.4.1: Chains

Chains are the smallest Abelian group containing all directed curves in an open set  $\Omega \subseteq \mathbb{C}$ .

In the formulation, a chain can be written as

$$\gamma = a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n \quad (4.42)$$

where  $a_j \in \mathbb{Z}$ ,  $\gamma_j$  are directed curves. The zero and negative of the group is easy to find. An integral of a function  $f$  on a chain  $\gamma$  is defined as

$$\int_{\gamma} f(z)dz = a_1 \int_{\gamma_1} f(z)dz + a_2 \int_{\gamma_2} f(z)dz + \cdots + a_n \int_{\gamma_n} f(z)dz \quad (4.43)$$

it is clear that this definition is well-defined: different representations of the same chain would yield the same integral.

#### Definition 4.4.2: Cycles

A **cycle** is a chain that can be represented as a sum of closed curves.

A chain is a cycle if in its representation, the initial and terminal points of each directed curve appear the same number of times. A simple corollary is that *the integral of an exact differential on a cycle is zero*.

The index of a point with respect to a cycle is defined similarly to closed curves. We have the obvious additivity property:

$$n(\gamma, z) = a_1n(\gamma_1, z) + a_2n(\gamma_2, z) + \cdots + a_nn(\gamma_n, z) \quad (4.44)$$

### 4.4.2 Simple Connectedness

This notion formulates our intuition that there are no holes.

#### Definition 4.4.3: Simple Connectedness

A region  $\Omega \subseteq \mathbb{C}$  is called **simply connected** if  $\mathbb{C}_{\infty} - \Omega$  is connected in  $\mathbb{C}_{\infty}$ .

In this case, a parallel strip is simply connected, but the outside of a circle is not.

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*Remark:*

This is not a common definition of simple connectedness. The common definition is that any closed curve in  $\Omega$  can be contracted to a point in  $\Omega$ . The two definitions are equivalent in  $\mathbb{C}$ ,

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but not in general topological spaces.

### Theorem 4.4.1: Criterion for Simply Connectedness

A region  $\Omega$  is simply connected iff  $\forall a \notin \Omega$  and  $\forall$  cycles  $\gamma \subseteq \Omega$  we have  $n(\gamma, a) = 0$ .

*Proof.* The  $\Rightarrow$  direction is obvious, as  $\mathbb{C}_{\pm\infty} - \Omega$  is connected, and  $\infty$  belongs to it, then  $\mathbb{C}_{\pm\infty} - \Omega$  is in the unbounded region determined by  $\gamma$ , thus  $n(\gamma, a) = 0$ .

To prove the  $\Leftarrow$  direction, we need an explicit construction. We assume  $\mathbb{C}_{\pm\infty} - \Omega = A \sqcup B$  for two disjoint closed sets, and  $\infty \in B$  while  $A$  is bounded. Let  $\delta > 0$  be the distance of  $A$  and  $B$ . Let  $a \in A$ , and  $Q$  be a net of squares covering the whole  $\mathbb{C}$  with edge length  $< \delta/\sqrt{2}$ , and  $a$  is a center of a square  $Q_0 \in Q$ .

Consider the cycle

$$\gamma = \sum_{j \in J} \partial Q_j, \quad J = \{j : Q_j \in Q, Q_j \cap A \neq \emptyset\} \quad (4.45)$$

consisting of all squares that intersects  $A$  (the sum is finite due to boundedness of  $A$ .) As  $a$  is contained in just one square in  $\gamma$ , we have  $n(\gamma, a) = 1$ . Also, we have  $\gamma$  does not intersects  $B$ , as the distance between  $A$  and  $B$  is greater than the diagonal of each square.  $\gamma$  does not meet  $A$  either, as every edge of square that intersects  $A$  are counted twice with opposite directions. Therefore,  $\gamma \subseteq \Omega$ , contradicting the assumption.  $\square$

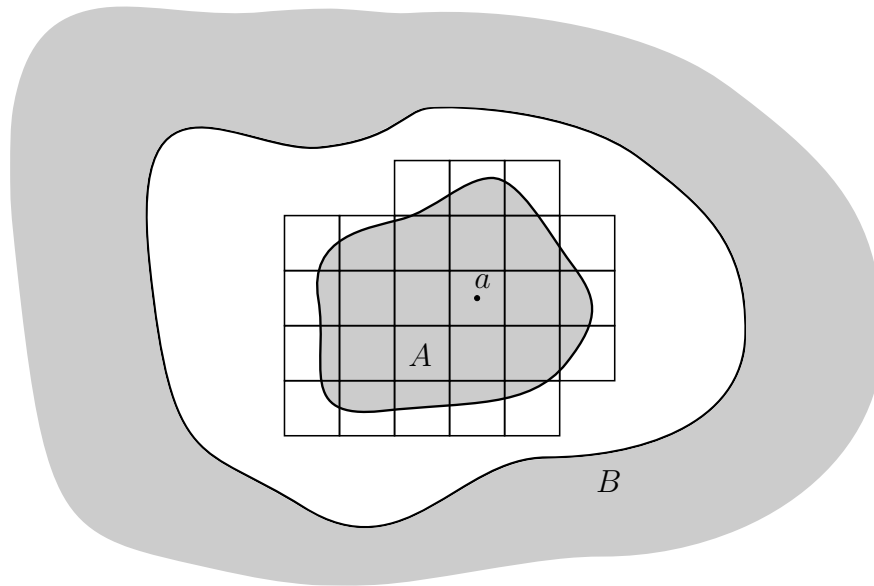


Figure 4.5: The Nets of Squares

Cauchy's theorem is not generally true for non-simply connected regions. For example, consider the function  $f(z) = 1/z$  on the region  $\Omega = \mathbb{C} - \{0\}$ . The function is analytic on  $\Omega$ , but for the



unit circle  $\gamma$ , we have

$$\int_{\gamma} f(z)dz = \int_{\gamma} \frac{1}{z}dz = 2\pi i$$

### 4.4.3 Homology

#### Definition 4.4.4: Homology

A cycle  $\gamma$  in an open set  $\Omega$  is said to be **homologous to zero** in  $\Omega$  if  $\forall a \notin \Omega$ , we have  $n(\gamma, a) = 0$ . Denoted as  $\gamma \sim 0(\text{mod } \Omega)$ .

Two cycles  $\gamma_1, \gamma_2$  in  $\Omega$  are said to be **homologous** in  $\Omega$  if  $\gamma_1 - \gamma_2 \sim 0(\text{mod } \Omega)$ . Denoted as  $\gamma_1 \sim \gamma_2(\text{mod } \Omega)$ .

For  $\Omega' \subseteq \Omega$ , it is obvious that  $\gamma \sim 0(\text{mod } \Omega')$  implies  $\gamma \sim 0(\text{mod } \Omega)$ . If  $\Omega$  is simply connected, then every cycle in  $\Omega$  is homologous to zero in  $\Omega$ .

### 4.4.4 The General Cauchy's Theorem

#### Theorem 4.4.2: The General Cauchy's Theorem

If  $f$  is analytic in an open set  $\Omega$  and  $\gamma$  is a cycle in  $\Omega$  that is homologous to zero in  $\Omega$ , then

$$\int_{\gamma} f(z)dz = 0 \quad (4.46)$$

Here are some frequently used corollaries:

- If  $f$  is analytic in a simply connected open set  $\Omega$ , then for all cycles  $\gamma$  in  $\Omega$ , we have  $\int_{\gamma} f(z)dz = 0$ .
- Every analytic function in a simply connected region is an exact differential. That is,  $\exists$  an analytic function  $F$  in  $\Omega$  such that  $F' = f$ .
- If  $f$  is analytic and  $\neq 0$  in a simply connected region  $\Omega$ , then it is possible to define a single-value branch of  $\log f(z)$  and  $\sqrt[n]{f(z)}$  in  $\Omega$ .

*Proof.* The first two are obvious, for the third statement, we have  $f'(z)/f(z)$  is analytic, so there is an analytic  $F$  in  $\Omega$  that  $F'(z) = f'(z)/f(z)$ . Thus we have

$$f(z)e^{-F(z)} \text{ has zero derivative so is constant.}$$

Let  $z_0 \in \Omega$  and choose one of the values of  $\log f(z_0)$ . We have

$$f(z_0)e^{-F(z_0)} = e^{\log f(z_0) - F(z_0)} = f(z)e^{-F(z)}$$

So we let

$$\log f(z) = F(z) - F(z_0) + \log f(z_0), \quad \sqrt[n]{f(z)} = \exp \frac{1}{n} \log f(z)$$

□

### 4.4.5 Proof of Cauchy's Theorem

We present a proof that is similar to the one of theorem 4.4.1.

First assume  $\Omega$  is a bounded open set. Given  $\delta > 0$ , we cover the whole  $\mathbb{C}$  with nets of squares. Denote  $Q_j, j \in J$  by those closed square regions  $Q_j \subseteq \Omega$ . Then the set  $J$  is finite. We denote

$$\Gamma_\delta = \sum_{j \in J} \partial Q_j, \quad \Omega_\delta = \text{Int} \bigcup_{j \in J} Q_j \quad (4.47)$$

Let  $\gamma$  be a circle homologous to zero in  $\Omega$ . Choose sufficiently small  $\delta$  that  $\gamma \subseteq \Omega_\delta$ . (This is possible for  $\gamma$  and  $\Omega^c$  are closed sets and have distance  $> 0$ .) Let  $\zeta \in \Omega - \Omega_\delta$ . (This is possible for  $\overline{\Omega_\delta} \subseteq \Omega$ .) Then  $\zeta \in Q$  for some  $Q \neq Q_j$ . Let  $\zeta_0 \in Q, \zeta_0 \notin \Omega$ . Draw a straight line  $\overline{\zeta\zeta_0}$  and it does not intersect  $\Omega_\delta$ . So we have  $n(\gamma, \zeta) = n(\gamma, \zeta_0) = 0$ . In particular, we have

$$\forall \zeta \in \Gamma_\delta, n(\gamma, \zeta) = 0$$

Suppose  $f$  is analytic in  $\Omega$ . Let  $z \in \text{Int } Q_{j_0}$  for some  $j_0 \in J$ . Then we have

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta) d\zeta}{\zeta - z} = \begin{cases} f(z), & \text{if } j_0 = j \\ 0, & \text{otherwise} \end{cases}$$

Thus we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta) d\zeta}{\zeta - z}$$

for every  $z$  in the interior of some square  $Q_j$ . As both side are continuous, we have this equation holds for all  $z \in \Omega_\delta$ . As a consequence, we have

$$\int_\gamma f(z) dz = \int_\gamma \left( \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta) d\zeta}{\zeta - z} \right) dz = \int_{\Gamma_\delta} \left( \frac{1}{2\pi i} \int_\gamma \frac{dz}{\zeta - z} \right) f(\zeta) d\zeta = 0$$

This interchange of integral is due to the continuity of both  $z$  and  $\zeta$  (Taking parametric form we can prove this using the real analysis theorem).

If  $\Omega$  is unbounded, replace it with a bounded subregion that contains  $\gamma$ , and we still have our result.

### 4.4.6 Local Exact Differential

#### Definition 4.4.5: Local Exact Differential

A differential form  $pdx + qdy$  is local exact in  $\Omega$  if  $\forall z \in \Omega$ , there is a neighborhood  $\Omega_z$  of  $z$  that  $pdx + qdy$  is an exact differential in  $\Omega_z$ .

A sufficient and necessary condition for local exactness is that  $\int_\gamma pdx + qdy = 0$  holds for all  $\gamma = \partial R$ , where  $R \subseteq \Omega$  is an open rectangle with edge parallel to the axes.

*Proof.* The sufficiency is obvious, taking a disk as the neighborhood. And define  $F$  similarly to the proof of theorem 4.1.4.

The necessity is also obvious, cover  $\overline{R}$  with each neighborhood  $U_x$  of  $x \in R$  that  $pdx + qdy$  is exact. As  $\overline{R}$  is compact, we can take a finite subcover  $U_1, \dots, U_n$ . Therefore, divide the rectangle  $R$  into enough small rectangles that each is fully in one of the  $U_j$ . Then we have our result.  $\square$

**Theorem 4.4.3: Locally Exact Implies Zero Cycle**

If  $pdx + qdy$  is locally exact in  $\Omega$ , then for any cycle  $\gamma \sim 0$  in  $\Omega$ , we have

$$\int_{\gamma} pdx + qdy = 0 \quad (4.48)$$

As we can see, the general Cauchy's theorem is a special case of this theorem. However, there is no obvious way to modify the proof of the general Cauchy's theorem to prove this theorem. So we would follow another way.

*Proof.* Without the loss of generality, we shall assume  $\Omega$  is bounded.

We first show that  $\gamma$  can be replaced by a polygon  $\sigma$ . Let the distance of  $\gamma$  to  $\Omega^c$  be  $\rho$ . Let  $\gamma : z = z(t), t \in [a, b]$ . Then  $z(t)$  is uniformly continuous on  $[a, b]$ . Take  $\delta > 0$  that

$$\forall t, t' \in [a, b], |t - t'| < \delta \rightarrow |z(t) - z(t')| < \rho.$$

Divide  $[a, b]$  into subintervals of length  $< \delta$ . Then each subarc  $\gamma_i$  of  $\gamma$  lies in a small disk  $\Delta_i$  of radius  $\rho$  that lies entirely in  $\Omega$ . Join the endpoints of each subarc  $\gamma_i$  to form a polygon  $\sigma_i$  with a vertical and a horizontal edge. Then  $\sigma_i \subseteq \Delta_i$  as well.

It is easy to see that  $pdx + qdy$  is exact in  $\Delta_i$  for each  $i$ . (This is a result by multivariable calculus). Thus

$$\int_{\sigma_i} pdx + qdy = \int_{\gamma_i} pdx + qdy$$

Taking the sum of all  $\sigma_i$ , we have

$$\int_{\sigma} pdx + qdy = \int_{\gamma} pdx + qdy$$

Now we extend each segment that form  $\sigma$  to a rectangular grid. There are finite rectangles  $R_i$  and unbounded rectangles(strips)  $R'_j$ . Choose ANY  $a_i \in R_i$  for all  $i$ , and we shall prove

$$\sigma = \sum_i n(\sigma, a_i) \partial R_i. \quad (4.49)$$

(Well this is quite obvious in an intuitive way, each rectangle inside contributes  $n(\sigma, a_i)$  times to the total cycle).

To prove this, we let  $\sigma_0 = \sum_i n(\sigma, a_i) \partial R_i$ . And let  $a'_j \in R'_j$ . It is clear that

$$n(\partial R_i, a_k) = \delta_{ik}, \quad n(\partial R_i, a'_j) = 0$$

Therefore, we have  $n(\sigma_0, a'_j) = 0$ . We also have  $n(\sigma, a_j) = 0$  for  $a_j$  belongs to the unbounded region determined by  $\sigma$ . So we have

$$n(\sigma - \sigma_0, a) = 0, \quad \forall a \text{ is in any } R_i \text{ or } R'_j$$

We shall conclude that  $\sigma = \sigma_0$  from here: Let  $\sigma_{ik}$  be the common side of  $R_i$  and  $R_k$ . Choose orientation so that  $R_i$  lies on the left of  $\sigma_{ik}$ . Suppose  $\sigma - \sigma_0$  contains  $c\sigma_{ik}$ . Then  $\sigma' = \sigma - \sigma_0 - c\partial R_i$  does not contain  $\sigma_{ik}$ , so we have  $n(\sigma', a_i) = n(\sigma', a_k)$ . Thus we have  $-c = 0$ . The same holds true for all  $R_i, R'_j$ .

Now we prove that all  $\overline{R_i}$  such that  $n(\sigma, a_i) \neq 0$  are contained in  $\Omega$ . If not so, take some  $a_i \in \overline{R_i} \cap \Omega^c$ , (then  $a_i$  do not lie on  $\sigma$ ) then take a very small neighborhood that  $b_i \in R_i$  and  $b_i, a_i$  are in the same region determined by  $\sigma$ . So we have  $n(\sigma, b_i) = n(\sigma, a_i) = 0$  for  $\sigma \sim 0$  in  $\Omega$ .

It is easy to show that  $pdx + qdy$  is exact in  $\overline{R_i}$ , so we have

$$\int_{\sigma} pdx + qdy = \sum_i n(\sigma, a_i) \int_{\partial R_i} pdx + qdy = 0$$

□

#### 4.4.7 Multiply Connected Regions

##### Definition 4.4.6: Multiply Connectedness

A region which is not simply connected is called **multiply connected**.

Let  $\Omega$  be an open set. If  $\mathbb{C}_{\infty} - \Omega$  has  $n$  components, then  $\Omega$  is said to be  **$n$ -connected**. If  $n = 1$ , then  $\Omega$  is simply connected. If  $\mathbb{C}_{\infty} - \Omega$  has infinitely many components, then  $\Omega$  is said to be **infinitely connected**.

In the case of finite connectedness, let  $A_1, \dots, A_n$  be the components of  $\mathbb{C}_{\infty} - \Omega$  and assume  $\infty \in A_n$ . If  $\gamma$  is any cycle in  $\Omega$ , then in each  $A_i$  we have  $n(\gamma, a_i)$  is constant for all  $a_i \in A_i$ , and  $n(\gamma, a) = 0$  for all  $a \in A_n$ .

Furthermore, duplicating the construction method used in the proof of theorem 4.4.1, we can find  $n - 1$  cycles  $\gamma_1, \dots, \gamma_{n-1}$  in  $\Omega$  such that  $n(\gamma_i, a_j) = \delta_{ij}$  for all  $a_j \in A_j$ . For a given cycle  $\gamma \in \Omega$ , let  $c_i = n(\gamma, a_i)$ . Now every point outside  $\Omega$  has zero index with respect to the cycle  $\gamma - \sum_{i=1}^{n-1} c_i \gamma_i$ . So we have

$$\gamma \sim \sum_{i=1}^{n-1} c_i \gamma_i \quad (\text{mod } \Omega) \quad (4.50)$$

As  $c_1 \gamma_1 + \dots + c_{n-1} \gamma_{n-1} \sim 0$  would imply  $c_1 = c_2 = \dots = c_{n-1} = 0$ , we have the uniqueness of this representation:

##### Theorem 4.4.4: The Representation of Cycle

Let  $\Omega$  be an  $n$ -connected region, and let  $\gamma_1, \dots, \gamma_{n-1}$  be cycles in  $\Omega$  such that  $n(\gamma_i, a_j) = \delta_{ij}$  for all  $a_j \in A_j$ . Then every cycle  $\gamma$  in  $\Omega$  can be uniquely represented as

$$\gamma \sim \sum_{i=1}^{n-1} c_i \gamma_i \quad (\text{mod } \Omega)$$

where  $c_i = n(\gamma, a_i)$  for some (and hence all)  $a_i \in A_i$ .

The  $\gamma_i$  are called the homology basis for the region  $\Omega$ .

Therefore, the general Cauchy's theorem would imply: for an analytic function  $f$  in an  $n$ -connected region  $\Omega$ , we have

$$\int_{\gamma} f(z) dz = \sum_{i=1}^{n-1} c_i \int_{\gamma_i} f(z) dz \quad (4.51)$$

We define

$$P_i = \int_{\gamma_i} f(z)dz \quad (4.52)$$

as long as  $\gamma_i$  satisfies theorem 4.4.4, then  $P_i$  does not depend on the choice of  $\gamma_i$ . We call  $P_i$  the **modulus of periodicity** of the differential  $f(z)dz$  with respect to the cycle  $\gamma_i$ , or the **periods** of the indefinite integral  $\int f(z)dz$ .

## 4.5 The Calculus of Residues

As it turns out, periods are likely easier to calculate.

### 4.5.1 The Residue Theorem

All theorems we derived from the Cauchy's Theorem on a disk is now valid in arbitrary regions for  $\gamma \sim 0$ .

#### Theorem 4.5.1: General Cauchy's Integral Formula

If  $f$  is analytic in an open set  $\Omega$  and  $\gamma$  is a cycle in  $\Omega$  that is homologous to zero in  $\Omega$ , then for all  $a \in \mathbb{C}, a \notin \gamma$ , we have

$$n(\gamma, a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz \quad (4.53)$$

*Remark:*

There is no need for a separate talk on exceptional points, for they are removable singularities which could be ignored.

Now we come a function  $f$  which is analytic in a region  $\Omega$  with finite isolated singularities  $a_1, \dots, a_n$ . Denote  $\Omega' = \Omega - \{a_1, \dots, a_n\}$ . For each  $a_i$ , there is a  $\delta_j > 0$  such that the doubly connected region  $0 < |z - a_j| < \delta_j$  is contained in  $\Omega'$ . Draw a circle  $C_j$  around each  $a_j$  with radius  $< \delta_j$ . Let

$$P_j = \int_{C_j} f(z)dz, \quad R_j = \frac{P_j}{2\pi i}$$

Thus the function

$$f(z) - \frac{R_j}{z - a_j}$$

has a vanishing period around  $C_j$ .

**Definition 4.5.1: Residue**

Let  $f$  be analytic in a region  $\Omega$  with isolated singularity  $a$ . The **residue** of  $f$  at  $a$  is the unique complex number  $R$  such that the function differential

$$\left(f(z) - \frac{R}{z-a}\right) dz$$

is exact in an annulus  $0 < |z-a| < \delta$  for some  $\delta > 0$ . Denoted  $R = \text{Res}_{z=a} f(z)$ .

If there are infinite many isolated singularities, there are only infinite pf them in a closed disk covering  $\gamma$ , so it doesn't matter.

**Theorem 4.5.2: The Residue Theorem**

Let  $f$  be analytic in a region  $\Omega$  with isolated singularities  $a_j, j \in J$ , and  $\gamma$  be a cycle in  $\Omega$  that does not pass  $a_j$ . Then

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{j \in J} n(\gamma, a_j) \text{Res}_{z=a_j} f(z) \quad (4.54)$$

And only a finite number of  $a_j$  contributes to the sum.

*Remark:*

Indeed, this result holds for every point set  $a_j$  as long as it contains the singularities, the non-singularity points would contribute zero of the sum.

There is no direct and universal way to find residues for essential singularities. However, for poles we can tell another story. If  $a$  is a pole, then

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + B_0 + \varphi(z).$$

where  $\varphi(z)$  is analytic in a neighborhood of  $a$  and  $B_h \neq 0$ . Well we can see that

$$\text{Res}_{z=a} f(z) = B_1 \quad (4.55)$$

for the remainder is certainly exact in a no-center neighborhood of  $a$ . For simple poles, we have

$$\text{Res}_{z=a} f(z) = \lim_{z \rightarrow a} (z-a)f(z) \quad (4.56)$$

*Remark:*

In presentations of Cauchy's theorem, the integral formula and the residue theorem which follow more classical lines, there is no mention of homology, nor is the notion of index used explicitly. Instead, the curve  $\gamma$  to which the theorems are applied is supposed to form the complete boundary of a subregion of  $\Omega$ , and the orientation is chosen so that the subregion lies to the left of  $\Omega$ . With the general point of view it is still possible to isolate the classical case. All that is needed is to accept the following definition:

**Definition 4.5.2: A Curve that Bounds a Region**

A cycle  $\gamma$  is said to bounded the region  $\Omega$  iff  $n(\gamma, a) = 1$  for all  $a \in \Omega$  and  $n(\gamma, a) = 0$  for all  $a \notin \Omega \wedge a \notin \gamma$ .

**4.5.2 The Argument Principle**

Cauchy's Integral Formula is a special case of the residue theorem, taking that the function  $f(z)/(z - a)$  has a simple pole at  $z = a$  with residue  $f(a)$ .

Another application is to determine the number of zeros of an analytic function in a region, as theorem 4.3.6 states, which we now generalize to meromorphic functions.

**Theorem 4.5.3: The Argument Principle**

Let  $f$  be a meromorphic function in a region  $\Omega$  with zeros  $a_j$  and poles  $b_k$ , with multiplicities  $m_j, n_k$ , then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_j m_j n(\gamma, a_j) - \sum_k n_k n(\gamma, b_k) \quad (4.57)$$

for any cycle  $\gamma \in \Omega, \gamma \sim 0$  which does not pass any zeros or poles.

*Proof.* The singularities of  $f'(z)/f(z)$  would lie in the singularities and zeros of  $f$ . For a zero  $a$  of order  $h$  we can write  $f(z) = (z - a)^h f_h(z)$ , and thus  $f'(z) = h(z - a)^{h-1} f_h(z) + (z - a)^h f'_h(z)$ . And we have

$$\frac{f'(z)}{f(z)} = \frac{h}{z - a} + \frac{f'_h(z)}{f_h(z)}$$

As  $f'_h(z)/f_h(z)$  is analytic around  $a$ , we have  $\text{Res}_{z=a} f'/f = h$ .

Taking  $h \rightarrow -h$  the result holds for poles. □

The left of equation (4.57) is  $n(\Gamma, 0)$  where  $\Gamma$  is the image  $f(\gamma)$ . We would have  $n(\Gamma, 0) = 0$  if  $\Gamma$  lies in a disk not containing the origin.

**Theorem 4.5.4: Rouché's Theorem**

Let  $\gamma \sim 0$  in  $\Omega$  and  $n(\gamma, z) = 0$  or  $1$  for  $\forall z \notin \gamma$ . Suppose  $f, g$  are analytic in  $\Omega$  and satisfy  $|f(z) - g(z)| < |f(z)|$  on  $\gamma$ . Then  $f, g$  has the same number of zeros enclosed by  $\gamma$ . (Enclosure means the zeros of  $f$  and  $g$  that has index 1 with respect to  $\gamma$ , counting multiplicities)

*Proof.* Well the assumption shows that  $f(z), g(z) \neq 0$  on  $\gamma$ . And we have

$$\left| \frac{g(z)}{f(z)} - 1 \right| < 1, \quad \forall z \in \gamma$$

Let  $F(z) = g(z)/f(z)$  then we have  $n(F(\gamma), 0) = 0$ , as the image curve lies on the disk radius 1 centered at 1. By the argument principle, we have our result. (That is,  $F$  has the same number of zeros and poles inside  $\gamma$ .) □

An application involves finding the number of zeros of  $f$  in a closed disk neighborhood of a point  $a$ . We approximate  $f$  by Taylor's Theorem:

$$f(z) = P_{n-1}(z - a) + (z - a)^n f_n(z)$$

where  $P_{n-1}$  is a polynomial of degree  $n - 1$  and  $f_n(z)$  is analytic in the disk radius  $R$ . If we have  $R^n |f_n(z)| < |P_{n-1}(z)|$  on  $|z - a| = R$ , then  $f$  and  $P_{n-1}$  has the same number of zeros inside the disk.

We can also generalize the argument principle. Take  $g$  be an analytic function in  $\Omega$  and we have

$$g(z) \frac{f'(z)}{f(z)} = \frac{hg(a)}{z-a} + g_1(z) + \frac{f'_n(z)}{f_n(z)} g(z)$$

Thus we have  $\text{Res}_{z=a} g(z)f'(z)/f(z) = hg(a)$  for a zero  $a$  of order  $h$ . For a pole  $b$  of order  $k$ , we have similar results.

### Theorem 4.5.5: Generalized Argument Principle

Let  $f$  be a meromorphic function in a region  $\Omega$  with zeros  $a_j$  and poles  $b_k$ , with multiplicities  $m_j, n_k$ . Let  $g$  be an analytic function in  $\Omega$ , then

$$\frac{1}{2\pi i} \int_{\gamma} g(z) \frac{f'(z)}{f(z)} dz = \sum_j m_j g(a_j) n(\gamma, a_j) - \sum_k n_k g(b_k) n(\gamma, b_k) \quad (4.58)$$

**The Representation of Inverse Functions** For an analytic function  $f$  in  $\Omega$  and  $f(z_0) = w_0$ , we have  $f(z) - w_0$  has a zero of order  $n$  at  $z_0$ . We know that for a small  $\epsilon, \delta > 0$ , the equation  $f(z) = w$  has  $n$  roots  $z_j(w)$  in the disk  $|z - z_0| < \epsilon$  for  $|w - w_0| < \delta$ . We apply  $g(z) = z^m$ , we have

$$\sum_{j=1}^n z_j(w)^m = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - w} z^m dz, \quad \forall |w - w_0| < \delta \quad (4.59)$$

where  $\gamma : |z - z_0| = \epsilon$ .

For  $j = 1, m = 1$  we have

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - w} z dz, \quad \forall |w - w_0| < \delta \quad (4.60)$$

Thus the power sums (elementary symmetric functions) of  $z_j(w)$  are analytic functions of  $w$  in a small neighborhood of  $w_0$ , thus we have  $z_j(w)$  is the roots of a polynomial

$$z^n + a_1(w)z^{n-1} + \cdots + a_{n-1}(w)z + a_n(w) = 0 \quad (4.61)$$

whose coefficients are analytic functions of  $w$  in a small neighborhood of  $w_0$ . The coefficients are given by elementary symmetric functions of  $z_j(w)$ .

### 4.5.3 The Definite Integrals

**The rational functions of  $\sin$  and  $\cos$**

$$I = \int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta \quad (4.62)$$

where  $R(x, y)$  is a rational function of  $x, y$ . We can write  $z = e^{i\theta}$  and thus

$$I = -i \int_{\gamma} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{z}, \quad \gamma : |z| = 1$$



**The rational infinite integrals**

$$I = \int_{-\infty}^{\infty} R(x) dx \quad (4.63)$$

where the degree of denominator of  $R(x)$  is at least 2 units greater than the degree of numerator, and we assume there is no poles on the real axis. We can take a segment  $(-\rho, \rho)$  and the semicircle on the upper half plane. As  $\rho \rightarrow \infty$  the integral on the semicircle vanishes, and we have

$$\int_{-\infty}^{\infty} R(x) dx = 2\pi i \sum_{y>0} \text{Res } R(z). \quad (4.64)$$

The same method also applies to

$$I = \int_{-\infty}^{\infty} R(x) e^{ix} dx = 2\pi i \sum_{y>0} \text{Res } R(z) e^{iz} \quad (4.65)$$

It is surprising to see that the same result holds for a simple zero at  $\infty$ .

**Theorem 4.5.6: Rational Infinite Integral with Exponential**

Let  $R(x)$  be a rational function of  $x$  with  $R(\infty) = 0$ , and there are no poles on the real axis. Then

$$\int_{-\infty}^{\infty} R(x) e^{ix} dx = 2\pi i \sum_{y>0} \text{Res } R(z) e^{iz} \quad (4.66)$$

*Proof.* We first prove the convergence of the integral. Consider

$$I(X_1, X_2) = \int_{-X_1}^{X_2} R(x) e^{ix} dx, \quad X_1, X_2 \in \mathbb{R}, -X_1 < X_2$$

for  $Y > 0$ , consider the rectangle

$$R = \{z = x + iy : -X_1 < x < X_2, 0 < y < Y\}$$

when  $X_1, X_2, Y$  are sufficiently large, this rectangle will cover all poles in the upper half plane. Also, we have  $|zR(z)|$  is bounded, so the integral over the right side would be

$$\left| \int_{X_2}^{X_2+iY} R(z) e^{iz} dz \right| \leq C \int_0^Y \frac{e^{-y}}{|X_2 + iy|} dy < \frac{C}{X_2} \int_0^Y e^{-y} dy < \frac{C}{X_2}$$

The same holds for the left side, and for the upper side, we have

$$\left| \int_{-X_1+iY}^{X_2+iY} R(z) e^{iz} dz \right| \leq C \int_{-X_1}^{X_2} \frac{e^{-Y}}{|x + iY|} dx < C \frac{X_1 + X_2}{Y} e^{-Y}$$

We also have the residue theorem

$$\int_{\partial R} R(z) e^{iz} dz = 2\pi i \sum_{y>0} \text{Res } R(z) e^{iz}$$

As  $Y \rightarrow \infty$ , the integral over the upper side would vanish, and we have

$$\left| \int_{-X_1}^{X_2} R(x)e^{ix} dx - 2\pi i \sum_{y>0} \text{Res } R(z)e^{iz} \right| < C \left( \frac{1}{X_1} + \frac{1}{X_2} \right)$$

This would imply that

$$\int_{-\infty}^{\infty} R(x)e^{ix} dx = 2\pi i \sum_{y>0} \text{Res } R(z)e^{iz}$$

□

Now, if  $R(z)$  has a simple pole at 0, we can adjust the rectangle a little to form a small semicircle of radius  $\delta$  at 0.

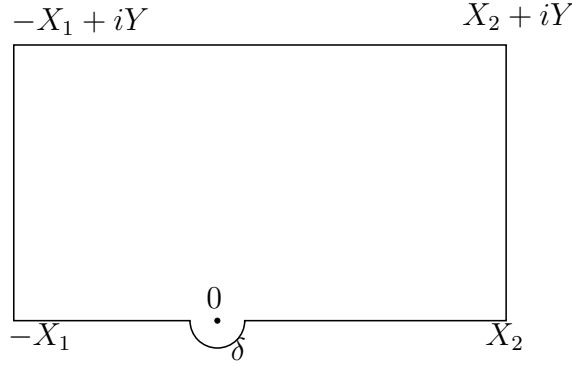


Figure 4.6: Rational Infinite Integral

If we write  $R(z)e^{iz} = \frac{B}{z} + R_0(z)$ , where  $R_0(z)$  is analytic in the neighborhood of 0, then the integration of the first term would be  $\pi i B$ , and the integration of the second term would tend to 0 as  $\delta \rightarrow 0$ . Thus we have

$$\lim_{\delta \rightarrow 0} \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} R(x)e^{ix} dx = 2\pi i \left( \sum_{y>0} \text{Res}_{y>0} R(z)e^{iz} + \frac{1}{2}B \right) \quad (4.67)$$

The limit on the left side is called the **Cauchy Principal Value** of the integral, denoted by P. V.  $\int_{-\infty}^{\infty} R(x)e^{ix} dx$ . So we have

$$\text{P. V.} \int_{-\infty}^{\infty} R(x)e^{ix} dx = 2\pi i \sum_{y>0} \text{Res}_{y>0} R(z)e^{iz} + \pi i \sum_{y=0} \text{Res}_{y=0} R(z)e^{iz} \quad (4.68)$$

---

**Example: Cauchy Principal Value**


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A direct application is

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i$$

Taking the imaginary part, we have

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$


---

Another type of interest is

$$I = \int_0^{\infty} x^{\alpha} R(x) dx, \quad 0 < \alpha < 1. \quad (4.69)$$

where  $R(z)$  has a zero of order  $\geq 2$  at  $\infty$ , and at most a simple pole at 0. (There are no poles on  $(0, \infty)$ ).

Take  $z \rightarrow t^2$ , we have

$$I = 2 \int_0^{\infty} t^{2\alpha+1} R(t^2) dt$$

As  $z^{2\alpha} = e^{2\alpha \text{Log } z}$ . We take the analytic branch, cutting the negative imaginary axis, and the imaginary part of  $\text{Log } z$  are taken to be in  $(-\pi/2, 3\pi/2)$ . Then we take the following cycle, with the small semicircle tends to 0 and the big semicircle tends to  $\infty$ . For a semicircle  $\gamma$  of radius  $r$ , we have  $|z^4 R(z^2)|$  is bounded as  $r \rightarrow \infty$ , and  $|z^2 R(z^2)|$  is bounded as  $r \rightarrow 0$ . Thus we have

$$\left| \int_{\gamma} z^{2\alpha+1} R(z^2) dz \right| \leq r^{2\alpha+1} \int_0^{\pi} \frac{C_1}{r^2} r d\theta = r^{2\alpha} \pi C_1 \rightarrow 0, \quad r \rightarrow 0$$

$$\left| \int_{\gamma} z^{2\alpha+1} R(z^2) dz \right| \leq r^{2\alpha+1} \int_0^{\pi} \frac{C_2}{r^4} r d\theta = r^{2\alpha-2} \pi C_2 \rightarrow 0, \quad r \rightarrow \infty$$

So we have

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) dz = 2\pi i \sum_{y>0} \text{Res}_{y>0} z^{2\alpha+1} R(z^2) \quad (4.70)$$

Also we have

$$\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) dz = \int_0^{\infty} (z^{2\alpha+1} + (-z)^{2\alpha+1}) R(z^2) dz = \frac{1}{2} (1 - e^{2\pi i \alpha}) I \quad (4.71)$$


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*Remark:*

We can also use the second cycle in figure 4.7 to the original integral with slight change in adjustment and justification, and the result would be the same.

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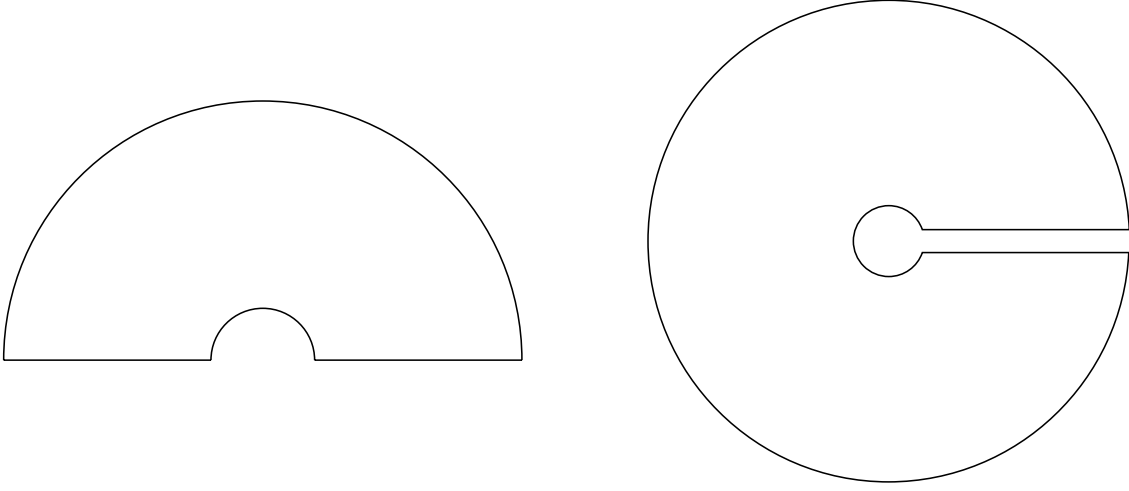


Figure 4.7: Infinite Rational Integral with Power

### The logarithmic integrals

$$I = \int_0^\pi \log(\sin \theta) d\theta \quad (4.72)$$

Consider the expression

$$1 - e^{2iz} = -2ie^{iz} \sin z = 1 - e^{-2y}(\cos 2x + i \sin 2x)$$

it falls on  $(-\infty, 0]$  when  $x = n\pi, y \leq 0$ . Omitting this lines, we have an analytic branch (the principal branch) for the function  $\log(1 - e^{2iz})$ . Let

$$R = \{z = x + iy : 0 < x < \pi, 0 < y < Y\}$$

And let  $\gamma = \partial R$ , with the vertices at  $0, \pi$  changed to small quarter circles of radius  $\delta$  at  $0$  and  $\pi$ . As  $Y \rightarrow \infty$ , the integral on the upper side tends to 0, and that

$$\lim_{z \rightarrow 0} \left| \frac{1 - e^{2iz}}{z} \right| = 2$$

which means that the integral  $\log(1 - e^{2iz})$  of quarter circles near  $0, \pi$  would behave like  $\delta \log \delta$  which tends to 0. Also, the side edge integral vanishes due to periodicity. Thus we have

$$\int_0^\pi \log(-2ie^{iz} \sin z) dz = 2\pi i \sum_{y>0, 0<x<\pi} \text{Res} \log(1 - e^{2iz}) = 0 \quad (4.73)$$

We have  $\log(-2ie^{ix}) = \log 2 + ix - \pi i/2$  (lying in the principal branch) so we have

$$\int_0^\pi \log(\sin x) dx = - \int_0^\pi \log(-2ie^{ix}) dx = -\pi \log 2 \quad (4.74)$$

**The Bergman Kernel** Sometimes complex integration can be used to evaluate area integrals.

**Theorem 4.5.7: Bergman's Kernel Formula**

If  $f$  is analytic and bounded for the unit disk  $|z| < 1$ , and let  $|\zeta| < 1$ , then we have

$$f(\zeta) = \frac{1}{\pi} \int_{|z|<1} \frac{f(z)}{(1 - \bar{z}\zeta)^2} dx dy \quad (4.75)$$

*Proof.* Write it in polar coordinates, we have

$$\text{RHS} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{f(re^{i\theta})}{(1 - re^{-i\theta}\zeta)^2} r dr d\theta = 2 \int_0^1 \left( \frac{1}{2\pi i} \int_{|z|=r} \frac{zf(z)}{(z - r^2\zeta)^2} dz \right) r dr$$

As  $|r^2\zeta| < |r|$ , we have

$$\text{RHS} = 2 \int_0^1 (f(r^2\zeta) + r^2\zeta f'(r^2\zeta)) r dr = r^2 f(r^2\zeta) \Big|_0^1 = \text{LHS}$$

□

## 4.6 Harmonic Functions

### 4.6.1 Definition and Basic Properties

**Definition 4.6.1: Harmonic Function**

A function  $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  defined on a region  $\Omega$  is called **harmonic** if it is twice continuously differentiable and satisfies the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (4.76)$$

In polar coordinates, the Laplace equation becomes

$$r^2 \Delta u = r \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (4.77)$$

This would imply that any analytic function depending only on  $r$  has the form  $u(r) = A + B \log r$ .

If  $u$  is harmonic in  $\Omega$ , then

$$f(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \quad (4.78)$$

is analytic in  $\Omega$ , according to theorem 2.1.4.

We now pass to the differential form

$$f(z) dz = \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + i \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right)$$

The first part is just  $du$ , and the second part is local exact, which potential is the conjugate harmonic function  $v$  of  $u$ .

*Remark:*

In a simple connected region  $\Omega$ , we can always find a harmonic conjugate  $v$  of  $u$  such that  $f(z)dz = du + idv$ . In this case, we have  $F = u + iv$  is analytic in  $\Omega$ , and  $f(z)dz = dF$ .

However in general, the second part is not exact (if  $\Omega$  is not single valued especially), we denote

$$*du = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

called the conjugate differential of  $u$ . Then we have

$$f dz = du + i *du \quad (4.79)$$

According to theorem 4.4.3, for any cycle  $\gamma \sim 0$  in  $\Omega$ , we have

$$\int_{\gamma} f dz = \int_{\gamma} du + i \int_{\gamma} *du = 0$$

which is consistent with Cauchy's theorem. Also, this would imply

$$\int_{\gamma} \frac{\partial u}{\partial n} |dz| = 0 \quad (4.80)$$

which is easily proved by the divergence theorem, where  $\partial u / \partial n$  is the normal derivative of  $u$  on  $\gamma$ .

In a simply connected region  $\Omega$ , we can define the harmonic conjugate  $v$  of  $u$  without difficulty. In multiple connected regions, the conjugate functions has periods:

$$\int_{\gamma_i} *du = \int_{\gamma_i} \frac{\partial u}{\partial n} |dz|$$

where  $\gamma_i$  are cycles in a homology basis.

#### Theorem 4.6.1: Pair Harmonic Functions

If  $u_1, u_2$  are harmonic in  $\Omega$ , and  $\gamma \sim 0$  in  $\Omega$ , then

$$\int_{\gamma} u_1 *du_2 - u_2 *du_1 = 0 \quad (4.81)$$

*Proof.* The simplest proof is by calculation, we have

$$\begin{aligned} u_1 *du_2 - u_2 *du_1 &= u_1 \left( -\frac{\partial u_2}{\partial y}dx + \frac{\partial u_2}{\partial x}dy \right) - u_2 \left( -\frac{\partial u_1}{\partial y}dx + \frac{\partial u_1}{\partial x}dy \right) \\ &= \left( -u_1 \frac{\partial u_2}{\partial y} + u_2 \frac{\partial u_1}{\partial y} \right) dx + \left( u_1 \frac{\partial u_2}{\partial x} - u_2 \frac{\partial u_1}{\partial x} \right) dy \end{aligned}$$

This is certainly a local exact differential, as we have

$$\frac{\partial}{\partial y} \left( -u_1 \frac{\partial u_2}{\partial y} + u_2 \frac{\partial u_1}{\partial y} \right) - \frac{\partial}{\partial x} \left( u_1 \frac{\partial u_2}{\partial x} - u_2 \frac{\partial u_1}{\partial x} \right) = -u_1 \frac{\partial^2 u_2}{\partial y^2} + u_2 \frac{\partial^2 u_1}{\partial y^2} - u_1 \frac{\partial^2 u_2}{\partial x^2} + u_2 \frac{\partial^2 u_1}{\partial x^2} = 0$$

□

### 4.6.2 The Mean Value Theorem

Apply theorem 4.6.1 to the pair  $u_1(z) = \log r$  and  $u_2(z) = u(z)$ , where  $u$  is harmonic in  $|z| < \rho$ . We choose  $\Omega = \{z : 0 < |z| < \rho\}$  and  $\gamma = C_1 - C_2$  where  $C_i : |z| = r_i < \rho$ . Thus

$$\log r_1 \int_{C_1} r_1 \frac{\partial u}{\partial r} d\theta - \int_{C_1} u d\theta = \log r_2 \int_{C_2} r_2 \frac{\partial u}{\partial r} d\theta - \int_{C_2} u d\theta$$

Therefore, we have

$$\log r \int_{|z|=r} r \frac{\partial u}{\partial r} d\theta - \int_{|z|=r} u d\theta = \text{const} \quad \forall r < \rho \quad (4.82)$$

If we take  $u_1 = 1, u_2 = u$ , we have

$$\int_{|z|=r} r \frac{\partial u}{\partial r} d\theta = \text{const}, \quad \forall r < \rho \quad (4.83)$$

#### Theorem 4.6.2: Mean Value of a Harmonic Function

Let  $u$  be a harmonic function in an annulus  $r_1 < |z| < r_2$ , then the arithmetic mean of  $u$  is

$$\frac{1}{2\pi} \int_{|z|=r} u d\theta = \alpha \log r + \beta. \quad (4.84)$$

where  $\alpha, \beta \in \mathbb{R}$  are constants. If  $u$  is harmonic in the whole disk  $r < \rho$ , then  $\alpha = 0, \beta = u(0)$ . If  $u$  is harmonic in the disk  $B(z_0, r_0)$ , then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta \quad (4.85)$$

*Proof.* This can also be directly proved by the proof of theorem 4.3.8. □

*Remark:*

For a slightly different case, if  $u$  is harmonic in  $r < \rho$ , and continuous in  $r \leq \rho$ , then the theorem also holds:

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\theta}) d\theta$$

The proof uses the uniform continuity due to the close disk.

#### Theorem 4.6.3: The Maximum Principle for Harmonic Functions

A nonconstant harmonic function  $u$  is defined on a region  $\Omega$ , then  $u$  does not have a maximum or minimum value on  $\Omega$ .

*Proof.* Using the similar method proving maximum value theorem of analytic functions 4.3.8 would do. Let  $A = \{z : u(z) = \max u\}$ , and  $B = \Omega - A$ . Then  $\forall z \in A$ , take a small disk and the mean value theorem says that  $u$  is constant in the disk, so  $A$  is open. As  $u$  is continuous,  $B = u^{-1}((-\infty, \max u))$  is also open, which contradicts to the connectedness of  $\Omega$ . □

### 4.6.3 Poisson's Formula

#### Theorem 4.6.4: The Uniqueness Theorem of Dirichlet Boundary

Let  $u, v$  be harmonic functions in a region  $\Omega$ , and continuous on  $\bar{\Omega}$ . Then if  $u = v$  on the boundary  $\partial\Omega$ , then  $u = v$  in  $\Omega$ .

*Proof.* We have  $u - v$  is harmonic in  $\Omega$  and continuous on  $\bar{\Omega}$ . The maximum and minimum of  $u - v$  takes place on  $\partial\Omega$ , which is constant in  $\Omega$ . Thus  $u = v$  in  $\Omega$ .  $\square$

We now want to solve  $u$  if its boundary value are given. For simplicity we consider disks first.

- First, we can use the mean value theorem to evaluate the center.
- Use a linear transform to carry any point to the center. Let

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}. \quad (4.86)$$

mapping  $|\zeta| \leq 1 \mapsto |z| \leq R, 0 \mapsto a$ . Now  $u(S(\zeta))$  is harmonic in  $|\zeta| < 1$ , and continuous on  $|\zeta| \leq 1$ . Thus

$$u(a) = \frac{1}{2\pi} \int_{|\zeta|=1} u(S(\zeta)) d\theta_\zeta$$

From

$$\zeta = \frac{R(z - a)}{R^2 - \bar{a}z}$$

we have

$$d\theta_\zeta = -i \frac{d\zeta}{\zeta} = -i \left( \frac{1}{z - a} + \frac{\bar{a}}{R^2 - \bar{a}z} \right) dz = \left( \frac{z}{z - a} + \frac{\bar{a}z}{R^2 - \bar{a}z} \right) d\theta_z$$

From  $R^2 = z\bar{z}$  we have

$$d\theta_\zeta = \frac{R^2 - |a|^2}{|z - a|^2} d\theta_z$$

#### Theorem 4.6.5: Poisson's Formula

Suppose  $u$  is harmonic for  $|z| < R$  and continuous on  $|z| \leq R$ , then for any  $a \in \mathbb{C}$  with  $|a| < R$ , we have

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} u(z) \frac{R^2 - |a|^2}{|z - a|^2} d\theta_z = \frac{1}{2\pi} \int_{|z|=R} \operatorname{Re} \frac{z + a}{z - a} u(z) d\theta_z \quad (4.87)$$

Poisson's Formula also gives us a explicit representation of the conjugate harmonic function  $v$  of  $u$  in the disk  $|z| < R$ : as

$$u(z) = \operatorname{Re} \left( \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} \right)$$

We have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC, \quad C \in \mathbb{R} \quad (4.88)$$



is an analytic function, and  $f = u + iv$ . This is known as the Schwarz integral formula.

As a special case, take  $u = 1$ , we have

$$\int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} d\theta_z = 2\pi \quad (4.89)$$

#### 4.6.4 Schwarz's Theorem

We can relax the continuity condition on the boundary a little. As long as  $u$  has sufficient regularity (say piecewise continuity) on the boundary, the integral in Poisson's formula still makes sense. We then define  $u = \operatorname{Re} f$  as in the Schwarz integral formula. Then  $u$  is harmonic in  $|z| < R$ . The question whether  $u$  matches the boundary condition or not is answered by the following theorem.

For simplicity, choose  $R = 1$ , define  $U(\theta)$ ,  $\theta \in [0, 2\pi)$  to be a piecewise continuous real function, and let

$$P_U(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z} U(\theta) d\theta, \quad |z| < 1 \quad (4.90)$$

Then  $U \mapsto P_U(z)$  is a linear positive functional given that  $U > 0 \rightarrow P_U(z) > 0$ .

As  $P_c = c$ , we have

$$U(\theta) \in [m, M] \rightarrow P_U(z) \in [m, M] \quad (4.91)$$

##### Theorem 4.6.6: Schwarz's Theorem

Let  $U(\theta)$  be a piecewise continuous real function on  $[0, 2\pi]$ , and let  $P_U(z)$  be defined as above. Then for any  $\theta_0 \in [0, 2\pi]$  where  $U$  is continuous, we have  $P_U(z)$  is harmonic in  $|z| < 1$ , and

$$\lim_{z \rightarrow e^{i\theta_0}} P_U(z) = U(\theta_0) \quad (4.92)$$

*Proof.* We've already proved that  $P_U(z)$  is harmonic in  $|z| < 1$ . Now we need to show the limit.

- Decompose the unit circle to open arcs  $\overline{C_1} \sqcup \overline{C_2}$ , and let  $U_i = U|_{C_i}$ . Then  $P_U = P_{U_1} + P_{U_2}$ .
- $P_{U_1}$  is harmonic on  $\mathbb{C} - C_1$ , being the real part of an analytic function. And we also have

$$\operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}.$$

So  $P_{U_1}(z) = 0$  for  $z \in C_2$ . Through continuity we conclude that  $P_{U_1}(z) \rightarrow 0$  as  $z \rightarrow e^{i\theta_0}$ , where  $e^{i\theta_0} \in C_2$ .

- Suppose  $U(\theta_0) = 0$ , otherwise we can replace  $U$  by  $U - U(\theta_0)$ . Given  $\epsilon > 0$ , find  $C_1, C_2$  that  $e^{i\theta_0} \in C_2$  and  $\forall e^{i\theta} \in C_2, |U(\theta)| < \epsilon/2$ . Due to equation 4.91, we have  $|P_{U_2}(z)| < \epsilon/2$  for  $|z| < 1$ . The continuity of  $P_{U_1}$  near  $e^{i\theta_0}$  implies that  $\exists \delta > 0$  such that  $|P_{U_1}(z)| < \epsilon/2$  for  $|z - e^{i\theta_0}| < \delta$ . Thus we have

$$|P_U(z) - U(\theta_0)| = |P_{U_1}(z) + P_{U_2}(z) - U(\theta_0)| < \epsilon, \quad |z - e^{i\theta_0}| < \delta$$

which implies the limit.

□

**A Geometric Interpretation of Poisson's Formula** Let  $\theta^*$  be such that on the unit circle  $e^{i\theta}, z, e^{i\theta^*}$  are collinear. Then

$$P_U(z) = \frac{1}{2\pi} \int_0^{2\pi} U(\theta) d\theta^* = \frac{1}{2\pi} \int_0^{2\pi} U(\theta^*) d\theta \quad (4.93)$$

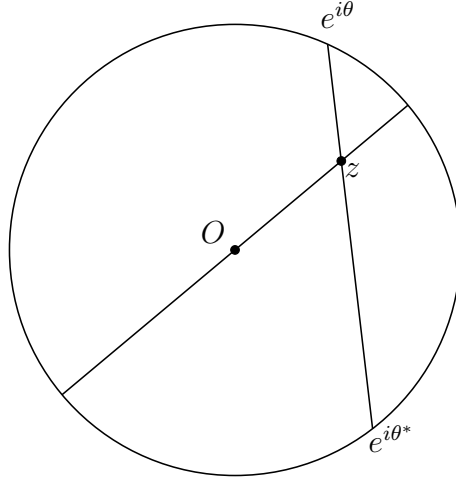


Figure 4.8: Geometry of Poisson's Formula

### 4.6.5 The Reflection Principle

We notice that if  $u(z)$  harmonic, then  $u(\bar{z})$  is also harmonic, and if  $f(z)$  is analytic, then  $\overline{f(\bar{z})}$  is also analytic.

*Proof.* This is quite straightforward. Let  $g(z) = \overline{f(\bar{z})}$ , then

$$g(z + \Delta z) - g(z) = \overline{f(\bar{z} + \overline{\Delta z})} - \overline{f(\bar{z})} = \overline{f'(\bar{z})\overline{\Delta z} + o(|\Delta z|)} = \overline{f'(\bar{z})}\Delta z + o(|\Delta z|)$$

which implies that  $g'(z) = \overline{f'(\bar{z})}$ . If  $u = \operatorname{Re} f$  then  $u(\bar{z}) = \operatorname{Re} g(z)$  so  $u(\bar{z})$  is harmonic.  $\square$

This can be simply interpreted as flipping both  $\Omega$  and the range about the real axis. Let  $\Omega^* = \{\bar{z} : z \in \Omega\}$ .

Consider when  $\Omega = \Omega^*$ , as  $\Omega$  is connected, it must contain some open interval  $I \subseteq \mathbb{R}$ . Assume  $f$  is real on  $I$ . As  $f(z) - \overline{f(\bar{z})}$  is analytic on  $I$ , which implies that  $f(z) = \overline{f(\bar{z})}$  on  $\Omega$ . Setting  $f = u + iv$ , we have  $u(z) = u(\bar{z})$  and  $v(z) = -v(\bar{z})$ .

**Theorem 4.6.7: The Reflection Principle**

Let  $\Omega$  be a symmetric region  $\Omega = \Omega^*$ , and  $\Omega^+ = \Omega \cap \{z : \text{Im } z > 0\}$  and  $\sigma = \Omega \cap \mathbb{R}$ . Suppose  $v$  is continuous in  $\Omega^+ \cup \sigma$ , harmonic in  $\Omega^+$ , and  $v = 0$  on  $\sigma$ . Then  $v$  can be extended to a harmonic function in  $\Omega$  by setting  $v(\bar{z}) = -v(z)$ .

If  $v$  is the imaginary part of an analytic function  $f$  in  $\Omega^+$ , then  $f$  has an analytic extension to  $\Omega$  by setting  $f(\bar{z}) = \overline{f(z)}$ .

*Proof.* We construct  $V(z)$  as the theorem states.

$$V(z) = \begin{cases} v(z), & z \in \Omega^+ \\ 0, & z \in \sigma \\ -v(\bar{z}), & z \in \Omega^- \end{cases}$$

and prove that  $V$  is harmonic in  $\Omega$ . For  $x_0 \in \sigma$  consider a disk  $\Delta \subseteq \Omega$  centered at  $x_0$ . Then let  $P_V$  be the Poisson Integral with boundary values of  $V$  on  $\partial\Delta$ . We have  $V - P_V$  being harmonic on  $\Delta \cap \Omega^+$ , and  $V(z) - P_V(z) = 0$  on  $\sigma$  due to obvious symmetry. Considering the upper half disk, we have a boundary 0 Dirichlet problem, which shows that  $V = P_V$  on the upper half disk, so  $V = P_V$  on  $\Delta$ , therefore,  $V$  is harmonic at  $x_0$ .

The next part of the theorem follows smoothly, as choosing a conjugate function on the disk that coincides with  $v$  on  $\Omega^+$  would do.  $\square$

Taking linear fractional transformations (to the domain and image), we can generalize the theorem to symmetry with respect to circles and other lines.

**Theorem 4.6.8: Generalized Reflection Principle**

Let  $C, C'$  be circles or lines in  $\mathbb{C}$ . Let  $\Omega$  be a region symmetric with respect to  $C$ ,  $\Omega = \Omega^*$ , and  $\Omega^+, \Omega^-$  be two sides of  $\Omega$  to  $C$ . If  $f$  is analytic in  $\Omega^+$ , continuous on  $\overline{\Omega^+}$ , and  $f(C) \subseteq C'$ , then  $f$  can be extended to a function  $F$  in  $\Omega$  by setting  $f(z^*[C]) = f(z)^*[C']$ . (where  $z^*[C]$  is the reflection of  $z$  with respect to  $C$ ).



# Chapter 5

## Series and Product Developments

A useful tool to explicitly represent an analytic function.

### 5.1 Power Series

#### 5.1.1 Weierstrass' Theorem

In analysis, uniform convergence is crucial for studying the regularity properties of a limit function.

**The Domain** We now have a sequence of analytic functions  $\{f_n\}$  defined on  $\Omega_n$ . As we want to consider the limit function  $f$  in  $\Omega$ , it would make sense that

$$\forall z \in \Omega, \exists n_0 \in \mathbb{N}, \forall n > n_0, z \in \Omega_n.$$

A typical case is that  $\Omega_n \subsetneq \Omega_{n+1}$  for all  $n \in \mathbb{Z}_+$  and  $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ . In this way, however, the convergence is not uniform as no  $f_n$  is defined on  $\Omega$ . So we consider a weaker assumption, but still strong enough: The inner compact uniform convergence.

#### Theorem 5.1.1: The Uniform Convergence Implies Analyticality

Let  $\{f_n\}$  be a sequence of analytic functions defined on regions  $\Omega_n$  with limit function  $f(z) = \lim_{n \rightarrow \infty} f_n(z)$  defined on  $\Omega$ . If for every compact set  $K \subset \Omega$ ,  $f_n$  is uniformly convergent on  $K$ , then  $f$  is analytic on  $\Omega$ .

Moreover,  $f'_n(z)$  converges uniformly to  $f'(z)$  on every compact  $K \subseteq \Omega$ .

*Proof.* Let  $\Delta : |z - a| \leq r$  be a closed disk in  $\Omega$ , then for enough large  $n$ , we have  $\Delta \subseteq \Omega_n$ , for any closed curve  $\gamma$  in  $\Delta$ , we have

$$\int_{\gamma} f(z) dz = \lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = 0.$$

due to uniform convergence. Thus, by Morera's theorem,  $f$  is analytic on  $\Delta$ . Since  $\Delta$  is arbitrary, we conclude that  $f$  is analytic on  $\Omega$ .

Explicitly, we can write

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)}{\zeta - z} d\zeta,$$

where  $C : |\zeta - a| = r$  and  $|z - a| < r$ . Due to uniform convergence, we can exchange the limit and the integral:

$$f(z) = \lim_{n \rightarrow \infty} f_n(z) = \frac{1}{2\pi i} \int_C \lim_{n \rightarrow \infty} \frac{f_n(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

showing  $f$  is analytic in the disk. Following the same route, we have

$$f'_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta,$$

So we have

$$\lim_{n \rightarrow \infty} f'_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta = f'(z). \quad (5.1)$$

In a compact set, convergence is uniform, thus we have finished the proof.  $\square$

Repeated applications suggest that  $f_n^{(k)}(z)$  converges uniformly to  $f^{(k)}(z)$  on every compact set  $K \subseteq \Omega$ .

Apply this to series, we naturally have the following theorem.

### Theorem 5.1.2: Weierstrass Theorem

Let  $\{f_n\}$  be a sequence of analytic functions with limit function  $f(z) = \sum_{n=1}^{\infty} f_n(z)$  defined on  $\Omega$ . If for every compact set  $K \subseteq \Omega$ ,  $\sum f_n$  is uniformly convergent on  $K$ , then  $f$  is analytic on  $\Omega$ .

Moreover,  $\sum f'_n(z)$  converges uniformly to  $f'(z)$  on every compact  $K \subseteq \Omega$ .

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### Proposition: Uniform Convergence on Compact Sets

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As  $K$  is compact so is closed, then the maximum and minimum of  $|f_n(z) - f_m(z)|$  is achieved on  $\partial K$ . So uniform convergence on  $K$  is equivalent to uniform convergence on  $\partial K$ .

---

### Theorem 5.1.3: Hurwitz Theorem

Let  $\{f_n\}$  be a sequence of analytic functions defined on  $\Omega$ ,  $f_n(z) \neq 0$ , with limit function  $f(z) = \lim_{n \rightarrow \infty} f_n(z)$  defined on  $\Omega$ . If for every compact set  $K \subset \Omega$ ,  $\{f_n\}$  converges uniformly to  $f$ , then  $f = 0$  or  $\forall z \in \Omega, f(z) \neq 0$ .

*Proof.* Suppose  $f$  is not identically zero, then the zeros are isolated.  $\forall z_0 \in \Omega$ , let  $r > 0$  that  $\forall 0 < |z - z_0| \leq r, f(z) \neq 0$ . Denote

$$C = \min \{f(z) : |z - z_0| = r\} > 0$$

then  $1/f_n$  converges uniformly to  $1/f$  on  $|z - z_0| = r$ . Thus, there exists  $n_0$  such that for all  $n > n_0$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_C \frac{f'_n(z)}{f'_n(z)} dz = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = 0.$$

From theorem 4.3.6, we know that the integral counts the number of zeros of  $f_n$  in  $|z - z_0| < r$ , which is zero. So  $f(z_0) \neq 0$ .  $\square$

### 5.1.2 The Taylor Series

According to theorem 4.3.2, let  $f$  be analytic in the region  $\Omega$ . For any  $z, z_0 \in \Omega$  we have

$$f(z) = \sum_{k=0}^n \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k + f_{n+1}(z)(z - z_0)^{n+1}.$$

Let  $D \subseteq \Omega$  be a disk containing  $z$  with center at  $z_0$  and radius  $\rho$ . Then we have

$$f_{n+1}(z) = \int_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)}.$$

Denote  $M = \max \{|f(z)| : z \in \partial D\}$ , then obtain

$$|f_{n+1}(z)(z - z_0)^{n+1}| \leq \frac{M |z - z_0|^{n+1}}{\rho^n (\rho - |z - z_0|)}. \quad (5.2)$$

which means that the remainder term inner compact uniform converges to zero in  $D$  as  $n \rightarrow \infty$ .

#### Theorem 5.1.4: Taylor's Series

Let  $f$  be analytic in the region  $\Omega$ . For any  $z_0 \in \Omega$  and any open disk  $D \subseteq \Omega$  centered at  $z_0$ , the Taylor series

$$S(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad z \in D$$

converges uniformly to  $f(z)$  on every compact subset of  $D$ .

Therefore, the radius of convergence is at least the distance from  $z_0$  to the boundary of  $\Omega$ . (Well, it can be larger)

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#### *Remark:*

The uniqueness of Taylor series is guaranteed by taking the derivative of the series term by term and evaluating at  $z_0$ .

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#### *Proposition:* The Derivatives and Integral of Taylor Series

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Let  $f$  be analytic in the region  $\Omega$  and  $z_0 \in \Omega$ . The Taylor series of  $f$  at  $z_0$  is

$$S(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad z \in D$$

if  $D$  is an open disk centered at  $z_0$  contained in  $\Omega$ . Then  $f(z) = S(z)$  for  $z \in D$  and  $S(z)$  converges uniformly to  $f(z)$  on every compact subset of  $D$ .

Independently speaking,  $S$  is a power series, which is inner compact uniform convergent and analytic in an open disk  $D$  of radius  $R$  centered at  $z_0$ , and diverges for  $|z - z_0| > R$ . Moreover,

---

- The derivative of the series is

$$S'(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{(k-1)!} (z - z_0)^{k-1}, \quad z \in D.$$

- The integral of the series is

$$\int_{z_0}^z S(\zeta) d\zeta = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{(k+1)!} (z - z_0)^{k+1}, \quad z \in D.$$

which have the same radius of convergence  $R$  as  $S$ .

*Proof.* This is the Abel Disk Theorem 2.2.3. □

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### Example: Taylor Series of Elementary Functions

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- Exponential Function

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots, \quad z \in \mathbb{C}$$

This is of course the definition of  $e^z$ , but it is also the Taylor series at  $z_0 = 0$  because of the uniqueness of Taylor series.

- Sine and Cosine Function

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots, \quad z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots, \quad z \in \mathbb{C}$$

- Power and logarithm: For a non- $\mathbb{Z}_+$  power or logarithm, we need first to choose an analytic branch near the origin. For  $(1+z)^\mu$  or  $\log(1+z)$ , we can choose the principal branch with a cut along  $(-\infty, -1]$ . Then the Taylor series at  $z_0 = 0$  is

$$(1+z)^\mu = 1 + \mu z + \binom{\mu}{2} z^2 + \cdots + \binom{\mu}{n} z^n + \cdots, \quad z \in \mathbb{C}, |z| < 1$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^{n-1} \frac{z^n}{n} + \cdots, \quad z \in \mathbb{C}, |z| < 1$$

The radius of convergence is 1 for both series if  $\mu \notin \mathbb{Z}_+$ , for if  $R > 1$ , then all the derivative would be bounded for  $|z| < 1$ , which is not the case. For  $\mu \in \mathbb{Z}_+$ , then the series converges in  $\mathbb{R}$ .

- Arctangent and Arcsine: We determine the branch by

$$\arctan z = \int_0^z \frac{1}{1+\zeta^2} d\zeta, \quad \arcsin z = \int_0^z \frac{1}{\sqrt{1-\zeta^2}} d\zeta, \quad z \in \mathbb{C}, |z| < 1.$$


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The path is taken to be any path from 0 to  $z$  that lies in the unit open disk. Through term-by-term integration, we have

$$\begin{aligned}\arctan z &= z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots + (-1)^{n-1} \frac{z^{2n-1}}{2n-1} + \cdots, \quad z \in \mathbb{C}, |z| < 1 \\ \arcsin z &= z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots + \frac{(2n-1)!!}{(2n)!!} \frac{z^{2n+1}}{2n+1} + \cdots, \quad z \in \mathbb{C}, |z| < 1\end{aligned}$$

The radius of convergence is 1 for both series.

#### Notation 5.1.1: $[z^n]$

Denote  $[z^n]$  to be any function which is analytic and has a zero of order at least  $n$  at  $z = 0$ . For any analytic function at the origin, we can say

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + [z^{n+1}], \quad (5.3)$$

Where the coefficients are uniquely determined by the Taylor series.

Therefore, finding the Taylor series coefficients of  $f(z)$  is equivalent to finding the polynomial  $P_n$  that  $f(z) - P_n(z) = [z^{n+1}]$  has a zero of order at least  $n+1$  at  $z = 0$ . We can say that no matter the degree of  $P_n$  is larger than  $n$ , the coefficients of  $P_n$  up to  $z^n$  are uniquely determined.

*Proof.* Assume there are two such polynomials  $P_n$  and  $Q_n$ , then  $P_n(z) - Q_n(z) = [z^{n+1}]$ . Assume that  $P_n(z) - Q_n(z) = a_kz^k + a_{k+1}z^{k+1} + \cdots + a_mz^m$  with  $a_k \neq 0$  and  $k \leq n$ , then  $P_n(z) - Q_n(z)$  has a zero of order  $k \leq n$  at  $z = 0$ , which is a contradiction.  $\square$

Therefore, we can easily say that if

$$\begin{aligned}f(z) &= P_n(z) + [z^{n+1}], & g(z) &= Q_n(z) + [z^{n+1}], \\ f(z)g(z) &= P_n(z)Q_n(z) + [z^{n+1}],\end{aligned}$$

So as to determine the Taylor series of  $fg$  at  $z = 0$  up to  $z^n$ , we only need to determine the product of the polynomials  $P_n$  and  $Q_n$  up to  $z^n$ . Explicitly, we have:

#### Proposition: Taylor Series of Operation of Functions

Let  $f$  and  $g$  be analytic in the region  $\Omega$  and  $z_0 \in \Omega$ . The Taylor series of  $f$  and  $g$  at  $z_0$  are

$$\begin{aligned}f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, & z &\in D \\ g(z) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z - z_0)^k, & z &\in D\end{aligned}$$

if  $D$  is an open disk centered at  $z_0$  contained in  $\Omega$ . Then  $f(z)$  and  $g(z)$  can be expressed as power series in  $D$ , which converge uniformly to  $f(z)$  and  $g(z)$  on every compact subset of  $D$ . Moreover,

- For any constant  $c \in \mathbb{C}$ , the Taylor series of  $cf$  at  $z_0$  is

$$cf(z) = \sum_{k=0}^{\infty} c \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad z \in D.$$

- The Taylor series of  $f + g$  at  $z_0$  is

$$f(z) + g(z) = \sum_{k=0}^{\infty} \left( \frac{f^{(k)}(z_0)}{k!} + \frac{g^{(k)}(z_0)}{k!} \right) (z - z_0)^k, \quad z \in D.$$

- The Taylor series of  $fg$  at  $z_0$  is

$$f(z)g(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^k \frac{f^{(j)}(z_0)}{j!} \frac{g^{(k-j)}(z_0)}{(k-j)!} \right) (z - z_0)^k, \quad z \in D.$$

- If  $g(z_0) \neq 0$ , then the Taylor series of  $f/g$  at  $z_0$  can be determined by:

First letting  $R_n$  be such that  $P_n = Q_n R_n + [z^{n+1}]$ , then taking the coefficients of  $R_n$  up to  $z^n$  to be the coefficients of the Taylor series of  $f/g$  at  $z_0$  up to  $z^n$ . Explicitly, we have

$$\frac{f(z)}{g(z)} = \sum_{k=0}^{\infty} r_k (z - z_0)^k, \quad z \in D,$$

This is because  $f - R_n g = [z^{n+1}]$ . And as  $g(z_0) \neq 0$ ,  $f/g - R_n = [z^{n+1}]$ .

- For the composite function  $f(g(z))$ , for simplicity let  $z_0 = 0$  and  $g(0) = 0$ , then set

$$f(w) = P_n(w) + [w^{n+1}], \quad g(z) = Q_n(z) + [z^{n+1}], \quad Q_n(0) = 0,$$

Then we have

$$f(g(z)) = P_n(Q_n(z) + [z^{n+1}]) + [z^{n+1}] = P_n(Q_n(z)) + [z^{n+1}].$$

- For the inverse function  $f^{-1}(z)$ , let  $z_0 = 0$  and  $f(0) = 0$ , assume  $f'(0) \neq 0$  so there is a local homeomorphism inverse function. Let  $f(z) = P_n(z) + [z^{n+1}]$  and  $f^{-1}(z) = Q_n(z) + [z^{n+1}]$  with  $P_n(0) = Q_n(0) = 0$ , then we have

$$z = f(f^{-1}(z)) = P_n(Q_n(z)) + [z^{n+1}].$$

The proof of existence and uniqueness of  $Q_n$  is due to the Lagrange Inversion Theorem.

**Theorem 5.1.5: Lagrange Inversion Theorem**

Let  $f$  be analytic in the region  $\Omega$  and  $z_0 \in \Omega$ . The Taylor series of  $f$  at  $z_0$  is

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad z \in D$$

if  $D$  is an open disk centered at  $z_0$  contained in  $\Omega$ . Then  $f(z)$  can be expressed as a power series in  $D$ , which converges uniformly to  $f(z)$  on every compact subset of  $D$ .

Moreover, if  $f'(z_0) \neq 0$ , then there exists a local inverse function  $f^{-1}$  such that  $f^{-1}(f(z)) = z$  for all  $z$  in a neighborhood of  $z_0$ . The Taylor series of  $f^{-1}$  at  $w_0 = f(z_0)$  is

$$f^{-1}(w) = z_0 + \sum_{n=1}^{\infty} \frac{(w - w_0)^n}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} \left( \frac{z - z_0}{f(z) - w_0} \right)^n \right]_{z=z_0}, \quad w \in D',$$

where  $D'$  is an open disk centered at  $w_0$  contained in the range of the local inverse function.

**5.1.3 The Laurent Series**

A series of the form

$$b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} + \cdots$$

can be viewed as a power series in  $1/z$ . So it will converge in a region of the form  $|z| > R$  for some  $R > 0$  and diverge for  $|z| < R$ .

For a more general form is called a Laurent series:

**Definition 5.1.1: Laurent Series**

A Laurent series centered at  $z_0$  is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{5.4}$$

where  $a_n \in \mathbb{C}$  for all  $n \in \mathbb{Z}$ .

A Laurent series can be viewed as the sum of two series: a power series in  $z - z_0$  and a power series in  $1/(z - z_0)$ . It would converge iff both series converge. Suppose the power series converges for  $|z - z_0| < R_2$  and the other converges for  $|z - z_0| > R_1$ , then the Laurent series converges in the annulus  $R_1 < |z - z_0| < R_2$ . Note that if  $R_1 \geq R_2$ , then there is no convergence region.

Conversely, we can start with an analytic function whose analyticity domain contains an annulus  $R_1 < |z - a| < R_2$ , then we can represent it as a Laurent series in that annulus.

**Theorem 5.1.6: Laurent's Theorem**

Let  $f$  be analytic in the annulus  $R_1 < |z - a| < R_2$ . Then  $f$  can be represented as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n, \quad R_1 < |z-a| < R_2,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta, \quad R_1 < r < R_2.$$

The series converges uniformly to  $f$  on every compact subset of the annulus.

*Proof.* We try to represent  $f = f_1 + f_2$ , and DEFINE

$$f_1(z) = \frac{1}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{\zeta-z} d\zeta, \quad |z-a| < r < R_2,$$

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{\zeta-z} d\zeta, \quad R_1 < r < |z-a|.$$

Both integral is irrelevant to the choice of  $r$  in the given range. By Cauchy's integral formula, we have  $f(z) = f_1(z) + f_2(z)$  for  $R_1 < |z-a| < R_2$ . (Adding the two integrals means integrate via two circles with opposite orientation, which  $\sim 0$ ).

Also, we can see that  $f_1$  is an analytic function in  $|z-a| < R_2$  and  $f_2$  is an analytic function in  $|z-a| > R_1$ . So we can expand them as power series:

$$f_1(z) = \sum_{n=0}^{\infty} A_n(z-a)^n, \quad |z-a| < R_2, \quad A_n = \frac{1}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta,$$

For  $f_2$  we consider the transformation  $z = 1/z' + a$ ,  $\zeta = 1/\zeta' + a$ , then

$$f_2(1/z' + a) = \frac{1}{2\pi i} \int_{|\zeta'|=1/r} \frac{f(1/\zeta' + a)}{\zeta' - z'} \frac{z'}{\zeta'} d\zeta' = \sum_{n=0}^{\infty} B_n(z')^n, \quad |z'| < 1/R_1,$$

where

$$\begin{aligned} B_n &= \frac{1}{n!} \left[ \frac{d^n}{d(z')^n} \int_{|\zeta'|=1/r} \frac{f(1/\zeta' + a)}{\zeta' - z'} \frac{z'}{\zeta'} d\zeta' \right]_{z'=0} \\ &= \frac{1}{2\pi i} \int_{|\zeta'|=1/r} \frac{f(1/\zeta' + a)}{(\zeta')^{n+1}} d\zeta' \\ &= \frac{1}{2\pi i} \int_{|\zeta-a|=r} f(\zeta)(\zeta-a)^{n-1} d\zeta. \end{aligned}$$

$$f_2(z) = \sum_{n=0}^{\infty} B_n(1/(z-a))^n = \sum_{n=-\infty}^{-1} B_{-n}(z-a)^n, \quad |z-a| > R_1.$$

In all, we can write:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n, \quad R_1 < |z-a| < R_2, \quad a_n = \frac{1}{2\pi i} \int_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta.$$

□

The situation can be easily extended to  $R_1 = 0$ .

## 5.2 Partial Fractions and Factorization

There are two ways to represent a rational function: partial fractions and factorization of both numerator and denominator. Both are based on the fundamental theorem of algebra. We shall do this in a more general setting, that is, meromorphic functions.

### 5.2.1 Partial Fractions

Assume  $f$  is meromorphic in a region  $\Omega$ , for each pole  $b_\nu$  there is a neighborhood that contains no other poles. So we can write the Laurent series of  $f$  at  $b_\nu$ :

$$P_\nu\left(\frac{1}{z-b_\nu}\right) + \text{positive power series of } (z-b_\nu),$$

where  $P_\nu$  is a polynomial in  $1/(z-b_\nu)$  without constant term. There are also finite terms in  $P_\nu$  because  $b_\nu$  is a pole and  $\lim_{z \rightarrow b_\nu} f(z)(z-b_\nu)^m = c \neq \infty$  for some  $m \in \mathbb{Z}_+$ .

Then we can subtract all these principal parts from  $f$ :

$$f(z) = \sum_{\nu} P_\nu\left(\frac{1}{z-b_\nu}\right) + g(z). \quad (5.5)$$

For there are at most countable poles in  $\Omega$ , the sum is well-defined. However, there are generally infinite poles and the sum may not converge. So we need to introduce some convergence factors to make the sum convergent. There is no harm done if we subtract some analytic functions from  $P_\nu$ , as long as we add them to  $g(z)$ .

#### Theorem 5.2.1: Mittag-Leffler Theorem

Let  $\{b_\nu\}$  be a sequence of complex numbers with  $\lim_{\nu \rightarrow \infty} b_\nu = \infty$ , and  $P_\nu$  be polynomials without constant term. Then there exists meromorphic function  $f$  in  $\mathbb{C}$  whose poles are exactly at  $b_\nu$  and the principal part of the Laurent series of  $f$  at  $b_\nu$  is  $P_\nu(1/(z-b_\nu))$ . And it can be expressed as

$$f(z) = \sum_{\nu=1}^{\infty} \left[ P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z) \right] + g(z), \quad (5.6)$$

where  $p_\nu(z)$  are polynomials chosen such that the series absolutely converges, and  $g(z)$  is an entire function (analytic in  $\mathbb{C}$ ).

*Proof.* Suppose  $b_\nu \neq 0$  for all  $\nu$ , otherwise we can just perform a slight translation. The function  $P_\nu(1/(z-b_\nu))$  is analytic in  $|z| < |b_\nu|$ . So we can expand it as a Taylor series at  $z = 0$ . Choose  $p_\nu(z)$  to be a partial sum of the Taylor series at  $z = 0$ , ending at the degree  $n_\nu$ .

From the proof of theorem 4.3.3, we know that if the maximum of  $|P_\nu|$  for  $|z| = |b_\nu|/2$  is  $M_\nu$ , then

$$\left| P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z) \right| \leq 2M_\nu \left( \frac{2|z|}{|b_\nu|} \right)^{n_\nu+1}, \quad |z| \leq \frac{|b_\nu|}{4}.$$

We choose  $n_\nu$  such that

$$2^{n_\nu} \geq M_\nu 2^\nu$$

For any  $R$ , there are only finite  $\nu$  such that  $|b_\nu| \leq 4R$ . For sufficiently large  $\nu$ , that all  $|b_\nu| > 4R$ , we have

$$\left| P_\nu\left(\frac{1}{z - b_\nu}\right) - p_\nu(z) \right| \leq 2M_\nu \left( \frac{2R}{|b_\nu|} \right)^{n_\nu+1} \leq 2M_\nu \left( \frac{1}{2} \right)^{n_\nu+1} \leq 2^{-\nu}.$$

So if we omit the first finite terms, the series converges uniformly and absolutely on  $|z| \leq R$ . As  $R$  is arbitrary, the series converges uniformly and absolutely on every compact subset of  $\mathbb{C}$ . So the sum is meromorphic in  $\mathbb{C}$  with poles exactly at  $b_\nu$  and the principal part of the Laurent series of  $f$  at  $b_\nu$  is  $P_\nu(1/(z - b_\nu))$ .  $\square$

*Remark:*

Conversely, we can say that given a meromorphic function  $f$  in  $\mathbb{C}$  with poles at  $b_\nu$ , and the principal part of the Laurent series of  $f$  at  $b_\nu$  is  $P_\nu(1/(z - b_\nu))$ . Then we can find a suitable  $g$  that has the same poles and principal parts, so  $f - g$  is entire. If we merge the  $f - g$  into the  $g$  in the theorem, then we have the same form as in the theorem expressing  $f$ .

For a slight clarification, if  $f$  is meromorphic in  $\mathbb{C}$  with infinitely many poles, then the poles must have a limit point at  $\infty$ . This is because any closed subset of  $\mathbb{C}$  is compact, so there are only finite poles in that.

### *Example:* The Mittag-Leffler Theorem

- Consider the function  $\frac{\pi^2}{\sin^2 \pi z}$ . It has poles at all integers, and the principal part of the Laurent series at  $n \in \mathbb{Z}$  is  $\frac{1}{(z - n)^2}$ . So we can express it as

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2} + g(z),$$

where  $g(z)$  is an entire function. We prove that  $g(z) = 0$ . First notice that both sides have period 1, so does  $g(z)$ . Also if we let  $z = x + iy$  then

$$|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x$$

which means that  $\pi^2/\sin^2 \pi z$  tends uniformly to 0 as  $y \rightarrow \pm\infty$ . And the series also tends uniformly to 0 as  $y \rightarrow \pm\infty$ . So  $g(z)$  tends to 0 uniformly as  $y \rightarrow \pm\infty$ . As  $g(z)$  is periodic, it is bounded in  $\mathbb{C}$ . By Liouville's theorem,  $g(z)$  is a constant. So  $g(z) = 0$ .

So we can infer that  $g$  is bounded in  $0 \leq x \leq 1$  and from periodicity, it is bounded in  $\mathbb{C}$ . By Liouville's theorem,  $g$  is a constant. As  $g(iy) \rightarrow 0$  as  $y \rightarrow \pm\infty$ , we have  $g(z) = 0$ . Therefore,

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}.$$

- Consider taking the integral of both sides, and the integral of the left side is  $-\pi \cot \pi z$ . The integral of the right side is inner compact uniform convergent in every domain that does not contain any integer. However, the series of  $1/(z - n)$  does not converge. So

we need to introduce some convergence factors. We can take

$$\sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right) + \frac{1}{z} = \sum_{n \neq 0} \frac{z}{n(z-n)} + \frac{1}{z}.$$

So we have

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \frac{z}{n(z-n)} + C = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} + C.$$

For both sides are odd functions,  $C = 0$ . Therefore,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

- Conversely, we now study the sum

$$\lim_{m \rightarrow \infty} \sum_{n=-m}^m \frac{(-1)^n}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2}.$$

Which obviously is a meromorphic function with poles at all integers. Separate the odd and even terms, we have

$$\sum_{n=-(2k+1)}^{2k+1} \frac{(-1)^n}{z-n} = \sum_{n=-k}^k \frac{1}{z-2n} - \sum_{n=-k-1}^k \frac{1}{z-(2n+1)}.$$

which has limit

$$\frac{\pi}{2} \cot \frac{\pi z}{2} - \frac{\pi}{2} \cot \frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z}.$$

So we have

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2} =_{m \rightarrow \infty} \sum_{n=-m}^m \frac{(-1)^n}{z-n}. \quad (5.7)$$

### 5.2.2 Infinite Products

An infinite product is an expression of the form

$$\prod_{n=1}^{\infty} (1 + a_n), \quad a_n \in \mathbb{C}, a_n \neq -1. \quad (5.8)$$

Taking the logarithm, we have

$$\sum_{n=1}^{\infty} \log(1 + a_n).$$

where the branch of the logarithm is chosen to be the principal branch. Denote

$$P_n = \prod_{k=1}^n (1 + a_k), \quad S_n = \sum_{k=1}^n \log(1 + a_k).$$

Then we have  $P_n = e^{S_n}$ . If  $S_n \rightarrow S$ , then  $P_n \rightarrow e^S$ . So the infinite product converges to  $e^S$ . Conversely, if  $P_n \rightarrow P$ , then it is not necessary that  $S_n \rightarrow \log P$  in the principle branch. But we shall prove that if  $P \neq 0$ , then there exists an integer  $m$  such that  $S_n \rightarrow \log P + 2m\pi i$ .

*Proof.* SORRY □

### Theorem 5.2.2: Infinite Products

The infinite product  $\prod_{n=1}^{\infty} (1 + a_n)$  converges to a nonzero limit iff the series  $\sum_{n=1}^{\infty} \log(1 + a_n)$  converges to a limit in the principal branch.

Furthermore, as

$$\lim_{z \rightarrow 0} \frac{\log(1 + z)}{z} = 1,$$

We say that the infinite product converges absolutely if  $\sum_{n=1}^{\infty} \log(1 + a_n)$  converges absolutely. And we have the following theorem:

### Theorem 5.2.3: Absolute Convergence of Infinite Products

The infinite product  $\prod_{n=1}^{\infty} (1 + a_n)$  converges absolutely iff the series  $\sum_{n=1}^{\infty} |a_n|$  converges.

*Remark:*

For uniform convergence we can say the same. We define that an infinite product  $\prod_{n=1}^{\infty} (1 + a_n(z))$  converges uniformly in a set  $E$  if the series  $\sum_{n=1}^{\infty} \log(1 + a_n(z))$  converges uniformly in  $E$ .

## 5.2.3 Canonical Products

### Definition 5.2.1: Entire Functions

A function  $f$  is entire if it is analytic in  $\mathbb{C}$ .

If  $g$  is an entire function so is  $f = e^g$  and  $\forall z, f(z) \neq 0$ . Conversely, if  $f$  is entire and  $\forall z, f(z) \neq 0$ , then we can say that there exists an entire function  $g$  such that  $f = e^g$ .

*Proof.* Consider  $f'/f$ , which is analytic in  $\mathbb{C}$ . So there exists an entire function  $g$  such that  $g' = f'/f$ . We have  $f(z)e^{-g(z)}$  has zero derivative, so  $f(z) = e^{g(z)+C}$ . □