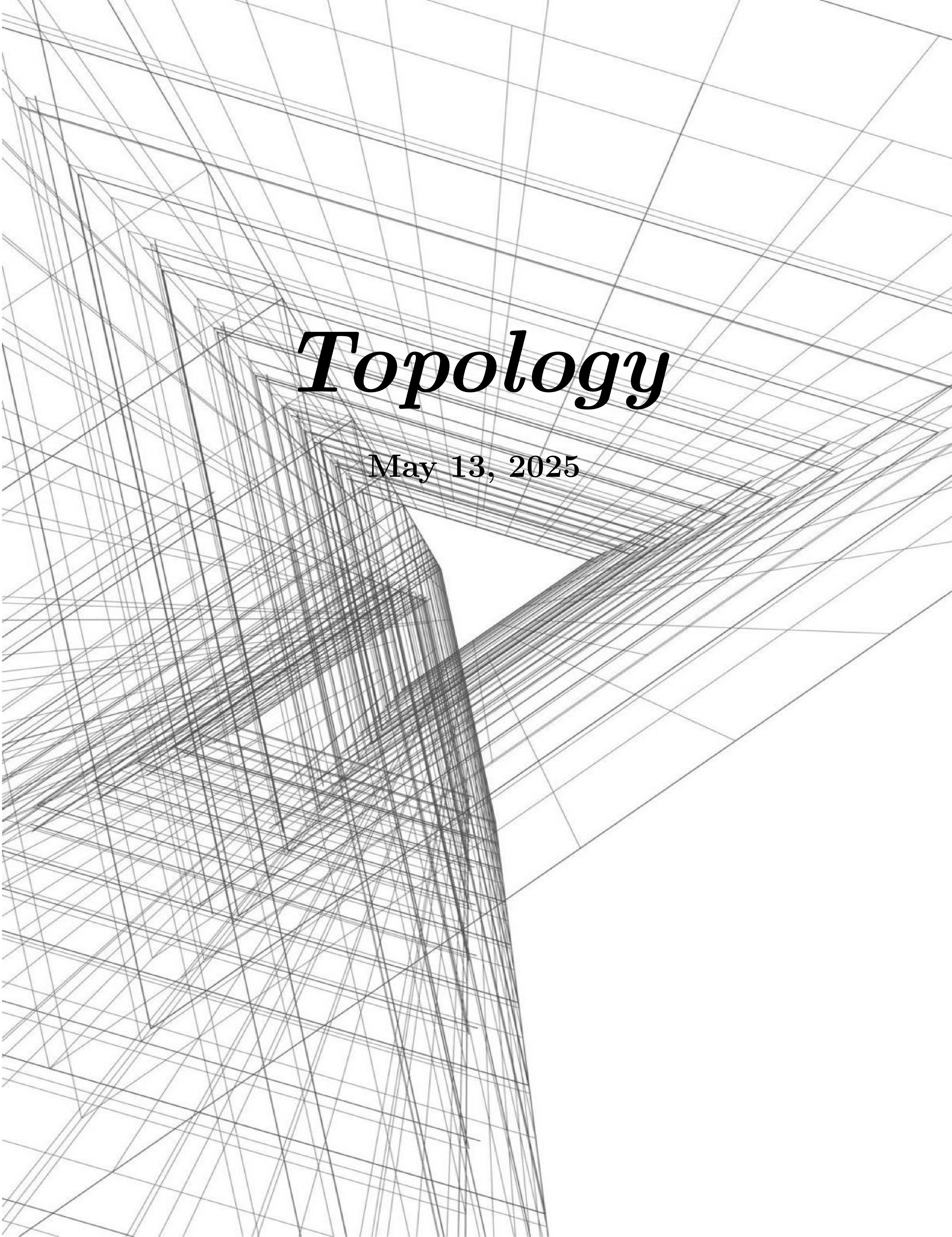

Topology

May 13, 2025

Contents

Chapter 1

Topological Spaces

The concept of topological space grew out of the study of the real line and Euclidean space and the study of continuous functions on these spaces.

1.1 Topological Spaces

Definition 1.1.1: Topology

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X (that is, $T \subseteq P(X)$) such that the following conditions hold:

- $\emptyset, X \in \mathcal{T}$.
- The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.
- The intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

Example: **Topologies**

- The *discrete topology* on a set X is the topology $\mathcal{T} = P(X)$, where $P(X)$ is the power set of X. In this topology, every subset of X is open.
- The *indiscrete topology*, or trivial topology, on a set X is the topology $\mathcal{T} = \{\emptyset, X\}$. In this topology, only the empty set and the entire set are open.
- Topologies on $\mathbb{R}$.
  1. $\mathcal{T}_1$ consisting of $\mathbb{R}$, $\emptyset$, and all open intervals (a, b).
  2. $\mathcal{T}_2$ consisting of $\mathbb{R}$, $\emptyset$, and all $[-n, n]$ for $n \in \mathbb{Z}_+$.
- Topologies on $\mathbb{N}$.
  1. The initial segment topology: $\mathcal{T}_1$ consisting of $\mathbb{N}$, $\emptyset$ and the set $\{1, \ldots, n\}$ for $n \in \mathbb{N}$.

2. The final segment topology: $\mathcal{T}_2$ consisting of $\mathbb{N}$, $\emptyset$ and the set $\{n, n+1\ldots\}$ for $n \in \mathbb{N}$.

Definition 1.1.2: Finer Topologies

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is said to be *finer* than $\mathcal{T}_1$. Similarly, $\mathcal{T}_1$ is said to be *coarser* than $\mathcal{T}_2$. If $\mathcal{T}_1$ is neither finer nor coarser than $\mathcal{T}_2$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be *incomparable*.

1.2 Open Sets, Closed Sets and Clopen Sets

The axioms of a topology is the extension of the concept of open sets in Euclidean space. As we can see, every open sets in Euclidean space satisfies the axioms of a topology. Therefore, we can define open sets in a topological space as follows.

Definition 1.2.1: Open Sets

If X is a set with a topology $\mathcal{T}$, a subset $U \subseteq X$ is *open* if $U \in \mathcal{T}$.

We define closed sets as the complements of open sets, which is a natural extension of the concept of closed sets in Euclidean space.

Definition 1.2.2: Closed Sets

Let $(X, \mathcal{T})$ be a topological space. A subset $S \subseteq X$ is a *closed set* if $X - S$ is open.

Theorem 1.2.1: Properties of Closed Sets

If $(X, \mathcal{T})$ is a topological space, then

1. $\emptyset$ and X are closed.
2. The intersection of any collection of closed sets is closed.
3. The union of any finite collection of closed sets is closed.

Proof. This is a rather straightforward consequence of the definition of closed sets. Using

$$X - \bigcap_{\alpha \in I} S_\alpha = \bigcup_{\alpha \in I} (X - S_\alpha) \quad \text{and} \quad X - \bigcup_{i=1}^{n} S_i = \bigcap_{i=1}^{n} (X - S_i),$$

would do. □

Remark:
Note that openness and closedness are not mutually exclusive. A set can be both open and closed, such as $\emptyset$ and X in any topological space. Also, there are sets that are neither open nor closed, just like $(0, 1]$ in the standard topology on $\mathbb{R}$.

Definition 1.2.3: Clopen Sets

A subset S in a topological space $(X, \mathcal{T})$ is a *clopen set* if it is both open and closed.

Example: **Clopen Sets**

- In every topological space, $\emptyset$ and X are clopen.
- In the discrete topology, every subset of X is clopen.
- In the indiscrete topology, only $\emptyset$ and X are clopen.

1.2.1 Distinct Topologies on Finite and Infinite Sets

Proposition: **Finite Set Topologies**

1. The number of topologies on a finite set increases as X increases.
2. if finite set X has $n \in \mathbb{N}$ points, then it has at least $(n-1)!$ distinct topologies.

Proof. Use mathematical induction. If $X = n$, and there are M different topologies. Let $Y = n+1$, the additional point is x. For any $\mathcal{T}$, let $\mathcal{T}' = \{U \cup \{x\} : U \in \mathcal{T}\}$ is a topology on Y ($\mathcal{T}'$ is constructed by adding x to every element of $\mathcal{T}$), and $\mathcal{T}'$ is a topology on Y. Then $\{\emptyset, Y\}$ is a new topology, thus Y has at least $M+1$ topologies.

Furthermore, Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{x_1, \ldots, x_{n+1}\}$. If $\mathcal{T}$ is any topology on X, fix $i \in \{1, \ldots, n\}$, define U_i by replacing any occurrences of x_i in Y with x_{n+1}, then $\mathcal{T}_i = \{U_i : U \in \mathcal{T}\}$ is a topology on Y. Then for any topology on X, we can construct at least n distinct topologies on Y. □

Proposition: **Infinite Set Topologies**

If X is any infinite set with cardinality $\aleph$, there are at least $2^{\aleph}$ distinct topologies on X.

Proof. As $P(X)$ has cardinality $2^{\aleph}$, there are at least $2^{\aleph}$ distinct topologies on X: $\forall U \in P(X), \{\emptyset, U, X\}$ is a topology on X. □

1.3 The Finite-Closed Topology

Sometimes, describing a topology in terms of closed sets is more convenient than describing it in terms of open sets.

Definition 1.3.1: Finite-Closed Topology

Let X be any nonempty set. A topology $\mathcal{T}_f$ on X is called the *finite-closed topology* if the closed sets in $\mathcal{T}_f$ are precisely the finite sets and X itself.

We shall prove that the finite-closed topology is indeed a topology.

Proof. We shall prove that the finite complement topology satisfies the axioms of a topology.

- $\emptyset, X \in \mathcal{T}_f$.
- Let $\{U_\alpha : \alpha \in I\}$ be a subset of $\mathcal{T}_f$, where I is an index set. Then
$$X - \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I} (X - U_\alpha)$$
is an intersection of finite sets, so it is finite. If all of U_α is $\emptyset$, then $X - \bigcup_{\alpha \in I} U_\alpha = X$. Therefore, $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_f$.
- Let $\{U_i\}_{i=1}^n$ be a subset of $\mathcal{T}_f$. Then
$$X - \bigcup_{i=1}^n U_i = \bigcap_{i=1}^n (X - U_i)$$
is a finite intersection of finite sets, so it is finite. If there is an i such that $U_i = \emptyset$, then $\bigcup_{i=1}^n U_i = \emptyset$. Therefore, $\bigcup_{i=1}^n U_i \in \mathcal{T}_f$.

$\square$

There are many other ways of constructing topologies.

Theorem 1.3.1: Topology of Preimages

Let $(Y, \mathcal{T})$ be a topological space and X be a nonempty set. Let $f : X \to Y$ be a function. Then $\mathcal{T}_1 = \{f^{-1}(S) : S \in \mathcal{T}\}$ is a topology on X.

We shall prove a lemma first.

Lemma 1.3.1: Preimage of Union and Intersection

Let $f : X \to Y$. Then we have
$$f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha) \tag{1.1}$$
and
$$f^{-1}\left(\bigcap_{\alpha \in I} U_i\right) = \bigcap_{\alpha \in I} f^{-1}(U_i) \tag{1.2}$$
for any $U_\alpha \subseteq Y$.

Proof.
- $\forall x \in f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right)$, we have $f(x) \in \bigcup_{\alpha \in I} U_\alpha$, then $f(x) \in U_\alpha$ for some $\alpha \in I$. Then $x \in f^{-1}(U_\alpha)$, thus $x \in \bigcup_{\alpha \in I} f^{-1}(U_\alpha)$. Other way is the same.

- $\forall x \in f^{-1}\left(\bigcap_{\alpha\in I} U_\alpha\right)$, we have $f(x) \in \bigcap_{\alpha\in I} U_\alpha$, then $f(x) \in U_\alpha$ for all $\alpha \in I$. Then $\forall \alpha \in I, x \in f^{-1}(U_\alpha)$, thus $x \in \bigcap_{\alpha\in I} f^{-1}(U_\alpha)$. Other way is the same.

□

Remark:
Note that the image does not preserve set operation as preimages. Like for $f : \mathbb{R} \to \mathbb{R}, x \mapsto 0$, we have $f(\{0\} \cap \{1\}) \neq f(\{0\}) \cap f(\{1\})$.

We now prove theorem 1.3.1

Proof.

- First we have $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, so $\emptyset, X \in \mathcal{T}_1$.
- Let $\{U_\alpha\}_{\alpha\in I}$ be a subset of $\mathcal{T}_1$. Then let $U_\alpha = f^{-1}(V_\alpha)$ where $V_\alpha \in \mathcal{T}$. Then

$$\bigcup_{\alpha\in I} U_\alpha = f^{-1}\left(\bigcup_{\alpha\in I} V_\alpha\right) \in \mathcal{T}_1$$

- Similarly, we have

$$\bigcap_{i=1}^{n} U_i = f^{-1}\left(\bigcap_{i=1}^{n} V_i\right) \in \mathcal{T}_1$$

□

1.3.1 T_1-Spaces

Definition 1.3.2: T_1-Spaces

A topological space $(X, \mathcal{T})$ is a T_1-space $\forall x \in X$, the set $\{x\}$ is closed in $\mathcal{T}$.

Of course, this means that any finite subset of X is closed, so the finite-closed topology is indeed a T_1-space.

We can also rephrase the definition of T_1-space by separation.

Theorem 1.3.2: Equivalent Definition of T_1-space

A topological space is a T_1-space iff

$$\forall x, y \in X, x \neq y, \exists U, V \in \mathcal{T}(x \in U \wedge y \notin U \quad \wedge \quad x \notin V \wedge y \in V) \tag{1.3}$$

Proof.

- The if part: Let $U = \{x\}^c$ and $V = \{y\}^c$ would do.
- The only if part: $\forall y \neq x$, let $V_y \in \mathcal{T}$ be such that $x \notin V_y \wedge y \in V_y$. Then $\bigcup_{y\neq x} V_y = \{x\}^c$ is open so $\{x\}$ is closed.

□

1.3.2 T_0-Spaces and the Sierpinski Space

Definition 1.3.3: T_0-Space

A topological space $(X, \mathcal{T})$ is a T_0-space if $\forall a, b \in X, a \neq b$, there is an open set containing a but not b, or there exists an open set containing b but not a. That is

$$\exists U \in \mathcal{T}(a \in U \wedge b \notin U) \quad \vee \quad \exists V \in \mathcal{T}(b \in V \wedge a \notin V)$$

Theorem 1.3.3

Every T_1-space is a T_0-space.

Proof. We assume that X has at least 2 points. If $(X, \mathcal{T})$ is a T_1-space, then $\forall a \neq b$, we have $\{b\}$ is closed, then $X - \{b\} \in \mathcal{T}$ contains a but not b. □

The other way is not correct.

Definition 1.3.4: Sierpinski Space

Let $X = \{0, 1\}$, $\mathcal{T} = \{\emptyset, \{0\}, X\}$ be a topology on X. This is called Sierpinski Space.

It is easy to say that the Sierpinski space is a T_0-space but not T_1-space.
Similar to arbitrary topologies we have

Proposition:

If $X = n$, then the number of T_0-space increases as n increases.

Proof. Again by induction. If $X = n$, and there are M different T_0-spaces. Let $Y = n + 1$, the additional point is x. For any $\mathcal{T}$ such that $(X, \mathcal{T})$ is a T_0-space. $\mathcal{T}' = \mathcal{T} \cup \{U \cup \{x\} : U \in \mathcal{T}\}$ is a topology on Y ($\mathcal{T}'$ is constructed by $\mathcal{T}$ and adding x to every element of $\mathcal{T}$), and $(Y, \mathcal{T}')$ is a T_0-space.

We now construct a T_0-space that does not have this structure. Let $Y = \{y_1, \ldots, y_{n+1}\}$, then Let $Y_i = \{y_1, \ldots, y_i\}$, then let $\mathcal{T} = \{Y_0, \ldots, Y_{n+1}\}$, then $(Y, \mathcal{T})$ is a T_0-space. □

Remark:
Our intuition of T_0 and T_1 spaces comes from "how nicely can we distinguish points in the space".
In T_0-spaces, we can "tell any two points apart", that is, there does not exist two points that are either in or not in the same open set. As the only properties of a topological space is the open sets, we can only distinguish two points if there exists an open set containing one but not the other.
In T_1-spaces, we can "tell any point apart from a closed set". This ensures that we can tell two points from both directions. Moreover, every point is a closed set gives us convenience for the definition of limits and continuity.

1.3.3 Countable-Closed Topology

Definition 1.3.5: Countable-Closed Topology

Let X be any infinite set. Then $\mathcal{T}_c = \{U \subseteq X : X - U \text{ is countable } \vee U = \emptyset\}$ is the countable-closed topology on X.

This is very similar to the Finite-Closed Topology 1.3.1.

1.3.4 Intersection of Two Topologies

Proposition: **Intersection of topologies**

Let $\mathcal{T}_\alpha, \alpha \in I$ be topologies on X. Then

1. $\mathcal{T} = \bigcap_{\alpha \in I} T_\alpha$ is a topology on X.
2. If $(X, \mathcal{T}_\alpha)$ are T_1-spaces, then $(X, \mathcal{T})$ is a T_1-space.

Proof. 1. First $\emptyset, X \in \mathcal{T}$. And we have if $U_\beta \in \mathcal{T}$, then $\bigcup U_\beta \in \mathcal{T}$. Intersection the same.

2. If $\forall x \in X, \{x\}$ is closed in $\mathcal{T}_\alpha$ for all $\alpha \in I$, then $X - \{x\} \in \mathcal{T}_\alpha$. Then $X - \{x\} \in \bigcap \mathcal{T}_\alpha$, meaning $\{x\}$ is closed in $\mathcal{T}$.

□

1.3.5 Door Space

Definition 1.3.6: Door Space

A topological space $(X, \mathcal{T})$ is a door space iff $\forall U \in \mathcal{T}$ is open or closed.

Example: **Door Spaces**

- The indiscrete space is a door space.

1.3.6 Saturated Set

Definition 1.3.7: Saturated Sets

If $(X, \mathcal{T})$ is a topological space and $S \subseteq X$, then S is a saturated set if it is the intersection of open sets

Remark:
Well, it can be the intersection of infinite number of open sets, so it is not necessarily open.

Example: **Saturated Sets**

- Every open set is a saturated set. In a finite set, every saturated set is open.
- In a T_1-space every set is a saturated set.

  Proof. For any set $S \subseteq X$, we have $\forall x \in X - S, \{x\}$ is closed, so $X - \{x\}$ is open. And

  $$S = \bigcap_{x \in X - S} (X - \{x\})$$

  is a saturated set. $\square$

- The other way is also true: If every subset is saturated, then it is a T_1-space.

  Proof. It is quite straightforward. $\forall A \subseteq X$, we have $X - A$ is a union of closed sets. Let $A = X - \{x\}$ would do. $\square$

There are obvious sets that are not saturated. Like the discrete topology of any nonempty set.

Chapter 2

Euclidean Topology

The Euclidean topology is a central character of the topology story, acting as an inspiration for future thoughts as well.

2.1 The Euclidean Topology on $\mathbb{R}$

Definition 2.1.1: The Euclidean Topology

A subset $S \subseteq \mathbb{R}$ is said to be open in Euclidean topology if $\forall x \in S$, $\exists a, b \in \mathbb{R}$ with $a < b$, such that $x \in (a, b) \subseteq S$.

We shall prove that the Euclidean topology is indeed a topology on $\mathbb{R}$.

Proof.

- Firstly, $\emptyset$ and $\mathbb{R}$ are open.
- Now let $\{A_i : i \in I\}$ be a collection of open sets. Let $A = \bigcup_{i \in I} A_i$. For $\forall x \in A$, we have $x \in A_i$ for some $i \in I$. Since A_i is open, $\exists a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ such that $x \in (a_i, b_i) \subseteq A_i$. Then $x \in (a_i, b_i) \subseteq A$. Hence A is open.
- Let $A_1, A_2, \ldots, A_n$ be a finite collection of open sets. Let $A = \bigcap_{i=1}^{n} A_i$. For $\forall x \in A$, we have $x \in A_i$ for all $i = 1, 2, \ldots, n$. Since A_i is open, $\exists a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ such that $x \in (a_i, b_i) \subseteq A_i$. Let $a = \max\{a_1, a_2, \ldots, a_n\}$ and $b = \min\{b_1, b_2, \ldots, b_n\}$. Then $x \in (a, b) \subseteq A$. Hence A is open.

$\square$

Remark:
This definition is a little bit complicated to understand. We shall clarify it by seeing that the openness and closedness are indeed what we mean in our usual sense.

1. Let $r, s \in \mathbb{R}$ with $r < s$. Then the open interval (r, s) and $(r, +\infty)$ and $(-\infty, s)$ are open and not closed in Euclidean topology.
2. Let $r, s \in \mathbb{R}$ with $r < s$. Then the closed interval $[r, s]$ and $[r, +\infty)$ and $(-\infty, s]$ are closed and not open in Euclidean topology.
3. Each singleton set $\{a\}$ is closed and not open in $\mathbb{R}$.

---

4. The only clopen sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$. (We shall prove this later)

The Euclidean topology also helps us to understand why do we define open sets to be closed under arbitrary union and finite intersection. An infinite intersection of open sets may be not open.

$$\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

We shall also clarify that not all open sets in $\mathbb{R}$ are open intervals. For example, $(0,1)\cup(2,3)$.

Example: **Integers and Rationals in $\mathbb{R}$**

- The set of integers $\mathbb{Z}$ is closed and not open in $\mathbb{R}$.

  Proof. Just make $\mathbb{Z} = \bigcup_{x\in\mathbb{Z}} \{x\}$ would do. □

- The set of rationals $\mathbb{Q}$ is neither open nor closed in $\mathbb{R}$.

  Proof. There is no open interval in $\mathbb{Q}$ and $\mathbb{Q}^c$ □

2.1.1 F_σ-Sets and G_δ-Sets

Definition 2.1.2: F_σ-Sets

Let $(X, \mathcal{T})$ be a topological space. A subset $S \subseteq X$ is said to be an F_σ-set if it is the union of a *countable* number of closed sets.

Definition 2.1.3: G_δ-Sets

Let $(X, \mathcal{T})$ be a topological space. A subset $S \subseteq X$ is said to be an G_δ-set if it is the intersection of a *countable* number of open sets.

Example:

- All intervals in $\mathbb{R}$ are F_σ-sets.

  Proof. Let $I = (a, b)$. Then $I = \bigcup_{n\in\mathbb{N}} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$. The other intervals can be proved similarly. □

- All intervals in $\mathbb{R}$ are G_δ-sets.

  Proof. Let $I = [a, b]$. Then $I = \bigcap_{n\in\mathbb{N}} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$. □

- $\mathbb{Q}$ is a F_σ-set but not a G_δ-set.

  Proof. $\mathbb{Q} = \bigcup_{x\in\mathbb{Q}} \{x\}$. (We shall prove that $\mathbb{Q}$ is not a G_δ-set later) □

Theorem 2.1.1: Complements of F_σ-sets and G_δ-sets

The complement of an F_σ-set is a G_δ-set.
The complement of an G_δ-set is a F_σ-set.

2.2 The Basis of a Topology

Our intuitive idea of open sets makes it easier for us to picture the Euclidean topology.

Proposition:

A subset $S \subseteq \mathbb{R}$ is open iff it is a union of open intervals.

Proof. If S is an open set, then $\forall x \in \mathbb{R}, \exists I_x = (a, b) \subseteq S$ such that $x \in I_x$. We now claim that $S = \bigcup_{x \in S} I_x$.

First we have $S \subseteq \bigcup_{x \in S} I_x$, and we have $\forall x, I_x \subseteq S$, so $\bigcup_{x \in S} I_x \subseteq S$. □

Therefore, to depict the open sets in Euclidean topology, it suffices to show that all open intervals are open sets.

Definition 2.2.1: Basis of a Topology

Let $(X, \mathcal{T})$ be a topological space. A basis for the topology $\mathcal{T}$ is a collection $\mathcal{B}$ of open sets such that every open set in $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

We can understand this by "$\mathcal{B}$ generates $\mathcal{T}$".

Example: **Basis of Topology**

- Let $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, then $\mathcal{B}$ is a basis for the euclidean topology on $\mathbb{R}$.

  In fact, $\mathcal{B}' = \{(a, b) : a, b \in \mathbb{Q}, a < b\}$ is also a basis for the euclidean topology on $\mathbb{R}$.

  Proof. We shall verify that every element in $\mathcal{B}$ is a union of open intervals in $\mathcal{B}'$. Let $I = (a, b) \in \mathcal{B}$, then
  $$I = \bigcup_{a<x<y<b, x,y \in \mathbb{Q}} (x, y)$$
  □

- Let $(X, \mathcal{T})$ be a discrete space and $\mathcal{B}$ be the collection of all singletons in X. Then $\mathcal{B} = \{\{x\}, x \in X\}$ is a basis for $\mathcal{T}$.

- For all topological spaces $(X, \mathcal{T})$, $\mathcal{B} = \mathcal{T}$ is a basis for $\mathcal{T}$.

We see that there can be many basis for the same topology. However not all collections of open sets can be a basis for a topology. For example, if there is an element that does not belong to any element in $\mathcal{B}$, then $\mathcal{B}$ cannot be a basis for any topology. Even if every element in X belongs to some element in $\mathcal{B}$, $\mathcal{B}$ may not be a basis for any topology.

Example:

Let $X = \{a, b, c\}$ and $\mathcal{B} = \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Then $\mathcal{B}$ is not a basis for any topology on X. Writing all possible unions, we have

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

But $\mathcal{T}$ is not a topology for $\{c\} = \{a, c\} \cap \{b, c\}$ is not in $\mathcal{T}$.

Remark:
The topology generated by a basis $\mathcal{B}$ is actually

$$\mathcal{T} = \left\{\bigcup S : S \subseteq \mathcal{B}\right\}$$

To verify that two basis generate the same topology, we only need to verify that every element in one basis can be written as a union of elements in the other basis. That is, the elements in one basis is open in the topology generated by the other basis, and vice versa.

Theorem 2.2.1: Conditions for a Collection to be a Basis

Let X be a nonempty set and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology on X iff

1. $X = \bigcup\limits_{B \in \mathcal{B}} B$
2. $\forall B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is a union of elements of $\mathcal{B}$.

Proof. This is really quite straightforward. The "if" part is just the definition of a basis. The "only if" part is also easy to prove. Let $\mathcal{T} = \{\bigcup S : S \subseteq \mathcal{B}\}$

First we have $\emptyset, X \in \mathcal{T}$.

Then let $U, V \in \mathcal{T}$. Then $U = \bigcup S$ and $V = \bigcup T$ for some $S, T \subseteq \mathcal{B}$. Then $U \cap V = \bigcup S \cap T$. Since $S \cap T \subseteq \mathcal{B}$, $U \cap V \in \mathcal{T}$.

Finally let $\{U_i\}_{i \in I}$ be a collection of elements in $\mathcal{T}$. Then $U_i = \bigcup S_i$ for some $S_i \subseteq \mathcal{B}$. Then $\bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup S_i = \bigcup S$ for some $S \subseteq \mathcal{B}$. Hence $\bigcup_{i \in I} U_i \in \mathcal{T}$. □

And now it is far more easier to define topologies: we only need to write a basis.

Definition 2.2.2: Euclidean Topology on $\mathbb{R}^n$

An open rectangle in $\mathbb{R}^n$ has the form $\{\langle x_1, \ldots, x_n \rangle : a_i < x_i < b_i\}$ for some $a_i, b_i \in \mathbb{R}$. We define $\mathcal{B}$ to be the collection of all open rectangles in $\mathbb{R}^n$. Then the topology generated by $\mathcal{B}$ is called the Euclidean topology on $\mathbb{R}^n$.

Proposition: **Disks of the Euclidean Topology**

The disc $D = \{\langle x_1, \ldots, x_n\rangle : x_1^2 + \ldots + x_n^2 < r^2\}$ is an open set in $\mathbb{R}^n$.
We also have a more general result: every disk $D = \{\langle x_1, \ldots, x_n\rangle : \sum_{i=1}^n (x_i - a_i)^2 < r^2\}$ is an open set in $\mathbb{R}^n$.

Proof. We do this by finding a small open rectangle in the disc that contains the point.

Let $x = \langle x_1, \ldots, x_n\rangle \in D$. Let $r' = r - \sqrt{x_1^2 + \ldots + x_n^2}$. Then the open rectangle $R_x = \left\{\langle y_1, \ldots, y_n\rangle : |y_i - x_i| < \frac{r'}{4n}\right\}$ is contained in the disc and contains x.

Thus we have the disc

$$D = \bigcup_{x \in D} R_x$$

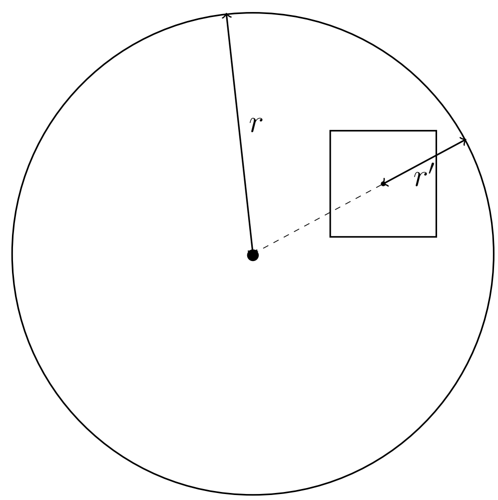

Figure 2.1: Finding Rectangle

$\square$

Theorem 2.2.2: Disks as Basis

The collection of all discs in $\mathbb{R}^n$ is a basis for the Euclidean topology on $\mathbb{R}^n$.

Proof. First we verify that all discs in $\mathbb{R}^n$ are indeed a basis. For convenience, we denote a disk with center a and radius r as $D(a, r)$.

- First $\mathbb{R}^n = \bigcup_{r>0} D(0, r)$.
- Then let $D_1(a_1, r_1), D_2(a_2, r_2)$ be any open disks with $D_1 \cap D_2 \neq \emptyset$, for any $a \in D_1 \cap D_2$, let $r = \min\{r_1 - |a - a_1|, r_2 - |a - a_2|\}$. Then $D(a, r) \subseteq D_1 \cap D_2$. So $D_1 \cap D_2 = \bigcup_{a \in D_1 \cap D_2} D(a, r)$.

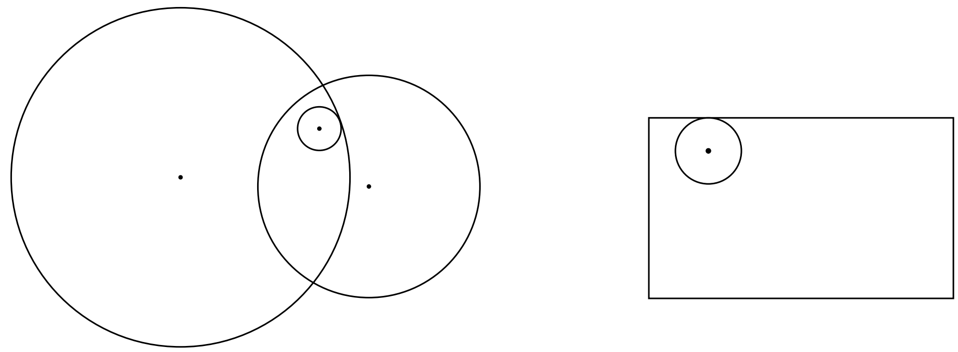

Figure 2.2: Disks as Basis

To prove that the topology generated by disks is indeed the Euclidean topology, we only need to show that every open rectangle is a union of disks. Let R be an open rectangle, $\forall a \in R$, let r be the least distance from a to the boundary of R. Then $D(a,r) \subseteq R$. Then $R = \bigcup_{a \in R} D(a,r)$. $\square$

2.2.1 Second Axiom of Countability

Definition 2.2.3: Second Axiom of Countability

A topological space $(X, \mathcal{T})$ is said to satisfy the second axiom of countability if there exists a countable basis for $\mathcal{T}$.

Example: **Second Axiom of Countability**

- The Euclidean topology on $\mathbb{R}$ satisfies the second axiom of countability. (For $\mathcal{B} = \{(a,b) : a, b \in \mathbb{Q}, a < b\}$)
- Similarly, the Euclidean topology on $\mathbb{R}^n$ satisfies the second axiom of countability. (For $\mathcal{B} = \{D(a,r) : a \in \mathbb{Q}^n, r \in \mathbb{Q}\}$)
- The discrete topology on an uncountable set does not satisfy the second axiom of countability.

  Proof. Well every basis must have every singleton set, so the basis must be uncountable. $\square$

Proposition: **Open Subsets of $\mathbb{R}$**

Every open subset of $\mathbb{R}$ are F_σ-sets and G_δ-sets.

Proof.
- Let $S \subseteq \mathbb{R}$ be open. Then because S can be written as a union of countability many of $\mathcal{B} = \{(a, b), a, b \in \mathbb{Q}, a < b\}$. And each (a, b) is a F_σ-set, so S is a F_σ-set.
- S is open, so it is a G_δ-set.

□

2.2.2 Product Topology

Definition 2.2.4: Product Topology

Let $\mathcal{B}_1$ be a basis for $(X, \mathcal{T}_1)$, and $\mathcal{B}_2$ be a basis for $(Y, \mathcal{T}_2)$. Then $\mathcal{B} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a basis for the product topology on $X \times Y$.
In fact, the product topology is just $\mathcal{T} = \{T_1 \times T_2 : T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2\}$.

We first prove that $\mathcal{B}$ is a basis.

Proof.
- First $\forall \langle x, y\rangle \in X \times Y$, we have $x \in B_x$ for some $B_x \in \mathcal{B}_1$ and $y \in B_y$ for some $B_y \in \mathcal{B}_2$. Then $\langle x, y\rangle \in B_x \times B_y \in \mathcal{B}$. So $X \times Y = \bigcup_{B \in \mathcal{B}} B$.
- For $B_1 \times B_2, B_1' \times B_2' \in \mathcal{B}$, we have $(B_1 \times B_2) \cap (B_1' \times B_2') = (B_1 \cap B_1') \times (B_2 \cap B_2')$. Since $B_1 \cap B_1'$ and $B_2 \cap B_2'$ are unions of elements in $\mathcal{B}_1$ and $\mathcal{B}_2$, $(B_1 \cap B_1') \times (B_2 \cap B_2')$ is a union of elements in $\mathcal{B}$.

□

2.3 Basis of a Given Topology

Theorem 2.3.1: Conditions for being a basis of given topology

Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B}$ of subsets of X is a basis for $\mathcal{T}$ iff

- $\forall B \in \mathcal{B}$ is open. That is, $\mathcal{B} \subseteq \mathcal{T}$.
- $\forall U \in \mathcal{T} \forall x \in U \exists B_x \in \mathcal{B}, x \in B_x \subseteq U$.

Proof. This is quite straightforward. The "if" part is just definition.
For the "only if" part, we have $\forall U \in \mathcal{T}$, $U = \bigcup_{x \in U} B_x$ for some $B_x \in \mathcal{B}$. □

Proposition:

Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $U \subseteq X$ is open in $\mathcal{T}$ iff $\forall x \in U$, $\exists B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$.

Note that this proposition is exactly how we came to define the Euclidean topology on $\mathbb{R}$.

Theorem 2.3.2: Verifying Same Topology for Different Basis

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ are basis for $\mathcal{T}_1$ and $\mathcal{T}_2$ on X, then $\mathcal{T}_1 = \mathcal{T}_2$ iff

- $\forall B \in \mathcal{B}_1$ and $\forall x \in B$, $\exists B_x \in \mathcal{B}_2$ such that $x \in B_x \subseteq B$.
- $\forall B \in \mathcal{B}_2$ and $\forall x \in B$, $\exists B_x \in \mathcal{B}_1$ such that $x \in B_x \subseteq B$.

Proof. This is exactly what we mean by saying one basis is open is the sense of the other. ☐

Proposition: **Larger Basis**

Let $\mathcal{B}$ be a basis for $(X, \mathcal{T})$, If $\mathcal{B}_1 \subseteq X$ with $\mathcal{B} \subseteq \mathcal{B}_1 \subseteq \mathcal{T}$, then $\mathcal{B}_1$ is also a basis for $\mathcal{T}$.

Proof. This is fairly easy. For $\bigcup \mathcal{B}_1 = \bigcup \mathcal{B} = X$ and intersection closed under $\mathcal{B}$ obviously implies intersection closed under $\mathcal{B}_1$. ☐

This would imply that there are uncountably many basis for the Euclidean topology on $\mathbb{R}$.

Example: **The Topology on $C[0,1]$**

$C[0,1]$ is the set of all continuous functions on $[0,1]$. We shall define a topology on $C[0,1]$ by using the following basis.

1. Let $M(f,\epsilon) = \left\{g : g \in C[0,1], \int_0^1 |f(x) - g(x)| \,\mathrm{d}x < \epsilon\right\}$. (This is the set of functions that are "close" to f)

   The collection $\mathcal{M} = \{M(f,\epsilon) : f \in C[0,1], \epsilon > 0\}$ is a basis for a topology $\mathcal{T}_1$ on $C[0,1]$.

   Proof.
   - First we have $f \in M(f,\epsilon)$ obviously. So $C[0,1] = \bigcup_{f \in C[0,1]} f$.
   - Let $M(f_1,\epsilon_1), M(f_2,\epsilon_2) \in \mathcal{M}$, then for $\forall g \in M(f_1,\epsilon_1) \cap M(f_2,\epsilon_2)$, we need to find an ϵ such that $M(g,\epsilon) \subseteq M(f_1,\epsilon_1) \cap M(f_2,\epsilon_2)$.

     Let $I_1 = \int_0^1 |f_1(x) - g(x)| \,\mathrm{d}x$, $I_2 = \int_0^1 |f_2(x) - g(x)| \,\mathrm{d}x$. We have $I_1 < \epsilon_1$ and $I_2 < \epsilon_2$.

     Let $\epsilon = \min\{\epsilon_1 - I_1, \epsilon_2 - I_2\}$. Then $\forall f \in M(g,\epsilon)$ we have $\int_0^1 |f(x) - g(x)| \,\mathrm{d}x < \epsilon$. So

$$
\begin{aligned}
I_1 = \int_0^1 |g(x) - f_1(x)| \,\mathrm{d}x &> \int_0^1 ||f(x) - g(x)| - |f(x) - f_1(x)|| \,\mathrm{d}x \\
&\geq \left| \int_0^1 |f(x) - g(x)| \,\mathrm{d}x - \int_0^1 |f(x) - f_1(x)| \,\mathrm{d}x \right|
\end{aligned}
$$

     Thus

$$
\int_0^1 |f(x) - g(x)| \,\mathrm{d}x < I_1 + \epsilon \leq \epsilon_1
$$

     Similarly we have $\int_0^1 |f(x) - g(x)| \,\mathrm{d}x < \epsilon_2$. So $M(g,\epsilon) \subseteq M(f_1,\epsilon_1) \cap M(f_2,\epsilon_2)$.

   ☐

2. Let $U(f, \epsilon) = \left\{g : g \in C[0,1], \sup_{x\in[0,1]} |f(x) - g(x)| < \epsilon\right\}$. (This is also a way to show the collection of functions that are close to f, but not the same as the one previously)

   The collection $\mathcal{U} = \{U(f, \epsilon) : f \in C[0,1], \epsilon > 0\}$ is a basis for a topology $\mathcal{T}_2$ on $C[0,1]$.

   Proof. Changing the integral to supremum, the proof is similar. □

3. We have $\mathcal{T}_1 \neq \mathcal{T}_2$.

   Proof. Intuitively, the first basis allows "sharp points" that is far from original function, while the second basis does not. Let $f \in U(f, \epsilon) \in \mathcal{U}$. For all $M(g, \epsilon') \in \mathcal{M}$ we have some function $h \in M(g, \epsilon')$ that $\sup h - \inf h > 2\epsilon$. So $h \notin U(f, \epsilon)$. Making $U(f, \epsilon)$ not open in the sense of $\mathcal{T}_1$.

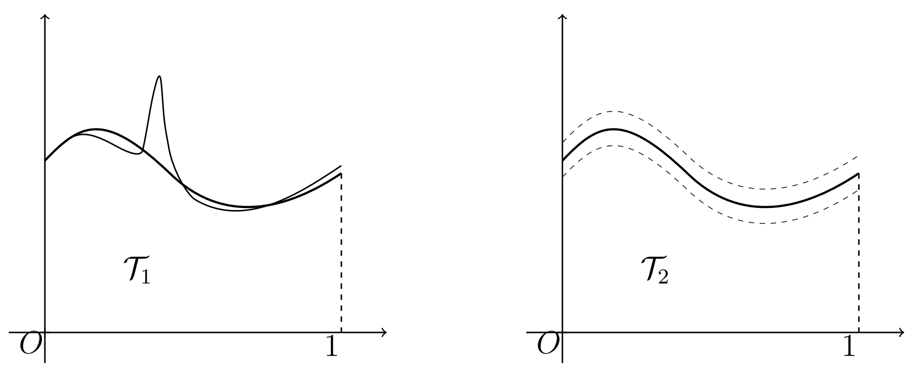

Figure 2.3: Sharp Points of Functions

□

2.3.1 Subbasis of a Topology

Definition 2.3.1: Subbasis of a Topology

Let $(X, \mathcal{T})$ be a topological space. A non-empty collection $\mathcal{S}$ of subsets of X is said to be a subbasis for $\mathcal{T}$ if the collection of all finite intersections of elements of $\mathcal{S}$ is a basis for $\mathcal{T}$.

Remark:
Unlike a basis, a subbasis does not require closure under finite intersections. This allows you to define a topology by specifying only the "essential" open sets you care about (e.g.,

preimages of projections in product topology). The topology then automatically incorporates the necessary intersections and unions. This avoids the need to specify all finite intersections upfront, making it a more economical starting point than a basis.

Example: **Subbasis**

- The collection of all open intervals of the form $(-\infty, a)$ or (b, ∞) is a subbasis for the Euclidean topology on $\mathbb{R}$. (The basis is the collection of all open intervals)

Chapter 3

Limit Points

In analysis we often consider sequences of points in a metric space. The limit of a sequence is a point that the sequence gets arbitrarily close to as the index goes to infinity. In topology however, sometimes it is not easy to depict a metric, so we need a more general concept of a limit. And this process will make us understand better of closed set as well.

Another important notion is connectedness. Intuitively, a set is connected if it is in one piece. We will see that this notion is closely related to the concept of limit points.

3.1 Limit Points and Closure

If $(X, \mathcal{T})$ is a topological space it is usual to refer elements of X as *points*.

Definition 3.1.1: Limit Point

Let A be a subset of a topological space $(X, \mathcal{T})$. A point $x \in X$ is a *limit point* of A if every open set containing x contains a point of A different from x. That is

$$\forall U \in \mathcal{T}(x \in U \to \exists a \neq x, a \in A \cap U) \to x \text{ is a limit point of } A.$$

Not a limit point mean that $\exists U \in \mathcal{T}, x \in U$ and $U - \{x\} \subseteq X - A$.

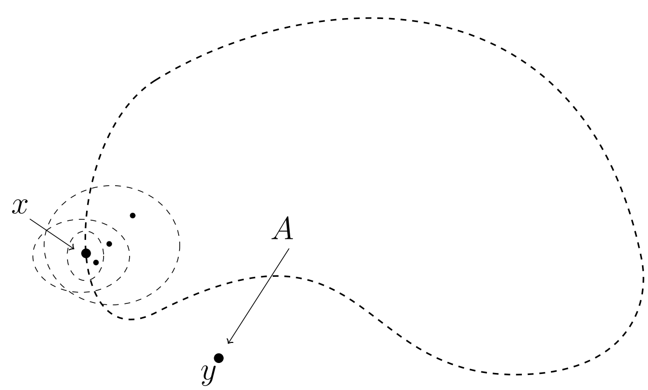

Figure 3.1: Limit Points

Remark:
This is what we mean by "there are points of A arbitrarily close to x" without explicitly representing distances.
On a more familiar note, there can be a limit point of a set that is not in the set itself, just like 0 of $(0, 1)$ in Euclidean topology of $\mathbb{R}$ (The x in figure 3.1).
Also, There are points in the set that are not limit points. Like 0 in $\{0\} \cup (1, 2)$, (The y in figure 3.1)

The next proposition is useful to identify closed sets.

Theorem 3.1.1: Closed Sets and Limit Points

Let A be a subset of the topological space $(X, \mathcal{T})$. Then A is closed if and only if it contains all its limit points.

Proof.

- Assume A is closed in $(X, \mathcal{T})$. If $p \in X - A$ is a limit point of A, then $X - A$ is an open set containing an element of A, contradicts.
- Assume A contains all its limit points. $\forall x \in X - A$ is not a limit point, so $\exists U \in \mathcal{T}$ such that $x \in U$ and $\forall a \neq x, a \notin A \cap U$. With $x \notin A$, this means $A \cap U = \emptyset$, thus $x \in U \subseteq X - A$, so $X - A$ is open.

□

Corollary 3.1.1

Let A be a subset of $(X, \mathcal{T})$, and A' be the set of all limit points of A, then $A \cup A'$ is closed. (Meaning that $A \cup A'$ would not produce any other limit points)

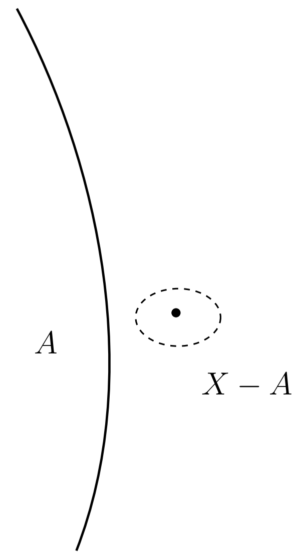

Figure 3.2: Closed Sets and Limit Points

Proof. Let $p \in X - (A \cup A')$, then p is not a limit point of A, so $\exists U \in \mathcal{T}$ with $p \in U$ and $U \cap A = \{p\}$ or $\emptyset$. But $p \notin A$, so $U \cap A = \emptyset$.

$\forall x \in U$, because U is an open set with $U \cap A = \emptyset$ so x is not a limit point of A, so $U \cap A' = \emptyset$, so $p \in U \subseteq X - (A \cup A')$. Therefore $A \cup A'$ is closed. □

Definition 3.1.2: Closure

Let A be a subset of $(X, \mathcal{T})$, and A' be the set of all limit points of A. Denote $\overline{A} = A \cup A'$ be the *closure* of A. ($\overline{A}$ is closed)

Remark:
In fact, $\overline{A}$ is the smallest closed set that contains A. That is, it is the intersection of all closed set containing A. We shall prove this.

Proposition: **Closure is Smallest**

Let S, T be non-empty subsets of $(X, \mathcal{T})$ with $S \subseteq T$.

1. If p is a limit point of S, then it is also a limit point of T.

   Proof. This is quite straightforward using $S \subseteq T$. □

2. Therefore, if B is a closed set containing A, then B must also contains A'.

Example: **Closure of Rationals**

In Euclidean topology, $\overline{\mathbb{Q}} = \mathbb{R}$.

Proof. Suppose $\exists x \in \mathbb{R} - \overline{\mathbb{Q}}$. As $\mathbb{R} - \overline{\mathbb{Q}}$ is open in $\mathbb{R}$ and open intervals are basis, so there is some open interval $x \in (a, b) \subseteq \mathbb{R} - \overline{\mathbb{Q}}$, contradicts. □

Note that this example really catches what we mean by "dense".

Definition 3.1.3: Dense

Let A be a subset of $(X, \mathcal{T})$. Then A is said to be *dense* in X if $\overline{A} = X$.

Example: **Dense sets**

- Let $(X, \mathcal{T})$ be a discrete space. Then the only dense subset of X is X itself. (Since every subset is closed)

Theorem 3.1.2: Condition for Dense sets

Let A be a subset of $(X, \mathcal{T})$, then A is dense in X if and only if $\forall U \in \mathcal{T}, U \neq \emptyset$ we have $A \cap U \neq \emptyset$.

Proof.

- If A is dense, and there is $U \in \mathcal{T}$ such that $U \neq \emptyset$ and $U \subseteq X - A$, Then $\forall x \in U$, x is not a limit point of A, and $x \notin A$, so $x \notin \overline{A}$, contradicts.

- Conversely, if $\forall U \in \mathcal{T}, U \neq \emptyset$ we have $A \cap U \neq \emptyset$, then $\forall x \in X$ is a limit point of A, just using the definition.

□

3.1.1 Intersections

Theorem 3.1.3: Intersections of closures

Let A and B be subsets of a topological space $(X, \mathcal{T})$. Then

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B} \tag{3.1}$$

Proof. $\forall x \in X - \overline{A}$, we have $U \in \mathcal{T}$ such that $x \in U \subseteq X - \overline{A}$. x is not a limit point of A, so it is not a limit point of $A \cap B$. So $x \in X - \overline{A \cap B}$. So $\overline{A}^c \subseteq \overline{A \cap B}^c$. Similarly $\overline{B}^c \subseteq \overline{A \cap B}^c$. So $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ □

Note that it can be $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$. For $A = \left\{\frac{1}{n} : n \in \mathbb{Z}_+\right\}$ and $B = \left\{-\frac{1}{n} : n \in \mathbb{Z}_+\right\}$ would do.

3.2 Neighbourhoods

Definition 3.2.1: Neighbourhoods

Let $(X, \mathcal{T})$ be a topological space. $N \subseteq X$ and $p \in N$. Then N is a neighbourhood of x if $\exists U \in \mathcal{T}$ such that $p \in U \subseteq N$.

Well, this actually depicts our intuition of p being "in" the interior of a set. Like $[-1, 1]$ is a neighbourhood of 0 but $[0, 1]$ is not in the sense of Euclidean topology. It is a slacker condition that open sets containing x.

We can reduce the condition of limit points to neighbourhoods.

Proposition: **Neighbourhoods and Limit Points**

Let A be a subset of a topological space $(X, \mathcal{T})$.

1. A point $x \in X$ is a *limit point* of A if and only if every neighbourhood containing x contains a point of A different from x.
2. A is closed iff $\forall x \in X - A$ there is a neighbourhood N of x such that $N \subseteq X - A$.
3. $A \in \mathcal{T}$ iff $\forall x \in A$ there exists a neighbourhood N of x such that $N \subseteq A$.
4. A is dense in X if and only if for every neighbourhood $N \neq \emptyset$ of x we have $A \cap N \neq \emptyset$

3.2.1 Separable Spaces

Definition 3.2.2: Separable Spaces

A topological space $(X, \mathcal{T})$ is said to be separable if it has a dense subset which is countable.

3.2.2 Interior of a Set

Definition 3.2.3: Interior of a Set

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The largest open set contained in A is called the *interior* of A, denoted $\operatorname{Int} A$. That is, it is the union of all open sets contained in A.

Theorem 3.2.1: Interior of a Set

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then

$$\operatorname{Int} A = X - \overline{X - A}$$

Proof. First we have $X - \overline{X - A}$ is an open set contained in A, so $X - \overline{X - A} \subseteq \operatorname{Int} A$.

Let $U \subseteq A$ be an open set. Then any point in U is not a limit point of $X - A$, so $U \subseteq X - \overline{X - A}$, so $\operatorname{Int} A \subseteq X - \overline{X - A}$. $\square$

Corollary 3.2.1: Density and Interior

A is dense in $(X, \mathcal{T})$ iff $\text{Int}(X - A) = \emptyset$.

This theorem gives an explicit expression of interior of a set. It can also be seen as a definition, thought less clear then definition 3.2.3.

Proposition: **Operations on Interiors of sets**

In a topological space $(X, \mathcal{T})$ let $A_i \subseteq X$.

1. $\text{Int}(A_1 \cap A_2) \subseteq \text{Int}\, A_1 \cap \text{Int}\, A_2$.
2. $\text{Int}\, A_1 \cup \text{Int}\, A_2 \subseteq \text{Int}(A_1 \cup A_2)$.

The next theorem we shall see that the dense part of a set has the same limit points as the original set.

Theorem 3.2.2: Dense part of a set

Let S be a dense set of $(X, \mathcal{T})$, then $\forall U \in \mathcal{T}, \overline{S \cap U} = \overline{U}$.

3.2.3 The Sorgenfrey Line

Let $\mathcal{B} = \{[a, b) : a \in \mathbb{R}, b \in \mathbb{Q}, a < b\}$. Then

1. $\mathcal{B}$ is the basis of a topology $\mathcal{T}_1$ on $\mathbb{R}$. The topological space is called the Sorgenfrey line.
2. If $\mathcal{T}$ is the Euclidean topology on $\mathbb{R}$, then $\mathcal{T} \subset \mathcal{T}_1$.
3. $\forall a, b \in \mathbb{R}$ with $a < b$, then $[a, b)$ is a clopen set in $(\mathbb{R}, \mathcal{T}_1)$.
4. The Sorgenfrey line is a separable space.
5. The Sorgenfrey line does not satisfies the second axiom of countability.

3.3 Connectedness

Looking more closely at clopen set, we shall observe that nontrivial clopen set would indicate some kind of separation. If $(X, \mathcal{T})$ is a topological space, and $U \in X$ is a nontrivial clopen sets, then U^c is also a clopen set. Each U and U^c can be seen as topological spaces themselves, satisfying the axioms at the own case. So U can be seen as the joint of two spaces.

Definition 3.3.1: Connectedness

Let $(X, \mathcal{T})$ be a topological space. Then it is said to be connected iff the only clopen sets are $\emptyset$ and X.

We have $\mathbb{R}$ is connected by following.

Proposition:

Let T be a clopen set of $\mathbb{R}$, then either $T = \mathbb{R}$ or $T = \emptyset$.

Proof. Suppose $T \neq \mathbb{R}$ and $T \neq \emptyset$, then there is an element $x \in T$ and $z \in \mathbb{R} - T$, assume $x < z$. Let $S = T \cap [x, z]$, then S is closed. (To limit our discussion).

Let $p = \sup S$, then $p \in S$. Also $z \in \mathbb{R} - S$ so $p < z$. But T is also an open set so $p \in (a, b) \subseteq T$, let $p < t < \min\{b, z\}$, so $t \in T$ and $t \in [p, z]$. Thus $t \in S$, contradicts. □

Theorem 3.3.1: Condition for Not Connected

Let $(X, \mathcal{T})$ be any topological space. Then $(X, \mathcal{T})$ is not connected iff it has proper non-empty disjoint open subsets A, B such that $X = A \cup B$.

Proof. This is obvious since $A = X - B$ is closed. □

Chapter 4

Homeomorphisms

In each branch of mathematics it is essential to recognize when two structures are equivalent. The equivalence of topological spaces are called homeomorphisms.

4.1 Subspaces

Definition 4.1.1: Subspaces

Let Y be a non-empty subset of a topological space $(X, \mathcal{T})$. The collection $\mathcal{T}_Y = \{O \cap Y : O \in \mathcal{T}\}$ is a topology on Y called the subspace topology. The topological space $(Y, \mathcal{T}_Y)$ is said to be a subspace of $(X, \mathcal{T})$.

Well it is not hard to show that if $\mathcal{B}$ is a basis of $(X, \mathcal{T})$, then the set $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis of $(Y, \mathcal{T}_Y)$.

Also we have

Proposition: **Chain of Subspaces**

Let $A \subseteq B \subseteq X$ where $(X, \mathcal{T})$ is a topological space. Let $\mathcal{T}_B$ be the topology induced on B by $\mathcal{T}$ and $\mathcal{T}_1, \mathcal{T}_2$ be topology induced on A by $\mathcal{T}_B$ and $\mathcal{T}$, then $\mathcal{T}_1 = \mathcal{T}_2$.
That is, a subspace of a subspace is a subspace.

4.1.1 Hausdorff Spaces or T_2-spaces

Definition 4.1.2: Hausdorff Spaces or T_2-spaces

A topological space $(X, \mathcal{T})$ is a Hausdorff space (T_2-space) if

$$\forall a, b \in X, a \neq b, \exists U, V \in \mathcal{T}(a \in U \wedge b \in V \wedge U \cap V = \emptyset)$$

This is a simultaneous 2-side separation compare to T_1-spaces, which only needs respective 2-side separation, and T_0-spaces only needs 1-side separation.

Unsurprisingly we have: every T_2-space is a T_1-space.

4.1.2 Regular Spaces and T_3-spaces

Definition 4.1.3: Regular Spaces or T_3-Spaces

A topological space $(X, \mathcal{T})$ is a regular space if

$$\forall A \subseteq X, A \notin \mathcal{T}, \forall x \in X - A, \exists U, V \in \mathcal{T}(x \in U \wedge A \subseteq V \wedge U \cap V = \emptyset)$$

This is a two side separation between a point and a non-open set.
If $(X, \mathcal{T})$ is regular and is a T_1-space, then is is said to be a T_3-space.

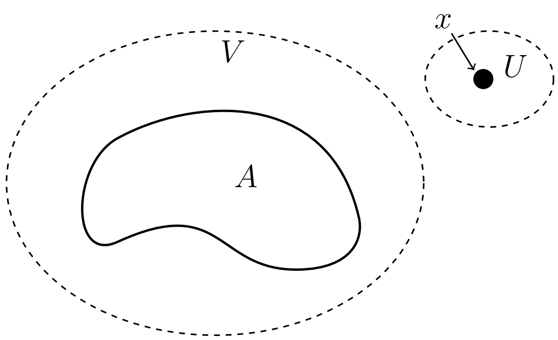

Figure 4.1: Regular Space

4.1.3 Homeomorphisms

We now turn to the notion of equivalent topological spaces.

Definition 4.1.4: Homeomorphisms

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. Then they are homeomorphic if there exists a bijection $f : X \to Y$ which has the following properties:

1. $\forall U \in \mathcal{T}_2, f^{-1}(U) \in \mathcal{T}_1$.
2. $\forall V \in \mathcal{T}_1, f(V) \in \mathcal{T}_2$.

f is said to be a homeomorphism between $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$. And we write $(X, \mathcal{T}_1) \cong (Y, \mathcal{T}_2)$.

Well the two conditions of the bijection f can be seen as a bijection $g : \mathcal{T}_1 \to \mathcal{T}_2$ with $V \mapsto f(V)$.

Catching our intuition of what it means to be identical.

Proposition: **Homeomorphisms is an equivalence relation**

Homeomorphisms is an equivalence relation.

Example: **Open Intervals in $\mathbb{R}$ is homeomorphic**

- Every two non-open intervals $(a, b), (c, d) \in \mathbb{R}$ are homeomorphic.
- $\mathbb{R}$ is homeomorphic to $(-1, 1)$ in Euclidean topology.

Proof. Give

$$f : (0, 1) \to (a, b), f(x) = a(1 - x) + bx$$

would enough. We also give

$$g : (-1, 1) \to \mathbb{R}, g(x) = \frac{x}{1 - |x|}$$

□

As you can see, the homeomorphism to the "infinite large plane" is quite useful. We can use this to prove that a disk and a rectangle is homeomorphic, for they both $\cong \mathbb{R}^2$.

4.1.4 Group of Homeomorphisms

Definition 4.1.5: Group pf Homeomorphisms

Let $(X, \mathcal{T})$ be a topological space, and G be the set of all homeomorphic of X to itself. Then G is a group under composition of functions.

4.2 Non-Homeomorphisms Spaces

It is usually harder to identify two spaces that are not homeomorphic to each other. We do this by observing some characteristic properties that two homeomorphic space have in common.

Proposition:

Any topological space homeomorphic to a connected space is connected.

The following are some properties preserved by homeomorphisms.

- Connectedness
- T_0-space, T_1-space, T_2-space, T_3-space.
- Regular space.
- Satisfying the second axiom of countability.

- Separable space.
- Discrete Space, indiscrete space.
- finite-closed topology, countable-closed topology.

We move on to identity the relationship of $\mathbb{R}$ and $\mathbb{R}^2$. The following states the connected subspaces of $\mathbb{R}$.

Definition 4.2.1: Interval

A subset $S \subseteq \mathbb{R}$ is said to be an interval if it has the property:

$$\forall x, z \in S, y \in \mathbb{R}(x < y < z \to y \in S)$$

Remark:
Note that every interval has the following form:

$$\{a\}, [a, b], (a, b], (a, b), [a, b), \text{ and change some to infinity }.$$

Proposition: **Connected Subspaces of $\mathbb{R}$**

A subspace $S \subseteq \mathbb{R}$ is connected iff it is an interval.

Proof. First all intervals are connected.

Conversely, let S be connected. Suppose $x, z \in S, y \in \mathbb{R}, x < y < z, y \notin S$, then $(-\infty, y) \cap S = (-\infty, y] \cap S$ is a clopen subset, which is not trivial. □

This illustrate what we mean by connectedness.

Now we show that $(0, 1)$ is not homeomorphic to $[0, 1]$. We give a result beneath.

Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ be a homeomorphism. Let $a \in X$, then $X - \{a\}$ is a subspace of X and has induced topology $\mathcal{T}_2$. Also $Y - \{f(a)\}$ has induced $\mathcal{T}_2$. Then $(X - \{a\}, \mathcal{T}_2)$ is homeomorphic to $(Y - \{f(a)\}, \mathcal{T}_3)$.

Corollary 4.2.1

If $a, b, c, d \in \mathbb{R}$ with $a < b, c < d$, then

1. $(a, b) \ncong [c, d)$.
2. $(a, b) \ncong [c, d]$.
3. $[a, b) \ncong [c, d]$

Proof. Let $(X, \mathcal{T}) = [c, d)$ and $(T, \mathcal{T}_1) = (a, b)$. If the two is homeomorphic, we have $X - \{c\} \cong Y - \{y\}$ for some $y \in Y$. But $X - \{c\} = (c, d)$ is connected, and $Y - \{y\}$ is not. □

4.2.1 Local Homeomorphisms

Definition 4.2.2: Local Homeomorphisms

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_1)$ be topological spaces. A map $f : X \to Y$ is said to be a local homeomorphism if each point $x \in X$ has an open neighborhood U such that the restriction $f|_U : U \to f(U)$ is a homeomorphism.

Remark:
A local homeomorphism is a map that, in small neighborhoods, behaves like a homeomorphism (locally preserving structure), even if it might not be globally bijective or continuous in the inverse sense. This is useful for capturing the idea that local behavior can be simpler and well-understood (like Euclidean space), even if the global structure of a space is more complicated.

4.2.2 Semi-Open sets

Definition 4.2.3: Semi-Open Sets

A subset A of $(X, \mathcal{T})$ is a semi-open set if $O \in \mathcal{T}$ such that $O \subseteq A \subseteq \overline{O}$.

There are 3 important ways of creating now topological spaces: forming subspaces, products and quotient spaces.

Chapter 5

Continuous Mappings

5.1 Continuous Mappings

We are already familiar with continuous mappings on $\mathbb{R} \to \mathbb{R}$. We do this my the so called $\epsilon - \delta$ language. We shall generalize the concept without the definition of a metric.

- Let $f : \mathbb{R} \to \mathbb{R}$, then f is continuous if and only if

  $$\forall a \in \mathbb{R}, \forall f(a) \in U \in \mathcal{T}, \exists a \in V \in \mathcal{T}, (f(V) \subseteq U)$$

  This change the way we say $f(a) - \epsilon$ to $f(a) + \epsilon$ to an open set containing $f(a)$.

We are tempted to define continuous from above pattern. But we have a more elegant equivalent condition as follows.

Lemma 5.1.1

Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Then the following are equivalent.

1. $\forall U \in \mathcal{T}', f^{-1}(U) \in \mathcal{T}$.
2. $\forall a \in X, \forall U \in \mathcal{T}', f(a) \in U, \exists V \in \mathcal{T}(a \in V \wedge f(V) \subseteq U)$.

Proof.
- Form (1) to (2) is quite straightforward, taking $V = f^{-1}(U)$.
- Let $U \in \mathcal{T}'$, if $f^{-1}(U) = \emptyset$ then $f^{-1}(U) \in \mathcal{T}$.
  Otherwise, $\forall a \in f^{-1}(U)$, then $\exists V \in \mathcal{T}, a \in V, f(V) \subseteq U$, thus $V \subseteq f^{-1}(U)$, so $f^{-1}(U) \in \mathcal{T}$.

$\square$

Definition 5.1.1: Continuous Function

Let $(X, \mathcal{T})$ and $(Y \in \mathcal{T}_1)$ be topological spaces and $f : X \to Y$. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ is said to be a continuous mapping if $\forall U \in \mathcal{T}_1, f^{-1}(U) \in \mathcal{T}$.

Proposition: **Composition of Continuous Mapping**

f, g are continuous mapping then $g \circ f$ is continuous.

The next result shows that we can also define continuous mapping via closed sets.

Theorem 5.1.1: Continuity and Closed Sets

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_1)$ be topological spaces. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ is continuous if and only if $\forall$ closed sets $S \subseteq Y$, we have $f^{-1}(S)$ is a closed subset of X.

Proof. $f^{-1}(S^c) = f^{-1}(S)^c$. □

The next result illustrate the relation between continuous mappings and homeomorphisms.

Theorem 5.1.2: Continuity and Homeomorphisms

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and $f : X \to Y$. Then f is a homeomorphism iff

- f is continuous.
- f is a bijection.
- f^{-1} is continuous.

Proof. This follows directly from the definition of continuity and homeomorphisms. □

Remark:
The need of f^{-1} being continuous is that the domain may have "more" open sets. While f^{-1} maps open sets to open set, f may not. For example

$$f : [0, 1) \to S^1, f(t) = (\cos 2\pi t, \sin 2\pi t)$$

Where $f([0, \frac{1}{2}))$ is not open. ($[0, \frac{1}{2})$ is open)

Also, the restriction of a continuous mapping is a continuous map.

Proposition: **Restriction of Continuous Mappings**

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces, and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ a continuous mapping, $A \subseteq X$, and $\mathcal{T}_2$ the induced topology on A. Then $f|_A$ is continuous.

5.1.1 Coarser Topology and Finer Topology

Definition 5.1.2: Coarser and Finer Topology

Let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on X. Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ ($\mathcal{T}_2$ is coarser than $\mathcal{T}_1$) if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

5.2 Intermediate Value Theorem

Proposition:

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_1)$ be topological space and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ surjective and continuous. If $(X, \mathcal{T})$ is connected, then $(Y, \mathcal{T}_1)$ is connected.
That is, any continuous image of a connected space is connected.

Proof. $U \in Y$ is clopen then $f^{-1}(U)$ is clopen. □

Remark:
The surjectivity means that we cannnot freely modify Y.

Definition 5.2.1: Path-Connected

A topological space $(X, \mathcal{T})$ is path-connected if for each pair of distinct point $a, b \in X$, there exists a continuous mapping $f : [0, 1] \to (X, \mathcal{T})$, such that $f(0) = a$ and $f(1) = b$.
f is a path joining a and b.

Proposition:

Every path-connected space is connected.

Proof. Let $(X, \mathcal{T})$ be a path-connected space and not connected. Then it has proper non-empty clopen subset U, let $a \in U$ and $b \in X - U$. f be a path joining a and b.
$f^{-1}(U)$ is a clopen set of $[0, 1]$ but it is neither $\emptyset$ nor $[0, 1]$, contradicts. □

Remark:
The converse of the proposition is false. For example, Let

$$X = \left\{ \langle x, y \rangle : y = \sin \frac{1}{x}, 0 < x \leq 1 \right\} \cup \{\langle 0, y \rangle : -1 \leq y \leq 1\}$$

Then X is connected but it is not path-connected.

We can now show that $\mathbb{R} \ncong \mathbb{R}^2$. Clearly $\mathbb{R}^2 - \{\langle 0, 0 \rangle\}$ is path connected and therefore is connected. But $\forall z \in \mathbb{R}, \mathbb{R} - \{a\}$ is not connected.

Similarly we have

Theorem 5.2.1: Continuous Image of a path-connected space

A continuous image of a path-connected space is path-connected.

Proof. Composition of continuous mapping would produce a new path. □

The next is a beautiful application of continuous mappings.

Theorem 5.2.2: the Weierstrass Intermediate Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) \neq f(b)$, then $\forall p$ between $f(a)$ and $f(b)$, $\exists c \in [a, b], f(c) = p$.

Proof. As $[a, b]$ is connected then $f([a, b])$ is connected, which is a interval. So $\forall p$ between $f(a)$ and $f(b)$ is in $f([a, b])$. □

Corollary 5.2.1: Fixed Point Theorem

Let $f : [0, 1] \to [0, 1]$ be continuous. Then $\exists z \in [0, 1]$ such that $f(z) = z$.
(This is a special case of the Brouwer Fixed Point Theorem, which deals with the n-dimensional cube)

Chapter 6

Metric Spaces

Now we have distances!

6.1 Metric Spaces

Definition 6.1.1: Metric Spaces

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ that satisfies:

- $\forall a, b \in X, d(a, b) \geq 0$ and $d(a, b) = 0 \leftrightarrow a = b$.
- $\forall a, b \in X, d(a, b) = d(b, a)$.
- $\forall a, b, c \in X, d(a, c) \leq d(a, b) + d(a, c)$.

Then d is said to be a metric on X. (X, d) is called a metric space.

Example: **Metrics**

1. $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, d(a, b) = |a - b|$.
2. The Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.
3. The discrete metric $d : X \times X \to \mathbb{R}$,
$$d(a, b) = \begin{cases} 0, & a = b \\ 1, & a \neq b \end{cases}$$
4. Let $f, g \in C[0, 1]$, then
$$d_1(f, g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x$$
$$d_2(f, g) = \sup \{|f(x) - g(x)| : x \in [0, 1]\}$$

Definition 6.1.2: Open Ball

Let (X, d) be a metric space and $r \in \mathbb{R}_+$. Then the open ball at $a \in X$ of radius r is that

$$B_r(a) = \{x \in X : d(a, x) < r\}.$$

Theorem 6.1.1: Metric Spaces and Topology

Let (X, d) be a metric space. Then the collection of open balls in (X, d) is a basis of a topology $\mathcal{T}$ on X.

Proof. Quite EASY. □

Definition 6.1.3: Equivalent Metrics

Metrics on a set X is said to be equivalent if they induce the same topology on X.

It is easy to check that no matter the balls are disks (Euclidean metric) or squares (the absolute value sum) all induce the Euclidean topology on $\mathbb{R}^2$.

This next result is a restatement of a previous theorem, but is also familiar to us in analysis, which gives the definition of open sets.

Proposition:

Let (X, d) be a metric space and $\mathcal{T}$ be the topology induced on X by d. The $U \subseteq X$ is open in $(X, \mathcal{T})$ if and only if

$$\forall a \in U, \exists \epsilon > 0, (B_\epsilon(a) \subseteq U).$$

We shall see that though every metric can induce a topology, there are topology that cannot be formed by metric. We see this by the connection of metrics and Hausdorff Spaces.

Theorem 6.1.2: Metric and Hausdorff Spaces

Let (X, d) be ant metric spaces and $\mathcal{T}$ is a topology induced on X by d. Then (X, d) is a Hausdorff space.

Proof. Let $a, b \in X, a \neq b$, then let $\epsilon = d(a, b) > 0$. Consider the open balls $B_{\epsilon/2}(a)$ and $B_{\epsilon/2}(b)$. We prove that $B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b) = \emptyset$, which is obvious for the triangle inequality. □

Definition 6.1.4: Metrizable

A space $(X, \mathcal{T})$ is said to be metrizable if there exists a metric d on a set X with the property that $\mathcal{T}$ is the topology induced by d.

It is obvious that every subspace of a metrizable space is metrizable.

6.1.1 Normal Spaces and T_4-space

Definition 6.1.5: Normal Space and T_4-space

A topological space $(X, \mathcal{T})$ is said to be normal space if for each pair of disjoint closed sets A, B, there exists disjoint open sets U, V such that $A \subseteq U, B \subseteq V$.
A Hausdorff Normal space is called T_4-space.

Proposition: **Normal Spaces**

Every metrizable space is a normal space.

Proof. First define the distance of a point and a set. $d(x, A) = \inf \{d(x, a) \mid a \in A\}$. Suppose A and B are disjoint closed sets, we let:

- $\forall a \in A, r = \frac{1}{3}d(a, B)$, and $U = \bigcup_{a \in A} B(a, r)$.
- $\forall b \in B, s = \frac{1}{3}d(b, A)$, and $V = \bigcup_{b \in B} B(b, s)$.

We shall prove $U \cap V = \emptyset$. If $\exists z \in U \cap V$, Then $\exists x \in A, y \in B$ such that $d(z, x) < \frac{1}{3}d(x, B)$ and $d(z, y) < \frac{1}{3}d(y, A)$. Then we have $d(z, x) < \frac{1}{3}d(x, y)$ and $d(z, x) < \frac{1}{3}d(y, x)$, which contradicts to the triangle inequality. □

Proposition:

Every T_4-space is Hausdorff.

Proof. Using the fact that in T_1-space every singleton is closed. □

6.1.2 Isometry

Definition 6.1.6: Isometric

Let (X, d) and (Y, d_1) be metric spaces. If there exists a surjective mapping $f : (X, d) \to (X, d_1)$ such that $\forall x_1, x_2 \in X$

$$d(x_1, x_2) = d_1(f(x_1), f(x_2)). \tag{6.1}$$

Then such a mapping is an isometry.

Proposition: **Isometric and Homeomorphic**

Isometric metric spaces are homeomorphic. Every isometry is a homeomorphism.

Definition 6.1.7: Isometric Embedding

Let (X, d) and (Y, d_1) be metric space and $f : X \to Y$. Let $Z = f(X)$ and $d_2 = d_1|_Z$. If $f : (X, d) \to (Z, d_2)$ is an isometry, then f is said to be an isometric embedding of (X, d) onto (Y, d_1).

6.1.3 First Axiom of Countability

Definition 6.1.8: First Axiom of Countability

A topological space $(X, \mathcal{T})$ us said to satisfy the first axiom of countability if $\forall x \in X$ there is a countable family $\{U_i(x)\} \subseteq \mathcal{T}$ of open sets containing x, such that $\forall x \in U \subseteq \mathcal{T}$, U has at least one of U_i as subsets.
(We can assume $U_{k+1} \subseteq U_k$ as explained below).

Remark:
This gives the notion that every point x can be infinitely-small approached via a countable many of open sets.
NOTE that we can choose another list $\{V_n\}$ with $V_1 = U_1$ and $V_n = \bigcap_{i=1}^n U_n$. Note that $\{V_n\}$ also satisfies the first axiom of countability with $V_{k+1} \subseteq V_k$ for all $k \in \mathbb{Z}_+$.

Theorem 6.1.3: Metrizable are First countable

Every metrizable space satisfies the first axiom of countability.

Proof. Let $U_n = B_{\frac{1}{n}}(x)$ would do. □

Theorem 6.1.4: Second Countable and First Countable

Every topological space that satisfies the second axiom of countability also satisfies the first axiom of countability.

Proof. If $(X, \mathcal{T})$ has a countable basis $\mathcal{B}$, let $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$, then $\mathcal{B}_x$ is countable. For all $x \in U \in \mathcal{T}$, we have $U = \bigcup_\alpha B_\alpha$ for some $B_\alpha \in \mathcal{B}$. Then $x \in B_\alpha \in B_x$ for some α. □

Remark:
The first countability represents some sense of "local countability". It only needs a point can be countably approached. In fact, taking the countable sequence $\{U_i\}$ as basis we can generate a "neighborhood topology". While the second countability represent "global countability".

6.1.4 Total Boundedness

Definition 6.1.9: Total Boundedness

A subset S of a metric space (X, d) is totally bound iff

$$\forall \epsilon > 0, \exists x_1, \ldots, x_n \in X, S \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

That is, S can be written as a finite number of open balls of radius ϵ.

6.1.5 Locally Euclidean Spaces and Topological Manifolds

Definition 6.1.10: Locally Euclidean

A topological space $(X, \mathcal{T})$ is said to be locally Euclidean if $\exists n \in \mathbb{Z}_+$ such that $\forall x \in X, \exists U(x) \in \mathcal{T}$, such that $x \in U(x)$ and $U(x)$ is homeomorphic to an open ball of 0 in $\mathbb{R}^n$ with the Euclidean metric.
A Hausdorff locally Euclidean space is said to be a topological manifold.

6.2 Convergence of Sequences

We are already very familiar of the notions of convergence.

Definition 6.2.1: Convergence

Let (X, d) be a metric space and $x_1, \ldots, x_n, \ldots$ a sequence in X. Then the sequence converge to $x \in X$ iff

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, d(x, x_n) < \epsilon.$$

Proposition: **Uniquely Convergence**

If $x_n \to x$ and $x_n \to y$ then $x = y$.

We can describe the topological structure solely by convergence.

Proposition: **Describe Topology with Convergence**

Let (X, d) be a metric space. Then a subset $A \subseteq X$ is closed in (X, d) iff every convergent sequence of points in A converges to a point in A.

Proof.

- Suppose A is closed and $x_n \to x$ with $x \notin A$. Then $\exists B_\epsilon(x) \cap A = \emptyset$, contradicts.
- If $X - A$ is not open, then $\exists x \in X - A$ such that $\forall \epsilon > 0, \exists y \in A, d(x, y) < \epsilon$. Letting $\epsilon = \frac{1}{n}$ we have $y_n \to x$ which contradicts.

$\square$

Proposition: **Describe Continuous Function with Convergence**

Let (X, d) and (Y, d_1) be metric spaces and f a mapping of X to Y. Let $\mathcal{T}$ and $\mathcal{T}_1$ be topologies determined by d and d_1.
Then f is continuous iff $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Proof. To verify f^{-1} : closed sets to closed sets would suffice. Let A be a closed set in X, let $x_1, \ldots x_n, \ldots$ be a sequence in $f^{-1}(A)$. As $x_n \to x$, we have $f(x_n) \to f(x)$. But $f(x) \in A$, then $x \in f^{-1}(A)$, so $f^{-1}(A)$ is continuous.

In the other direction, if f is continuous and $x_n \to x$. Then $\forall \epsilon > 0, \exists \delta > 0$ such that

$$x \in B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))).$$

For $\exists N \in \mathbb{N}, \forall n > N, x_n \in B_\delta(x)$, then $f(x_n) \in B_\epsilon(f(x))$, then $f(x_n) \to f(x)$. $\square$

Corollary 6.2.1: Continuous by $\epsilon - \delta$ language

Let (X, d) and (Y, d_1) be metric spaces, $f : X \to Y$ and $\mathcal{T}, \mathcal{T}_1$are topologies determined by d, d_1. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ is continuous iff

$$\forall x_0 \in X, d(x, x_0) < \delta \, (d_1(f(x), f(x_0))) < \epsilon.$$

6.2.1 Distance of two sets

Definition 6.2.2: Distance of two Sets

Let A, B be nonempty sets in (X, d). Define

$$\rho(A, B) = \inf \{d(a, b) \mid a \in A, b \in B\} \tag{6.2}$$

be the distance of A and B.

Proposition: **Closure by Distance of Sets**

If S is an nonempty subset of (X, d), then $\overline{S} = \{x \mid x \in X, \rho(\{x\}, S) = 0\}$.

6.2.2 Convergence in a General Point of View

We now discuss convergence in an arbitrary topological space.

Definition 6.2.3: Convergence in General

Let $(X, \mathcal{T})$ be a topological space and $x_1, \ldots, x_n, \ldots$ be a sequence of X. We say that $x_n \to x$ if

$$\forall U \in \mathcal{T}, x \in U, \exists N \in \mathbb{Z}_+, \forall n > N, x_n \in U.$$

6.2.3 Sequentially Closed Sets

Definition 6.2.4: Sequentially Closed

Let S be a subset of topological space $(X, \mathcal{T})$, then S is sequentially closed iff every convergent sequence in S converge to a point in S.
S is sequentially open iff $X - S$ is sequentially closed.
(Note that in a metric space, sequentially closed is just closed.)

Example: **Sequentially Closed and Closed**

For the indiscrete topology on $\mathbb{R}$, every nontrivial subset is sequentially closed but not closed. (Just note that if $x \in U \in \mathcal{T}$ then $U = X$.)

Definition 6.2.5: Sequential Space

A topological space $(X, \mathcal{T})$ is a sequential space if every sequentially closed set is closed.

Definition 6.2.6: Frechet-Urysohn Spaces

A topological space $(X, \mathcal{T})$ is a Frechet-Urysohn Space if $\forall S \subseteq X$, and $\forall a \in \overline{S}$, there is a sequence $s_n \to a$ in S.

Remark:
The intuition here is that the boundary can be approached by sequences of points, apart from open sets, which is looser then the first axiom of countability.

Proposition: **About Frechet-Urysohn Space**

- Every first countable space is Frechet-Urysohn Space. (So is every metric space)

  Proof. If X is first-countable, then $\forall S \subseteq X, a \in \overline{S}$, let $\{U_i\}$ be a countable many of open sets that follows the first axiom of countability, and $U_{i+1} \subseteq U_i$, then $U_i \cap S \neq \emptyset$. So we let $x_i \in U_i \cap S$. To prove that $x_n \to a$, $\forall a \in V \in \mathcal{T}$, we have $V \subseteq U_N$ for some N, and $\forall n > N, x_n \in U_n \subseteq U_N$. □

- Every Frechet-Urysohn space is a sequential space.

  Proof. For every sequentially closed set S in a Frechet-Urysohn space, $\forall a \in \overline{S}$, there is a sequence $x_n \to a$ in S, so $a \in S$, thus is a closed set. □

- Every subspace of a Frechet-Urysohn space is a Frechet-Urysohn space.

  Proof. For any $(Y, \mathcal{T}') \subseteq (X, \mathcal{T})$ is a subspace, Let $S \subseteq Y$, and $\overline{S}_Y \subseteq \overline{S}_X \subseteq X$. (We can do this for every closed set in subspace is obtained by intersection of Y and a closed set in X.) then $\forall a \in \overline{S}_Y \subseteq Y, a \in \overline{S}_X \subseteq X$, applying Frechet-Urysohn on X would do. □

6.2.4 Countable Tightness

Definition 6.2.7: Countable Tightness

A topological space $(X, \mathcal{T})$ has countable tightness if for each $S \subseteq X, x \in \overline{S}$, there exists a countable $C \subseteq S$ such that $x \in \overline{C}$.

Proposition: **Countable Tightness and Sequential Space**

If $(X, \mathcal{T})$ is a sequential space, then it has countable tightness.

Proof. SORRY □

We have the following chain of implications:

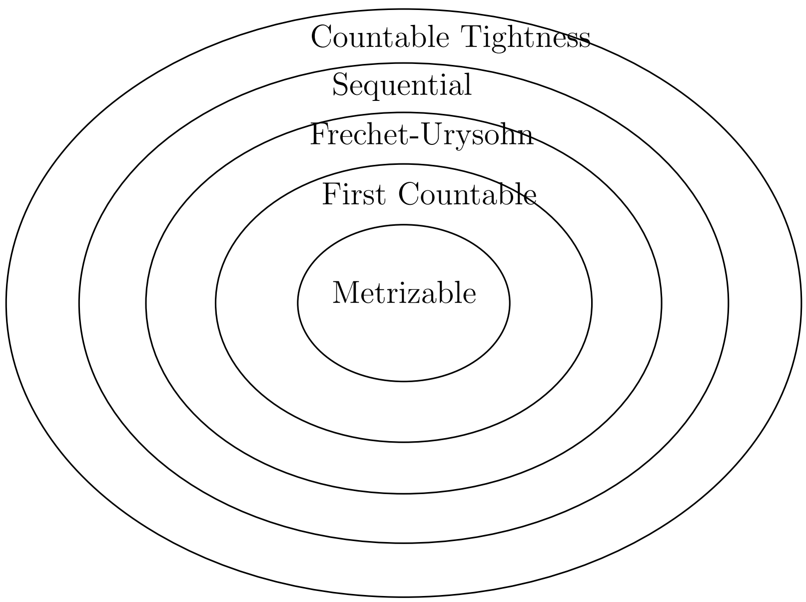

Figure 6.1: Implications

6.3 Completeness

Completeness is what we mean by every "convergent" sequence has a limit inside the range.

Definition 6.3.1: Cauchy Sequences

A sequence $x_1, \ldots, x_n, \ldots$ of points in a metric space (X, d) is a Cauchy sequence if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, d(x_m, x_n) < \epsilon. \tag{6.3}$$

It is well-known in analysis that every convergent sequence is a Cauchy sequence.

Proposition: **Convergence and Cauchy Sequence**

Let (X, d) be a metric space and $x_1, \ldots, x_n, \ldots$ a sequence of X. If $\exists a \in X, x_n \to a$, then the sequence is a Cauchy sequence.

The other way around is not true: Think about $\mathbb{Q}$ in $\mathbb{R}$.

Definition 6.3.2: Completeness

A metric space (X, d) is complete if every Cauchy sequence converges to a point in X.

Well $\mathbb{R}$ is complete in the Euclidean metric.

Lemma 6.3.1: Monotonic Subsequence

Any sequence in $\mathbb{R}$ has a monotonic subsequence.

Proof. We first define a peak point: Let $\{x_n\}$ be a sequence, then n_0 is a peak point if $\forall n > n_0, x_n \leq x_{n_0}$.

Assume $\{x_n\}$ has infinite number of peak points. Then the subsequence of peak points is a decreasing sequence.

Otherwise if there exists an $N \in \mathbb{N}$ such that $\forall n > N, n$ is not a peak point. Take $n_1 > n_0$, take $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$ and so on, we get an increasing sequence. $\square$

Proposition: **Bounded Monotonic Sequence Converges**

Let $\{x_n\}$ be a monotonic sequence in $\mathbb{R}$ with the Euclidean metric. Then $\{x_n\}$ converges to a point in $\mathbb{R}$ iff $\{x_n\}$ is bounded.

Proof. Suppose $\{x_n\}$ is increasing. Clearly if $\{x_n\}$ is not bounded then $\{x_n\}$ diverges. Otherwise let $L = \sup\{x_n\}$. Then $\forall \epsilon > 0, \exists x_n \in \{L - \epsilon, L\}$. Then obviously $x_n \to L$. $\square$

Theorem 6.3.1: Bolzano-Weiertrass Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Corollary 6.3.1: Completeness of $\mathbb{R}$

The metric space $\mathbb{R}$ with Euclidean metric is a complete metric space.
Also the result is true for $\mathbb{R}^m$.

Proposition: **Subspaces of Complete Spaces**

Let (X, d) be a complete metric space, Y a subset of X and $d_1 = d|_Y$. Then

$$Y \text{ is closed } \Leftrightarrow (Y, d_1) \text{ is complete .}$$

Proof. If Y is closed, let $\{x_n\}$ be a Cauchy sequence in Y. Then $x_n \to x$ for some $x \in X$, but as Y is closed, the $x \in Y$. The other way just use proposition 6.2. □

Remark:
We we know, $\mathbb{R}$ is complete while $(0, 1)$ is not. Therefore, completeness is not preserved by homeomorphisms. This is quite obvious since Cauchy sequences are based on metric which is not a topology property. So it is preserved by isometry.

Definition 6.3.3: Completely Metrizable

A topology is said to completely metrizable if there exists a metric d on X such that (X, d) is complete.

Definition 6.3.4: Separable

A topological space is separable if it has a countable dense subset.

It is seen that $\mathbb{R}$ and every countable topological space is separable.

Definition 6.3.5: Polish Space

A topological space $(X, \mathcal{T})$ is Polish space if it is separable and complete metrizable.

It is shown that $\mathbb{R}^n$ is Polish space.

Definition 6.3.6: Souslin Space

A topological space $(X, \mathcal{T})$ is a Souslin space (Suslin Space) if it is Hausdorff and a continuous image of a Polish space.
If $(Y, \mathcal{T}_1)$ is a topological space and $A \subseteq Y$. If $(A, \mathcal{T}_1|_A)$ is a Souslin space, then A is said to be an analytic set.

Back to isometry, we can think that $\mathbb{R}$ is a supplement of $\mathbb{Q}$ that mend the holes that are not limits of Cauchy sequences. We call such supplement a completion. We now give this intuition a formal definition.

Definition 6.3.7: Completion

Let (X, d) and (Y, d_1) be metric spaces and $f : X \to Y$. If Y is a complete metric space, $f : (X, d) \to (Y, d_1)$ is an isometric embedding and $f(X)$ is dense in Y. Then (Y, d_1) is said to be a completion of (X, d).

Remark:
The completion is a process of "filling the loopholes". Therefore some natural questions arouse:

- Every metric space has a completion?
- Is there a "smallest" completion that fills just enough loopholes without introducing

---

other points? Is the smallest filling unique?

The answer is yes.

Theorem 6.3.2: Every Metric Space has a Completion

If (X, d) is a metric space, then it has a completion.

Proof. SORRY $\square$

Theorem 6.3.3: Uniqueness of Completion

Let (A, d_1) and (B, d_2) be complete metric spaces. Let $(X, d_3) \subseteq (A, d_1), (Y, d_4) \subseteq (B, d_2)$. And X dense in A and Y in B. If there is an isometry $f : (X, d_3) \to (Y, d_4)$ then there exists an isometry $g : (A, d_1) \to (B, d_2)$ such that $g(x) = f(x), \forall x \in X$.
So we say that, up to isometry, the completion is unique.

Embedding the concept of complete metric space into normed vector spaces, we get what is called a Banach space.

Definition 6.3.8: Banach space

Let $(N, \|\|)$ be a normed vector space and d the associated metric on set N. Then $(N, \|\|)$ is said to be a Banach space if (N, d) is a complete metric space.

As we know that every incomplete normed vector space can be extended to a Banach space. And we are glad to say that the completion is also a Banach space.

Theorem 6.3.4: The completion of a normed vector space is a Banach space

Let X be any normed vector space. Then it is possible to put a normed vector space structure on $\tilde{X}$, the complete metric space constructed by X.

6.4 Contraction Mapping

Definition 6.4.1: Fixed Point

let $f : X \to X$. Then a point $x \in X$ is a fixed point iff $f(x) = x$.

Definition 6.4.2: Contraction Mapping

Let (X, d) be a metric space and $f : X \to X$. Then f is a contraction mapping if $\exists r \in (0, 1)$, such that

$$\forall x_1, x_2 \in X, d(f(x_1), f(x_2)) \leq r \cdot d(x_1, x_2) \tag{6.4}$$

This is Lipchistz Continuity.

Proposition: **Contraction Mapping is Continuous**

Let f be a contraction mapping of (X,d). Then f is a continuous mapping.

Theorem 6.4.1: Contraction Mapping Theorem (Banach Fixed Point Theorem)

Let (X,d) be a Banach space and $f : X \to X$ a contraction mapping. Then f has precisely one fixed point.

Proof. Let $x \in X$ be any point, then consider the sequence

$$x, f(x), f^{[2]}(x), \ldots, f^{[n]}(x), \ldots$$

This is a Cauchy sequence for $d(f(x), f(f(x))) \leq r \cdot d(x, f(x))$. Then

$$\begin{aligned} d(f^{[m]}(x), f^{[n]}(x)) &\leq r^m \cdot d(x, f^{[n-m]}(x)) \\ &\leq r^m \cdot \left(f(x, f(x)) + f(f(x), f^{[2]}(x)) + \ldots + d(f^{[n-m-1]}(x), f^{[n-m]}(x))\right) \\ &\leq r^m \cdot d(x, f(x)) \left(1 + r + r^2 + \ldots + r^{n-m-1}\right) \\ &\leq \frac{r^m f(x, f(x))}{1-r} \end{aligned}$$

Then the sequence tends to z, which is a fixed point.

Uniqueness is proved by $d(t,z) = d(f(t), f(z)) \leq r \cdot d(t,z)$ making $t = z$, for fixed points t, z. □

6.5 Baire Space

Theorem 6.5.1: Baire Category Theorem

Let (X,d) be a complete metric space. If $X_1, \ldots, X_n, \ldots$ is a sequence of open dense subsets of X, then the set $\bigcap_{n=1}^{\infty} X_n$ is dense in X.

Proof. It suffice to show that if U is any non-empty open subset of (X,d), then $U \cap \bigcup_{n=1}^{\infty} X_n \neq \emptyset$.

As X_1 is open and dense in X, the $U \cap X_1$ is an nonempty open subset of (X,d). Let U_1 be an open ball of radius ≤ 1 such that $\overline{U_1} \subseteq U \cap X_1$. Define inductively that U_n is an open ball of radius $\leq \frac{1}{n}$ such that $\overline{U_n} \subseteq U_{n-1} \cap X_n$.

Let $x_n \in U_n$, then the sequence $\{x_n\}$ is a Cauchy sequence. Then we have $x_n \to x$ for some $x \in X$.

Note that $\forall m \in \mathbb{Z}_+, \forall n > m, x_m \in \overline{U_m}$, therefore $x \in \overline{U_m}$. Therefore, $\forall n \in \mathbb{N}, x \in \overline{U_n}$, thus $x \in \bigcap_{n=1}^{\infty} \overline{U_n}$. Also we have $\bigcap_{i=1}^{\infty} \overline{U_i} \subseteq U \cap X_n, \forall n \in \mathbb{Z}_+$, then we have $x \in \bigcap_{n=1}^{\infty} \overline{U_n} \subseteq U \cap \bigcap_{n=1}^{\infty} X_n$. □

Definition 6.5.1: Interior, Boundary and Exterior Points

Let $(X, \mathcal{T})$ be any topological space and $A \subseteq X$.

- The largest open subset contained in A is called the interior of A, denoted $\operatorname{Int} A$. Each point $x \in \operatorname{Int} A$ is called the interior point.
- The set $\operatorname{Int}(X - A)$ is called the exterior of A, denoted $\operatorname{Ext} A$.
- Then set $\overline{A} - \operatorname{Int} A$ is called the boundary point of A.

Remark:
Well, we can see that:

- An interior point x, is that $\exists U \in \mathcal{T}, x \in U \subseteq A$.
- An exterior point is a interior point in $X - A$.
- The other points are boundary points.

Definition 6.5.2: Nowhere dense

A subset A of a topological space $(X, \mathcal{T})$ is said to be nowhere dense if $\operatorname{Int} \overline{A} = \emptyset$.

This formalize our intuition of not having a part that is dense.
We'll rephrase the Baire category theorem.

Corollary 6.5.1: Baire Category Theorem

Let (X, d) is a complete metric space. If $X_1, \dots, X_n, \dots$ is a sequence of subsets of X such that $X = \bigcup_{n=1}^{\infty} X_n$. Then $\exists n \in \mathbb{Z}_+, X_n$ is not nowhere dense.

Proof. SORRY □

Definition 6.5.3: Baire Space

A topological space $(X, \mathcal{T})$ is said to be a Baire space if $\forall \{X_n\}$ of open dense subsets of X, the set $\bigcap_{n=1}^{\infty} X_n$ is also dense in X.

Corollary 6.5.2

Every complete metrizable spaces are Baire spaces.

Remark:
Note that the definition of Baire space do not depend on metric structure, it is a topological definition.
The completeness is very important, the set $\mathbb{Q}$ is not a Baire space.

Definition 6.5.4: First and Second Category

Let Y be a subset of $(X, \mathcal{T})$. If Y is a union of a countable many nowhere dense subsets of X, then Y is said to be a set of the first category of meager in $X, \mathcal{T})$. If Y is not first category, then it is second category.

Chapter 7

Compactness

The most important topological property is compactness.

7.1 Compact Space

Definition 7.1.1: Compact Sets

Let A be a subset of a topological space $(X, \mathcal{T})$. Then A is compact iff every open cover of A has a finite subcover. That is,

$$\forall A \subseteq \bigcup_{i \in I} O_i, \exists \text{ a finite family } O_{i_1}, \ldots, O_{i_n}, A = \bigcup_{k=1}^{n} O_{i_k}$$

If X is compact, we say that $(X, \mathcal{T})$ is a compact space.

Example: **Compact Sets**

- If $(X, \mathcal{T}) = \mathbb{R}$ and $A = (0, \infty)$, then A is not compact.
- Any finite set is compact.

Remark:
As we can see, compactness is some sense of generalization of finiteness. But of course, there are infinite sets that are compact.

Theorem 7.1.1: The Internal Property of Compactness

Let A be a subset of $(X, \mathcal{T})$ and $\mathcal{T}_1$ the topology induced on A. Then A is a compact set iff $(A, \mathcal{T}_1)$ is a compact space.

Proof. □

- If $(A, \mathcal{T}_1)$ is a compact space, the let O_i be an open cover of A, then $O_i \cap S$ is an open cover of A in $(A, \mathcal{T}_1)$, which has a finite subcover $O_{i_n} \cap S$, then O_{i_n} is a finite subcover of A is $(X, \mathcal{T})$.

- If A is a compact set, then for all open cover of $(A, \mathcal{T}_1)$, similarly we have $O_i' = O_i \cap S$ for some open set $O_i \in \mathcal{T}$. The next step proceeds the same as above.

Proposition: **Closed Interval is Compact**

The closed interval $[0,1]$ is compact.

Proof. We see $[0,1]$ as a space. Let $O_i, i \in I$ be any covering of $[0,1]$. Then $\forall x \in [0,1], \exists i, x \in O_i$. As O_i is open, there exists an open interval $x \in U_x \subseteq O_i$.

Define $S \subseteq [0,1]$ by

$$S = \{z : [0,z] \text{ can be covered by a finite number of sets } U_x\}$$

If $x \in S, y \in U_x$, then $[x,y] \subseteq U_x$ (assuming $x \leq y$). So

$$[0,y] \subseteq U_{x_1} \cup \cdots \cup U_{x_n} \cup U_x$$

So $y \in S$. This implies that $\forall x \in [0,1], U_x \cap S = \emptyset$ or U_x, which means

$$S = \bigcup_{x \in S} U_x, \quad [0,1] \backslash S = \bigcup_{x \notin S} U_x$$

Thus S is clopen in $[0,1]$, but $[0,1]$ is connected and $0 \in S$, we have only $S = [0,1]$. □

7.1.1 Alexander Subbasis Theorem

Theorem 7.1.2: Alexander Subbasis Theorem

A topological space $(X, \mathcal{T})$ is compact, then every subbasis cover has a finite subcover. That is, if $\mathcal{S}$ is a subbasis of $\mathcal{T}$, and $\{O_i\} \subseteq \mathcal{S}$ is a cover of X, then O_i has a finite subcover.

7.2 The Heine-Borel Theorem

The next result states that "a continuous image of a compact space is compact".

Theorem 7.2.1

Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_1)$ be a continuous surjective map. If $(X, \mathcal{T})$ is compact then $(Y, \mathcal{T}_1)$ is compact.

Proof. If O_i is an open covering of T, then $f^{-1}(O_i)$ is an open coving of X, and that will do. □

Corollary 7.2.1: Homeomorphism and Compactness

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_1)$ be homeomorphic topological space. If $(X, \mathcal{T})$ is compact iff $(Y, \mathcal{T}_1)$ is compact.

Corollary 7.2.2: Open Intervals are not Compact

For $a, b \in \mathbb{R}$ with $a < b$, $[a,b]$ is compact while (a,b) is not compact.

Proof. $[a,b]$ is homeomorphic to $[0,1]$ and (a,b) is homeomorphic to $(0,\infty)$. □

Theorem 7.2.2: Closed Subsets of Compact Spaces

Every closed subset of a compact space is compact.

Proof. Let A be a closed subset of a compact space $(X,\mathcal{T})$. Let $U_i \in \mathcal{T}$ be an open covering of A. Then

$$X = \left(\bigcup_{i\in I} U_i\right) \cup (X - A)$$

This is an open cover of X, and thus has a subcover. □

Theorem 7.2.3: Compact sets in Hausdorff Space is Closed

A compact subset of a Hausdorff space is closed.

Proof. Let A be a compact set of a Hausdorff space $(X,\mathcal{T})$. We shall show that A contains all its limit points. Let $p \in X - A$, then $\forall a \in A, \exists U_a, V_a \in \mathcal{T}, a \in U_a, p \in V_a, U_a \cap V_a = \emptyset$.

Then we have

$$A \subseteq \bigcup_{a\in A} U_a$$

is a open covering of A. Therefore, we have

$$A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}$$

Put $U = \bigcup_{i=1}^n U_{a_i}$ and $V = \bigcap_{i=1}^n V_{a_i}$. Then $p \in V$ and $V \cap U = \emptyset$. So $V \cap A = \emptyset$. Therefore, p is not a limit point of A. □

Remark:
In a compact Hausdorff space, compact sets $\Leftrightarrow$ closed sets.

Proposition: **Compact and Bounded**

A compact subset of $\mathbb{R}$ is bounded.

Theorem 7.2.4: Heine-Borel Theorem

A is a closed bounded subset of $\mathbb{R} \Leftrightarrow A$ is compact. $(n \geq 1)$

Proof. Every closed bounded subset $A \subseteq \mathbb{R}$ has $A \subseteq [a,b]^n$ for some a, b. And A is a closed set of the compact set $[a,b]^n$. □

Similar to $\mathbb{R}^n$ we defined boundedness on a metric space as

Definition 7.2.1: Boundedness on a metric space

A subset $A \subseteq (X,d)$ is bounded if $\exists r \in \mathbb{R}, \forall a, b \in A, d(a,b) < r$.

Theorem 7.2.5: Compact subset of a metric space

Let A be a compact subset of a metric space (X, d), then A is closed and bounded

Proof. As a metric space is Hausdorff, we have A is closed. Now fix $x_0 \in X$ and define $f : (A, \mathcal{T}) \to \mathbb{R}$ by

$$f(a) = d(a, r_0), \forall a \in A.$$

Then f is continuous so $f(A)$ is compact. Thus, $f(A)$ is bounded so $f(a) = d(a, r_0) \leq M, \forall a \in A$, thus A is bounded. □

We can generalize the Heine-Borel Theorem to $\mathbb{R}^n$:

Theorem 7.2.6: Generalized Heine-Borel Theorem

A is a closed bounded subset of $\mathbb{R}^n \Leftrightarrow A$ is compact. $(n \geq 1)$

Proof. Postponed. □

Remark:
The Heine-Borel Theorem cannot be generalized to arbitrary metric spaces. As closed bounded subsets may not be compact.

Proposition: **Continuous Functions on Intervals**

Let $a, b \in \mathbb{R}$ and f is a continuous function from $[a, b]$ to $\mathbb{R}$. Then $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$.

Proof. As $[a, b]$ is connected, then $f([a, b])$ is connected hence is an interval. Also $[a, b]$ is compact so $f([a, b])$ is compact thus is a closed interval. □

7.2.1 Closed and Open Mapping

Definition 7.2.2: Closed and Open Mapping

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_1)$ be topological spaces. A mapping $f : X \to Y$ is a *closed mapping* if for every closed subset $A \subseteq X$, $f(A)$ is closed in Y. Open mappings are similar.

Proposition:

Continuous mappings on compact Hausdorff spaces are closed.

Proof. If $A \subseteq X$ is closed, then A is compact, so $f(A)$ is compact, so $f(A)$ is closed. □

Theorem 7.2.7: Compact Hausdorff Spaces are Normal

Every compact Hausdorff space is a normal space.

Proof. SORRY □

7.2.2 Other Compactness

Definition 7.2.3: Relative Compact

A subset $A \subseteq (X, \mathcal{T})$ is said to be relatively compact if $\overline{A}$ is compact.

Definition 7.2.4: Supercompact

A topological space $(X, \mathcal{T})$ is supercompact if there is a subbasis $\mathcal{S}$ of $\mathcal{T}$ such that if $\{O_i : i \in I\}$ with $O_i \in \overline{S}$ is any open cover, then there exists $j, k \in I$ such that $X = O_j \cup O_k$.

Example: **Supercompact**

[0, 1] with the Euclidean topology is supercompact.

Definition 7.2.5: Countably Compact

A topological space $(X, \mathcal{T})$ is countably compact if every countable open covering of X has a finite subcover. (Note that this is weaker than compactness).

Proposition: **Properties of Countably Compactness**

- A metrizable space is countably compact $\Leftrightarrow$ it is compact.
- The continuous image of a countably compact space is countably compact.

  Proof. Similar to the proof of 7.2.1. □

Definition 7.2.6: Locally Compact

A topological space $(X, \mathcal{T})$ is said to be locally compact if each point $x \in X$ has a compact neighborhood.

7.2.3 Generalized Convergent Sequences

We generalize the concept of convergent sequences to arbitrary topological space.

Definition 7.2.7: Convergence Generalized

A topological space $(X, \mathcal{T})$. Let $x_1, \ldots, x_n, \ldots$ be points in X. Then the sequence converge to $x \in X$ iff $\forall U \in \mathcal{T}, \exists N \in \mathbb{N}, \forall n \geq N, x_n \in U$.

Proposition: **Properties of Convergences**

- In a Hausdorff space $(X, \mathcal{T})$, every convergent sequence converge to uniquely one point.
- A sequence can converge to infinitely many points. The indiscrete topology would do.

Definition 7.2.8: Sequentially Compact

A topological space $(X, \mathcal{T})$ is sequentially compact iff every sequence has a convergent subsequence.

Definition 7.2.9: Pseudocompact

A topological space $(X, \mathcal{T})$ is pseudocompact if every continuous function $X \to \mathbb{R}$ is bounded.

Proposition: **Properties of Pseudocompactness**

- Every compact space is pseudocompact.

  Proof. A compact space in $\mathbb{R}$ is bounded. □

- Any countably compact space is pseudocompact.
- The continuous image of a pseudocompact space is pseudocompact.

  Proof. SORRY □

Review

Topological Spaces

T_x-spaces

1. **T_0-spaces:** (Definition 1.3.3) $\forall a, b \in X, a \neq b$, there is an open set containing a but not b, or there exists an open set containing b but not a. That is

$$\exists U \in \mathcal{T}(a \in U \wedge b \notin U) \quad \vee \quad \exists V \in \mathcal{T}(b \in V \wedge a \notin V)$$

2. **T_1-spaces:** (Definition 1.3.2)