Introduction to Functional Analysis

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Chapter 1

Metric Spaces

1.1 Metric Spaces

Definition 1.1.1: Metric Spaces

A **metric space** is a set X together with a function $d : X \times X \to \mathbb{R}$, called a **metric**, that satisfies the following properties for all $x, y, z \in X$:

1. **Non-negativity:** $d(x, y) \geq 0$.
2. **Identity of indiscernibles:** $d(x, y) = 0$ if and only if $x = y$.
3. **Symmetry:** $d(x, y) = d(y, x)$.
4. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$.

Example: **Metric Spaces**

- The Euclidean space $\mathbb{R}^n$ with the standard metric $d(x, y) = \|x - y\|_2$, where $\| \cdot \|_2$ is the Euclidean norm.

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

- The bounded sequence space ℓ^∞, consisting of all bounded sequences of complex numbers, or $\ell^\infty = \{x = (x_1, x_2, \ldots) \in \mathbb{C}^\infty : \sup_i |x_i| < \infty\}$, with the metric defined by

$$d(x, y) = \sup_i |x_i - y_i|$$

- The function space $C[a, b]$, consisting of all continuous functions on the interval $[a, b]$, with the metric defined by

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$$

- The discrete metric space, where the metric is defined as

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

(This can be thought of an n-dimensional tetrahedron with edge length 1 with the Euclidean metric.)

- Hamming metric, which is used in coding theory, defined for two strings of equal length as the number of positions at which the corresponding symbols are different. For example, for two binary strings x and y of length n, the Hamming distance is given by

$$d(x, y) = \sum_{i=1}^{n} \not\Vdash_{\{x_i \neq y_i\}}$$

where $\not\Vdash_{\{.\}}$ is the indicator function.

- The sequence space s, consisting of all sequences of complex numbers. The metric is defined by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

- The bounded function space $B(X)$, consisting of all bounded functions from a set X to $\mathbb{C}$. The metric is defined by

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

- The space ℓ^p, for $1 \leq p < \infty$, consisting of all sequences of complex numbers $x = (x_1, x_2, \ldots)$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$. The metric is defined by

$$d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}$$

When $p = 2$, this space is known as the Hilbert sequence space ℓ^2.

Proposition: **Some Inequalities**

- The Young's inequality: For $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

- The Hölder's inequality: For sequences $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in ℓ^p and

ℓ^q respectively, where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{1/q}$$

when $p = q = 2$, this reduces to the Cauchy-Schwarz inequality:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{1/2}$$

- The Minkowski inequality: For sequences $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in ℓ^p, where $1 \leq p < \infty$, we have

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{1/p}$$

1.2 Convergence and Completeness

Definition 1.2.1: Convergence

A sequence (x_n) in a metric space (X, d) is said to **converge** to a point $x \in X$ if for every $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, we have $d(x_n, x) < \epsilon$. This is denoted as $x_n \to x$ as $n \to \infty$.

$$\lim_{n \to \infty} x_n = x$$

If (X, d) is a metric space, we say that a sequence (x_n) in X is **bounded** if there exists a constant $M > 0$ such that for all n, $d(x_n, x_0) < M$ for some fixed point $x_0 \in X$.

- (Uniqueness of limits) If a sequence converges, it is bounded and its limit is unique.

- If $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$.

Definition 1.2.2: Cauchy's Sequence and Completeness

A sequence (x_n) in a metric space (X, d) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$, we have $d(x_m, x_n) < \epsilon$.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, d(x_m, x_n) < \epsilon$$

A metric space is said to be **complete** if every Cauchy sequence in the space converges to a limit in the space.

A convergent sequence is a Cauchy sequence, but the converse is not necessarily true.

Theorem 1.2.1: Closure in Metric Spaces

Let (X, d) be a metric space and let $A \subseteq X$ is not empty.

- $x \in \overline{M}$ iff there is a sequence (x_n) in M such that $x_n \to x$.
- M is closed iff every convergent sequence in M converges to a point in M. That is, $M = \overline{M}$.

Theorem 1.2.2: Subspace of a Complete Space

Let (X, d) be a complete metric space and let $Y \subseteq X$ be a non-empty subset. Then Y is a complete metric space with the induced metric $d_Y(x, y) = d(x, y)$ for all $x, y \in Y$ iff Y is closed in X.

Theorem 1.2.3: Sequences and Continuous Mapping

A function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is continuous at a point $x_0 \in X$ if and only if for every sequence (x_n) in X that converges to x_0, the sequence $(f(x_n))$ converges to $f(x_0)$ in Y.

1.3 Completeness Proofs

Completeness of $\mathbb{R}$ and $\mathbb{C}$ The spaces $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces with the standard metric.

As each component is a Cauchy sequence, and $\mathbb{C}$ is complete, each component converges. Thus, the sequence converges in $\mathbb{C}$ to the limit constructed from the limits of the components.