Introduction to Smooth Manifolds

February 8, 2026

Contents

Chapter 1

Smooth Manifolds

In simple terms, smooth manifolds are spaces that locally look like $\mathbb{R}^n$, and on which we can do calculus. We can visualize them like smooth plane curves like circles and parabolas.

The simplest manifold and topological manifolds, which encode just the properties of what we mean by "locally look like $\mathbb{R}^n$ ". However, to do calculus (volume, curvature, etc.), we need a stronger restriction – the notion of smoothness. Intuitively, we can describe smoothness by having a tangent structure that moves continuously from point to point. For more sophisticated applications we can restrict it to be embedded in some ambient Euclidean vector space. The structure of this ambient space is superfluous that is not guaranteed by the internal structure of the manifold itself.

Also, it is evidently that we cannot define smoothness solely by topological structure. A circle and a square are homeomorphic topological space, but we all agree that square is not smooth but circle is. Therefore, we should think a smooth manifold has two layers of structure: topological manifolds and smoothness.

1.1 Topological Manifolds

Definition 1.1.1: Topological Manifolds

Suppose $(M, \mathcal{T})$ is a topological space, we say that M is a topological manifold of dimension n if it has the following property:

- M is a Hausdorff space. $\forall p \neq q$ in M, there are disjoint open sets $U, V \subseteq M$ such that $p \in U, q \in V$.
- M is second-countable. There exists a countable basis for the topology of M.
- M is locally Euclidean of dimension n : Each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$, in the Euclidean topology. We call the n here the dimension of the topological manifold, denoted $\dim M$.

The last property can be expressed explicitly as: $\forall p \in M, \exists$ open set $U \subseteq M, p \in U$ and $\hat{U} \subseteq \mathbb{R}^n$ such that $U \cong \hat{U}$.

Remark:
We can change the definition to letting U to be homeomorphic to some open balls in $\mathbb{R}^n$. This is equivalent to the original definition.

Proof. If we have a neighborhood that is homeomorphic to a open subspace of $\mathbb{R}^n$, then we have an open ball subspace that would do. □

We also abbreviate M being a topological manifold of dimension n by M^n. It is worth mentioning that we do not consider spaces with mixed dimensions, like a disjoint union of a plane and a line. The dimension here is global to all the point in the space.

Theorem 1.1.1: Topological Invariant of Dimension

A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.

Remark:
The empty set satisfies the definition of a topological manifold of dimension n for every n. But in most circumstances we shall just ignore the trivial case.

A basic example of an n-dimensional topological space is $\mathbb{R}^n$ itself. As every metrizable space is Hausdorff and $\{B(a, r) \mid a \in \mathbb{Q}^n, r \in \mathbb{Q}\}$ is a countable basis.

1.1.1 Coordinate Chart

Definition 1.1.2: Coordinate Chart

Let M be a topological manifold of dimension n, a coordinate chart on M is a pair (U, φ), where U is an open set of M and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.
An *atlas* on M is a collection of coordinate charts $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$.

By the definition of a topological manifold, $\forall p \in M$, we can find a neighborhood where we can define a (U, φ).

- If $\varphi(p) = 0$, we say that the chart is centered at p. (We can always find a chart centered at p by subtracting $\varphi(p)$.
- Given a (U, φ), we say U a coordinate domain. If $\varphi(U)$ is a ball, we say U a coordinate ball.
- φ is called a (local) coordinate map. And the component functions $(x^1, \ldots, x^n)$ of φ are called local coordinates on U. We have $\varphi(p) = (x^1(p), \ldots, x^n(p))$.

1.1.2 Examples

Example: **Graphs of Continuous Functions**

Let $U \subseteq \mathbb{R}^n$ be an open set. And $f : U \to \mathbb{R}^k$ be a continuous function. The graph of f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \wedge y = f(x)\right\} \tag{1.1}$$

with the subspace topology. Let $\pi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection map, and let $\varphi : \Gamma(f) \to U$ be the restriction of π to $\Gamma(f)$.

$$\varphi(x, y) = x, (x, y) \in \Gamma(f)$$

Then $\Gamma(f)$ is a topological manifold of dimension n. $(\Gamma(f), \varphi)$ is a global coordinate chart.

Example: **Spheres**

For each $n \in \mathbb{N}$, the uni sphere $\mathbb{S}^n$ is a subspace of $\mathbb{R}^{n+1}$, and a local part (hemisphere would do) is the graph of a continuous mapping.

Example: **Projective Spaces**

The n-dimensional real projective space $\mathbb{RP}^n$, is defined as $(X, \mathcal{T})$, where

- X is the 1-dimensional linear subspaces of $\mathbb{R}^n$. (The lines that cross the origin)
- $\mathcal{T}$ is the quotient topology.

Example: **Product Manifold**

Suppose $M_1, \ldots, M_k$ are topological manifolds of dimension $n_1, \ldots, n_k$ respectively. Then the product space $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.

Proof. The Hausdorff and second-countable properties follows from the product topology itself. Given any point $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$, we can find a neighborhood U_i of p_i such that $U_i \cong \hat{U}_i \subseteq \mathbb{R}^{n_i}$. Then $U = U_1 \times \ldots \times U_k$ is a neighborhood of p and homeomorphic to $\hat{U}_1 \times \ldots \times \hat{U}_k \subseteq \mathbb{R}^{n_1 + \ldots + n_k}$. $\square$

1.1.3 Topological Properties of Manifolds

We shall see that manifolds have a well-behaved topological structure, thanks to the Hausdorff and second-countable properties.

Lemma 1.1.1: Precompact Coordinate Balls

Every topological manifold has a countable basis of precompact coordinate balls. (Precompact means its closure is compact)

First we shall show that every second countable space is Lindelöf(every open cover has a countable subcover).

Proof. First, let $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$ be a countable basis of the topology of M. Given any open cover $\{U_\alpha \mid \alpha \in A\}$ of M, for each B_i, we can find a U_{α_i} such that $B_i \subseteq U_{\alpha_i}$. Then $\{U_{\alpha_i} \mid i \in \mathbb{N}\}$ is a countable subcover of M. □

Now we prove the lemma. For any chart (U, φ), as $\varphi(U)$ is an open subset of $\mathbb{R}^n$, we can find a countable basis of precompact balls $\{B_i \mid i \in \mathbb{N}\}$ of $\varphi(U)$. Then $\{\varphi^{-1}(B_i) \mid i \in \mathbb{N}\}$ is a countable basis of precompact coordinate balls of U. As M is Lindelöf, we can find a countable collection of charts that cover M. The union of the countable bases of precompact coordinate balls of these charts is a countable basis of precompact coordinate balls of M.

Connectedness Topological manifolds also have nice connectedness properties.

Proposition: **Connectedness Properties of Manifolds**

Let M be a topological manifold, then

- M is locally path-connected.
- M is connected iff it is path-connected.
- The components of M are the same as its path components.
- M has countably many components, each is an open subset of M and a topological manifold itself.

Proof. Since each coordinate ball is path-connected, M has a basis of path-connected neighborhoods, so it is locally path-connected. The second and third properties follows from general topology. The openness of components follows from local path-connectedness. The countability of components follows from second-countability and the disjointness of components.(The components are an open cover of M, so we can find a countable subcover. As the components are disjoint, the only subcover is itself.) □

Local Compactness and Paracompactness Topological manifolds are also locally compact and paracompact.

Definition 1.1.3: Exhaustion

Let X be a topological space, an exhaustion of X is a sequence of compact sets $\{K_j\}_{j\in\mathbb{Z}}$ such that

- $K_j \subseteq \operatorname{Int} K_{j+1}$ for all $j \in \mathbb{Z}$.

- $\bigcup_{j\in\mathbb{Z}} K_j = X$.

We say that X is *exhausted* by $\{K_j\}_{j\in\mathbb{Z}}$.

We can see that for a second-countable locally compact Hausdorff space, we can find a countable exhaustion. This is because we can find a countable basis of compact sets, and we can take the union of these compact sets to form an exhaustion.

Proposition: **Local Compactness and Paracompactness of Manifolds**

Let M be a topological manifold, then

- M is locally compact.
- M is paracompact. In fact, given any open cover $\mathcal{X}$ of M and any basis $\mathcal{B}$, there is a countable, locally finite open refinement of $\mathcal{X}$ by elements of $\mathcal{B}$.

Proof. Local compactness follows from the fact that each point has a precompact coordinate ball neighborhood. As second-countable Hausdorff spaces are normal, and every regular Lindelöf space is paracompact, M is paracompact. For a construction, let $\{K_j\}_{j\in\mathbb{Z}}$ be an exhaustion of M by compact sets. For each j, let $V_j = K_{j+1} - \operatorname{Int} K_j$ and $W_j = \operatorname{Int} K_{j+2} - K_{j-1}$. Then □

Fundamental Groups of Manifolds The topological restrictions on manifolds also limit their fundamental groups, which is of great importance when we study covering spaces of manifolds.

Theorem 1.1.2: Fundamental Groups of Manifolds

The fundamental group of a topological manifold is at most countable.

Proof. SORRY, but fairly obvious due to the countability of coordinate balls. □

1.2 Smooth Structures

If we only have the topological structure of a manifold, we cannot do calculus on it. One may try to define derivatives of functions on the manifold by using coordinate charts, but the problem is that this definition is not invariant under homeomorphisms.

For example, the map given by

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \varphi(x, y) = (x^{1/3}, y^{1/3})$$

is a homeomorphism, but it is easy to construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that f is differentiable at 0, but $f \circ \varphi$ is not differentiable at 0.

The smooth structure allows us to formalize the idea of smooth transition between different coordinate charts, so that we can define derivatives of functions on the manifold in an invariant way. Let $U \in \mathbb{R}^n$, and $V \in \mathbb{R}^m$ be two open sets, a map $F : U \to V$ is said to be *smooth* (or C^∞, infinitely differentiable) if all its component functions have continuous partial derivatives of all orders. If F is bijective and both F and F^{-1} are smooth, then F is called a *diffeomorphism*.

Let M be a n-dimensional topological manifold, and for $p \in M$, take a coordinate chart (U, φ) with $p \in U$. We would think that a function $f : U \to \mathbb{R}$ is smooth if $f \circ \varphi^{-1} : \hat{U} \to \mathbb{R}$ is smooth (here $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$). But this would only make sense if this is independent of the choice of the chart (U, φ). Therefore, we need to impose some restrictions on the charts, called *smooth charts*. As this is not preserved by arbitrary homeomorphisms, we should thought this as a new structure on the manifold, called *smooth structure*.

Definition 1.2.1: Transition Map

For an n-dimensional topological manifold M, let (U, φ) and (V, ψ) be two coordinate charts such that $U \cap V \neq \emptyset$. Then the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \tag{1.2}$$

is called a *transition map* from (U, φ) to (V, ψ). It is a composition of homeomorphisms, so it is a homeomorphism itself.
Two coordinate charts (U, φ) and (V, ψ) are said to be *smoothly compatible* if either $U \cap V = \emptyset$, or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. A smooth atlas on M is an atlas whose charts are pairwise smoothly compatible.

However, there may be many different smooth atlases that gave the same set of smooth functions on M. We could define an equivalence relation on the set of smooth atlases, but a more straightforward way is to define a maximal smooth atlas: A smooth atlas $\mathcal{A}$ is said to be *maximal* or *complete* if any coordinate chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.

Definition 1.2.2: Smooth Structure

Let M be a topological manifold. A *smooth structure* on M is a maximal smooth atlas $\mathcal{A}$ on M. A smooth manifold is a pair $(M, \mathcal{A})$.

It is not convenient to work with maximal smooth atlases directly, so we have the following theorem that allows us to work with arbitrary smooth atlases.

Proposition: **Existence of Maximal Smooth Atlas**

Let M be a topological manifold,

- Every smooth atlas $\mathcal{A}$ on M is contained in a unique maximal smooth atlas, called the maximal smooth atlas determined by $\mathcal{A}$.
- Two smooth atlases $\mathcal{A}$ and $\mathcal{A}'$ on M determine the same smooth structure if and only if their union $\mathcal{A} \cup \mathcal{A}'$ is a smooth atlas.

Remark:
Intuitively, this means that we can define an equivalence relation on the set of smooth atlases, where two atlases are equivalent if they can be combined to form a larger smooth atlas. Each equivalence class has a unique maximal element, and all elements in the equivalence class are

just the sub-atlases of this maximal element.

Proof. Let $\mathcal{A}$ be a smooth atlas on M. Let $\overline{\mathcal{A}}$ be the set of all coordinate charts that are smoothly compatible with every chart in $\mathcal{A}$. We claim that $\overline{\mathcal{A}}$ is a maximal smooth atlas containing $\mathcal{A}$.

First, let $(U,\varphi),(V,\psi) \in \overline{\mathcal{A}}$, for $x = \varphi(p) \in \varphi(U \cap V)$, we have some chart $(W,\theta) \in \mathcal{A}$ with $p \in W$. Therefore, we have

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$$

is smooth in a neighborhood of x, so we have $\overline{\mathcal{A}}$ is a smooth atlas. Moreover, every chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ is also smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. Therefore, $\overline{\mathcal{A}}$ is maximal.

For the second part, if $\mathcal{A}$ and $\mathcal{A}'$ determine the same smooth structure, then they are both contained in the same maximal smooth atlas, so their union is a smooth atlas. Conversely, if their union is a smooth atlas, then every chart in $\mathcal{A}'$ is smoothly compatible with every chart in $\mathcal{A}$, so $\mathcal{A}' \subseteq \overline{\mathcal{A}}$. Then both $\mathcal{A}$ and $\mathcal{A}'$ are contained in $\overline{\mathcal{A}}$, so they determine the same smooth structure. □

Remark:
There exists topological manifolds that do not admit any smooth structure. For example, the E8 manifold in dimension 4. The first such example was constructed by Kervaire in 1960. On the other hand, there are also topological manifolds that admit more than one smooth structure. The first such example is the 7-sphere, discovered by Milnor in 1956. In fact, it is known that for every $n \geq 7$, there exist topological manifolds of dimension n that admit more than one smooth structure.
NOTE that different smooth manifold can be diffeomorphic, which we shall justify later.

We can produce various kinds of structures by changing the requirements on the transition maps:

- If we require the transition maps to be homeomorphisms, we get the notion of a topological manifold.
- If we require the transition maps to be diffeomorphisms of class C^k (i.e., having continuous derivatives up to order k), we get the notion of a C^k-manifold.
- If we require the transition maps to be real-analytic (can be expanded as a convergent power series around each point) diffeomorphisms, we get the notion of a real-analytic manifold.
- If we have even dimension, we can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and require the transition maps to be holomorphic (analytic) diffeomorphisms, we get the notion of a complex manifold.

1.2.1 Local Coordinate Representation

If M is a smooth manifold, any chart (U,φ) in the smooth structure is called a smooth chart, and the coordinate map φ is called a smooth coordinate map.

We say a set $B \subseteq M$ is *Regular coordinate ball* if there is a larger coordinate ball $B' \subseteq M$ such that $\overline{B} \subseteq B'$ and a smooth coordinate map $\varphi : B' \to \mathbb{R}^n$ such that for some positive number $r < r'$, we have

$$\varphi(B) = B_r(0), \qquad \varphi(\overline{B}) = \overline{B_r(0)}, \qquad \varphi(B') = B_{r'}(0) \tag{1.3}$$

Therefore, the regular coordinate ball is precompact.

Remark:
This is not true for arbitrary coordinate balls, take $M = \mathbb{R} - \{0\}$, and $B = B_{(1)}(1)$, there is no larger coordinate ball that contains the closure of B, and it is not precompact.

Proposition: **Countabe Basis of Regular Coordinate Balls**

Every smooth manifold has a countable basis of regular coordinate balls.

Proof. This is a slight improvement of lemma 1.1.1. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ be a countable atlas of smooth charts that cover M. For each $\alpha \in A$, as $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$, we can find a countable basis of regular balls $\{B_{\alpha,i} \mid i \in \mathbb{N}\}$ of $\varphi_\alpha(U_\alpha)$. Then $\{\varphi_\alpha^{-1}(B_{\alpha,i}) \mid \alpha \in A, i \in \mathbb{N}\}$ is a countable basis of regular coordinate balls of M. □

If we have a chart (U, φ), we can simply identify U with $\varphi(U) \subseteq \mathbb{R}^n$. Therefore, for simplicity we shall say that a point $p \in M$ has coordinates $(x^1(p), \dots, x^n(p))$ instead of writing $\varphi(p) = (x^1(p), \dots, x^n(p))$.

A simple example is the polar coordinate on an open set of $\mathbb{R}^2$.

1.3 Examples of Smooth Manifolds

0-dimensional Smooth Manifolds 0-dimensional topological manifolds are just countable discrete spaces. Therefore, the only smooth structure on a 0-dimensional topological manifold is the trivial one, where every chart is a homeomorphism to an open subset of $\mathbb{R}^0 = \{0\}$.

Euclidean Spaces For each $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is a smooth manifold of dimension n with the smooth structure given by the *standard smooth atlas*, which consists of the single global chart $(\mathbb{R}^n, {}_{\mathbb{R}^n})$.

There are other smooth structures on $\mathbb{R}^n$. For example, consider $\psi(x) = x^3$, then $(\mathbb{R}, \psi)$ determines a smooth structure on $\mathbb{R}$ that is different from the standard one. However, it can be shown that there are diffeomorphic.

Finite-Dimensional Vector Spaces Let V be a finite-dimensional real vector space of dimension n. Then V is isomorphic to $\mathbb{R}^n$ as a vector space if we take a basis. It is fairly obvious that all basis give the same smooth structure on V, making it a smooth manifold of dimension n, called the *standard smooth structure* on V.

1.3.1 Einstein Summation Convention

In differential geometry, we often deal with objects that have multiple components, such as vectors and tensors. To simplify the notation, we use the Einstein summation convention, which states that when an index appears twice in a single term, once as a subscript and once as a superscript, it implies summation over all possible values of that index. For example,

$$E(x) = x^i e_i = \sum_{i=1}^{n} x^i e_i$$

To be consistent, we shall use superscripts for components of vectors and subscripts for basis vectors.

Space of Matrices Let $m, n \in \mathbb{N}$, the space of all $m \times n$ real matrices, denoted by $\mathbb{R}^{m\times n}$, is a finite-dimensional vector space of dimension mn. Therefore, it has a standard smooth structure, making it a smooth manifold of dimension mn.

Open Submanifolds Let M be a smooth manifold of dimension n, and let $U \subseteq M$ be an open subset. Then U is a topological manifold of dimension n with the subspace topology. Define a smooth structure on U by

$$\mathcal{A}_U = \{(V, \varphi) \in \mathcal{A}_M, V \subseteq U\} \tag{1.4}$$

Then $\mathcal{A}_U$ is a smooth atlas on U, making it a smooth manifold of dimension n, called an *open submanifold* of M.

Remark:
As $\mathcal{A}$ is maximal, for any chart, its subchart is also in $\mathcal{A}$. Therefore, our requirement is sufficient for a mere inclusion.

The General Linear Group Let $n \in \mathbb{N}$, the general linear group $(n, \mathbb{R})$ is the set of all invertible $n \times n$ real matrices, which is an open subset of $\mathbb{R}^{n\times n}$ (the determinant function is continuous, and $(n, \mathbb{R})$ is the preimage of $\mathbb{R} - \{0\}$). Therefore, it is a smooth manifold of dimension n^2, with the smooth structure induced from the standard smooth structure on $\mathbb{R}^{n\times n}$.

Full Rank Matrices Let $m < n$ be two natural numbers, the set of all $m \times n$ real matrices of rank m, denoted by $M_m(m \times n, \mathbb{R})$, is an open subset of $\mathbb{R}^{m\times n}$ (the map that sends a matrix to the maximum absolute value of its $m \times m$ minors is continuous, and $M_m(m \times n, \mathbb{R})$ is the preimage of $(0, \infty)$). Therefore, it is a smooth manifold of dimension mn, with the smooth structure induced from the standard smooth structure on $\mathbb{R}^{m\times n}$.

For $m = n$, we have $M_n(n \times n, \mathbb{R}) = (n, \mathbb{R})$.

Linear Map Spaces Let V and W be finite-dimensional real vector spaces of dimension m and n respectively. The set of all linear maps from V to W, denoted by $\mathcal{L}(V, W)$, is a finite-dimensional vector space of dimension mn. Therefore, it has a standard smooth structure, making it a smooth manifold of dimension mn.

Graphs of Smooth Functions Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^k$ be a smooth function. The graph of f is a n-dimensional smooth manifold, by the projection map as a global smooth chart.

Example: **Spheres**

For each $n \in \mathbb{N}$, the unit sphere $\mathbb{S}^n$ is a topological n-manifold. Each hemisphere is the graph of a smooth mapping, and it is fairly easy to check that the transition maps are all smooth. Therefore, $\mathbb{S}^n$ is a smooth manifold of dimension n, called the *standard smooth structure* on $\mathbb{S}^n$.

Level Sets of Smooth Functions Suppose $U \in \mathbb{R}^n$ is an open set, and $\Phi : U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set

$$M_c = \Phi^{-1}(c) = \{x \in U \mid \Phi(x) = c\} \tag{1.5}$$

is called a *level set* of Φ. Suppose $M_c \neq \emptyset$, and for every $a \in M_c$, the derivative $D\Phi(a) : \mathbb{R}^n \to \mathbb{R}$ is non zero. Then by the implicit function theorem, take $\partial\Phi/\partial x^i(a) \neq 0$, we can find a neighborhood U_a of a such that $M_c \cap U_a$ is the graph of a smooth function from an open subset of $\mathbb{R}^{n-1}$ to $\mathbb{R}$. Therefore, M_c is a topological manifold of dimension $n-1$. By checking the transition maps, we can see that M_c is a smooth manifold of dimension $n-1$.

Projective Spaces The n-dimensional real projective space $\mathbb{RP}^n$ can be given a smooth structure by using standard charts.

Proposition: **Smooth Product Manifolds**

Suppose $M_1, \ldots, M_k$ are smooth manifolds of dimension $n_1, \ldots, n_k$ respectively. Then the product space $M_1 \times \ldots \times M_k$ is a smooth manifold of dimension $n_1 + \ldots + n_k$, with the smooth structure determined by charts of the form

$$(U_1 \times \ldots \times U_k, \varphi_1 \times \ldots \times \varphi_k)$$

where (U_i, φ_i) is a smooth chart on M_i.

Up to now, we construct smooth manifolds from topological manifolds. By the following lemma, we can construct smooth manifolds directly from smooth atlases.

Lemma 1.3.1: The Smooth Manifold Chart Lemma

Let M be a set, and let $\{U_\alpha\}_{\alpha\in A}$ be a collection of subsets of M and $\varphi_\alpha : U_\alpha \to \hat{U}_\alpha \subseteq \mathbb{R}^n$, such that

- For each $\alpha \in A$, φ_α is a bijection from U_α to an open subset $\hat{U}_\alpha$ of $\mathbb{R}^n$.
- For each $\alpha, \beta \in A$, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$, and the map

  $$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

  is a diffeomorphism.
- Countable many U_α cover M.

- For any two distinct points $p, q \in M$, either there is an $\alpha \in A$ such that $p, q \in U_\alpha$, or there are $\alpha, \beta \in A$ such that $p \in U_\alpha, q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.

Then there is a unique topology and smooth structure on M such that $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ is a smooth atlas on M, making M a smooth manifold of dimension n.

Remark:
The sets U_α gives local properties of every point in M, so we can define a topology on M by declaring open sets of M by inverses of open sets in $\mathbb{R}^n$ via the maps φ_α. The second requirement ensures that the charts are smoothly compatible, so we can define a smooth structure on M. The third requirement ensures that M is second-countable, and the fourth requirement ensures that M is Hausdorff.

Example: **Grassmann Manifolds**

Let V be a finite-dimensional real vector space of dimension n. For each $k \leq n$, the Grassmannian $G_k(V)$ is the set of all k-dimensional linear subspaces of V. We can give $G_k(V)$ a smooth structure.

Proof. SORRY □

1.4 Manifolds with Boundary

Many spaces, like closed balls and half-spaces, are not manifolds in the usual sense, because they have "edges". However, we can generalize the notion of manifolds to include such spaces because they still locally resemble Euclidean spaces, except at the boundary points.

Definition 1.4.1: Manifold with Boundary

An n-dimensional topological manifold with boundary is a Hausdorff, second-countable topological space M such that for every point $p \in M$, there exists a neighborhood U of p that is either homeomorphic to an open subset of $\mathbb{R}^n$ or to an (relative) open subset of the closed half-space $\mathbb{H}^n$, where

$$\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\right\}$$

We call a chart (U, φ) a *boundary chart* if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ with $\varphi(U) \cap \partial\mathbb{H}^n \neq \emptyset$, and an *interior chart* if $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

A point $p \in M$ is called an *interior point* if there is an interior chart (U, φ) with $p \in U$. p is a *boundary point* if there is a boundary chart (U, φ) with $p \in U$ and $\varphi(p) \in \partial\mathbb{H}^n$.

The set of all boundary points of M is called the *boundary* of M, denoted by ∂M. The set of all interior points of M is called the *interior* of M, denoted by $\operatorname{Int} M$.

Remark:
A point must be either a boundary point or an interior point. If p is not a boundary point, then either it is in the domain of an interior chart, or it is in the domain of a boundary chart

but mapped to the interior of $\mathbb{H}^n$. In the latter case, we can shrink the domain to get an interior chart containing p.

The following theorem shows that a point cannot be both a boundary point and an interior point.

Theorem 1.4.1: Topological Invariance of Boundary

Let M be a topological manifold with boundary, then each point $p \in M$ is either a boundary point or an interior point, but not both. Thus

$$M = \partial M \cup \operatorname{Int} M, \qquad \partial M \cap \operatorname{Int} M = \emptyset \tag{1.6}$$

Remark:
NOTE that here the concept of boundary is not the same as the boundary of a subspace in topology. When in confusion, we shall call the former the *manifold boundary* and the latter the *topological boundary*.
Manifold boundary is a local, absolute concept, while topological boundary is a global, relative concept. For example, consider the closed unit disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ as a manifold with boundary. The manifold boundary of D is the unit circle $\mathbb{S}^1$, while the topological boundary of D in $\mathbb{R}^2$ is also $\mathbb{S}^1$. However, if we consider D as a subspace of itself, then its topological boundary is empty, since D has no points outside itself.

Proposition: **Manifold Structure on Interior and Boundary**

Let M be a topological manifold with boundary of dimension n. Then

- The interior $\operatorname{Int} M$ is an n-dimensional topological manifold (without boundary), with the subspace topology.
- The boundary ∂M is an $(n-1)$-dimensional topological manifold (without boundary), with the subspace topology.
- M is a topological manifold (without boundary) iff $\partial M = \emptyset$.
- If $n = 0$, then $\partial M = \emptyset$ and M is a 0-dimensional topological manifold (without boundary).

Proof. SORRY □

Proposition: **Topological Properties of Manifolds with Boundary**

Let M be a topological manifold with boundary, then

- M has countable basis of precompact coordinate balls and half-balls.
- M is locally compact.

- M is paracompact.
- M is locally path-connected.
- M has countablely many components, each is an open subset of M and a connectd topological manifold with boundary itself.
- The fundamental group of M is at most countable.

1.4.1 Smooth Structure on Manifolds with Boundary

First we shall define smooth functions on arbitrary subset of $\mathbb{R}^n$:

Definition 1.4.2: Smooth Maps on subset of $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be an arbitrary subset, a map $f : A \to \mathbb{R}^k$ is said to be *smooth* if for every point $p \in A$, there is an open neighborhood U of p in $\mathbb{R}^n$ and a smooth map $\tilde{f} : U \to \mathbb{R}^k$ such that $\tilde{f}|_{U\cap A} = f|_{U\cap A}$.

The definition of smooth atlases and smooth structures on manifolds with boundary are similar to those on manifolds without boundary, except that we now allow charts to be homeomorphisms to open subsets of $\mathbb{H}^n$, and tweak the definition of smooth compatibility accordingly.

Proposition: **Properties of Smooth Manifolds with Boundary**

Let M be a smooth manifold with boundary of dimension n. Then

- The interior $\operatorname{Int} M$ is an n-dimensional smooth manifold (without boundary), with the subspace topology and the smooth structure induced from M.
- The boundary ∂M is an $(n-1)$-dimensional smooth manifold (without boundary), with the subspace topology and the smooth structure induced from M.
- Every smooth manifold with boundary has a countable basis of regular coordinate balls and half-balls.
- The smooth manifold chart lemma 1.3.1 also holds for smooth manifolds with boundary. Just replace $\mathbb{R}^n$ by $\mathbb{R}^n$ or $\mathbb{H}^n$ accordingly.

As a product of $\mathbb{H}^m$ and $\mathbb{H}^n$ is not a half space, the product of two manifolds with boundary is not a manifold with boundary in general. (It is a smooth manifold with corners, which we shall not discuss here.)

Proposition: **Products of Smooth Manifold with Boundary**

Suppose $M_1, M_2, \ldots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then the product space $M_1 \times M_2 \times \ldots \times M_k \times N$ is a smooth manifold with boundary, with the boundary

$$\partial(M_1 \times M_2 \times \ldots \times M_k \times N) = M_1 \times M_2 \times \ldots \times M_k \times \partial N$$

Chapter 2

Smooth Maps

We shall do calculus on smooth manifolds via smooth maps between them.

2.1 Smooth Functions and Smooth Maps

Although formally maps and functions are the same thing, we shall technically denote functions as maps from a manifold to $\mathbb{R}^n$ and maps as maps between manifolds.

2.1.1 Smooth Functions on Manifolds

Definition 2.1.1: Smooth Functions on Manifolds

Let M be a smooth n-manifold and $k \in \mathbb{N}$. A function $f : M \to \mathbb{R}^k$ is a **smooth function** if for every $p \in M$, there exists a smooth chart (U, φ) containing p the corresponding coordinate representation $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is a smooth function (in the usual sense) on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.
The definition for manifolds with boundary is similar.
We denote all smooth functions from M to $\mathbb{R}^k$ by $C^\infty(M, \mathbb{R}^k)$ or simply $C^\infty(M)$ when $k = 1$. It is a vector space over $\mathbb{R}$.

Remark:
If $M \subseteq \mathbb{R}^n$, the definition coincide with the usual definition of smooth functions on subsets of $\mathbb{R}^n$, obviously.

We shall see that the definition holds for all charts containing p if it holds for one chart containing p, thanks to the smoothness of transition maps.

Proposition: **Smoothness is Chart-Independent**

Let M be a smooth manifold, with or without boundary, and let $f : M \to \mathbb{R}^k$ be a function. Then f is a smooth function if and only if for every $p \in M$ and every smooth chart (U, φ) containing p, the coordinate representation $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is a smooth function (in the usual sense) on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.

Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) on M, the function

$$\hat{f} : \varphi(U) \to \mathbb{R}^k, \quad \hat{f} = f \circ \varphi^{-1} \tag{2.1}$$

is called the **coordinate representation** of f with respect to the chart (U, φ). Then the definition just says that f is smooth if and only if for every $p \in M$, there exists a chart (U, φ) containing p such that the coordinate representation $\hat{f}$ is smooth in the usual sense.

2.1.2 Smooth Maps Between Manifolds

Definition 2.1.2: Smooth Maps Between Manifolds

Let M and N be smooth manifolds. A map $F : M \to N$ is a **smooth map** if for every $p \in M$, there exist smooth charts (U, φ) on M containing p and (V, ψ) on N containing $F(p)$ such that $F(U) \subseteq V$ and the coordinate representation

$$\hat{F} : \varphi(U) \to \psi(V), \quad \hat{F} = \psi \circ F \circ \varphi^{-1} \tag{2.2}$$

is a smooth map (in the usual sense) between the open subsets $\varphi(U) \subseteq \mathbb{R}^m$ and $\psi(V) \subseteq \mathbb{R}^n$.
The definition for manifolds with boundary is similar.
We denote all smooth maps from M to N by $C^\infty(M, N)$.

Our previous definition of smooth functions is a special case of this definition when $N = \mathbb{R}^k$.

Remark:
The requirement that $F(U) \subseteq V$ is crucial, as we need to make F completely in control when we express it in coordinates. So we can identify F with its coordinate representation $\hat{F}$ on U.

Proposition: **Smooth Maps are Continuous**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a map. If F is smooth, then it is continuous.

Proof. As F is smooth, for every $p \in M$, there exist smooth charts (U, φ) on M containing p and (V, ψ) on N containing $F(p)$ such that the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth in the usual sense.

$$F_U = \psi^{-1} \circ \hat{F} \circ \varphi : U \to V$$

is continuous as a composition of continuous maps. Since F agrees with F_U on U, F is continuous at p. As p is arbitrary, F is continuous. □

Proposition: **Characterization of Smooth Maps**

Suppose M and N are smooth manifolds, with or without boundary, and let $F : M \to N$ be a map. Then F is smooth if and only if one of the following equivalent conditions holds:

- For every $p \in M$ there exist smooth charts (U, φ) on M containing p and (V, ψ) on N

containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ is smooth in the usual sense.

- F is continuous and there exists smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ on M and $\{(V_\beta, \psi_\beta)\}$ on N such that for every α and β, the composite map $\psi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to \psi_\beta(V_\beta)$ is smooth in the usual sense.

It is also obvious that smooth maps does not depend on the choice of charts, thanks to the smoothness of transition maps.

Proposition: **Smoothness is Local**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a map. Then F is smooth if and only if for every $p \in M$ and every smooth chart (U, φ) on M containing p and every smooth chart (V, ψ) on N containing $F(p)$ such that $F(U) \subseteq V$, the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth in the usual sense.

Proposition: **Algebra of Smooth Maps**

Let M, N, P be smooth manifolds, with or without boundary.

- The constant map $C : M \to N$ defined by $C(p) = q$ for some fixed $q \in N$ is smooth.
- The identity map $\mathrm{Id}_M : M \to M$ is smooth.
- If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota : U \hookrightarrow M$ is smooth.
- If $F : M \to N$ and $G : N \to P$ are smooth maps, then the composition $G \circ F : M \to P$ is smooth.

Proposition: **Smooth Maps by Components**

Suppose $M_1, \dots, M_k$ and N are smooth manifolds, with or without boundary (at most one of $M_1, \dots, M_k$ has nonempty boundary), and let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ be the projection map onto the i-th factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each component map $F_i = \pi_i \circ F : N \to M_i$ is smooth for $i = 1, \dots, k$.

Example: **Smooth Maps**

- If M is a 0-manifold, then every map $F : M \to N$ is smooth.
- The wrapping map $\epsilon : \mathbb{R} \to S^1$ defined by $\epsilon(t) = \exp(2\pi i t)$ is smooth. So is $\epsilon^n : \mathbb{R}^n \to T^n$ defined by $\epsilon^n(t_1, \dots, t_n) = (\exp(2\pi i t_1), \dots, \exp(2\pi i t_n))$.
- The inclusion map $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.
- The quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ defined by $\pi(x) = [x]$ is smooth.

- If $M_1, \dots, M_k$ are smooth manifolds, then each projection map $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth.

2.1.3 Diffeomorphisms

Definition 2.1.3: Diffeomorphisms

A **diffeomorphism** is a smooth map $F : M \to N$ that is a bijection and whose inverse $F^{-1} : N \to M$ is also smooth. If such a map exists, we say that M and N are **diffeomorphic**, denoted by $M \cong N$.

Remark:
Diffeomorphisms are isomorphisms in the category of smooth manifolds, so diffeomorphic manifolds are "the same" from the smooth manifold point of view.

Diffeomorphisms give an equivalence relation on the class of smooth manifolds. And it is fairly interesting to ask whether a given manifold has multiple smooth structures that are not diffeomorphic to each other. As it turns out, for $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure up to diffeomorphism, while for $n = 4$, there are uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$!

2.2 Partitions of Unity

The Gluing lemma in topology states that

> Let X, Y be topological spaces, and if one of the following holds:
>
> - X is the union of finitely many closed subsets $A_1, \dots, A_n$.
> - X is the union of open subsets $\{U_\alpha\}_{\alpha \in A}$.
>
> If we are given continuous maps $f_i : A_i \to Y$ (or $f_\alpha : U_\alpha \to Y$) that agree on the overlaps, then there exists a unique continuous map $f : X \to Y$ such that $f|_{A_i} = f_i$ (or $f|_{U_\alpha} = f_\alpha$).

We can glue smooth maps for the open cover case, but not for the closed cover case. This is fairly obvious, Take $f(x) = |x|$ on $\mathbb{R}$, and cover $\mathbb{R}$ by the two closed sets $(-\infty, 0]$ and $[0, \infty)$. The restrictions $f|_{(-\infty,0]}$ and $f|_{[0,\infty)}$ are both smooth, but f is not smooth at 0.

A slight disadvantage of gluing smooth maps over open covers is that we need to make sure the maps agree on the overlaps. To get around this, we introduce partitions of unity, which allow us to glue local smooth properties into global smooth properties without worrying about the overlaps.

Our discussion is based on the existence of smooth bump functions that are positive in a specific part and vanish outside a slightly larger part. Take the function

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

on $\mathbb{R}$ for example.

Lemma 2.2.1: Smooth Bump on $\mathbb{R}^n$

Given any $0 < r_1 < r_2$, there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H(x) = 1$ for $||x|| \leq r_1$, $H(x) = 0$ for $||x|| \geq r_2$, and $0 < H(x) < 1$ for $r_1 < ||x|| < r_2$.

Proof. Using f to patch the two regions together would do. □

Definition 2.2.1: Partition of Unity

Suppose M is a topological space and $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ is an open cover of M. A **partition of unity subordinate to $\mathcal{X}$** is a collection of continuous functions $\{\varphi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

- For each $\alpha \in A$, $0 \leq \varphi_\alpha(p) \leq 1$ for all $p \in M$.
- $\operatorname{supp} \varphi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.
- The family of supports $\{\operatorname{supp} \varphi_\alpha\}_{\alpha \in A}$ is locally finite.
- For every $p \in M$, $\sum_{\alpha \in A} \varphi_\alpha(p) = 1$ (only finitely many terms are nonzero by local finiteness).

If each φ_α is smooth, we say it is a **smooth partition of unity**.

Theorem 2.2.1: Existence of Smooth Partitions of Unity

Let M be a smooth manifold, with or without boundary, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be any open cover of M. Then there exists a smooth partition of unity $\{\varphi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ subordinate to $\mathcal{U}$.

Proof. SORRY □

As you can see, we can use smooth partitions of unity to glue local smooth properties into global smooth properties. This is extremely useful in differential geometry.

Definition 2.2.2: Bump functions

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be closed and $A \subseteq U \subseteq M$ for some open set U. A **bump function** for A supported in U is a continuous function $\psi : M \to \mathbb{R}$ such that

- $0 \leq \psi(p) \leq 1$ for all $p \in M$.
- $\psi(p) = 1$ for all $p \in A$.
- $\operatorname{supp} \psi \subseteq U$.

Proposition: **Existence of Smooth Bump Functions**

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be closed and $A \subseteq U \subseteq M$ for some open set U. Then there exists a smooth bump function for A

supported in U.

Proof. Use the existence of smooth partitions of unity. Let $U_0 = U, U_1 = M - A$. □

Now we deal with smooth maps on arbitrary subsets of manifolds. Suppose M, N are smooth manifolds, with or without boundary, and $A \subseteq M$ is arbitrary. A map $F : A \to N$ is **smooth** if for every $p \in A$, there exists an open neighborhood U of p in M and a smooth map $\tilde{F} : U \to N$ such that $\tilde{F}|_{U \cap A} = F|_{U \cap A}$.

Lemma 2.2.2: Extension Lemma for Smooth Functions

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be closed and $f : A \to \mathbb{R}^k$ be a smooth function. Then for any open set U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\operatorname{supp} \tilde{f} \subseteq U$.

Proof. For each $p \in A$, take an open neighborhood $W_p \subseteq U$ and a smooth function $\tilde{f}_p : W_p \to \mathbb{R}^k$ such that $\tilde{f}_p|_{W_p \cap A} = f|_{W_p \cap A}$. Then the family $\{W_p\}_{p \in A} \cup \{M - A\}$ is an open cover of M. Let $\{\varphi_p : M \to \mathbb{R}\}_{p \in A} \cup \{\varphi_0\}$ be a smooth partition of unity subordinate to this cover. Define

$$\tilde{f}(x) = \sum_{p \in A} \varphi_p(x) \tilde{f}_p(x).$$

From local finiteness, the sum is well-defined and smooth. Also, $\operatorname{supp} \tilde{f} \subseteq U$ and for any $x \in A$,

$$\tilde{f}(x) = \sum_{p \in A} \varphi_p(x) \tilde{f}_p(x) = \sum_{p \in A} \varphi_p(x) f(x) = f(x).$$

Remark:
Note that the codomain is $\mathbb{R}^k$ here, this lemma would fail for arbitrary manifolds.

□

Definition 2.2.3: Exhaustion Functions

If M is a topological space, a continuous function $f : M \to \mathbb{R}$ is an **exhaustion function** if for every $c \in \mathbb{R}$, the sublevel set $M_c = f^{-1}((-\infty, c])$ is compact.

Well, as $n \in \mathbb{Z}_+$, the sets M_n forms an exhaustion of M by compact sets, hence the name.

Proposition: **Existence of Smooth Exhaustion Functions**

Every smooth manifold M without boundary admits a smooth positive exhaustion function.

Proof. SORRY □

Theorem 2.2.2: Level Sets of Smooth Functions

Let M be a smooth manifold. If K is a closed subset of M, then there exists a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

Proof. SORRY □

Chapter 3

Tangent Vectors

The basic idea of calculus is linear approximation.

In analysis, we come across the idea of geometric tangent vectors in $\mathbb{R}^n$, which are used for "directional derivatives" of multivariable functions. We shall follow this path initially, and then move to a more abstract definition of tangent vectors as derivations.

3.1 Tangent Vectors

Take $S^{n-1} \subseteq \mathbb{R}^n$ for example. For a point $x \in S^{n-1}$, usually we think it as a location, expressed by coordinates $(x^1, x^2, \ldots, x^n)$. But when doing calculus, we sometimes need to think of it as a vector. Geometrically, we can think of a vector as an arrow which has arbitrary start point. We imagine tangent vectors as arrows starting from the point x. That is to say, they live in a copy of $\mathbb{R}^n$ that is "attached" to the point x.

3.1.1 Geometric Tangent Vectors

Given $a \in \mathbb{R}^n$, define the geometric tangent space to $\mathbb{R}^n$ at a as the vector space

$$\mathbb{R}^n_a = \{a\} \times \mathbb{R}^n = \{(a, v) : v \in \mathbb{R}^n\}, \quad (a, v) + (a, w) = (a, v + w), \quad c(a, v) = (a, cv). \tag{3.1}$$

A geometric tangent vector in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$. We shall denote $v_a = (a, v)$

From this perspective, we can think of the tangent space of S^{n-1} at $a \in S^{n-1}$ as a subspace of $\mathbb{R}^n_a$: As all vectors in $\mathbb{R}^n_a$ that are perpendicular to the radius vector from the origin to a. To do this, we must have an inner product inherited from $\mathbb{R}^n$ via the natural isomorphism between $\mathbb{R}^n_a$ and $\mathbb{R}^n$.

This cannot be generalized to arbitrary manifolds, since there is no ambient Euclidean space to provide such a notion of perpendicularity. We shall use smooth structures to define tangent vectors in a more abstract way.

We turn to directional derivatives to motivate our definition. Every geometric tangent vector $v_a \in \mathbb{R}^n_a$ defines a map

$$D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad D_v|_a f = D_v f(a) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(a + tv). \tag{3.2}$$

This map is linear and satisfies the Leibniz product rule:

$$D_v|_a(fg) = f(a) D_v|_a g + g(a) D_v|_a f.$$

If $v_a = v^i e_{i,a}$ in the standard basis, then we have

$$D_v|_a(f) = v^i \frac{\partial f}{\partial x^i}(a).$$

We now reverse the process.

Definition 3.1.1: Derivation

A **derivation** at $a \in \mathbb{R}^n$ is a linear map

$$w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

that satisfies the Leibniz product rule:

$$w(fg) = f(a)w(g) + g(a)w(f)$$

for all $f, g \in C^\infty(\mathbb{R}^n)$. The set of all derivations at a is denoted by $T_a\mathbb{R}^n$. Then $T_a\mathbb{R}^n$ is a vector space under the operations

$$(w_1 + w_2)(f) = w_1(f) + w_2(f), \quad (cw)(f) = cw(f).$$

It is fairly surprising that $T_a\mathbb{R}^n$ is isomorphic to the geometric tangent space $\mathbb{R}^n_a$.

Lemma 3.1.1: Property of Derivations

Suppose $a \in \mathbb{R}^n$ and $w \in T_a\mathbb{R}^n$, $f, g \in C^\infty(\mathbb{R}^n)$.

- If f is constant, then $w(f) = 0$.
- If $f(a) = g(a) = 0$, then $w(fg) = 0$.

Proof. If $f(x) = 1$, then $wf = w(ff) = f(a)wf + f(a)wf = 2wf$, so $wf = 0$. If $f(x) = c$, then $wf = w(cf_1) = cw(f_1) = 0$. □

Proposition: **The Structure of $T_a\mathbb{R}^n$**

Let $a \in \mathbb{R}^n$. Then

- For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a$ defined above is a derivation at a.
- The map $v_a \mapsto D_v|_a$ is a vector space isomorphism from $\mathbb{R}^n_a$ to $T_a\mathbb{R}^n$.

Proof. To prove isomorphism:

- Linearity: we have
$$D_{c_1v+c_2w}|_a f = (c_1v + c_2w)^i \frac{\partial f}{\partial x^i}(a) = c_1 v^i \frac{\partial f}{\partial x^i}(a) + c_2 w^i \frac{\partial f}{\partial x^i}(a) = c_1 D_v|_a f + c_2 D_w|_a f.$$
- Injectivity: if $D_v|_a = 0$, then for all $f \in C^\infty(\mathbb{R}^n)$, $D_v|_a f = v^i \dfrac{\partial f}{\partial x^i}(a) = 0$. Taking $f(x) = x^j$, we have $v^j = 0$ for all j, so $v = 0$.

- Surjectivity: let $w \in T_a\mathbb{R}^n$. Define $v^i = w(x^i)$, and let $v_a = v^i e_i|_a$. For any $f \in C^\infty(\mathbb{R}^n)$, by Taylor's theorem, we have

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + R(x),$$

$$R(x) = \sum_{i,j=1}^{n} (x^i - a^i)(x^j - a^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))\, \mathrm{d}t.$$

  As $R(x)$ is the sum of products of functions vanishing at a, by the previous lemma we have $w(R) = 0$. Thus,

$$w(f) = w\left(f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) w(x^i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) v^i = D_v|_a f.$$

□

We have thus established the equivalence. And this definition can be generalized to arbitrary smooth manifolds.

3.1.2 Tangent Vectors on Manifolds

Definition 3.1.2: Tangent Vectors on Manifolds

Let M be a smooth manifold, with or without boundary, and let $p \in M$. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a **derivation at** p if it satisfies the Leibniz product rule:

$$v(fg) = f(p)v(g) + g(p)v(f), \quad \forall f, g \in C^\infty(M).$$

The set of all derivations at p is denoted by T_pM and called the **tangent space** of M at p. Its elements are called **tangent vectors** to M at p.

Proposition: **Property of Tangent Vectors on Manifolds**

Let M be a smooth manifold, with or without boundary, and let $p \in M$. If $v \in T_pM$ and $f, g \in C^\infty(M)$, then

- If f is constant, then $v(f) = 0$.
- If $f(p) = g(p) = 0$, then $v(fg) = 0$.

3.2 The Differential of a Smooth Map

We talk about the differential in analysis as linear approximations of functions at a given point. In the manifold case, there makes no sense to talk about linear transformations between manifolds, so we do it in terms of tangent spaces.

Definition 3.2.1: Differential on Manifolds

If M, N are smooth manifolds, with or without boundary, and $F : M \to N$ is a smooth map, then for each $p \in M$, we define a map

$$\mathrm{d}F_p : T_pM \to T_{F(p)}N \tag{3.3}$$

to be the **differential** of F at p, defined by

$$(\mathrm{d}F_p(v))(f) = v(f \circ F), \quad \forall f \in C^\infty(N), v \in T_pM. \tag{3.4}$$

Remark:
This is quite natural. To give a geometric intuition, take a curve γ in M with $\gamma(0) = p$ and $\gamma'(0) = v$. Then $F \circ \gamma$ is a curve in N with $(F \circ \gamma)(0) = F(p)$, and the tangent vector of $F \circ \gamma$ at 0 is $\mathrm{d}F_p(v)$.
Here v is a directional derivative operator acting on functions on M. which is given: $v(g) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} g(\gamma(t))$ for $g \in C^\infty(M)$. Then $\mathrm{d}F_p(v)$ is also a directional derivative operator acting on functions on N: for $f \in C^\infty(N)$,

$$(\mathrm{d}F_p(v))(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f((F \circ \gamma)(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ F)(\gamma(t)) = v(f \circ F).$$

The operator $\mathrm{d}F_p$ is linear, as for $v, w \in T_pM$, $c \in \mathbb{R}$, we have

$$(\mathrm{d}F_p(cv + w))(f) = (cv + w)(f \circ F) = cv(f \circ F) + w(f \circ F) = c(\mathrm{d}F_p(v))(f) + (\mathrm{d}F_p(w))(f).$$

Is also follows the Leibniz product rule:

$$\begin{aligned}
(\mathrm{d}F_p(v))(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\
&= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\
&= f(F(p))(\mathrm{d}F_p(v))(g) + g(F(p))(\mathrm{d}F_p(v))(f).
\end{aligned}$$

Proposition: **Properties of Differential**

Let M, N, P be smooth manifolds, with or without boundary, and let $F : M \to N$ and $G : N \to P$ be smooth maps. Then for each $p \in M$,

- $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is a linear map.
- (Chain Rule) $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$.
- If $\mathrm{id}_M : M \to M$ is the identity map, then $\mathrm{d}(\mathrm{id}_M)_p$ is the identity map on T_pM.
- If F is a diffeomorphism, then $\mathrm{d}F_p$ is an isomorphism, and $(\mathrm{d}F_p)^{-1} = \mathrm{d}(F^{-1})_{F(p)}$.

Our first application of differentials is to relate tangent spaces of manifoldes to those of Euclidean spaces via charts. But first we shall prove that tangent vectors are local-behaved, for charts only give local information.

Proposition: **Locality of Tangent Vectors**

Let M be a smooth manifold, with or without boundary, and let $p \in M$. If $v \in T_pM$ and $f, g \in C^\infty(M)$ agree on an open neighborhood of p, then $v(f) = v(g)$.

Proof. Let $f, g \in C^\infty(M)$ agree on an open neighborhood U of p. Then $h = f - g$ vanishes on U. Let $\psi \in C^\infty(M)$ be a smooth bump function that is 1 on $\operatorname{supp} h$ and $\operatorname{supp} \psi \subseteq M - \{p\}$. Then $h = h\psi$, so by the previous proposition, we have

$$v(h) = v(h\psi) = h(p)v(\psi) + \psi(p)v(h) = 0.$$

Thus, $v(f) = v(g)$. □

Proposition: **Tangent Space to Open Subsets**

Let M be a smooth manifold, with or without boundary, and let $U \subseteq M$ be an open subset. Let $\iota : U \hookrightarrow M$ be the inclusion map. Then for $p \in U$, the differential

$$\mathrm{d}\iota_p : T_pU \to T_pM$$

is an isomorphism.

Proof. Via the extension lemma, every $f \in C^\infty(U)$ can be extended to a function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_U = f$. Thus, with the locality of tangent vectors, we can easily see the result. □

Therefore, it is safe to identify T_pU with T_pM via the inclusion map.

Theorem 3.2.1: Dimension of Tangent Space

Let M be a smooth manifold of dimension n, and let $p \in M$. Then T_pM is an n-dimensional real vector space.

Proof. Take a chart (U, φ) containing p. Then as φ is a diffeomorphism from U to an open subset $\hat{U} \subseteq \mathbb{R}^n$, by the previous proposition, we have an isomorphism

$$T_pM \cong T_pU \cong T_{\varphi(p)}\hat{U} \cong T_{\varphi(p)}\mathbb{R}^n.$$

Thus, by the earlier result on $\mathbb{R}^n$, we have $\dim T_pM = n$. □

Now we address points on the boundary of manifolds with boundary. The situation is similar. First, we shall relate the tangent spaces of $T_a\mathbb{H}^n$ to those of $\mathbb{R}^n$ when $a \in \partial\mathbb{H}^n$. As $\mathbb{H}^n$ is not an open subset of $\mathbb{R}^n$, we cannot use the previous proposition 3.2 directly. However, we have the following result.

Lemma 3.2.1: Inclusion of $\mathbb{H}^n$

Let $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ be the inclusion map. Then for each $a \in \partial\mathbb{H}^n$, the differential $\mathrm{d}\iota_a : T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ is a linear isomorphism.

Proof. Assume $\mathrm{d}\iota_a(v) = 0$, then for all $f \in C^\infty(\mathbb{H}^n)$, we let $\tilde{f} \in C^\infty(\mathbb{R}^n)$ be an extension of f. Thus, $\tilde{f} \circ \iota = f$, and we have

$$v(f) = v(\tilde{f} \circ \iota) = (\mathrm{d}\iota_a(v))(\tilde{f}) = 0.$$

So $v = 0$ and $\mathrm{d}\iota_a$ is injective.

For surjectivity, let $w \in T_a\mathbb{R}^n$. Define $v : C^\infty(\mathbb{H}^n) \to \mathbb{R}$ by $v(f) = w(\tilde{f})$, where $\tilde{f} \in C^\infty(\mathbb{R}^n)$ is any extension of f. Thus

$$v(f) = w^i \frac{\partial \tilde{f}}{\partial x^i}(a).$$

From continuity, this does not depend on the choice of extension $\tilde{f}$, as we can get the result by limiting process from points in the interior of $\mathbb{H}^n$. So we have $\mathrm{d}\iota_a(v) = w$. ☐

Therefore, it is safe to identify $T_a\mathbb{H}^n$ with $T_a\mathbb{R}^n$ via the inclusion map, even for $a \in \partial\mathbb{H}^n$.

Proposition: **Dimension of Tangent Space with Boundary**

Let M be a smooth manifold of dimension n with boundary, and let $p \in M$. Then T_pM is an n-dimensional real vector space.

Next, as we know that for a finite-dimensional vector space, there exists a natural smooth structure on it. We shall see that the tangent space to a vector space at any point is naturally isomorphic to the vector space itself.

Proposition: **Tangent Space to a Vector Space**

Let V be a finite-dimensional real vector space with the standard smooth structure, and let $v \in V$. Then there is a natural isomorphism $V \cong T_vV$, defined by

$$v \mapsto D_v|_a, \quad D_v|_a(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a + tv), \quad \forall f \in C^\infty(V).$$

For any liner transformation $T : V \to W$ between finite-dimensional real vector spaces, we have

$$L \cong \mathrm{d}L_a, \qquad \mathrm{d}L_a(D_v|_a) = D_{L(v)}|_{L(a)}. \tag{3.5}$$

Therefore, we can identify T_vV with V itself via the above isomorphism. For example, since $GL(n, \mathbb{R})$ is an open subset of the vector space $M_{n\times n}(\mathbb{R})$, we can identify $T_AGL(n, \mathbb{R})$ with $M_{n\times n}(\mathbb{R})$ for each $A \in GL(n, \mathbb{R})$.

For products, we have the following result.

Theorem 3.2.2: Tangent Space to Product Manifolds

Let $M_1, \ldots, M_k$ be smooth manifolds, at most one have boundary. Let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection map onto the j-th factor. Then for each $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha_p : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k, \quad \alpha_p(v) = (\mathrm{d}\pi_1)_p(v), \ldots, (\mathrm{d}\pi_k)_p(v)$$

Is an isomorphism.

Therefore, we can identify $T_p(M_1 \times \cdots \times M_k)$ with $T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ via the above isomorphism.

3.3 Computation in Coordinates

We shall use charts to compute tangent vectors and differentials in coordinates.

Suppose M is a smooth manifold of dimension n (without boundary for simplicity), and (U, φ) is a chart containing $p \in M$. Then $\varphi : U \to \hat{U} \subseteq \mathbb{R}^n$ is a diffeomorphism, thus $\mathrm{d}\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism.

In $\mathbb{R}^n$, we have the standard basis

$$\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} : f \mapsto \frac{\partial f}{\partial x^i}(\varphi(p)), \quad i = 1, \ldots, n.$$

Therefore, the preimages of these basis vectors under $\mathrm{d}\varphi_p$ form a basis of T_pM, denoted by

$$\left.\frac{\partial}{\partial x^i}\right|_p = (\mathrm{d}\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = \mathrm{d}(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right), \quad i = 1, \ldots, n. \tag{3.6}$$

Acting on $f \in C^\infty(M)$, we have

$$\left.\frac{\partial}{\partial x^i}\right|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}), \quad \hat{f} = f \circ \varphi^{-1}, \quad \hat{p} = \varphi(p).$$

which is the coordinate expression of f and p in $\mathbb{R}^n$. We call $\left\{\left.\frac{\partial}{\partial x^i}\right|_p : i = 1, \ldots, n\right\}$ the **coordinate basis** of T_pM induced by the chart (U, φ).

Remark:
In $\mathbb{R}^n$, the coordinate basis vectors are just the partial derivative operators along the coordinate axes.

For points on the boundary of manifolds with boundary, the situation is similar, just replacing $\mathbb{R}^n$ with $\mathbb{H}^n$, and using the inclusion isomorphism between $T_a\mathbb{H}^n$ and $T_a\mathbb{R}^n$ for $a \in \partial\mathbb{H}^n$.

Theorem 3.3.1: The Coordinate Basis

Let M be a smooth manifold of dimension n, with or without boundary, and let $p \in M$. Then take any chart (U, φ) containing p. Then the coordinate vectors

$$\left.\frac{\partial}{\partial x^i}\right|_p = \mathrm{d}(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right), \quad i = 1, \ldots, n$$

form a basis of T_pM.

This a tangent vector $v \in T_pM$ can be expressed in coordinates as

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p, \quad v^i = v(x^i), \tag{3.7}$$

where $x^i = \pi_i \circ \varphi$ are the coordinate functions on U. The numbers v^i are called the **components** of v with respect to the coordinate basis induced by the chart (U, φ).

3.3.1 The Differential in Coordinates

Now, we shall do computations of differentials of smooth maps in coordinates form. First, for simplicity consider $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets, and let $F : U \to V$ be a smooth map. For $p \in U$, we have $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ being a linear map. In the standard coordinate bases, we have

$$\mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) = \left(\frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f.$$

Thus, we have

$$\mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \tag{3.8}$$

Writing in matrix form, we have

$$\mathrm{d}F_p = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \frac{\partial F^m}{\partial x^2}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix} \tag{3.9}$$

which is just the Jacobian matrix of F at p. The same can be said if U is an open subset of $\mathbb{H}^n$, so do V.

For a more general case, let M, N be smooth manifolds of dimension n, m respectively, with or without boundary, and let $F : M \to N$ be a smooth map. Take charts (U, φ) and (V, ψ) containing $p \in M$ and $F(p) \in N$ respectively. Then we have $\mathrm{d}\hat{F}_{\hat{p}} : T_{\hat{p}}\mathbb{R}^n \to T_{\hat{F}(\hat{p})}\mathbb{R}^m$ being the differential of the smooth map $\hat{F} = \psi \circ F \circ \varphi^{-1} : \hat{U} \to \hat{V}$ at $\hat{p} = \varphi(p)$. In the coordinate bases, we have

$$\begin{aligned} \mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= \mathrm{d}(\psi^{-1})_{\hat{F}(\hat{p})} \circ \mathrm{d}\hat{F}_{\hat{p}} \circ \mathrm{d}\varphi_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) \\ &= \mathrm{d}(\psi^{-1})_{\hat{F}(\hat{p})}\left(\mathrm{d}\hat{F}_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right) \\ &= \mathrm{d}(\psi^{-1})_{\hat{F}(\hat{p})}\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\left.\frac{\partial}{\partial y^j}\right|_{\hat{F}(\hat{p})}\right) \\ &= \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \end{aligned} \tag{3.10}$$

Which is just the pushforward of the Jacobian matrix of $\hat{F}$ at $\hat{p}$ via the charts.

3.3.2 Change of Coordinates

Suppose (U, φ) and (V, ψ) are two smooth charts on M and $p \in U \cap V$. Denote the coordinate functions of φ by $x^i = \pi_i \circ \varphi$ and those of ψ by $\tilde{x}^i = \pi_i \circ \psi$. Therefore, any tangent vector

$v \in T_pM$ can be expressed in both coordinate bases, and we want to find the relation between the components.

To do it, consider the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$, and we write its coordinate functions by

$$\varphi \circ \psi^{-1}(x) = (\tilde{x}^1(x), \ldots, \tilde{x}^n(x)).$$

We have identified V with $\psi(V) \subseteq \mathbb{R}^n$ via the chart ψ, so we use the same notation $\tilde{x}^i$ for the coordinate functions on V for simplicity. Then we have the differential

$$\mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} : T_{\varphi(p)}\mathbb{R}^n \to T_{\psi(p)}\mathbb{R}^n.$$

by the previous result, we have

$$\mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_{\psi(p)}. \tag{3.11}$$

So we have pull back to T_pM via the charts:

$$\begin{aligned} \left. \frac{\partial}{\partial x^i} \right|_p &= \mathrm{d}(\varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) \\ &= \mathrm{d}(\psi^{-1})_{\psi(p)} \circ \mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right) \\ &= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_p . \end{aligned} \tag{3.12}$$

Therefore, the components of v in the two coordinate bases are related by

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) v^i. \tag{3.13}$$

3.4 The Tangent Bundle

Definition 3.4.1: The Tangent Bundle

Let M be a smooth manifold, with or without boundary. The **tangent bundle** of M is the disjoint union

$$TM = \bigsqcup_{p \in M} T_pM = \{(p, v) : p \in M, v \in T_pM\}.$$

The map $\pi : TM \to M$ defined by $\pi(p, v) = p$ is called the **bundle projection**.

For example, the tangent bundle of $\mathbb{R}^n$ is naturally isomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ via the isomorphism

$$\mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n, \quad (a, v) \mapsto (a, D_v|_a).$$

But for general manifolds, we cannot identify TM with $M \times \mathbb{R}^n$ globally because we cannot have a natural way to identify each tangent space T_pM with each other.

Theorem 3.4.1: Structure of the Tangent Bundle

Let M be a smooth manifold of dimension n. Then TM has a natural topology and smooth structure such that TM is a smooth manifold of dimension $2n$. With this structure, the bundle projection $\pi : TM \to M$ is a smooth map.

Proof. The ultimate intuition is to do it locally via charts. For each smooth chart (U, φ) on M, note that $\pi^{-1}(U) = \bigsqcup_{p \in U} T_pM$. Define a map $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{\varphi}(p, v) = \tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\right), \tag{3.14}$$

So the image set is $\hat{U} \times \mathbb{R}^n$, being an open subset of $\mathbb{R}^{2n}$. It is also a bijection from $\pi^{-1}(U)$ to $\hat{U} \times \mathbb{R}^n$, because

$$\tilde{\varphi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = v_i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x^1, \ldots, x^n)}.$$

Now suppose we have two smooth charts (U, φ) and (V, ψ) on M and let $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ be the corresponding charts on TM. Then the sets

$$\tilde{\varphi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \varphi(U \cap V) \times \mathbb{R}^n, \quad \tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \psi(U \cap V) \times \mathbb{R}^n$$

are both open subsets of $\mathbb{R}^{2n}$. The transition map is given by

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = \left(\tilde{x}^1(x), \ldots, \tilde{x}^n(x), \frac{\partial \tilde{x}^j}{\partial x^1}(x)v^i, \ldots, \frac{\partial \tilde{x}^j}{\partial x^n}(x)v^i\right),$$

which is smooth.

Finally, choose a countable cover of charts $\{(U_\alpha, \varphi_\alpha)\}$ of M, then the corresponding charts $\{(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha)\}$ form an atlas of TM. The conditions of 1.3.1 are easily verified. □

Remark:
For smooth manifolds with boundary, the construction is similar, just replacing $\mathbb{R}^n$ with $\mathbb{H}^n$ in the above proof. We note that the only "half-ness" happens in the base manifold M, while each tangent space T_pM is a full n-dimensional vector space, so no harm is done to the tangent bundle structure.

Proposition: **Single-Chart Tangent Bundle**

If M is a smooth manifold of dimension n (with or without boundary) that can be covered by a single chart (M, φ), then the tangent bundle TM is diffeomorphic to $M \times \mathbb{R}^n$.

Proof. Obvious. □

Remark:
NOTE that although we can locally view TM as $U \times \mathbb{R}^n$ via charts, there is no natural way to identify TM with $M \times \mathbb{R}^n$ globally in general. In fact, this may not be true in many cases.

Putting all pointwise differentials together, we have a map

$$\mathrm{d}F : TM \to TN, \quad \mathrm{d}F(p, v) = (F(p), \mathrm{d}F_p(v)),$$

called the global differential of F.

Theorem 3.4.2: Global Differential is Smooth

Let M, N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. Then the global differential

$$\mathrm{d}F : TM \to TN$$

is a smooth map.

Proof. From the coordinate expression, we have

$$\mathrm{d}F(p, v) = \left(F(p), \frac{\partial F^j}{\partial x^i}(p) v^i \right),$$

which is smooth for F is. □

Proposition: **Properties of Global Differential**

Let M, N, P be smooth manifolds, with or without boundary, and let $F : M \to N$ and $G : N \to P$ be smooth maps. Then for each $(p, v) \in TM$,

- (Chain Rule) $\mathrm{d}(G \circ F) = \mathrm{d}G \circ \mathrm{d}F$.
- If $\mathrm{id}_M : M \to M$ is the identity map, then $\mathrm{d}(\mathrm{id}_M) = \mathrm{id}_{TM}$.
- If F is a diffeomorphism, then $\mathrm{d}F$ is a diffeomorphism, and $(\mathrm{d}F)^{-1} = \mathrm{d}(F^{-1})$.

Just using proposition 11.2 would do. From now we may denote $\mathrm{d}F^{-1}$ for either $\mathrm{d}(F^{-1})$ or $(\mathrm{d}F)^{-1}$, when F is a diffeomorphism.

3.5 Velocity Vectors of Curves

Definition 3.5.1: Curves

Let M be a manifold, with or without boundary. A **curve** in M is a continuous map $\gamma : J \to M$, where $J \subseteq \mathbb{R}$ is an open interval. Sometimes we may want J to have one or both endpoints, in which case slight modifications are needed.

Definition 3.5.2: Velocity

Let M be a smooth manifold, with or without boundary, and let $\gamma : J \to M$ be a smooth curve. The **velocity** of γ at $t_0 \in J$ is the tangent vector

$$\gamma'(t_0) = \mathrm{d}\gamma_{t_0} \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t_0} \right) \in T_{\gamma(t_0)} M. \tag{3.15}$$

Other notations include

$$\dot{\gamma}(t_0) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t_0) = \left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_{t=t_0}$$

The tangent vector $\gamma'(t_0)$ acts on functions by

$$\gamma'(t_0)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} f(\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (f \circ \gamma)(t).$$

which is the rate of change of f along the curve γ at t_0. For a smooth chart (U, φ) containing $\gamma(t_0)$, we can express the velocity in coordinates as

$$\gamma'(t_0) = \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)} = \left(\frac{\mathrm{d}\gamma^1}{\mathrm{d}t}(t_0), \ldots, \frac{\mathrm{d}\gamma^n}{\mathrm{d}t}(t_0) \right),$$

which is familiar in Euclidean space.

Next, we shall see that every tangent vector can be expressed as the velocity of some curve, which will lead us to an equivalent definition of tangent vectors.

Proposition: **Tangent Vector as Velocity**

Let M be a smooth manifold, with or without boundary, and let $p \in M$. Then for any tangent vector $v \in T_pM$, there exists a smooth curve $\gamma : J \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Proof. First suppose $p \in \operatorname{Int} M$, then let (U, φ) be a smooth chart centering p. Then we write $v = v^i \partial/\partial x^i|_p$. For sufficiently small ϵ, we have a smooth curve $\gamma : (-\epsilon, \epsilon) \to U$ defined by

$$\gamma(t) = \varphi^{-1}\left(tv^1, \ldots, tv^n\right)$$

which is smooth because φ^{-1} is smooth.

Now if $p \in \partial M$, then let (U, φ) be a smooth boundary chart centering p. We can similarly define a smooth curve $\gamma : [0, \epsilon) \to U$ or $(-\epsilon, 0] \to U$ by the same formula for sufficiently small $\epsilon > 0$, depending on the sign of the first component of v. □

For composition, we have the following result.

Proposition: **Velocity under Composition**

Let M, N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. If $\gamma : J \to M$ is a smooth curve, then for each $t_0 \in J$,

$$(F \circ \gamma)'(t_0) = \mathrm{d}F_{\gamma(t_0)}\left(\gamma'(t_0)\right).$$

Proof. Just the chain rule:

$$(F \circ \gamma)'(t_0)(f) = \mathrm{d}(F \circ \gamma)_{t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right)(f) = \mathrm{d}F \circ \mathrm{d}\gamma_{t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right)(f) = \mathrm{d}F_{\gamma(t_0)}\left(\gamma'(t_0)\right)(f).$$

□

We can also use curve velocity to compute differentials: Suppose M, N are smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map, then to compute $\mathrm{d}F_p(v)$ for $p \in M$ and $v \in T_pM$, we can first find a smooth curve $\gamma : J \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, then we have

$$\mathrm{d}F_p(v) = \mathrm{d}F_p\left(\gamma'(0)\right) = (F \circ \gamma)'(0).$$

3.6 Alternative Definition of Tangent Vectors

3.6.1 Derivations of the Space of Germs

A smooth function element on M is an ordered pair (f, U), where $U \subseteq M$ is an open set and $f \in C^\infty(U)$. Two smooth function elements (f, U) and (g, V) are said to be equivalent at $p \in U \cap V$ if there exists an open neighborhood $W \subseteq U \cap V$ of p such that $f|_W = g|_W$. The equivalence class of (f, U) at p is called the **germ** of f at p, and the set if all germs of smooth functions at p is denoted by $C_p^\infty(M)$.

Remark:
Intuitively, $C_p^\infty(M)$ of a smooth function at p contains all distinguishable smooth functions locally around p.

We notice that $C_p^\infty(M)$ is a real vector space and an associative algebra under operations defined by

- Addition: $[(f, U)] + [(g, V)] = [(f + g, U \cap V)]$.
- Scalar Multiplication: $c[(f, U)] = [(cf, U)]$.
- Multiplication: $[(f, U)] \cdot [(g, V)] = [(fg, U \cap V)]$.

Now we denote the germ of f at p simply by $[f]_p$ when there is no confusion.

A derivation of $C_p^\infty(M)$ is a linear map $v : C_p^\infty(M) \to \mathbb{R}$ such that for all $[f]_p, [g]_p \in C_p^\infty(M)$,

$$v([f]_p \cdot [g]_p) = f(p)v([g]_p) + g(p)v([f]_p). \tag{3.16}$$

The set of all derivations of $C_p^\infty(M)$ is denoted by $\mathcal{D}_p(M)$. And it is simple to verify that $\mathcal{D}_p(M)$ is naturally isomorphic to T_pM.

3.6.2 Equivalent Class of Curves

This definition captures the intuitive idea of tangent vectors as "directions" at a point. Suppose p is a point of M, and consider all smooth curves $\gamma : J \to M$ such that $\gamma(0) = p$. We say two such curves γ_1 and γ_2 are equivalent at p if for any smooth function $f : M \to \mathbb{R}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \gamma_1)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \gamma_2)(t). \tag{3.17}$$

The equivalence classes are denoted by $[\gamma]$, and all such equivalence classes form a set denoted by $\mathcal{V}_p(M)$, which is naturally isomorphic to T_pM.

3.7 Categories and Functors

A category $\mathcal{C}$ consists of

- A class $\mathrm{Ob}(\mathcal{C})$, whose elements are called objects of $\mathcal{C}$.
- A class $\mathrm{Hom}(\mathcal{C})$, whose elements are called morphisms of $\mathcal{C}$.
- For each morphism $f \in \mathrm{Hom}(\mathcal{C})$, there are two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ called the source and target of f, denoted by $f : X \to Y$.
- For each triplet of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, there is a mapping called composition
$$\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z), \quad (g, f) \mapsto g \circ f.$$
where $\mathrm{Hom}(A, B)$ is the class of all morphisms from A to B.

The morphisms and objects must satisfy the following axioms:

- (Associativity) For each $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For each object $X \in \mathrm{Ob}(\mathcal{C})$, there exists an identity morphism $\mathrm{id}_X : X \to X$ such that for each $f : X \to Y$.
$$\mathrm{id}_Y \circ f = f, \quad f \circ \mathrm{id}_X = f.$$

A morphism $f : X \to Y$ is called an isomorphism if there exists a morphism $g : Y \to X$ such that
$$g \circ f = \mathrm{id}_X, \quad f \circ g = \mathrm{id}_Y.$$

Chapter 4

Submersions, Immersions, and Embeddings

We shall study the geometric properties of smooth maps by there differential.

4.1 Maps of Constant Rank

Suppose M and N are smooth manifolds, with or without boundary, and $F : M \to N$ is a smooth map. For each point $p \in M$, we define the **rank** of F at p to be the rank of the linear map $\mathrm{d}F_p : T_pM \to T_{F(p)}N$, which is just the rank of the Jacobian matrix of F in any coordinate chart containing p and $F(p)$, or just $\dim \operatorname{range} \mathrm{d}F_p$. If the rank of F is the same at every point of M, we say that F is a **map of constant rank**.

The maximum possible rank of F at any point is $\min\{\dim M, \dim N\}$. If the rank of F at p is equal to this maximum value, we say that F has **full rank** at p. If F has full rank at every point of M, we say that F has **constant full rank**.

Definition 4.1.1: smooth Submersions and Immersions

A smooth map $F : M \to N$ between smooth manifolds is a **smooth submersion** if $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is surjective for every point $p \in M$.
A smooth map $F : M \to N$ between smooth manifolds is a **smooth immersion** if $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is injective for every point $p \in M$.

From the continuity of $\mathrm{d}F$, we have the following result.

Proposition: **Local Surjectivity and Local Injectivity**

If $F : M \to N$ is a smooth map and $p \in M$. If $\mathrm{d}F_p$ is surjective, then there exists an open neighborhood U of p such that $F|_U : U \to N$ is a submersion. If $\mathrm{d}F_p$ is injective, then there exists an open neighborhood U of p such that $F|_U : U \to N$ is an immersion.

Proof. Choose any smooth coordinate chart (U, φ) on M containing p and any smooth coordinate chart (V, ψ) on N containing $F(p)$. Then the Jacobian matrix of F has full rank at p. As $\mathrm{d}F$ is continuous, there exists an open neighborhood $U' \subset U$ of p such that the Jacobian matrix of F has full rank at every point of U'. □

Example: **Submersion and Immersion**

- Suppose $M_1, \dots, M_k$ are smooth manifolds, then the projection map
$$\pi_i : M_1 \times \cdots \times M_k \to M_i, \quad (p_1, \dots, p_k) \mapsto p_i$$
is a smooth submersion for each $1 \le i \le k$.
- If $\gamma : J \to M$ is a smooth curve on a smooth manifold M, with or without boundary, then γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.
- If M is a smooth manifold then the tangent bundle projection $\pi : TM \to M$ is a smooth submersion.

Proposition: **Properties of Submersions and Immersions**

- If $F : M \to N$ and $G : N \to P$ are smooth submersions, then $G \circ F : M \to P$ is a smooth submersion.
- If $F : M \to N$ and $G : N \to P$ are smooth immersions, then $G \circ F : M \to P$ is a smooth immersion.
- The composition of maps of constant rank need not have constant rank.

Proof. For the third claim, take
$$f : \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (t, t^2)$$
and
$$g : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto y.$$
Then both f and g have constant rank 1, but the composition
$$g \circ f : \mathbb{R} \to \mathbb{R}, \quad t \mapsto t^2$$
does not have constant rank. □

4.1.1 Local Diffeomorphisms

If M, N are smooth manifolds with or without boundary, a smooth map $F : M \to N$ is a **local diffeomorphism** if for each point $p \in M$, there exists an open neighborhood U of p such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 4.1.1: The Inverse Function Theorem

Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ and $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is invertible, then there exist connected open neighborhoods U_0 of p in M and V_0 of $F(p)$ in N such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

Proof. Firstly, this implies M, N have the same dimension n, then choose smooth charts (U, φ) centering at p and (V, ψ) centering at $F(p)$ with $F(U) \subseteq V$. Then $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$

is a smooth map between open subsets of $\mathbb{R}^n$ with invertible Jacobian matrix at $\varphi(p)$. By the Euclidean version of the Inverse Function Theorem, there exist open neighborhoods U' of $\varphi(p)$ and V' of $\psi(F(p))$ such that $\hat{F}|_{U'} : U' \to V'$ is a diffeomorphism. Let $U_0 = \varphi^{-1}(U')$ and $V_0 = \psi^{-1}(V')$, then $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism. □

Remark:
NOTE that this is true only for manifolds without boundary.

Proposition: **Properties of Local Diffeomorphisms**

- Compositions of local diffeomorphisms are local diffeomorphisms.
- Finite products of local diffeomorphisms are local diffeomorphisms.
- The restriction of a local diffeomorphism to an open submanifold (with or without boundary) is a local diffeomorphism.
- Every diffeomorphism is a local diffeomorphism. Every bijective local diffeomorphism is a diffeomorphism.
- A map between smooth manifolds, with or without boundary, is a local diffeomorphism if and only if it has a local diffeomorphism coordinate representation at each point.

Proposition: **Local Diffeomorphisms, Submersions, and Immersions**

A smooth map between smooth manifolds (without boundary), is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.
Moreover, if $\dim M = \dim N$, then a smooth map $F : M \to N$ is a smooth submersion if and only if it is a smooth immersion, and in either case it is a local diffeomorphism.

Example: **Local Diffeomorphisms**

The map $\mathbb{R} \to S^1$ defined by $t \mapsto (\cos t, \sin t)$ is a local diffeomorphism, but not a diffeomorphism.

4.1.2 The Rank Theorem

Theorem 4.1.2: The Rank Theorem

Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F : M \to N$ is a smooth map of constant rank k. Then for each point $p \in M$, there exist smooth coordinate charts (U, φ) on M centered at p and (V, ψ) on N centered at $F(p)$ such that

$$\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is given by

$$\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

The linear version of the Rank Theorem is that under certain choice of basis, any linear map can be represented by a matrix of the form

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

Proof. From locality, just replace M, N by open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ containing p and $F(p)$, respectively. We also assume $p = 0$ and $F(p) = 0$.

SORRY □

The following corollary is an immediate consequence and also can be viewed as a restatement of the Rank Theorem.

Corollary 4.1.1: Local Linearity

Let M and N be smooth manifolds of dimensions m and n, respectively, and let $F : M \to N$ be a smooth map. If M is connected, then F has constant rank k if and only if for each point $p \in M$, there exist smooth coordinate charts (U, φ) on M centered at p and (V, ψ) on N centered at $F(p)$ such that the coordinate representation

$$\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is linear.

Theorem 4.1.3: Global Rank Theorem

Let M and N be smooth manifolds and let $F : M \to N$ be a smooth map of constant rank k. Then

- If F is surjective, then it is a smooth submersion.
- If F is injective, then it is a smooth immersion.
- If F is bijective, then it is a diffeomorphism.

Proof. SORRY □

4.1.3 The Rank Theorem with Boundary

The rank theorem does not generalize to manifolds with boundary in full generality. However, we do have the following partial result.

Theorem 4.1.4: The Local Immersion Theorem with Boundary

Suppose M is a smooth manifold with boundary of dimension m, N is a smooth manifold of dimension n, and $F : M \to N$ is a smooth immersion. Then for each point $p \in \partial M$, there exist smooth boundary charts (U, φ) on M centered at p and smooth chart (V, ψ) on N centered at $F(p)$ such that $F(U) \subseteq V$ and the coordinate representation

$$\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

Proof. SORRY □

4.2 Embeddings

Definition 4.2.1: Smooth Embeddings

Let M and N be smooth manifolds, with or without boundary. A **smooth embedding** is a smooth immersion $F : M \to N$ that is also a topological embedding; that is, F is a homeomorphism onto its image $F(M)$, where $F(M)$ is given the subspace topology inherited from N.

Proposition: **Compositions of Embeddings**

If $F : M \to N$ and $G : N \to P$ are smooth embeddings, then $G \circ F : M \to P$ is a smooth embedding.

Example: **Smooth Embeddings**

- Let M be a smooth manifold with or without boundary and $U \subseteq M$ be an open submanifold. Then the inclusion map $\iota : U \hookrightarrow M$ is a smooth embedding.
- If $M_1, \ldots, M_k$ are smooth manifolds and $p_i \in M_i$ for each $1 \leq i \leq k$, then each of

$$\iota_i : M_i \hookrightarrow M_1 \times \cdots \times M_k, \quad q \mapsto (p_1, \ldots, p_{i-1}, q, p_{i+1}, \ldots, p_k)$$

is a smooth embedding. Indeed, $\mathbb{R}^n \iota \mathbb{R}^{n+k}$ defined by $x \mapsto (x, 0)$ is a smooth embedding.

Here are some counterexamples that illustrate the definition.

- The map $\mathbb{R} \to \mathbb{R}^2$ defined by $t \mapsto (t^3, 0)$ is a smooth map and a topological embedding, but not a smooth immersion at $t = 0$, so it is not a smooth embedding.
- The figure-eight curve $\mathbb{R} \to \mathbb{R}^2$ defined by $t \mapsto (\sin t, \sin 2t)$ is a smooth immersion, but not a topological embedding because it is not a homeomorphism onto its image. (Compactness fails)
- Now is an interesting example that shows mere injectivity is not enough for a smooth embedding. Consider $\gamma : \mathbb{R} \to \mathbb{T}^2$ by

$$\gamma(t) = (e^{it}, e^{i\alpha t}),$$

where α is an irrational number. Then γ is a smooth immersion, injective (because α is irrational), but not a smooth embedding because it is not a homeomorphism onto its image. Because $\gamma(\mathbb{Z})$ is dense in $\mathbb{T}^2$, but $\mathbb{Z}$ is not dense in $\mathbb{R}$.

Proposition: **Criterion for Smooth Embeddings**

Suppose M and N are smooth manifolds, with or without boundary, and $F : M \to N$ is an injective smooth immersion. Then F is a smooth embedding if one of the following equivalent conditions holds:

- F is an open or closed map.
- F is a proper map, that is, for every compact subset $K \subseteq M$, the image $F(K)$ is compact in N.
- M is compact.
- M has empty boundary and $\dim M = \dim N$.

Proof. The first claim is just the definition of a topological embedding. The second claim follows from the fact that a proper map is a closed map, and the third claim follows from the fact that a compact set is closed. The fourth claim follows from the fact that if M has empty boundary, then it is an open submanifold of itself, and if $\dim M = \dim N$, then F is a local diffeomorphism, hence an open map. □

Theorem 4.2.1: Local Embedding Theorem

Suppose M and N are smooth manifolds, with or without boundary, and $F : M \to N$ is a smooth map. Then F is a smooth immersion iff every point $p \in M$ has a neighborhood U such that $F|_U : U \to N$ is a smooth embedding.

Proof. If F is a local smooth embedding on every point, then it has full rank everywhere, hence it is a smooth immersion. Conversely, if F is a smooth immersion, then for each point $p \in M$:

If $F(p) \notin \partial N$, then by the Rank Theorem, there exist coordinate charts (U, φ) on M centered at p and (V, ψ) on N centered at $F(p)$ such that the coordinate representation

$$\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

Then restricting U if necessary, we have that $F|_U : U \to N$ is injective. If $F(p) \in \partial N$, then the same argument applies slight adjustment by $\mathbb{H}^n$ coordinate charts.

Next, let U_1 be a precompact neighborhood of p in U such that $F|_{\overline{U_1}}$ is injective and has compact domain. Then $F|_{\overline{U_1}} : \overline{U_1} \to N$ is a topological embedding. □

Remark:
This gives a direct notion for topological immersions: a continuous map between topological spaces that is a local topological embedding at every point.

4.3 Submersions

Definition 4.3.1: Section

Let M and N be topological spaces, and let $\pi : M \to N$ be a continuous map. A **section** of π is a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{id}_N$.
A local section of π over an open set $U \subseteq N$ is a continuous map $\sigma : U \to M$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Really, sections are right inverses of the map π, so it is injective.

Note that global sections need not exist, for example, consider the S^1 projection onto RP^1. However, local sections always exist for submersions, as the following result shows.

Theorem 4.3.1: Local Section Theorem

Let M and N be smooth manifolds, and $\pi : M \to N$ be a smooth map. Then π is a smooth submersion if and only if for each point $p \in M$, it is in an image of a smooth local section of π.

Proof. Suppose π is a smooth submersion, and given $p \in M$. By the Rank Theorem, there exist smooth coordinate charts (U, φ) on M centered at p and (V, ψ) on N centered at $\pi(p)$ such that the coordinate representation $\pi(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$. For sufficiently small ϵ, the coordinate cube

$$C_\epsilon = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : |x^i| < \epsilon, 1 \leq i \leq m\}$$

Then $\pi(C_\epsilon)$ is also a coordinate cube in $\mathbb{R}^n$. The coordinate map by

$$\sigma(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$$

is a smooth local section.

Conversely, from $\pi \circ \sigma = \mathrm{id}_N$, we have that $\mathrm{d}\pi_{\sigma(q)} \circ \mathrm{d}\sigma_q = \mathrm{id}_{T_qN}$ for each $q \in N$. Hence, $\mathrm{d}\pi_{\sigma(q)}$ is surjective for each $q \in N$, so π is a smooth submersion. $\square$

Remark:
This theorem motivates the definition of a topological submersion: a continuous map between topological spaces such that every point in the domain is in the image of a local section.

Proposition: **Properties of Smooth Submersions**

Let M and N be smooth manifolds, and let $\pi : M \to N$ be a smooth submersion. Then

- π is an open map.
- If π is surjective, then it is a quotient map.

Proof. For the first claim, let $W \subseteq M$ be open and $q \in \pi(W)$, take any $p \in W$ such that $\pi(p) = q$. By the Local Section Theorem, there exists a smooth local section $\sigma : U \to M$ of π such that $\sigma(q) = p$. Then $\sigma^{-1}(W)$ is an open neighborhood of q contained in $\pi(W)$, so $\pi(W)$ is open in N.

The second claim follows from the first claim and the definition of a quotient map. $\square$

We can see that smooth submersions plays a similar role to that of quotient maps in topology.

Theorem 4.3.2: Passing Smoothly to Quotient

Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective smooth submersion. If P is a smooth manifold, with or without boundary, then a map $F : N \to P$ is smooth if and only if $F \circ \pi : M \to P$ is smooth.
Moreover, if $G : M \to P$ is a smooth map that is constant on each fiber of π, then there exists a unique smooth map $\tilde{G} : N \to P$ such that $G = \tilde{G} \circ \pi$.

Proof. If $F : N \to P$ is smooth, then $F \circ \pi : M \to P$ is smooth by composition of smooth maps. Conversely, suppose that $F \circ \pi : M \to P$ is smooth. Given any point $q \in N$, take any $p \in M$ such that $\pi(p) = q$. By the Local Section Theorem, there exists a smooth local section $\sigma : U \to M$ of π such that $\sigma(q) = p$. Then the restriction $F|_U = (F \circ \pi) \circ \sigma : U \to P$ is smooth. Since q was arbitrary, F is smooth. □

Theorem 4.3.3: Uniqueness of Smooth Quotients

Suppose M and N_1, N_2 are smooth manifolds, and $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ are surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F : N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$.

4.4 Smooth Covering Maps

In general topology, a **covering map** is a continuous surjective map $\pi : X \to Y$ such that for each point $y \in Y$, there exists an open neighborhood U of y such that $\pi^{-1}(U)$ is a disjoint union of open sets in X, each of which is homeomorphic to U via π.

In the context of smooth manifolds, we have the following definition.

Definition 4.4.1: Smooth Covering Map

Let E and M be connected smooth manifolds, with or without boundary. A **smooth covering map** is a smooth surjective map $\pi : E \to M$ such that for each point $p \in M$, there exists an open neighborhood U of p such that $\pi^{-1}(U)$ is a disjoint union of open sets in E, each of which is diffeomorphic to U via π.
We say that M is the **base space**, E is the **covering space**, and if E is simply connected, we say that E is the **universal covering space** of M.

Proposition: **Properties of Smooth Covering Maps**

Let E and M be connected smooth manifolds, with or without boundary, and let $\pi : E \to M$ be a smooth covering map. Then

- π is a local diffeomorphism, a smooth submersion, an open map and a quotient map.
- An injective smooth covering map is a diffeomorphism.
- A topological covering map between smooth manifolds is a smooth covering map if and only if it is a local diffeomorphism.

- Every local section of π is a smooth local section.

Example: **Smooth Covering Maps**

- The map $\mathbb{R} \to S^1$ defined by $t \mapsto (\cos t, \sin t)$ is a smooth covering map. Its universal covering space is $\mathbb{R}$ itself.
- For $n \geq 1$, the map $q : S^n \to \mathbb{R}P^n$ defined by $q(x) = [x]$ is a two-to-one smooth covering map.

For smooth covering maps, we also have the local section theorem strengthened.

Theorem 4.4.1: Local Section Theorem for Smooth Covering Maps

Let E and M be connected smooth manifolds, with or without boundary, and let $\pi : E \to M$ be a smooth covering map. Then for each evenly covered open set $U \subseteq M$, a point $q \in U$, and $p \in \pi^{-1}(q)$, there exists a unique smooth local section $\sigma : U \to E$ of π such that $\sigma(q) = p$.

Proof. This is quite obvious from the definition of smooth covering maps. □

Proposition: **Products of Covering Maps**

Let $E_1, \ldots, E_k$ and $M_1, \ldots, M_k$ be smooth manifolds without boundary, and let $\pi_i : E_i \to M_i$ be smooth covering maps for each $1 \leq i \leq k$. Then the product map

$$\pi_1 \times \cdots \times \pi_k : E_1 \times \cdots \times E_k \to M_1 \times \cdots \times M_k$$

is a smooth covering map.

Theorem 4.4.2: Covering Space of Smooth manifolds

Suppose M is a connected smooth n-manifold, and $\pi : E \to M$ is a topological covering map, then E is a topological n-manifold and has a unique smooth structure such that $\pi : E \to M$ is a smooth covering map.
Moreover, if M is smooth manifold with boundary, then E is a topological manifold with boundary such that $\partial E = \pi^{-1}(\partial M)$, and has a unique smooth structure with boundary such that $\pi : E \to M$ is a smooth covering map.

Proof. SORRY. □

Corollary 4.4.1: Existence of Universal Covering Manifold

If M is a connected smooth manifold, there exists a simply connected smooth manifold $\tilde{M}$ called the **universal covering manifold** of M, and a smooth covering map $\pi : \tilde{M} \to M$. The universal covering manifold is unique up to diffeomorphism. That is, if $\tilde{M}'$ is another simply connected smooth manifold with a smooth covering map $\pi' : \tilde{M}' \to M$, then there exists a diffeomorphism $F : \tilde{M} \to \tilde{M}'$ such that $\pi' \circ F = \pi$.

There are not many simple criterion for a smooth map to be a smooth covering map, but we do have the following sufficient (not necessary) condition.

Proposition: **Local Proper Diffeomorphism is a Smooth Covering Map**

Let E and M be connected smooth manifolds, with or without boundary, and let $\pi : E \to M$ be a smooth map. If π is a local diffeomorphism and a proper map, then it is a smooth covering map.

Chapter 5

Submanifolds

We have already seen that open subsets of manifolds are themselves manifolds. But the range of possible submanifolds is much broader.

5.1 Embedded Submanifolds

Definition 5.1.1: Embedded Submanifold

Suppose M is a smooth manifold, with or without boundary. An embedded submanifold of M is a subset $S \subseteq M$ equipped with the subspace topology and a smooth structure such that the inclusion map $\iota_S : S \hookrightarrow M$ is a smooth embedding.

If S is an embedded submanifold of M, then the difference $\dim M - \dim S$ is called the **codimension** of S in M. M is called the ambient manifold of S. An embedded hypersurface is an embedded submanifold of codimension 1.

Proposition: **Open Submanifolds**

Suppose M is a smooth manifold. The embedded submanifolds of M of codimension 0 are precisely the open subsets of M.

Proof. Suppose $U \subseteq M$ is open. Then it has the same dimension as M, and the inclusion map $\iota_U : U \hookrightarrow M$ is a smooth embedding, so U is an embedded submanifold of codimension 0.

Conversely, suppose U is an embedded submanifold of codimension 0. Then the inclusion map $\iota_U : U \hookrightarrow M$ is a smooth embedding, so it is a local diffeomorphism and an open map. Thus, U is open in M. ☐

We can produce embedded submanifolds using images of embeddings.

Proposition: **Images of Embeddings**

Suppose M is a smooth manifold, with or without boundary, and N is a smooth manifold without boundary. If $F : N \to M$ is a smooth embedding, then $S = F(N)$ with the subspace topology has a unique smooth structure making it into an embedded submanifold of M such that $F : N \to S$ is a diffeomorphism.

Remark:
As by definition, embedded submanifolds are images of embeddings, this proposition shows that embedded submanifolds are exactly images of embeddings.

Proposition: **Slices of Products**

Suppose M and N are smooth manifolds. For each $p \in N$, the subset $M \times \{p\} \subseteq M \times N$ is an embedded submanifold that is diffeomorphic to M.

Proposition: **Graphs as Submanifolds**

Suppose M is a smooth m-manifold without boundary, and N is a smooth n-manifold with or without boundary. If $U \subseteq M$ be open, and $f : U \to N$ is a smooth map, then let $\Gamma(f)$ denote the **graph** of f,

$$\Gamma(f) = \{(p, f(p)) \in M \times N : p \in U\}.$$

Then $\Gamma(f)$ is an embedded m-submanifold of $M \times N$ that is diffeomorphic to U.

Sometimes, merely being an embedded submanifold is not enough. An embedded submanifold S of M is said to be properly embedded if the inclusion map $\iota_S : S \hookrightarrow M$ is a proper map.

Proposition: **Criterion for Properly Embedded Submanifolds**

Suppose M is a smooth manifold, with or without boundary, and S is an embedded submanifold of M. Then S is properly embedded if and only if S is a closed subset of M.
Therefore, we have every compact embedded submanifold is properly embedded.

Proof. Suppose S is properly embedded. Then the inclusion map $\iota_S : S \hookrightarrow M$ is a proper map, so the preimage of every compact set in M is compact in S. In particular, the preimage of every closed set in M is closed in S. Since S has the subspace topology, this implies that S is closed in M.

Conversely, suppose S is closed in M. Then for any compact set $K \subseteq M$, the intersection $K \cap S$ is closed in K, and since K is compact, $K \cap S$ is also compact. Thus, the preimage of every compact set in M under the inclusion map ι_S is compact in S, so ι_S is a proper map. Therefore, S is properly embedded. $\square$

Proposition: **Global Graphs are Properly Embedded**

Suppose M is a smooth m-manifold without boundary, and N is a smooth n-manifold with or without boundary. If $f : M \to N$ is a smooth map, then the graph $\Gamma(f)$ is a properly embedded m-submanifold of $M \times N$ that is diffeomorphic to M.

5.1.1 Slice Charts for Embedded Submanifolds

We will show that embedded submanifolds are modeled locally by the standard embedding R^k into $\mathbb{R}^n$ as the first k-coordinates:

$$\mathbb{R}^k \hookrightarrow \mathbb{R}^n, \quad (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

More generally, a k-slice of $U \subseteq \mathbb{R}^n$ is any subset of the form

$$S = \{(x^1, \dots, x^n) \in U : x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$

for some constants $c^{k+1}, \dots, c^n \in \mathbb{R}$. Note that every k-slice is diffeomorphic to an open subset of $\mathbb{R}^k$ via the projection map onto the first k-coordinates.

Now, let M be a smooth manifold and (U, φ) be a smooth chart on M. If $S \subseteq U$ that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that S is a **k-slice of U**. In general, we allow slices have any constant values in their last $n-k$ coordinates.

Given a subset $S \subseteq M$, and $k \geq 0$, we say S satisfies the local k-slice condition if each point $p \in S$ has a smooth chart (U, φ) for M such that $p \in U$ and $S \cap U$ is a k-slice of U. Any such chart is called a **slice chart** for S in M and the corresponding coordinates $(x^1, \dots, x^n)$ are called **slice coordinates** for S in M.

Theorem 5.1.1: Local Slice Criterion for Embedded Submanifolds

Let M be a smooth n-manifold, and if $S \subseteq M$ is an embedded k-submanifold, then S satisfies the local k-slice condition.
Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace topology S is a topological manifold of dimension k, and there is a smooth structure making S into an embedded k-submanifold of M.

Proof. SORRY □

Later we shall see that the smooth structure constructed in the theorem is the unique one in which S is an embedded submanifold of M.

Also, if M is a smooth manifold with boundary, and $S \subseteq M$ is an embedded submanifold, then S might interment ∂M in complicated ways. However, if $S = \partial M$ itself, then the boundary charts for M is just slice charts for S in M, and we do have the following proposition.

Theorem 5.1.2: Boundary as Embedded Submanifold

If M is a smooth n-manifold with boundary, then ∂M with the subspace topology is a topological $(n-1)$-manifold without boundary, and there is a smooth structure making ∂M a properly embedded $(n-1)$-submanifold of M.

Later, we shall see that this smooth structure is the unique one in which ∂M is an embedded submanifold of M.

Remark:
In order to study submanifolds of manifolds with boundary in greater generality, a typical approach is to find an embedding of M into a larger smooth manifold $\tilde{M}$ without boundary.

5.1.2 Level Sets

In practice, many embedded submanifolds arise as solution sets of systems of equations. If $\Phi : M \to N$ be any map and $c \in N$, we call the set

$$\Phi^{-1}(c) = \{p \in M : \Phi(p) = c\}$$

the **level set** of Φ at c. In the special case where $N = \mathbb{R}^k$ and $c = 0$, we call $\Phi^{-1}(0)$ the **zero set** of Φ.

It is easy to find examples where level sets of smooth functions that are not smooth submanifold. As we previously saw, all closed subset of M can be expressed as the zero set of some smooth function $M \to \mathbb{R}$. However, we have

Theorem 5.1.3: Constant Rank Level Set Theorem

Suppose M, N are smooth manifolds, and $\Phi : M \to N$ is a smooth map with constant rank r. Then each level set of Φ is a properly embedded submanifold of M with codimension r. Specifically, if Φ is a submersion, then each level set is a properly embedded submanifold of codimension $\dim N$.

Proof. Let $\dim M = m$ and $\dim N = n$, and $k = m - r$ be the codimension. $\forall c$, let $S = \Phi^{-1}(c)$. From the rank theorem, $\forall p \in S$, there exist smooth charts (U, φ) around p in M and (V, ψ) around c in N such that Φ has the local representation

$$\Phi : (x^1, \dots, x^m) \mapsto (x^1, \dots, x^r, 0, \dots, 0).$$

So we have

$$S \cap U = \{(x^1, \dots, x^m) \in U : x^1 = 0, \dots, x^r = 0\},$$

Hence, S satisfies the local k-slice condition. By the Local Slice Criterion for Embedded Submanifolds, S is an embedded k-submanifold of M.

Finally, to see that S is properly embedded, note that Φ is continuous, so $S = \Phi^{-1}(c)$ is closed in M. By the Criterion for Properly Embedded Submanifolds, S is properly embedded. $\square$

Remark:
This corresponds to the familiar rank-nullity theorem from linear algebra.

Definition 5.1.2: Regular and Critical Point

Suppose M, N are smooth manifolds, and $\Phi : M \to N$ is a smooth map. A point $p \in M$ is called a **regular point** of Φ if the differential $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is surjective. If p is not a regular point, then it is called a **critical point** of Φ. The point $c \in N$ is called a **regular value** of Φ if every point in the preimage $\Phi^{-1}(c)$ is a regular point (we include $\Phi^{-1}(c) = \emptyset$). If c is not a regular value, then it is called a **critical value** of Φ.

Now, we weakens the hypothesis of the Constant Rank Level Set Theorem to only require that c is a regular value.

Corollary 5.1.1: Regular Level Set Theorem

Suppose M, N are smooth manifolds, and $\Phi : M \to N$ is a smooth map. If $c \in N$ is a regular value of Φ, then the level set $\Phi^{-1}(c)$ is a properly embedded submanifold of M with codimension $\dim N$.

Proof. From proposition 4.1, The set

$$U = \{p \in M : d\Phi_p \text{ is surjective}\}$$

is open in M and contains $\Phi^{-1}(c)$. The restriction $\Phi|_U : U \to N$ is a submersion, so by the Constant Rank Level Set Theorem, $\Phi^{-1}(c)$ is a properly embedded submanifold of U with codimension $\dim N$. Then take the composition $\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$ would would do. □

Not all embedded submanifolds arise as level sets of smooth maps. However, we do know that they at least locally do.

Proposition: **Local Level Set Representation of Embedded Submanifolds**

Suppose M is a smooth m-manifold, and $S \subseteq M$, then S is a embedded k-submanifold of M if and only if $\forall p \in S$, there exist an open neighborhood U of p in M and a smooth submersion $\Phi : U \to \mathbb{R}^{m-k}$ such that $S \cap U = \Phi^{-1}(c)$ for some $c \in \mathbb{R}^{m-k}$.

Definition 5.1.3: Defining Map

If $S \subseteq M$ is an embedded submanifold, then a smooth map $\Phi : M \to N$ that has S as a regular level set is called a **defining map** for S in M. If $N = \mathbb{R}^{m-k}$, we say Φ is a **defining function** for S in M. For example, $f(x) = |x|^2$ is a defining function for the sphere in $\mathbb{R}^n$. Generally, if $U \subseteq M$ is open and $\Phi : U \to N$ is a smooth map that has $S \cap U$ as a regular level set, then we say Φ is a **local defining map** for S in M.

The last proposition shows that every embedded submanifold has local defining function.

Example: **Surface of Revolution**

Let H be the half plane $\{(r, z) \in \mathbb{R}^2 : r > 0\}$, and $C \subseteq H$ be a one-dimensional embedded submanifold. Then the **surface of revolution** generated by C is the subset

$$S_C = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2}, z) \in C\}. \tag{5.1}$$

If $\varphi : U \to \mathbb{R}$ is any locally defining function for C in H, then the map

$$\Phi : \tilde{U} \to \mathbb{R}, \quad \Phi(x, y, z) = \varphi(\sqrt{x^2 + y^2}, z) \tag{5.2}$$

where $\tilde{U} = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2}, z) \in U\}$ is a local defining function for S_C in $\mathbb{R}^3$. Thus, by the Local Level Set Representation of Embedded Submanifolds, S_C is an embedded submanifold of $\mathbb{R}^3$.

5.2 Immersed Submanifolds

Definition 5.2.1: Immersed Submanifold

Suppose M is a smooth manifold, with or without boundary. An immersed submanifold of M is a subset $S \subseteq M$ equipped with a topology (not necessarily the subspace topology) making it a topological manifold without boundary, and a smooth structure such that the inclusion map $\iota_S : S \hookrightarrow M$ is a smooth immersion.

Remark:
This is a rather larger class of submanifolds, as every embedded submanifold is an immersed submanifold, but not conversely. We shall simply denote "smooth submanifold" to mean "immersed submanifold" unless otherwise specified. A smooth hypersurface is an immersed submanifold of codimension 1.
We can also define a immersed topological submanifold similarly, as a subset $S \subseteq M$ equipped with a topology making it a topological manifold (not necessarily the subspace topology) such that the inclusion map $\iota_S : S \hookrightarrow M$ is a topological immersion.

Usually, immersed submanifolds arise as images of immersions.

Proposition: **Images of Immersions as Submanifolds**

Suppose M is a smooth manifold, with or without boundary, and N is a smooth manifold without boundary. If $F : N \to M$ is an injective smooth immersion, then $S = F(N)$ has a unique topology and smooth structure making it into an immersed submanifold of M such that $F : N \to S$ is a diffeomorphism.

Proof. We shall define the topology on S to be $\{U \cap S : F^{-1}(U) \text{ is open in } N\}$. And the smooth structure is defined by the charts $\{F(U), \varphi \circ F^{-1}\}$ where (U, φ) are charts of N. □

Example: **Immersed Submanifolds**

The figure eight curve and the dense curve on the torus are examples of immersed submanifolds that are not embedded submanifolds.

Remark:
In fact, suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. Then every subset of S that is open in the subspace topology is open in the topology of S, but the converse is not necessarily true. The converse holds if and only if S is an embedded submanifold of M.

Proposition: **From Immersed to Embedded Submanifolds**

Suppose M is a smooth manifold, with or without boundary, and S is an immersed submanifold of M. If one of the following conditions holds, then S is an embedded submanifold of

M:

- S has codimension 0 in M.
- The inclusion map $\iota_S : S \hookrightarrow M$ is a proper map.
- S is compact.

Proposition: **Locally Embeddedness of Immersed Submanifolds**

Suppose M is a smooth manifold, with or without boundary, and S is an immersed submanifold of M. Then $\forall p \in S$, there exists an open neighborhood U of p in S that U is an embedded submanifold of U.

Remark:
This does NOT mean that we can find an open neighborhood W of p in M such that $S \cap W$ is an embedded submanifold of W.

Definition 5.2.2: Local Parametrization

Suppose M is a smooth manifold, with or without boundary, and S is an immersed k-submanifold of M. A **local parametrization** for S in M is a continuous map $X : U \to M$ such that

- U is an open subset of $\mathbb{R}^k$,
- $X(U)$ is an open subset of S (in the topology of S),
- $X : U \to X(U)$ is a homeomorphism (in the topology of S),

It is called a **smooth local parametrization** if $X : U \to X(U)$ is a diffeomorphism onto its image (with the smooth structure of S). If $X(U) = S$, then X is called a **global parametrization** of S in M.

Proposition: **Criterion for Local Parametrization**

Suppose M is a smooth manifold, with or without boundary, and S is an immersed k-submanifold of M. Let $\iota : S \hookrightarrow M$ be the inclusion map. A map $X : U \to M$ is a local parametrization for S in M if and only if there is a smooth coordinate chart (V, φ) for S that $X = \iota \circ \varphi^{-1}$.
Therefore, every point in S is in the image of a local parametrization.

Example: **Parametrizations**

- Graph parametrizations: Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is a smooth function. Then the map

$$\gamma_f : U \to \mathbb{R}^{n+k}, \quad \gamma_f(x) = (x, f(x))$$

is a global parametrization for the graph $\Gamma(f)$ of f in $\mathbb{R}^{n+k}$.

- Figure-eight curve: Let $S \subseteq \mathbb{R}^2$ be the figure-eight curve, Then the map

$$\beta : (-\pi, \pi) \to \mathbb{R}^2, \quad \beta(t) = (\sin t, \sin(2t))$$

is a global parametrization for S in $\mathbb{R}^2$.

5.3 Restricting Maps to Submanifolds

Theorem 5.3.1: Restricting the Domain of a Smooth Map

Suppose M, N are smooth manifolds, with or without boundary, and S is an immersed submanifold of M. If $F : M \to N$ is a smooth map, then the restriction $F|_S : S \to N$ is a smooth map.

Proof. The inclusion map $\iota_S : S \hookrightarrow M$ is smooth. And $F_S = F \circ \iota_S$, so F_S is smooth. □

But we cannot generally restrict the codomain of a smooth map to an immersed submanifold. For example, take

$$G : \mathbb{R} \to \mathbb{R}^2, \quad G(t) = (\sin t, \sin 2t),$$

it is a smooth map whose image is the figure-eight curve S. However, if we consider G as a map from $\mathbb{R}$ to S, then it is not even continuous at π.

Theorem 5.3.2: Restricting the Codomain of a Smooth Map

Suppose M, N are smooth manifolds without boundary, and S is an immersed submanifold of M, and $F : N \to M$ is a smooth map such that $F(N) \subseteq S$. If $F : N \to S$ is continuous in the topology of S, then it is smooth.
The result also holds when M has nonempty boundary. If S is an embedded submanifold of M, then the continuity hypothesis is automatically satisfied.

However, there are certain immersed but not embedded submanifolds that the result automatically holds without the continuity hypothesis. To distinguish them, we introduce the following definition.

Definition 5.3.1: Weakly Embedded

Suppose M is a smooth manifold, and S is an immersed submanifold of M. Then S is said to be **weakly embedded** if for every smooth manifold N and every smooth map $F : N \to M$ such that $F(N) \subseteq S$, the induced map $F : N \to S$ is smooth (without any additional continuity hypothesis).

5.3.1 Uniqueness of Smooth Structures on Submanifolds

SORRY

5.3.2 Extending Functions from Submanifolds

Lemma 5.3.1: Extension Lemma For Submanifolds

Suppose M is a smooth manifold, and $S \subseteq M$ is an immersed submanifold. If $f : S \to \mathbb{R}$ is a smooth function on the submanifold structure, denote $f \in C^\infty(S)$. Then

- If S is embedded, then there exist a neighborhood U of S in M and a smooth function $\tilde{f} \in C^\infty(U)$ such that $\tilde{f}|_S = f$.
- If S is properly embedded, then U can be taken to be all of M.

5.4 The Tangent Space to a Submanifold

Suppose M is a smooth manifold, with or without boundary, and S is an immersed submanifold of M. Since the inclusion map $\iota : S \hookrightarrow M$ is a smooth immersion, for each $p \in S$, we can identify the tangent space T_pS as a subspace of T_pM $\mathrm{d}\iota_p$:

$$\mathrm{d}\iota_p(v)f = v(f \circ \iota) = v(f|_S), \quad \forall v \in T_pS, \forall f \in C^\infty(M).$$

Proposition: **Identify Submanifold Tangent Space**

Suppose M is a smooth manifold, with or without boundary, and S is an immersed submanifold of M, and $p \in S$. Then a vector $v \in T_pM$ is in the subspace $T_pS \subseteq T_pM$ if and only if there exists a smooth curve $\gamma : J \to M$ such that

- $\gamma(J) \subseteq S$,
- γ is smooth as a map into S,
- $0 \in J$ and $\gamma(0) = p$,
- $\gamma'(0) = v$.

If S is an embedded submanifold, then we have

$$T_pS = \{v \in T_pM : \forall f \in C^\infty(M), f|_S = 0, vf = 0\}. \tag{5.3}$$

Proof. Quite clear from the local embeddedness of immersed submanifolds. $\square$

We can also characterize tangent spaces via defining maps.

Proposition: **Tangent Space via Defining Maps**

Suppose M is a smooth manifold, and S is an embedded submanifold of M. If $\Phi : U \to N$ is any local defining map for S in M then

$$T_pS = \ker \mathrm{d}\Phi_p : T_pM \to T_{\Phi(p)}N, \tag{5.4}$$

for each $p \in S \cap U$.
Specifically, if S is a level set of a smooth submersion $\Phi : M \to \mathbb{R}^k$, then

$$T_pS = \{v \in T_pM : v\Phi^i = 0, i = 1, \dots, k\}, \tag{5.5}$$

If M is a smooth manifold with boundary, and $p \in \partial M$, intuitively, we expect that we can classify the tangent vectors to three categories: those that point inward to M, those that point outward to M, and those that are tangent to the boundary ∂M itself. This is indeed the case. We interpret the boundary as an embedded submanifold of M from theorem 5.1.2.

Definition 5.4.1: Tangent Vectors On the Boundary

If $p \in \partial M$, then a vector $v \in T_pM - T_p(\partial M)$ is said to be **inward-pointing** if $\exists \epsilon > 0$, there is a smooth curve $\gamma : [0, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. It is said to be **outward-pointing** if the same holds for a smooth curve $\gamma : (-\epsilon, 0] \to M$.

Proposition: **Boundary Tangent Vectors from Coordinates**

Let M be a smooth n-manifold with boundary, and $p \in \partial M$. If (U, φ) is a boundary chart for M around p, with coordinates $(x^1, \dots, x^n)$, then a vector $v \in T_pM$ is inward-pointing if and only if it has positive x^n-component, outward-pointing if and only if it has negative x^n-component, and tangent to the boundary if and only if its x^n-component is zero.
This gives a partition of T_pM by

$$T_pM = \{\text{inward-pointing}\} \sqcup T_p(\partial M) \sqcup \{\text{outward-pointing}\}. \tag{5.6}$$

Definition 5.4.2: Boundary Defining Function

Suppose M is a smooth manifold with boundary. A **boundary defining function** for M is a smooth function $f : M \to [0, \infty)$ such that $\partial M = f^{-1}(0)$ and $\mathrm{d}f_p \neq 0$ for all $p \in \partial M$.

For example, the defining function for a closed unit ball in $\mathbb{R}^n$ is $f(x) = 1 - |x|^2$.

Proposition: **Existence of Boundary Defining Functions**

Suppose M is a smooth manifold with boundary. Then there exists a boundary defining function for M.

Proof. Let $\{U_\alpha, \varphi_\alpha\}$ be a collection of smooth charts covering M, define $f_\alpha : U_\alpha \to [0, \infty)$ by

- If U_α is an interior chart then $f_\alpha = 1$.
- If U_α is a boundary chart with coordinates $(x^1, \dots, x^n)$, then $f_\alpha = x^n$.

Thus $f_\alpha > 0$ in the interior and $f_\alpha = 0$ on the boundary. Take any partition of unity $\{\psi_\alpha\}$ subordinate to the cover and taking $f = \sum_\alpha \psi_\alpha f_\alpha$ would do. To see $\mathrm{d}f_p \neq 0$ for all $p \in \partial M$, we have

$$\mathrm{d}f_p(v) = \sum_\alpha (f_\alpha \mathrm{d}\psi_\alpha|_p(v) + \psi_\alpha(p)\mathrm{d}f_\alpha|_p(v)) = \sum_\alpha \psi_\alpha(p)\mathrm{d}f_\alpha|_p(v),$$

□

Remark:
Usually, it is fairly easy to say that if a subset of M is an embedded submanifold for they are exactly those satisfying the local slice condition. However, it is often much more difficult to determine whether a subset is an immersed submanifold. A common technique is to first assume it is, then derive a contradiction from some phenomenon:

- $\forall p \in S$, the tangent space T_pS is a linear subspace of T_pM with constant dimension.
- $\forall p \in S$, it is in the image of a local parametrization.
- Each vector tangent to S at p is the velocity vector of a smooth curve in S through p.
- Each vector tangent to S at p annihilates all smooth functions on M that vanish on S.

Example: **Proving Smooth Submanifolds**

Consider

$$S = \{(x, y) : y = |x|, x \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$

It is easy that $S - \{(0, 0)\}$ is an embedded submanifold of $\mathbb{R}^2$. If S is a smooth submanifold of $\mathbb{R}^2$, then it must be one-dimensional from local embeddedness. Then $T_{(0,0)}S$ must be a one-dimensional linear subspace of $T_{(0,0)}\mathbb{R}^2 \cong \mathbb{R}^2$. This means that there is a smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = (0, 0)$ and $\gamma'(0) \neq 0$. However, the only such curve is $\gamma(t) = (0, 0)$ for all t, which contradicts $\gamma'(0) \neq 0$. Hence, S is not a smooth submanifold of $\mathbb{R}^2$.

5.5 Submanifolds with Boundary

The definition is very similar to the case without boundary.

Definition 5.5.1: Submanifold with Boundary

Suppose M is a smooth manifold with or without boundary. A submanifold with boundary of M is a subset $S \subseteq M$ equipped with a topology making it a topological manifold with boundary, and a smooth structure such that the inclusion map $\iota_S : S \hookrightarrow M$ is a smooth immersion.
If the inclusion map is a smooth embedding, then S is called an **embedded submanifold with boundary** of M, and in the general case, it is called an **immersed submanifold with boundary** of M.
A regular domain of M is a properly embedded submanifold with boundary of codimension 0.

Proposition: **Topological Boundary and Manifold Boundary**

Suppose M is a smooth manifold without boundary, and $D \subseteq M$ is a regular domain. Then the topological boundary and interior of D in M coincide with the manifold boundary and interior of D as a manifold with boundary, respectively.

Proof. Simply due to D having the subspace topology from M. $\square$

Proposition: **Generating Regular Domains**

Suppose M is a smooth manifold without boundary, and $f \in C^\infty(M)$, then

- For each regular value $b \in \mathbb{R}$ of f, the set $f^{-1}((-\infty, b])$ is a regular domain in M. It is called a **sublevel set** of f. And if D is a regular domain that $D = f^{-1}((-\infty, b])$ for some f and b, then f is called a **defining function** for D in M.
- For each regular value $a < b$ in f, then the set $f^{-1}([a, b])$ is a regular domain in M.

Theorem 5.5.1: Existence of Sublevel Defining Functions

If M is a smooth manifold without boundary, and $D \subseteq M$ is a regular domain, then there exists a smooth function $f \in C^\infty(M)$ being a defining function for D in M. If D is compact, then f can be chosen to be a smooth exhaustion function on M.

Proposition: **Properties of Submanifolds with Boundary**

Suppose M is a smooth manifold with or without boundary, then

- Every open subset of M is an embedded submanifold with or without boundary of codimension 0.
- If N is a smooth manifold with boundary, and $F : N \to M$ is a smooth embedding, then $F(N)$ is an embedded submanifold with boundary of M, with the subspace topology and smooth structure.
- If $S \subseteq M$ is an immersed submanifold with boundary of M, then for each $p \in S$ there exists a neighborhood U of p in S such that U is an embedded submanifold with boundary of M.

A k-dimensional half-slice of U is a subset

$$\{(x^1, \dots, x^n) \in U : x^{k+1} = c^{k+1}, \dots, x^n = c^n, x^n \geq 0\},$$

We say that $S \subseteq M$ satisfies the **local half-slice condition** if $\forall p \in S$, there exist a chart (U, φ) for M around p such that $S \cap U$ is a k-dimensional usual slice or half-slice of U. In the former case it is called the interior slice chart of S in M, and in the latter case it is called the boundary slice chart of S in M.

Theorem 5.5.2: Local Half-Slice Criterion for Embedded Submanifolds with Boundary

Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$. Then S is an embedded submanifold with boundary of M if and only if it satisfies the local half-slice condition.

Theorem 5.5.3: Restricting the Domain of a Smooth Map to a Submanifold with Boundary

Suppose M, N are smooth manifolds with boundary and $S \subseteq M$ is an embedded submanifold with boundary.

- Restricting the domain: If $F : M \to N$ is a smooth map, then the restriction $F|_S : S \to N$ is a smooth map.
- Restricting the codomain: If $\partial M \neq \emptyset$, and $F : N \to M$ is a smooth map such that $F(N) \subseteq S$, then the induced map $F : N \to S$ is smooth.

Remark:
Actually, the requirement $\partial M \neq \emptyset$ is not necessary.

Chapter 6

Sard's Theorem

We study the behavior of critical values of smooth maps between manifolds. Sard's Theorem states that the set of critical values has measure zero in the target manifold.

6.1 The Sard's Theorem

Theorem 6.1.1: The Sard's Theorem

Suppose M, N are smooth manifolds, with or without boundary, and $F : M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N.

This shows that if $\dim M < \dim N$, then $F(M)$ has measure zero in N. Because each point of M is critical for F.

6.2 The Whitney Embedding Theorem

Now we formalize out intuition that smooth manifolds are smoooth "surfaces" in Euclidean space.

Firstly, we show that an injective immersion of an n-dimensional manifold into $\mathbb{R}^N$ can be turned into a lower dimensional immersion if $N > 2n + 1$.

Lemma 6.2.1: Lower the Immersion Dimension

Suppose $M \subseteq \mathbb{R}^N$ is a smooth n-dimensional submanifold with or without boundary. Let $\mathbb{R}^{N-1}$ be the subspace of $\mathbb{R}^N$ with last coordinate zero. For any $v \in \mathbb{R}^N - \mathbb{R}^{N-1}$, let $\pi_v : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the projection with kernel $\mathbb{R}v$. If $N > 2n+1$, then there exists a dense set of $v \in \mathbb{R}^N - \mathbb{R}^{N-1}$ such that $\pi_v|_M : M \to \mathbb{R}^{N-1}$ is an injective immersion.

Lemma 6.2.2: Lowering to $2n + 1$

Let M be a smooth n-dimensional manifold with or without boundary. If M has a smooth embedding into $\mathbb{R}^N$ for some $N > 2n + 1$, then it has a smooth embedding into $\mathbb{R}^{2n+1}$.

Theorem 6.2.1: Whitney Embedding Theorem

Every smooth n-dimensional manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.

Theorem 6.2.2: Whitney Immersion Theorem

Every smooth n-dimensional manifold with or without boundary admits a smooth immersion into $\mathbb{R}^{2n}$.

Theorem 6.2.3: Strong Whitney Embedding Theorem

Every smooth n-dimensional manifold with or without boundary admits a smooth embedding into $\mathbb{R}^{2n}$.

Theorem 6.2.4: Strong Whitney Immersion Theorem

Every smooth n-dimensional manifold with or without boundary admits a smooth immersion into $\mathbb{R}^{2n-1}$.

6.3 The Whitney Approximation Theorem

If $\delta : M \to \mathbb{R}$ is a positive continuous function, then we say two functiona $F_1, F_2 : M \to \mathbb{R}^k$ are **δ-close** if for all $p \in M$, we have

$$\|F_1(p) - F_2(p)\| < \delta(p).$$

Theorem 6.3.1: Whitney Approximation Theorem for Functions

Let M be a smooth manifold with or without boundary, and let $F : M \to \mathbb{R}^k$ be a continuous map. Given a positive continuous function $\delta : M \to \mathbb{R}$, there exists a smooth map $G : M \to \mathbb{R}^k$ that is δ-close to F. If F is smooth on a closed subset $A \subseteq M$, then we can choose G so that $G|_A = F|_A$.

6.3.1 Tabular Neighborhoods

We need to generalized the Whitney Approximation Theorem to maps between manifolds. If $F : N \to M$ is smooth, then by the Whitney Embedding Theorem, we can embed M into some $\mathbb{R}^n$, and approximate F by a smooth map into $\mathbb{R}^n$. However, the image may not lie in M. To fix this, we use **tabular neighborhoods**.

Definition 6.3.1: Normal Space

Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold. For each $p \in M$, the **normal space** to M at p is the subspace $N_pM \subseteq T_x\mathbb{R}^n$ that are orthogonal to T_pM via the inherited Euclidean inner product on $\mathbb{R}^n$ itself.
The **normal bundle** of M is the set

$$NM = \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n : p \in M, v \in N_pM\}. \tag{6.1}$$

with a natural projection map $\pi_{NM} : NM \to M$ defined as the restriction of the projection map $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ to NM.

Theorem 6.3.2: Structure of Normal Bundle

If $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, then the normal bundle NM is an embedded n-dimensional submanifold of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

Definition 6.3.2: Tabular Neighborhoods

Think NM as a submanifold in $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$E : NM \to \mathbb{R}^n, \quad E(p, v) = p + v. \tag{6.2}$$

Then E is smooth. A **tabular neighborhood** of M is a neighborhood $U \subseteq \mathbb{R}^n$ of M such that it is diffeomorphic image under E of an open subset $V \subseteq NM$ that

$$V = \{(p, v) \in NM : \|v\| < \delta(p)\} \tag{6.3}$$

for some positive continuous function $\delta : M \to \mathbb{R}$.

Theorem 6.3.3: Tabular Neighborhood Theorem

Every embedded submanifold $M \subseteq \mathbb{R}^n$ has a tabular neighborhood.

Definition 6.3.3: Retraction

A **retraction** of a topological space X onto a subspace $A \subseteq X$ is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

Proposition: **Tabular Neighborhood to Retraction**

If $M \subseteq \mathbb{R}^n$ is an embedded submanifold with a tabular neighborhood U, then there exists a smooth retraction $r : U \to M$ that is also a smooth submersion.

6.3.2 Smooth Approximation between Manifolds

Theorem 6.3.4: Whitney Approximation Theorem

Let N be smooth manifolds with or without boundary, M be a smooth manifold without boundary, and let $F : N \to M$ be a continuous map, then F is homotopic to a smooth map $G : N \to M$. Furthermore, if F is a smooth map on a closed subset $A \subseteq N$, then the homotopy can be taken to be relative to A.

Corollary 6.3.1: Extension Lemma for Smooth Maps

Suppose N is a smooth manifold with or without boundary, and M is a smooth manifold without boundary. If $A \subseteq N$ is closed and $F : A \to M$ is a smooth map, then f has a smooth extension to N if and only if it has a continuous extension to N.

Definition 6.3.4: Smooth Homotopy

A **smooth homotopy** between smooth maps $F_0, F_1 : M \to N$ is a smooth map $H : M \times [0,1] \to N$ such that $H(p, 0) = F_0(p)$ and $H(p, 1) = F_1(p)$ for all $p \in M$.
If N, M are smooth manifolds with or without boundary, it is easy to see that smooth homotopy is an equivalence relation on the set of smooth maps from N to M.

Theorem 6.3.5: Homotopy and Smooth Homotopy

Suppose N is a smooth manifold with or without boundary, and M is a smooth manifold without boundary. $F, G : N \to M$ are smooth maps. Then if F is homotopic to G, then they are smoothly homotopic. If F, G are homotopic relative to a closed subset $A \subseteq N$, then they are smoothly homotopic relative to A.

6.4 Transversality

Vector space intersects nicely: If $V_1, V_2 \subseteq W$ are subspaces of a vector space W, then $V_1 \cap V_2$ is also a subspace of W. For manifolds, it is not always true: There are smooth submanifolds whose intersection is not a submanifold.

Theorem 6.4.1: Transversality

Suppose M is a smooth manifold. Two embedded submanifolds $S_1, S_2 \subseteq M$ intersect **transversely** if for every $p \in S_1 \cap S_2$, we have

$$T_pS_1 + T_pS_2 = T_pM.$$

in linear algebraic sense, where we consider T_pS_1, T_pS_2 as subspaces of T_pM.
If $F : N \to M$ is a smooth map and $S \subseteq M$ is an embedded submanifold, then we say F is **transverse** to S if for every $p \in F^{-1}(S)$, we have

$$T_{F(p)}S + \mathrm{d}F_p(T_pN) = T_{F(p)}M. \tag{6.4}$$

Remark:
The first definition is easy to understand: At each intersection point, the two submanifolds' tangent spaces cross nicely to span the whole tangent space of the ambient manifold. The second definition generalizes the first: It means that the image of N and the submanifold S intersect nicely in M.
Specially, if F is a submersion, then it is transverse to every embedded submanifold of M. And two embedded submanifolds $S_1, S_2 \subseteq M$ intersect transversely if and only if the inclusion

map $i : S_1 \hookrightarrow M$ is transverse to S_2.

Theorem 6.4.2: Generalized Level Set Theorem

Suppose M, N are smooth manifolds and $S \subseteq M$ is an embedded submanifold.

- If $F : N \to M$ is a smooth map that is transverse to S, then $F^{-1}(S)$ is an embedded submanifold of N with codimension equal to that of S in M.
- If $S' \subseteq M$ is another embedded submanifold that intersects S transversely, then $S \cap S'$ is an embedded submanifold of M with codimension equal to the sum of the codimensions of S and S' in M.
- If $F : N \to M$ is a smooth submersion, then for any embedded submanifold S with codimension k in M, $F^{-1}(S)$ is an embedded submanifold of N with codimension k.

This shows in $\mathbb{R}^3$, a smooth surface and a smooth curve intersecting transversely will yield a discrete set of points, and two smooth surfaces intersecting transversely will yield a smooth curve.

Theorem 6.4.3: Global Characterization of Graphs

Suppose M, N are smooth manifolds and $S \subseteq M \times N$ is an immersed submanifold. Let π_M and π_N be the projection maps from $M \times N$ to M and N respectively. Then the following are equivalent:

- S is the graph of a smooth map $f : M \to N$.
- $\pi_M|_S$ is a diffeomorphism from S to M.
- For each $p \in M$, the submanifolds S and $\pi_M^{-1}(p)$ intersect transversely in $M \times N$ at exactly one point.
- S is the graph of $f = \pi_N \circ (\pi_M|_S)^{-1}$.

Corollary 6.4.1: Local Characterization of Graphs

Suppose M, N are smooth manifolds and $S \subseteq M \times N$ is an immersed submanifold. If $(p, q) \in S$, S intersects $\pi_M^{-1}(p)$ transversely at (p, q), then there exists an open neighborhood U of p in M and V of (p, q) in S such that V is the graph of a smooth map $f : U \to N$.

Now, we generalize the concept of smooth homotopy:

Suppose N, M, S are smooth manifolds and $\forall s \in S$, we have a smooth map $F_s : N \to M$. If the map $F : N \times S \to M$ defined as $F(p, s) = F_s(p)$ is smooth, then we say $\{F_s\}_{s \in S}$ is a **smooth family** of smooth maps from N to M. This is just a higher dimensional generalization of smooth homotopy.

Proposition: **Smooth Family and Homotopy**

If $\{F_s\}_{s\in S}$ is a smooth family of smooth maps from N to M, if S is connected, then for any $s_1, s_2 \in S$, F_{s_1} is smoothly homotopic to F_{s_2}.

Proof. S is connected so it is path connected. Let $\gamma : [0,1] \to S$ be a smooth path with $\gamma(0) = s_1$ and $\gamma(1) = s_2$. Define $H(p,s) = F(p, \gamma(s))$ is a smooth homotopy between F_{s_1} and F_{s_2}. ☐

Theorem 6.4.4: Parametric Transversality Theorem

Suppose N, M are smooth manifolds, $X \subseteq M$ is an embedded submanifold, and $\{F_s\}_{s\in S}$ is a smooth family of smooth maps from N to M. If the map $F : N \times S \to M$ defined as $F(p,s) = F_s(p)$ is transverse to X, then for almost every $s \in S$, the map $F_s : N \to M$ is transverse to X.

Theorem 6.4.5: Transversality Homotopy Theorem

Suppose N, M are smooth manifolds and $X \subseteq M$ is an embedded submanifold. Every smooth map $F : N \to M$ is homotopic to a smooth map $g : N \to M$ that is transverse to X.

Chapter 7

Lie Groups

7.1 Definition and Examples

Definition 7.1.1: Lie Group

A **Lie group** is a smooth manifold G without boundary that is also a group, such that the group operations (multiplication and inversion) are smooth maps.

Proposition: **Identify Lie Groups**

If G is a smooth manifold and a group such that the map

$$f : G \times G \to G, \quad (g, h) \mapsto gh^{-1} \tag{7.1}$$

is smooth, then G is a Lie group.

Proof. We have

$$g \cdot h = f(g, h^{-1}), \quad h^{-1} = f(e, h).$$

□

If G is a Lie group, $\forall g \in G$, define

$$L_g : G \to G, \quad h \mapsto gh, \quad R_g : G \to G, \quad h \mapsto hg.$$

These are called **left** and **right translations** by g, respectively. Both are diffeomorphisms with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$, obviously.

Example: **Lie Groups**

- The general linear group $GL(n, \mathbb{R})$ is a Lie group under matrix multiplication. It is an open submanifold of $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$.

  Some subsets of $GL(n, \mathbb{R})$: $GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) > 0\}$, $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) = 1\}$, $O(n) = \{A \in GL(n, \mathbb{R}) : A^T A = I\}$, $SO(n) = O(n) \cap SL(n, \mathbb{R})$.

  Similarly, we have complex versions $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $U(n)$, $SU(n)$.

- Generally, for any vector space V over $\mathbb{R}$ or $\mathbb{C}$ with finite dimension, the group of all automorphisms $Aut(V)$ is a Lie group isomorphic to $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.
- If G is a Lie group, and $H \subseteq G$ be an open subgroup, then H is also a Lie group.
- The $\mathbb{R}$ and $\mathbb{C}$ under addition are Lie groups. The $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ under multiplication are also Lie groups.
- The S^1 under complex multiplication is a Lie group, called the **circle group**.
- Given $G_1, \ldots, G_k$ Lie groups, their product $G_1 \times \cdots \times G_k$ is also a Lie group under component-wise multiplication:

$$(g_1, \ldots, g_k) \cdot (h_1, \ldots, h_k) = (g_1 h_1, \ldots, g_k h_k). \tag{7.2}$$

  So the n-torus $T^n = S^1 \times \cdots \times S^1$ is a Lie group.
- Any group with discrete topology is a 0-dimensional Lie group.

7.2 Lie Group Homomorphisms

Definition 7.2.1: Lie Group Homomorphism

A **Lie group homomorphism** is a smooth map $F : G \to H$ between Lie groups that is also a group homomorphism, i.e., $F(gh) = F(g)F(h)$ for all $g, h \in G$. It is called a **Lie group isomorphism** if it is a diffeomorphism as well.

Example: **Lie Group Homomorphisms**

- The inclusion map $S^1 \hookrightarrow \mathbb{C}^*$ is a Lie group homomorphism.
- The map $\exp : (\mathbb{R}, +) \to (\mathbb{R}^*, \cdot)$ is a Lie group homomorphism. And $\exp : (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ is a Lie group isomorphism. The same goes for $\mathbb{C}$.
- The map $\epsilon : \mathbb{R} \to S^1$ defined by $\epsilon(t) = e^{2\pi i t}$ is a Lie group homomorphism with kernel $\mathbb{Z}$. Same goes for $\epsilon^n : \mathbb{R}^n \to T^n$ defined by $\epsilon^n(t_1, \ldots, t_n) = (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$ with kernel $\mathbb{Z}^n$.
- The determinant map $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$ is a Lie group homomorphism.
- For any Lie group G, the conjugation map $C_g : G \to G$ defined by $C_g(h) = ghg^{-1}$ is a Lie group isomorphism for each fixed $g \in G$.

Theorem 7.2.1: Constant Rank for Lie Group Homomorphisms

Every Lie group homomorphism has constant rank. A Lie group homomorphism is an isomorphism if and only if it is a bijection.

Proof. Let $F : G \to H$ be a Lie group homomorphism, and e_g and e_h be the identity elements. $\forall g_0 \in G$, we have

$$F(L_{g_0}(g)) = F(g_0 g) = F(g_0)F(g) = L_{F(g_0)}(F(g)), \qquad F \circ L_{g_0} = L_{F(g_0)} \circ F.$$

Taking the differential at e_g, we get

$$dF_{g_0} \circ dL_{g_0}|_{e_g} = dL_{F(g_0)}|_{e_h} \circ dF_{e_g}.$$

As L_{g_0} and $L_{F(g_0)}$ are diffeomorphisms, $dL_{g_0}|_{e_g}$ and $dL_{F(g_0)}|_{e_h}$ are isomorphisms. Thus, dF_{g_0} has the same rank as dF_{e_g} for all $g_0 \in G$.

The second claim follows from the global rank theorem. □

7.2.1 Universal Covering Groups

Theorem 7.2.2: Existence of Universal Covering Groups

Every connected Lie group G has a simply connected Lie group $\tilde{G}$ called the **universal covering group** of G, that has a smooth covering map $\pi : \tilde{G} \to G$ which is also a Lie group homomorphism.
Also, the universal covering group is unique up to isomorphism.

Example: **Universal Covering Group**

- $\epsilon^n : \mathbb{R}^n \to T^n$ by
$$\epsilon^n(t_1, \ldots, t_n) = (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$$
is a Lie group homomorphism and a smooth covering map. So $\mathbb{R}^n$ is the universal covering group of T^n.
- $\exp : \mathbb{C} \to \mathbb{C}^*$ is a Lie group homomorphism and a smooth covering map. So $\mathbb{C}$ is the universal covering group of $\mathbb{C}^*$.

7.3 Lie Subgroups

Definition 7.3.1: Lie Subgroup

A **Lie subgroup** of a Lie group G is a subgroup H of G with a topology and smooth structure such that H is a Lie group and an immersed submanifold of G.

The following shows that embedded subgroups are automatically Lie subgroups.

Proposition: **Embedded Subgroup is Lie Subgroup**

If H is an embedded submanifold of a Lie group G and a subgroup of G, then H is a Lie subgroup.

Proof. We only need to show the multiplication and inversion maps on H are smooth. □

The possibility of open subgroups as a candidate is limited:

Lemma 7.3.1: Open Subgroups as Lie Subgroups

Suppose G is a Lie group and H is an open subgroup of G. Then H is an embedded Lie subgroup of G. In addition H is also closed, and thus a union of connected components of G.

Proof. If H is open, then it is embedded from proposition 5.1. Then every left coset gH is also open. As $G - H$ is a union of left cosets, it is also open. Thus, H is closed. □

If G is a group and $S \subseteq G$, then the subgroup generated by S is the intersection of all subgroups of G containing S.

Proposition: **Generating Lie Subgroups**

Suppose G is a Lie group and $W \subseteq G$ is any neighborhood of the identity element e.

- W generates an open subgroup of G.
- If W is connected, then the subgroup generated by W is also connected.
- If G is connected, then W generates G.

Proposition: **The Identity Component**

Let G be a Lie group, and let G_0 be the connected component of the identity element $e \in G$, called the **identity component** of G. Then G_0 is a normal subgroup of G, and it is the only connected open subgroup of G. Every connected component of G is a coset of G_0, thus diffeomorphic to G_0.

Now lets move on to Lie subgroups that are not open subgroups.

Proposition: **Kernel as Lie Subgroup**

If $F : G \to H$ is a Lie group homomorphism, then $\ker F$ is a properly embedded Lie subgroup of G, with codimension equal to the rank of F.

Proposition: **Image as Lie Subgroup**

If $F : G \to H$ is an injective Lie group homomorphism, then $F(G)$ has a unique smooth structure making it a Lie subgroup of H, such that $F : G \to F(G)$ is a Lie group isomorphism.

Example: **Embedded Lie Subgroups**

- $GL^+(n,\mathbb{R})$ is an open subgroup of $GL(n,\mathbb{R})$, thus an embedded Lie subgroup.
- S^1 is an embedded Lie subgroup of $\mathbb{C}^*$.
- $SL(n,\mathbb{R})$ is the kernel of the determinant map $\det : GL(n,\mathbb{R}) \to \mathbb{R}^*$, thus a properly embedded Lie subgroup of $GL(n,\mathbb{R})$ with codimension 1. As determinant is a smooth submersion, $SL(n,\mathbb{R})$ has dimension n^2-1.

On the torus, let $\gamma : \mathbb{R} \to T^2$ be defined by $\gamma(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$ for some irrational number α. Then γ is a Lie group homomorphism from $\mathbb{R}$ to T^2. Its image is a Lie subgroup of T^2 that is not an embedded submanifold. In fact, its image is dense in T^2.

More interesting, in general smooth submanifolds can be closed without being embedded submanifolds, like figure-eight curve in $\mathbb{R}^2$, and can be embedded without being closed. However, for Lie subgroups, we have the following result:

Theorem 7.3.1: Embeddedness and Closeness of Lie Groups

Suppose G is a Lie group and H is a Lie subgroup of G. Then H is closed in G if and only if H is an embedded submanifold of G.

7.4 Group Actions and Equivariant Maps

Lie groups often act on smooth manifolds in a smooth way. Generally, if G is a group and M is a set, a **left group action** of G on M is a map $G \times M \to M$, $(g,x) \mapsto g \cdot x$ such that $\forall g,h \in G$ and $x \in M$, $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$. A **right group action** is defined similarly.

Now, let G be a Lie group and M be a smooth manifold. A **smooth left group action** of G on M is a left group action such that the map $\theta : G \times M \to M$, $(g,x) \mapsto g \cdot x$ is smooth.

We denote the left action by $\theta_g : M \to M$, $x \mapsto g \cdot x$. For a smooth left group action, each θ_g is a diffeomorphism with inverse $\theta_{g^{-1}}$. Some frequently used notions are listed below.

- Orbit: For each $p \in M$, the **orbit** of p is
$$G \cdot p = \{g \cdot p : g \in G\}.$$
- Isotropy Group: The **isotropy group** or **stabilizer** of p is
$$G_p = \{g \in G : g \cdot p = p\}.$$
The stabilizer is a subgroup of G.
- Transitive Action: If for every $p,q \in M$, there exists $g \in G$ such that $g \cdot p = q$, then the action is called **transitive**. Equivalently, the only orbit is M itself.
- Free Action: If for every $p \in M$, the only $g \in G$ such that $g \cdot p = p$ is $g = e$. Equivalently, the isotropy group G_p is trivial for all $p \in M$.

Example: **Lie Group Action**

- Trivial Action: G is any Lie group, and M is any smooth manifold. The trivial action is defined by $g \cdot p = p$ for all $g \in G$ and $p \in M$. This is a smooth left group action. Every orbit is a single point, and the isotropy group is the whole group G.
- Natural action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$: For $A \in GL(n, \mathbb{R})$ and $v \in \mathbb{R}^n$, the action is defined by $A \cdot v = Av$. This is a smooth left group action. The orbit of v is the line spanned by v, and the isotropy group is the set of all matrices that scale v. There are two orbits: $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$.
- Every Lie Group acts smoothly on itself by left multiplication. The action is both free and transitive. Also, every Lie group acts smoothly on itself by conjugation, i.e., $g \cdot h = ghg^{-1}$.
- A discrete group G acts smoothly on a smooth manifold M if and only if for each $g \in G$, the map $\theta_g : M \to M$ is a smooth map from M to itself. For example, $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by translation, i.e., $(m_1, \ldots, m_n) \cdot (x_1, \ldots, x_n) = (x_1 + m_1, \ldots, x_n + m_n)$.

Another important situation is covering maps. Suppose E, M be topological spaces and $\pi : E \to M$ is a topological covering map. An automorphism of π (or **covering transformation** or **deck transformation**) is a homeomorphism $\varphi : E \to E$ such that $\pi \circ \varphi = \pi$. The set of all automorphisms of π forms a group under composition, denoted by $\mathrm{Aut}_\pi(E)$, left acting on E.

Proposition: **Automorphisms as Lie Group**

Suppose E, M are smooth manifolds, with or without boundary, and $\pi : E \to M$ is a smooth covering map. With the discrete topology, $\mathrm{Aut}_\pi(E)$ is a zero-dimensional Lie group that acts smoothly and freely on E.

7.4.1 Equivariant Maps

Definition 7.4.1: Equivariant Maps

Suppose G is a Lie group acting smoothly on smooth manifolds M and N. A map $F : M \to N$ is called G-**equivariant** if $\forall g \in G$ and $p \in M$,

$$F(g \cdot p) = g \cdot F(p).$$

We also say that F intertwines the actions of G on M and N.

Theorem 7.4.1: Equivariant Rank Theorem

Let M, N be smooth manifolds, and G be a Lie group. Suppose $F : M \to N$ is a G-equivariant smooth map. If G acts transitively on M, then F has constant rank.

Proof. Let θ and φ be the actions of G on M and N, respectively. $\forall p, q \in M$, choose $g \in G, \theta_g(p) =$

q. Then from $\varphi_g \circ F = F \circ \theta_g$, we have

$$d\varphi_g|_{F(p)} \circ dF_p = dF_q \circ d\theta_g|_p.$$

So dF_q has the same rank as dF_p. □

Definition 7.4.2: Orbit Map

Suppose G is a Lie group acting smoothly on a smooth manifold M. For each $p \in M$, the **orbit map** at p is the map $\theta^p : G \to M$ defined by $\theta^p(g) = g \cdot p$. The image of θ^p is the orbit $G \cdot p$.

Proposition: **Properties of Orbit Maps**

Suppose θ is a smooth left group action of a Lie group G on a smooth manifold M. For each $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is smooth and has constant rank. So the isotropy group $G_p = (\theta^{(p)})^{-1}(p)$ is a properly embedded Lie subgroup of G. If $G_p = e$, then $\theta^{(p)}$ is an injective smooth immersion, so the orbit $G \cdot p$ is an immersed submanifold of M.

Remark:
In fact, every orbit is an immersed submanifold of M. But we shall postpone the proof until we develop more tools.

Proof. The orbit map $\theta^{(p)}$ is smooth as

$$G \cong G \times \{p\} \hookrightarrow G \times M \xrightarrow{\theta} M.$$

As G acts transitively on itself by left multiplication, by the equivariant rank theorem, $\theta^{(p)}$ has constant rank. Thus, G_p is a properly embedded Lie subgroup of G.

Also, when $G_p = e$, $d\theta^{(p)}|_e$ is injective, so $\theta^{(p)}$ is an injective smooth immersion. Thus the orbit $G \cdot p$ is an immersed submanifold of M. □

Example: **The Orthogonal Group**

$O(n)$ is naturally a subgroup of $GL(n, \mathbb{R})$. Define a smooth map $\Phi : GL(n, \mathbb{R}) \to M(n, \mathbb{R})$ by $\Phi(A) = A^T A$. Then $O(n) = \Phi^{-1}(I)$.
Define an action of $GL(n, \mathbb{R})$ on $M(n, \mathbb{R})$ by $A \cdot X = A^T X A$. This is a smooth left group action. And $GL(n, \mathbb{R})$ acts on itself by left multiplication. The map Φ is $GL(n, \mathbb{R})$-equivariant, since

$$\Phi(A \cdot B) = \Phi(AB) = (AB)^T (AB) = B^T (A^T A) B = A \cdot \Phi(B).$$

So by the equivariant rank theorem, Φ has constant rank. Thus, $O(n)$ is a properly embedded Lie subgroup of $GL(n, \mathbb{R})$. It is compact for it is closed and bounded in $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$.
To compute the dimension of $O(n)$, take $B \in T_I GL(n, \mathbb{R}) \cong M(n, \mathbb{R})$. Then take $\gamma(t) = I + tB$, we have

$$d\Phi_I(B) = \frac{d}{dt}\Big|_{t=0} \Phi(I + tB) = \frac{d}{dt}\Big|_{t=0} (I + tB)^T (I + tB) = B^T + B.$$

So $\mathrm{d}\Phi_I(T_I GL(n,\mathbb{R}))$ is the space of symmetric matrices, which has dimension $\frac{n(n+1)}{2}$. Then, $O(n)$ is a properly embedded Lie subgroup of $GL(n,\mathbb{R})$ with dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. The special orthogonal group $SO(n)$ is the intersection of $O(n)$ and $SL(n,\mathbb{R})$. It is also a properly embedded Lie subgroup of $GL(n,\mathbb{R})$ with dimension $\frac{n(n-1)}{2}$.

Example: **Unitary Group**

The unitary group $U(n)$ is defined as

$$U(n) = \{A \in GL(n,\mathbb{C}) : A^*A = I\},$$

where A^* is the conjugate transpose of A. Similar to the orthogonal group, the unitary group is a properly embedded Lie subgroup of $GL(n,\mathbb{C})$ with dimension n^2. The special unitary group $SU(n) = U(n) \cap SL(n,\mathbb{C})$ is also a properly embedded Lie subgroup of $GL(n,\mathbb{C})$ with dimension $n^2 - 1$.

7.4.2 Semidirect Products

Definition 7.4.3: Semidirect Product

Suppose H, N are Lie groups, and $\theta : H \times N \to N$ is a smooth left group action of H on N. It is said to be by **automorphisms** if $\forall h \in H$, the map $\theta_h : N \to N$, $n \mapsto h \cdot n$ is a Lie group automorphism of N.
Now we define a new Lie group $G = N \rtimes_\theta H$, called the **semidirect product** of N and H with respect to θ. As a smooth manifold, $G = N \times H$. The group operation is defined by

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1(h_1 \cdot n_2), h_1 h_2). \tag{7.3}$$

Remark:
Intuitively, a semidirect product is a generalization of a direct product. If the action θ is trivial, i.e., $h \cdot n = n$ for all $h \in H$ and $n \in N$, then the semidirect product reduces to the direct product $N \times H$. Its like first twisting N by the action of H, then taking the product.

Example: **The Euclidean Group**

Consider $\mathbb{R}^n$ as a Lie group under addition, and $O(n)$ as a Lie group under matrix multiplication. Then the natural action of $O(n)$ on $\mathbb{R}^n$ by matrix multiplication is by automorphisms. The semidirect product $E(n) = \mathbb{R}^n \rtimes O(n)$ is called the **Euclidean group**, which is the group of all isometries of $\mathbb{R}^n$ that preserve distances. The group acting on $\mathbb{R}^n$ is given by

$$(b, A) \cdot x = Ax + b, \quad (b, A)(b', A') = (b + Ab', AA'),$$

Proposition: **Properties of Semidirect Products**

Suppose H, N are Lie groups, and $\theta : H \times N \to N$ is a smooth left group action of H on N by automorphisms. Let $G = N \rtimes_\theta H$ be the semidirect product of N and H with respect to θ. Then

- The subsets $\tilde{N} = N \times \{e_H\}$ and $\tilde{H} = \{e_N\} \times H$ are closed Lie subgroups of G that are isomorphic to N and H, respectively.
- $\tilde{N}$ is a normal subgroup of G.
- $\tilde{N} \cap \tilde{H} = \{e_G\}$, where $e_G = (e_N, e_H)$ is the identity element of G.
- $\tilde{N}\tilde{H} = G$.

Theorem 7.4.2: Characterization of Semidirect Products

Suppose G is a Lie group, and $N, H \subseteq G$ are closed Lie subgroups such that N is normal. Also, suppose $N \cap H = \{e_G\}$ and $NH = G$. Then the map $(n, h) \mapsto nh$ is a Lie group isomorphism from the semidirect product $N \rtimes_\theta H$ to G, where $\theta : H \times N \to N$ is the action by conjugation, i.e., $\theta(h, n) = hnh^{-1}$.

7.4.3 Representations of Lie Groups

SORRY

Chapter 8

Vector Fields

8.1 Vector Fields on Manifolds

Definition 8.1.1: Vector Fields

If M is a smooth manifold, with or without boundary, a **vector field** on M is a smooth section of the map $\pi : TM \to M$. In other words, a vector field is a continuous map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$. This means that for each point $p \in M$, the vector field X assigns a tangent vector $X(p) \in T_pM$.
Usually, we are interested in smooth vector fields, meaning that the map X is smooth. If X is not even necessarily continuous, we say that X is a **rough vector field**.

We shall denote $X(p)$ by X_p for each $p \in M$, to be more readable. For any vector field X on M, we can write its component functions

$$X_p = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

according to some local smooth chart $(U, \varphi = (x^1, \dots, x^n))$ around p.

Proposition: **Smoothness Criterion for Vector Fields**

Let M be a smooth manifold, with or without boundary, and let $X : M \to TM$ be a rough vector field on M. If $(U, \varphi = (x^1, \dots, x^n))$ is a smooth chart on M, then the restriction of X to U is smooth if and only if the component functions $X^i : U \to \mathbb{R}$ is smooth for each i.

Example: **Vector Fields**

- Now if $(U, \varphi = (x^i))$ is any smooth chart on M, then we can define the **coordinate vector fields** on U by
$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p . \tag{8.1}$$
called the i-th coordinate vector field on U.

- Euler vector field: The vector field V on $\mathbb{R}^n$ by

  $$V_x = x^i \left.\frac{\partial}{\partial x^i}\right|_x = x^1 \left.\frac{\partial}{\partial x^1}\right|_x + \cdots + x^n \left.\frac{\partial}{\partial x^n}\right|_x$$

  is called the **Euler vector field** on $\mathbb{R}^n$.

- The angle vector field: Let θ be any angle coordinate on a proper open subset $U \subseteq S^1$, then let $\mathrm{d}/\mathrm{d}\theta$ be the corresponding coordinate vector field on U. Any other angle coordinate only differs from θ by an additive constant, so the vector field $\mathrm{d}/\mathrm{d}\theta$ is independent of the choice of angle coordinate. Thus, we can define a vector field on all of S^1 by defining it to be $\mathrm{d}/\mathrm{d}\theta$ on U at each proper open subset $U \subseteq S^1$. This vector field is called the **angle vector field** on S^1.

  The same goes for tori T^n.

We can see that tangent spaces behave locally. So we can identify T_pU with T_pM without ambiguity.

Definition 8.1.2: Vector Field Along a Subset

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be any subset. A **vector field along** A is a continuous map $X : A \to TM$ such that $\pi \circ X = \mathrm{id}_A$. We call it a smooth vector field along A if for each $p \in A$, there is an open neighborhood V of p in M and a smooth vector field $\tilde{X}$ on V such that $\tilde{X}|_{V\cap A} = X|_{V\cap A}$.

Lemma 8.1.1: Extension Lemma for Vector Fields

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be any closed subset. If X is a smooth vector field along A, then for any open neighborhood U of A in M, there is a smooth vector field $\tilde{X}$ on M such that $\tilde{X}|_A = X$ and $\mathrm{supp}(\tilde{X}) \subseteq U$.

Specifically, any vector at a point $p \in M$ can be extended to a smooth vector field on M that vanishes outside a small neighborhood of p.

Definition 8.1.3: Vector Field Spaces

If M is a smooth manifold, with or without boundary, we denote the space of all smooth vector fields on M by $\mathfrak{X}(M)$. It is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication.

In addition, smooth vector fields can be multiplied by smooth functions to produce new smooth vector fields. If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, then we define a new vector field $fX \in \mathfrak{X}(M)$ by

$$(fX)_p = f(p)X_p \tag{8.2}$$

Proposition: **Properties of $\mathfrak{X}(M)$**

Let M be a smooth manifold, with or without boundary.

- If $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$, then $fX + gY \in \mathfrak{X}(M)$.
- $\mathfrak{X}(M)$ is a module over the ring $C^\infty(M)$.

8.1.1 Local and Global Frames

Definition 8.1.4: Local and Global Frames

Let M be a smooth manifold, with or without boundary, and an ordered k-tuple $(X_1, \dots, X_k)$ of vector fields on some subset $A \subseteq M$. We say that they are linearly independent if $\forall p \in A$, the vectors $(X_1)_p, \dots, (X_k)_p$ are linearly independent in T_pM, and it spans the tangent bundle over A if $\forall p \in A$, the vectors $(X_1)_p, \dots, (X_k)_p$ span T_pM.
A **local frame** on M is an ordered n-tuple of vector fields $(E_1, \dots, E_n)$ on some open subset $U \subseteq M$ that is linearly independent and spans the tangent bundle over U. The vectors $(E_1)_p, \dots, (E_n)_p$ then form a basis for T_pM for each $p \in U$. It is called a **global frame** if $U = M$, and a **smooth frame** if each E_i is a smooth vector field.
A smooth manifold M is called **parallelizable** if it admits a smooth global frame.

Example: **Local and Global Frames**

- The standard coordinate vector fields $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ form a smooth global frame on $\mathbb{R}^n$.
- For any smooth chart $(U, \varphi = (x^1, \dots, x^n))$ on a smooth manifold M, the coordinate vector fields $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ form a smooth local frame on U.
- The angle vector field $\mathrm{d}/\mathrm{d}\theta$ on S^1 is a smooth global frame. And the n angle vector fields $\partial/\partial\theta^1, \dots, \partial/\partial\theta^n$ form a smooth global frame on the torus T^n.

Proposition: **Completion of Local Frames**

Let M be a n-smooth manifold, with or without boundary,

- Let $U \subseteq M$ be any open subset, and let $(X_1, \dots, X_k)$ be a linearly independent k-tuple of smooth vector fields on U, where $k < n$. Then there exist smooth vector fields $X_{k+1}, \dots, X_n$ on U such that $(X_1, \dots, X_n)$ is a smooth local frame on U.
- If $(v_1, \dots, v_k)$ is any linearly independent k-tuple of vectors in T_pM at some point $p \in M$, where $k \leq n$, then there exist a smooth local frame (X_i) on some open neighborhood U of p such that $X_i|_p = v_i$ for each $1 \leq i \leq k$.
- If $X_1, \dots, X_n$ is a linearly independent n-tuple of smooth vector fields on some closed subset $A \subseteq M$, then there exist smooth local frame $(\tilde{X}_i)$ on some open neighborhood U of A such that $\tilde{X}_i|_A = X_i$ for each $1 \leq i \leq n$.

Specially for subsets of $\mathbb{R}^n$, we can use the Euclidean inner product to define orthonormal frames. For example, the standard coordinate vector fields on $\mathbb{R}^n$ or the polar coordinate vector fields on $\mathbb{R}^2 \setminus \{0\}$.

Lemma 8.1.2: Gram-Schmidt Algorithm for Frames

Suppose (X_j) is a smooth local frame for $T\mathbb{R}^n$ over some open subset $U \subseteq \mathbb{R}^n$. Then there exists a smooth orthonormal local frame (E_j) for $T\mathbb{R}^n$ over U such that

$$\operatorname{span}\{X_1, \ldots, X_k\} = \operatorname{span}\{E_1, \ldots, E_k\} \quad \text{for each } 1 \leq k \leq n.$$

Generally speaking, parallelizable manifolds are rare. For example, spheres S^n are only parallelizable for $n = 1, 3, 7$. We shall later see that all Lie groups are parallelizable.

8.1.2 Vector Fields as Derivations

Vector fields define operators on smooth functions. If $X \in \mathfrak{X}(M)$ and f is a smooth function on some open subset $U \subseteq M$, then we can define a new smooth function Xf on U by

$$(Xf)(p) = X_p(f) \quad \forall p \in U. \tag{8.3}$$

Remark:
Note the difference between Xf and fX. The former is a smooth function on U, while the latter is a smooth vector field on M.

It is quite direct that Xf is defined locally, for any open subset $V \subseteq U$, we have

$$(Xf)|_V = X|_V(f|_V).$$

Proposition: **Properties of Vector Field Derivations**

Let M be a smooth manifold, with or without boundary, and let $X : M \to TM$ be a rough vector field on M. Then the following are equivalent:

- X is smooth.
- For every $f \in C^\infty(M)$, the function $Xf : M \to \mathbb{R}$ is smooth.
- For every open subset $U \subseteq M$ and every $f \in C^\infty(U)$, the function $Xf : U \to \mathbb{R}$ is smooth.

Proof. Quite obvious, just taking a local chart around each point. □

Recall the definition of a derivation at a point 3.1.1. Now we present the global version.

Definition 8.1.5: Derivations

Let M be a smooth manifold, with or without boundary. A **derivation** on M is a linear map $D : C^\infty(M) \to C^\infty(M)$ such that for all $f, g \in C^\infty(M)$,

$$D(fg) = fD(g) + gD(f).$$

The derivation at a point $p \in M$ is just the composition of D with the evaluation map at p:

$$D_p : C^\infty(M) \to \mathbb{R}, \quad D_p(f) = D(f)(p).$$

We can see that for a vector field $X \in \mathfrak{X}(M)$, we have

$$X(fg)(p) = X_p(fg) = f(p)X_p(g) + g(p)X_p(f) = f(p)(Xg)(p) + g(p)(Xf)(p) = (fXg + gXf)(p).$$

which matches our definition.

Theorem 8.1.1: Vector Fields and Derivations

Let M be a smooth manifold, with or without boundary. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation on M if and only if there exists a smooth vector field $X \in \mathfrak{X}(M)$ such that $D(f) = Xf$ for all $f \in C^\infty(M)$.
Thus, we can identify the space of derivations on M with the space $\mathfrak{X}(M)$ of smooth vector fields on M.

Proof. We have already shown the "if" part. For the "only if" part, for each $p \in M$, define $X_p : C^\infty(M) \to \mathbb{R}$ by $X_p(f) = D(f)(p)$ would do. Smoothness follows from the fact that $D(f)$ is smooth for each $f \in C^\infty(M)$. □

8.2 Vector Fields and Smooth Maps

Given $F : M \to N$ a smooth map and X a vector field on M, we can try to "push forward" X to a vector field on N by $\mathrm{d}F_p(X_p) \in T_{F(p)}N$. However, this does not necessarily define a vector field on N. If F is not surjective, then there may be points in N that are not in the image of F, and if F is not injective, then there may be points in N that have multiple preimages in M with different pushed-forward vectors.

Definition 8.2.1: F-related Vector Fields

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. If X is a vector field on M and Y is a vector field on N, we say that X and Y are **F-related** if for every $p \in M$,

$$\mathrm{d}F_p(X_p) = Y_{F(p)}.$$

Proposition: **F-related Vector Fields on Smooth Functions**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, then they are F-related if and only if for every $g \in C^\infty(N)$,

$$X(g \circ F) = (Yg) \circ F. \tag{8.4}$$

Proof. We have

$$X(g \circ F)(p) = X_p(g \circ F) = \mathrm{d}F_p(X_p)(g) = Y_{F(p)}(g) = (Yg)(F(p)) = ((Yg) \circ F)(p).$$

□

Example: **F-related Vector Fields**

Let $F : \mathbb{R} \to \mathbb{R}^2, t \mapsto (\cos t, \sin t)$ be the standard embedding of the unit circle. Then the vector field $X = \mathrm{d}/\mathrm{d}t$ on $\mathbb{R}$ is F-related to the angle vector field

$$Y_{(x,y)} = -y \left.\frac{\partial}{\partial x}\right|_{(x,y)} + x \left.\frac{\partial}{\partial y}\right|_{(x,y)}$$

For an arbitrary smooth map $F : M \to N$, there may not exist any nontrivial F-related vector fields. But for diffeomorphisms, we have the following result.

Proposition: **Existence of Related Vector Fields for Diffeomorphisms**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a diffeomorphism. Then for every vector field $X \in \mathfrak{X}(M)$, there exists a unique vector field $Y \in \mathfrak{X}(N)$ that is F-related to X, and vice versa.
We often denote Y by $F_* X$, called the **pushforward** of X by F. Explicitly, for each $q \in N$,

$$(F_* X)_q = \mathrm{d}F_{F^{-1}(q)}(X_{F^{-1}(q)}). \tag{8.5}$$

and we have

$$((F_* X)g) \circ F = X(g \circ F) \quad \forall g \in C^\infty(N). \tag{8.6}$$

8.2.1 Vector Fields and Submanifolds

If $S \subseteq M$ is an immersed or embedded submanifold, with or without boundary, then in general, a vector field on M does not restrict to a vector field on S, because the vectors may not lie in the tangent spaces of S.

Definition 8.2.2: Tangent to a Submanifold

Let M be a smooth manifold, with or without boundary, and let $S \subseteq M$ be an immersed or embedded submanifold, with or without boundary. A vector field $X \in \mathfrak{X}(M)$ is said to be **tangent to** S if for every $p \in S$, $X_p \in T_p S \subseteq T_p M$.

Proposition: **Criterion for Tangency to a Submanifold**

Let M be a smooth manifold, with or without boundary, and let $S \subseteq M$ be an embedded submanifold, with or without boundary. A vector field $X \in \mathfrak{X}(M)$ is tangent to S if and only if for every smooth function $f \in C^\infty(M)$ that vanishes on S, the function Xf also vanishes on S.

If $S \subseteq M$ is an immersed submanifold, with or without boundary, and $Y \in \mathfrak{X}(S)$, then if there is a vector field $X \in \mathfrak{X}(S)$ that is $\iota : S \hookrightarrow M$-related to Y, then clearly Y is tangent to S. Because for each $p \in S$, $Y_p = \mathrm{d}\iota_p(X_p) = X_p \in T_p S$. We shall see that the converse is also true.

Proposition: **Restricting Vector Fields to Submanifolds**

Let M be a smooth manifold, and $S \subseteq M$ be an immersed submanifold, with or without boundary. Let $\iota : S \hookrightarrow M$ be the inclusion map. A vector field $Y \in \mathfrak{X}(M)$ is tangent to S if and only if there exists a vector field $X \in \mathfrak{X}(S)$ that is ι-related to Y. In this case, X is unique, and we often denote it by $Y|_S$, called the **restriction** of Y to S.

8.3 Lie Brackets

Now, we introduce an important operation on vector fields, joining two vector fields to produce a new vector field, called the Lie bracket.

Let $X, Y \in \mathfrak{X}(M)$ be two smooth vector fields on a smooth manifold M. Given a $f \in C^\infty(M)$, we can successively apply X and Y to f to get a new smooth function $Y(Xf)$. However, this operatio $f \mapsto YXf$n does not satisfy the Leibniz rule, so is not a vector field. To fix this, we introduce the Lie bracket.

Definition 8.3.1: Lie Bracket

Let M be a smooth manifold, with or without boundary, and let $X, Y \in \mathfrak{X}(M)$ be two smooth vector fields on M. The **Lie bracket** of X and Y is the map $[X, Y] : C^\infty(M) \to C^\infty(M)$ defined by

$$[X, Y]f = X(Yf) - Y(Xf) \quad \forall f \in C^\infty(M). \tag{8.7}$$

Then $[X, Y]$ is a smooth vector field on M.

Proof. We shall show that $[X, Y]$ is a derivation on M. For any $f, g \in C^\infty(M)$, we have

$$\begin{aligned}
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= fX(Yg) + (Xf)(Yg) + gX(Yf) + (Xg)(Yf) \\
&\quad - fY(Xg) - (Yf)(Xg) - gY(Xf) - (Yg)(Xf) \\
&= f(X(Yg) - Y(Xg)) + g(X(Yf) - Y(Xf)) \\
&= f[X, Y]g + g[X, Y]f.
\end{aligned}$$

□

Theorem 8.3.1: Coordinate Formula for Lie Bracket

Let M be a smooth manifold, with or without boundary, and let $(U, \varphi = (x^1, \ldots, x^n))$ be a smooth chart on M. If $X, Y \in \mathfrak{X}(M)$ are two smooth vector fields on M, then on U, we can write

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j},$$

where $X^i, Y^j \in C^\infty(U)$ are the component functions of X and Y with respect to the coordinate vector fields. Then the Lie bracket $[X, Y]$ on U is given by

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} = \left(X(Y^j) - Y(X^j) \right) \frac{\partial}{\partial x^j}.$$

Proof. Mere computation: take $f \in C^\infty(U)$,

$$\begin{aligned}
[X, Y]f &= X(Yf) - Y(Xf) \\
&= X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f}{\partial x^j}\right) - Y^i \frac{\partial}{\partial x^i}\left(X^j \frac{\partial f}{\partial x^j}\right) \\
&= X^i \left(\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right) - Y^i \left(\frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j} + X^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right) \\
&= \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} + \left(X^i Y^j - Y^i X^j\right)\frac{\partial^2 f}{\partial x^i \partial x^j} \\
&= \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}.
\end{aligned}$$

□

A trivial example is that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0, \quad \forall 1 \le i, j \le n.$$

This is only that mixed partial derivatives commute for smooth functions.

Proposition: **Properties of Lie Bracket**

Let M be a smooth manifold, with or without boundary, and let $X, Y, Z \in \mathfrak{X}(M)$ be three smooth vector fields on M, and let $f, g \in C^\infty(M)$ be two smooth functions on M. Then the Lie bracket satisfies the following properties:

- Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$ for all $a, b \in \mathbb{R}$.
- Antisymmetry: $[X, Y] = -[Y, X]$.
- Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.
- Leibniz rule: $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

Theorem 8.3.2: Naturality of Lie Bracket

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. If $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ are vector fields such that X_i is F-related to Y_i for $i = 1, 2$, then the Lie bracket $[X_1, X_2]$ is F-related to the Lie bracket $[Y_1, Y_2]$.

Proof. We have

$$X_1 X_2 (f \circ F) = X_1\left((Y_2 f) \circ F\right) = (Y_1(Y_2 f)) \circ F, \qquad X_2 X_1 (f \circ F) = (Y_2(Y_1 f)) \circ F.$$

So putting them together,

$$[X_1, X_2](f \circ F) = (Y_1(Y_2 f) - Y_2(Y_1 f)) \circ F = ([Y_1, Y_2]f) \circ F.$$

□

Corollary 8.3.1: Pushforward of Lie Bracket

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a diffeomorphism. If $X_1, X_2 \in \mathfrak{X}(M)$ then

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]. \tag{8.8}$$

Corollary 8.3.2: Brackets Tangent to Submanifolds

Let M be a smooth manifold and S be an immersed submanifold, with or without boundary. If $X, Y \in \mathfrak{X}(M)$ are vector fields that are tangent to S, then the Lie bracket $[X, Y]$ is also tangent to S.

8.4 The Lie Algebra of Lie Groups

Definition 8.4.1: Left-Invariant

Let G be a Lie group, and let $X \in \mathfrak{X}(G)$ be a smooth vector field on G. We say that X is **left-invariant** if for every $g \in G$, X is L_g-related to itself. In other words

$$\forall g, h \in G, \quad \mathrm{d}L_g(X_h) = X_{gh}. \tag{8.9}$$

Sine L_g is a diffeomorphism for each $g \in G$, this means $(L_g)_*X = X$ for all $g \in G$.
From linearity, the set of all left-invariant vector fields on G forms a vector subspace of $\mathfrak{X}(G)$, and it is closed under the Lie bracket.

Proof. We have from $(L_g)_*X = X$ and $(L_g)_*Y = Y$ that

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y].$$

□

Definition 8.4.2: Lie Algebra

A Lie algebra over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ equipped with a map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the bracket, such that for all $X, Y, Z \in \mathfrak{g}$ and $a, b \in \mathbb{R}$, the following properties hold:

- Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$.
- Antisymmetry: $[X, Y] = -[Y, X]$.
- Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

If $\mathfrak{g}$ is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a **Lie subalgebra** if it is closed under the bracket operation.
If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** if for all $X, Y \in \mathfrak{g}$, $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$. It is called a **Lie algebra isomorphism** if it is bijective, and we say that $\mathfrak{g}$ and $\mathfrak{h}$ are **isomorphic** Lie algebras.

Example: **Lie Algebras**

- The space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M, with or without boundary, is a Lie algebra under the Lie bracket.
- If G is a Lie group, then the set of all left-invariant vector fields on G forms a Lie subalgebra of $\mathfrak{X}(G)$, denoted by $\mathrm{Lie}(G)$.
- The vector space $M(n, \mathbb{R})$ of all $n \times n$ real matrices is a Lie algebra under the bracket operation defined by the commutator:

$$[A, B] = AB - BA \quad \forall A, B \in M(n, \mathbb{R}).$$

  denoted by $\mathfrak{gl}(n, \mathbb{R})$. Similarly, $\mathfrak{gl}(n, \mathbb{C})$ is a $2n^2$-dimensional real Lie algebra.
- Generally, if V is a vector space over $\mathbb{R}$, the space $\mathfrak{gl}(V)$ of all linear operators on V is a Lie algebra under the commutator bracket.
- Any vector space V becomes a Lie algebra under the trivial bracket operation $[X, Y] = 0$ for all $X, Y \in V$. Such a Lie algebra is called an **abelian** Lie algebra. (This name reflects that most cases Lie algebras are just commutators like above.)

For $\mathrm{Lie}(G)$, intuitively, if we know the vector at any point on the manifold, then we can determine the whole vector field by left translations, so the dimension of $\mathrm{Lie}(G)$ should be the same as that of G.

Theorem 8.4.1: Structure of $\mathrm{Lie}(G)$

Let G be a Lie group of dimension n. Then the evaluation map $\epsilon : \mathrm{Lie}(G) \to T_eG$ defined by $\epsilon(X) = X_e$ is a vector space isomorphism. In particular, $\dim \mathrm{Lie}(G) = \dim G = n$.
The inversion map is given by

$$T_eG \to \mathrm{Lie}(G), \quad v \mapsto v^L|_g = \mathrm{d}(L_g)_e(v). \tag{8.10}$$

Proof. It is clear that ϵ is linear. To show it is injective, suppose $X \in \mathrm{Lie}(G)$ such that $X_e = 0$. Then for any $g \in G$, $X_g = \mathrm{d}(L_g)_e(X_e) = \mathrm{d}(L_g)_e(0) = 0$. So X is the zero vector field, and hence ϵ is injective. Surjectivity follows from the construction in the second part of the theorem, and smoothness is clear: take any smooth curve $\gamma : (-\delta, \delta) \to G$ with $\gamma(0) = e$ and $\gamma'(0) = v$, then

$$(v^L f)(g) = v^L|_g(f) = \mathrm{d}(L_g)_e(v)(f) = v(f \circ L_g) = \gamma'(0)(f \circ L_g) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ L_g \circ \gamma)(0).$$

If we define $\varphi : (-\delta, \delta) \times G \to \mathbb{R}$ by $\varphi(t, g) = f \circ L_g \circ \gamma(t) = f(g\gamma(t))$, then φ is smooth, so $(v^L f)(g) = \partial\varphi/\partial t(0, g)$ is smooth in g. Thus v^L is a smooth vector field, and hence $v^L \in \mathrm{Lie}(G)$. ☐

Therefore, given any vector $v \in T_eG$, there exists a unique left-invariant vector field $X \in \mathrm{Lie}(G)$ such that $X_e = v$, denoted by X^L.

We shall see that the smoothness condition in the definition of left-invariant vector fields is actually superfluous.

Corollary 8.4.1: Left-Invariant Rough Field

Let G be a Lie group. For any rough vector field $X : G \to TG$ that is left-invariant, X is smooth, and hence $X \in \mathrm{Lie}(G)$.

Corollary 8.4.2: Parallelizability of Lie Groups

Every Lie group admits a left-invariant smooth global frame, and is therefore parallelizable.

Example: **Lie Algebras of Lie Groups**

- $\mathbb{R}^n$: as a Lie group under addition, so a vector field X is left-invariant if and only if $X^i\partial/\partial x^i$ has constant component functions X^i. Thus $\mathrm{Lie}(\mathbb{R}^n) \cong T_0\mathbb{R}^n \cong \mathbb{R}^n$.
- S^1: as a Lie group under multiplication, the basis is just $\mathrm{d}/\mathrm{d}\theta$, so $\mathrm{Lie}(S^1) \cong T_1S^1 \cong \mathbb{R}$. The same goes for the torus T^n, the basis is $\partial/\partial\theta^1, \dots, \partial/\partial\theta^n$, so $\mathrm{Lie}(T^n) \cong T_eT^n \cong \mathbb{R}^n$.

We notice that the group $\mathbb{R}^n, S^1$ are abelian, and their Lie algebras are also abelian. This is not a coincidence. Every abelian Lie group has an abelian Lie algebra.

Proof. If G is abelian, then for any $X, Y \in \mathrm{Lie}(G)$, we have SORRY □

We shall also see that the converse holds when G is connected.

Now we come to inspect the Lie algebra of $GL(n,\mathbb{R})$. Consider $GL(n,\mathbb{R})$ is an open subset of $\mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$, so its tangent space is naturally isomorphic to $\mathfrak{gl}(n,\mathbb{R})$ itself. Also, from the structure theorem of $\mathrm{Lie}(G)$ 8.4.1, we have a vector space isomorphism $\mathrm{Lie}(GL(n,\mathbb{R})) \cong T_IGL(n,\mathbb{R}) \cong \mathfrak{gl}(n,\mathbb{R})$. However, note that $\mathrm{Lie}(GL(n,\mathbb{R}))$ and $\mathfrak{gl}(n,\mathbb{R})$ have independently defined brackets, the former is defined by Lie brackets of left-invariant vector fields, while the latter is defined by commutators of matrices. We shall see that these two brackets actually agree under the above isomorphism.

Theorem 8.4.2: Lie Algebra of $GL(n,\mathbb{R})$

The composition of natural vector space isomorphisms

$$\mathrm{Lie}(GL(n,\mathbb{R})) \xrightarrow{\epsilon} T_IGL(n,\mathbb{R}) \xrightarrow{\cong} \mathfrak{gl}(n,\mathbb{R}) \tag{8.11}$$

is a Lie algebra isomorphism between the left-invariant Lie algebra $\mathrm{Lie}(GL(n,\mathbb{R}))$ and the matrix Lie algebra $\mathfrak{gl}(n,\mathbb{R})$. This can also be generalized to any finite-dimensional real vector space V to give a Lie algebra isomorphism between $\mathrm{Lie}(GL(V))$ and $\mathfrak{gl}(V)$.

Proof. Using the matrix entries X^i_j as global coordinates on $GL(n,\mathbb{R}) \subseteq \mathfrak{gl}(n,\mathbb{R})$, the isomorphism

$$T_IGL(n,\mathbb{R}) \xrightarrow{\cong} \mathfrak{gl}(n,\mathbb{R}), \quad A^i_j \frac{\partial}{\partial X^i_j}\bigg|_I \mapsto A = (A^i_j)$$

So the isomorphism from $\mathfrak{gl}(n,\mathbb{R})$ to $\mathrm{Lie}(GL(n,\mathbb{R}))$ is given by, take any $A \in \mathfrak{gl}(n,\mathbb{R})$, then the corresponding left-invariant vector field $X^A \in \mathrm{Lie}(GL(n,\mathbb{R}))$ is given by

$$A^L|_X = \mathrm{d}(L_X)_I\left(A^i_j \frac{\partial}{\partial X^i_j}\bigg|_I\right) = A^i_j \mathrm{d}(L_X)_I\left(\frac{\partial}{\partial X^i_j}\bigg|_I\right) = A^i_j X^k_i \frac{\partial}{\partial X^k_j}\bigg|_X = X^i_j A^j_k \frac{\partial}{\partial X^i_k}\bigg|_X.$$

Next is pure computation: Take any $A, B \in \mathfrak{gl}(n, \mathbb{R})$, then

$$\begin{aligned}
[A^L, B^L] &= \left[X^i_j A^j_k \frac{\partial}{\partial X^i_k}, X^p_q B^q_r \frac{\partial}{\partial X^p_r}\right] \\
&= X^i_j A^j_k \frac{\partial}{\partial X^i_k}\left(X^p_q B^q_r\right)\frac{\partial}{\partial X^p_r} - X^p_q B^q_r \frac{\partial}{\partial X^p_r}\left(X^i_j A^j_k\right)\frac{\partial}{\partial X^i_k} \\
&= X^i_j A^j_k B^k_r \frac{\partial}{\partial X^i_r} - X^p_q B^q_r A^r_k \frac{\partial}{\partial X^p_k} \\
&= X^i_j (A^j_k B^k_r - B^j_k A^k_r)\frac{\partial}{\partial X^i_r} \\
&= [A, B]^L.
\end{aligned}$$

□

8.4.1 Induced Lie Algebra Homomorphisms

We shall see that Lie group homomorphisms induce Lie algebra homomorphisms.

Theorem 8.4.3: Induced Lie Algebra Homomorphisms

Let G and H be Lie groups, and let $\mathfrak{g}, \mathfrak{h}$ be their respective Lie algebras. Suppose $F : G \to H$ is a Lie group homomorphism. For every $X \in \mathfrak{g}$, there exists a unique vector field $Y \in \mathfrak{h}$ that is F-related to X, denoted by $F_* X$. The map $F_* : \mathfrak{g} \to \mathfrak{h}$ defined by $X \mapsto F_* X$ is a Lie algebra homomorphism.

Proof. By the spirit of one point generates all, the only choice for Y is given by

$$Y = (\mathrm{d}F_e(X_e))^L.$$

And we have

$$\begin{aligned}
\mathrm{d}F_g(X_g) &= \mathrm{d}F_g\left(\mathrm{d}(L_g)_e(X_e)\right) = \mathrm{d}(F \circ L_g)_e(X_e) = \mathrm{d}(L_{F(g)} \circ F)_e(X_e) \\
&= \mathrm{d}(L_{F(g)})_{F(e)}\left(\mathrm{d}F_e(X_e)\right) = (\mathrm{d}F_e(X_e))^L|_{F(g)} = Y_{F(g)}.
\end{aligned}$$

where as $F(L_g g') = F(gg') = F(g)F(g') = L_{F(g)}(F(g'))$ for all $g' \in G$, so we have $F \circ L_g = L_{F(g)} \circ F$. So precisely, Y is F-related to X. Next from naturality of Lie bracket, F_* is a Lie algebra homomorphism. □

Remark:
Note that here we only require F to be a Lie group homomorphism, not necessarily a diffeomorphism. So the induced map F_* may not be an isomorphism.

Proposition: **Properties of Induced Lie Algebras**

- The identity homomorphism $\mathrm{id}_G : G \to G$ induces the identity Lie algebra homomorphism $\mathrm{id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$.
- Transitive property: if $F : G \to H$ and $H \to K$ are Lie group homomorphisms, then

the composition $K \circ F : G \to K$ induces the composition of Lie algebra homomorphisms $K_* \circ F_* : \mathfrak{g} \to \mathfrak{k}$.

- Isomorphic Lie groups have isomorphic Lie algebras.

8.4.2 Lie Algebra of Lie Subgroups

Intuitively, if H is a subgroup of a Lie group G, then the Lie algebra of H should be a Lie subalgebra of that of G. There is a slight confusion however, for elements of $\mathrm{Lie}(H)$ are vector fields on H, not on G. We propose a small patch here nevertheless.

Theorem 8.4.4: Lie Algebra of Lie Subgroups

Suppose $H \subseteq G$ is a Lie subgroup of a Lie group G, and $\iota : H \hookrightarrow G$ is the inclusion map. There is a Lie algebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$ isomorphic to $\mathrm{Lie}(H)$, given by

$$\mathfrak{h} = \iota_*(\mathrm{Lie}(H)) = \{X \in \mathrm{Lie}(G) : X \in T_eH \subseteq T_eG\}. \tag{8.12}$$

This is quite natural, as both can be generated by the tangent space at the identity. This gives a nice way to identify the Lie algebra of a Lie subgroup as a Lie subalgebra of that of the bigger Lie group.

Example: **Lie Algebra of $O(n)$**

The orthogonal group $O(n)$ is a Lie subgroup of $GL(n, \mathbb{R})$. For $\Phi(A) = A^TA$, it is equal to the level set $\Phi^{-1}(I)$. We have

$$T_IO(n) = \{B \in \mathfrak{gl}(n, \mathbb{R}) : B^T + B = 0\},$$

consisting of all skew-symmetric matrices. It is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ under the commutator bracket, denoted by $\mathfrak{o}(n)$.

We can do the same for $GL(n, \mathbb{C})$ viewed as a real Lie group.

Definition 8.4.3: Representation of Lie Algebra

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{R}$. A **representation** of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\varphi : \mathfrak{g} \to \mathfrak{gl}(V), \tag{8.13}$$

for some finite-dimensional real vector space V. If such a representation is injective, we say that $\mathfrak{g}$ is **faithfully represented** on V, in this case it is isomorphic to a Lie subalgebra of $\mathfrak{gl}(V)$.

Theorem 8.4.5: Ado's Theorem

Every finite-dimensional Lie algebra over $\mathbb{R}$ admits a faithful finite-dimensional representation, so it is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n with the commutator bracket.

Chapter 9

Integral Curves and Flows

9.1 Integral Curves

Suppose M is a smooth manifold, with or without boundary, and $\gamma : J \to M$ is a smooth curve, then $\forall t \in J$, the velocity vector $\gamma'(t) \in T_{\gamma(t)}M$. Now we want to work in the reverse direction.

Definition 9.1.1: Integral Curve

If V is a vector field on M, then a differentiable curve $\gamma : J \to M$ is called an **integral curve** of V if $\forall t \in J$,

$$\gamma'(t) = V_{\gamma(t)}.$$

Usually if $0 \in J$, we say γ is an integral curve of V **starting at** $\gamma(0)$.

Suppose V is a smooth vector field on M. For a smooth chart $U \subseteq M$, in local coordinates, we can write $\gamma(t) = (\gamma^i(t))$, and $V = (V^i)$, then the integral curve equation becomes

$$\frac{d\gamma^i}{dt}(t) = V^i(\gamma(t)), \quad i = 1, \dots, n. \tag{9.1}$$

From ODE theory, we have

Proposition: **Existence of Integral Curve**

Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exists $\epsilon > 0$ and an integral curve $\gamma : (-\epsilon, \epsilon) \to M$ of V such that $\gamma(0) = p$.

Next, we investigate some reparametrization properties of integral curves.

Proposition: **Reparametrization of Integral Curves**

Let V be a smooth vector field on a smooth manifold M, and let $\gamma : J \to M$ be an integral curve of V.

- Rescaling: For any $a \in \mathbb{R}$, the curve $\tilde{\gamma} : \tilde{J} \to M$ defined by $\tilde{\gamma}(s) = \gamma(as)$, where $\tilde{J} = \{s \in \mathbb{R} : as \in J\}$, is an integral curve of the vector field aV.
- Translation: For any $b \in \mathbb{R}$, the curve $\hat{\gamma} : \hat{J} \to M$ defined by $\hat{\gamma}(u) = \gamma(u + b)$, where

$\hat{J} = \{u \in \mathbb{R} : u + b \in J\}$, is an integral curve of the vector field V.

Proposition: **Naturality of Integral Curves**

Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if for every integral curve γ of X, the curve $F \circ \gamma$ is an integral curve of Y.

Proof. First suppose X and Y are F-related, and let $\gamma : J \to M$ be an integral curve of X. Then define $\sigma = F \circ \gamma : J \to N$. For any $t \in J$,

$$\sigma'(t) = \mathrm{d}F_{\gamma(t)}(\gamma'(t)) = \mathrm{d}F_{\gamma(t)}(X_{\gamma(t)}) = Y_{F(\gamma(t))} = Y_{\sigma(t)},$$

so σ is an integral curve of Y.

Conversely, suppose that for every integral curve γ of X, the curve $F \circ \gamma$ is an integral curve of Y. Let $p \in M$, and let $\gamma : (-\epsilon, \epsilon) \to M$ be an integral curve of X with $\gamma(0) = p$. Then $\sigma = F \circ \gamma : (-\epsilon, \epsilon) \to N$ is an integral curve of Y. Evaluating at $t = 0$, we have

$$Y_{F(p)} = (F \circ \gamma)'(0) = \mathrm{d}F_{\gamma(0)}(\gamma'(0)) = \mathrm{d}F_p(X_p),$$

so X and Y are F-related. □

9.2 Flows

We come by another way to look at integral curves. Suppose for each point $p \in M$, we have a unique integral curve $\theta^{(p)} : \mathbb{R} \to M$ (Actually, it may always be defined on the whole $\mathbb{R}$, but for simplicity we assume this here) starting at p. Then we can define a map

$$\theta_t : M \to M, \quad \theta_t(p) = \theta^{(p)}(t).$$

which sends each point p to the point that slides along the integral curve starting at p for time t. It is easy to see that

$$\theta_t \circ \theta_s = \theta_{t+s}, \quad \theta_0 = \mathrm{id}_M.$$

well, this is just the property of group action. So the map $\theta : \mathbb{R} \times M \to M$ defined by $\theta(t, p) = \theta_t(p)$ is an action of the group $\mathbb{R}$ on M.

Definition 9.2.1: Global Flow

Let M be a smooth manifold. A **global flow** on M is a continuous left action $\theta : \mathbb{R} \times M \to M$ of the group $\mathbb{R}$ on M.

The geometric meaning of a global flow is find where each point goes as time passes.

- For each fixed $t \in \mathbb{R}$, the map $\theta_t : M \to M$ defined by $\theta_t(p) = \theta(t, p)$ is a homeomorphism, and if the flow is smooth, then θ_t is a diffeomorphism.

- For each fixed $p \in M$, the map $\theta^{(p)} : \mathbb{R} \to M$ defined by $\theta^{(p)}(t) = \theta(t, p)$ is the orbit of p under the action.

Every smooth global flow can be associated with integral curves of a smooth vector field. For any $\theta : \mathbb{R} \times M \to M$ smooth global flow, we can define a vector field V on M by

$$V_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta_t(p).$$

called the **infinitesimal generator** of the flow.

Proposition: **Infinitesimal Generator**

Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M, and let V be its infinitesimal generator. Then V is a smooth vector field on M, and for each $p \in M$, the orbit $\theta^{(p)} : \mathbb{R} \to M$ is the unique integral curve of V starting at p.

Proof. Take any $f \in C^\infty(M)$ on some open neighborhood of p. Then

$$Vf(p) = V_p f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \theta_t)(p) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} (f \circ \theta)(t, p),$$

So it is smooth. The second part follows according to the definition of integral curves. □

9.2.1 The Fundamental Theorem of Flows

We cannot say that every smooth vector field on a smooth manifold generates a global flow because integral curves may not be defined for all $\mathbb{R}$. So we introduce a small patch here.

Definition 9.2.2: Flow

Let M be a manifold. A **flow domain** in M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ such that for each $p \in M$, the set $\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$ is an open interval containing 0. A **flow** or **local flow** on M is a continuous map $\theta : \mathcal{D} \to M$ such that

- $\forall p \in M$, we have $\theta(0, p) = p$;
- For any $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$, such that $s + t \in \mathcal{D}^{(p)}$, we have

$$\theta(t, \theta(s, p)) = \theta(s + t, p). \tag{9.2}$$

If θ is a flow, we define $\theta_t : M_t \to M$ by $\theta_t(p) = \theta(t, p)$, where $M_t = \{p \in M : (t, p) \in \mathcal{D}\}$. Then if θis smooth, the infinitesimal generator of θ is defined as before.

Proposition: **Infinitesimal Generator of a Flow**

Let $\theta : \mathcal{D} \to M$ be a smooth flow on a smooth manifold M, and let V be its infinitesimal generator. Then V is a smooth vector field on M, and for each $p \in M$, the orbit $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is an integral curve of V starting at p.

Definition 9.2.3: Maximal Integral Curve

Let V be a smooth vector field on a smooth manifold M. An integral curve $\gamma : J \to M$ of V is called a **maximal integral curve** of V if there is no integral curve $\tilde{\gamma} : \tilde{J} \to M$ of V such that $J \subsetneq \tilde{J}$ and $\tilde{\gamma}|_J = \gamma$.
A maximal flow of V is a flow $\theta : \mathcal{D} \to M$ that cannot be extended to a larger flow domain.

Theorem 9.2.1: The Fundamental Theorem of Flows

Let V be a smooth vector field on a smooth manifold M. Then there exists a unique maximal flow $\theta : \mathcal{D} \to M$ on M whose infinitesimal generator is V. And θ has the following properties:

- For each $p \in M$, the map $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ defined by $\theta^{(p)}(t) = \theta(t,p)$ is the unique maximal integral curve of V starting at p.
- If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))} = \mathcal{D}^{(p)} - s = \{t - s : t \in \mathcal{D}^{(p)}\}$.
- For each $t \in \mathbb{R}$, the set $M_t = \{p \in M : (t,p) \in \mathcal{D}\}$ is open in M, and the map $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

Proposition: **Naturality of Flows**

Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be F-related smooth vector fields, and let θ and η be their respective maximal flows. Then for all $t \in \mathbb{R}$, $F(M_t) \subseteq N_t$, and the following diagram commutes:

$$\begin{tikzcd}
M_t \arrow[r, "\theta_t"] \arrow[d, "F"] & M_{-t} \arrow[d, "F"] \\
N_t \arrow[r, "\eta_t"] & N_{-t}
\end{tikzcd}$$

Proof. For any $p \in M$, the curve $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F(p)$. So by the uniqueness of maximal integral curves, $\eta^{(F(p))}$ extends $F \circ \theta^{(p)}$ and must be defined at least on $\mathcal{D}^{(p)}$. Thus, $F(M_t) \subseteq N_t$ for all $t \in \mathbb{R}$. Moreover,

$$F(\theta^{(p)}(t)) = F \circ \theta^{(p)}(t) = \eta^{(F(p))}(t) = \eta_t(F(p)),$$

thus $\eta_t \circ F = F \circ \theta_t$ on M_t. □

9.2.2 Complete Vector Fields

As we have seen, not every smooth vector field generates a global flow.

Definition 9.2.4: Complete Vector Field

A smooth vector field V on a smooth manifold M is called **complete** if its maximal flow $\theta : \mathcal{D} \to M$ has flow domain $\mathcal{D} = \mathbb{R} \times M$; that is, it generates a global flow on M.

We will show that compactly supported vector fields are complete.

Lemma 9.2.1: Uniform Time Lemma

Let V be a smooth vector field on a smooth manifold M, and let $\theta : \mathcal{D} \to M$ be its maximal flow. If there exists $\epsilon > 0$ such that for each $p \in M$, the interval $(-\epsilon, \epsilon) \subseteq \mathcal{D}^{(p)}$, then V is complete.

Proof. Suppose for some $p \in M$, the domain $\mathcal{D}^{(p)}$ is bounded above, let $b = \sup \mathcal{D}^{(p)}$ and let $b - \epsilon < t_0 < b$, and $q = \theta(t_0, p)$. Then by the hypothesis, $\theta^{(q)}$ is defined on $(-\epsilon, \epsilon)$, so $\gamma : (-\epsilon, t_0 + \epsilon) \to M$ defined by

$$\gamma(t) = \begin{cases} \theta^{(p)}(t), & t \in (-\epsilon, b) \\ \theta^{(q)}(t - t_0), & t \in (t_0 - \epsilon, t_0 + \epsilon) \end{cases}$$

is an integral curve of V extending $\theta^{(p)}$, contradicting the maximality of $\theta^{(p)}$. Thus, $\mathcal{D}^{(p)}$ is unbounded above. A similar argument shows that $\mathcal{D}^{(p)}$ is unbounded below, so $\mathcal{D}^{(p)} = \mathbb{R}$ for all $p \in M$, and hence V is complete. □

Theorem 9.2.2: Completeness of Compactly Supported Vector Fields

Let M be a smooth manifold, and let V be a smooth vector field on M with compact support. Then V is complete.
Therefore, on a compact smooth manifold, every smooth vector field is complete.

Proof. Obvious. □

Theorem 9.2.3: Completeness of Left-Invariant Vector Fields

Let G be a Lie group, and let V be a left-invariant vector field on G. Then V is complete.

Proof. There is some ϵ that $\theta^{(e)}$ is defined on $(-\epsilon, \epsilon)$, where e is the identity element of G. For any $g \in G$, the integral curve $\theta^{(g)}$ starting at g is given by $L_g \circ \theta^{(e)}$, which is defined on $(-\epsilon, \epsilon)$ as well. Thus, by the Uniform Time Lemma, V is complete. □

Lemma 9.2.2: Escape Lemma

Suppose M is a smooth manifold, and $V \in \mathfrak{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of V and J is bounded above, let $b = \sup J$. Then for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in any compact subset of M.

9.3 Flowouts

Flows provide some technique for geometric constructions on manifolds.

Theorem 9.3.1: Flowout Theorem

Let M be a smooth manifold, let $S \subseteq M$ be an embedded k-dimensional submanifold, and let V be a smooth vector field on M that is nowhere tangent to S. Let $\theta : \mathcal{D} \to M$ be the maximal flow of V. Then let $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$ be the submanifold part of the flow domain. Let $\Phi = \theta|_{\mathcal{O}} : \mathcal{O} \to M$ be the restriction of the flow to $\mathcal{O}$.

- $\Phi : \mathcal{O} \to M$ is an immersion;
- $\partial/\partial t \in \mathfrak{X}(\mathcal{O})$ is Φ-related to $V \in \mathfrak{X}(M)$;
- There exists a smooth positive function $\delta : S \to \mathbb{R}$ such that the restriction $\Phi|_{\mathcal{O}_\delta} : \mathcal{O}_\delta \to M$ is injective, where
$$\mathcal{O}_\delta = \{(t, p) \in \mathbb{R} \times S : |t| < \delta(p)\}.$$
Thus $\Phi(\mathcal{O}_\delta)$ is an immersed submanifold of M containing S, and V is tangent to $\Phi(\mathcal{O}_\delta)$.
- If S has codimension 1, then $\Phi|_{\mathcal{O}_\delta} : \mathcal{O}_\delta \to \Phi(\mathcal{O}_\delta)$ is a diffeomorphism onto an open submanifold of M.

The submanifold $\Phi(\mathcal{O}_\delta)$ is called the **flowout** of S along V.

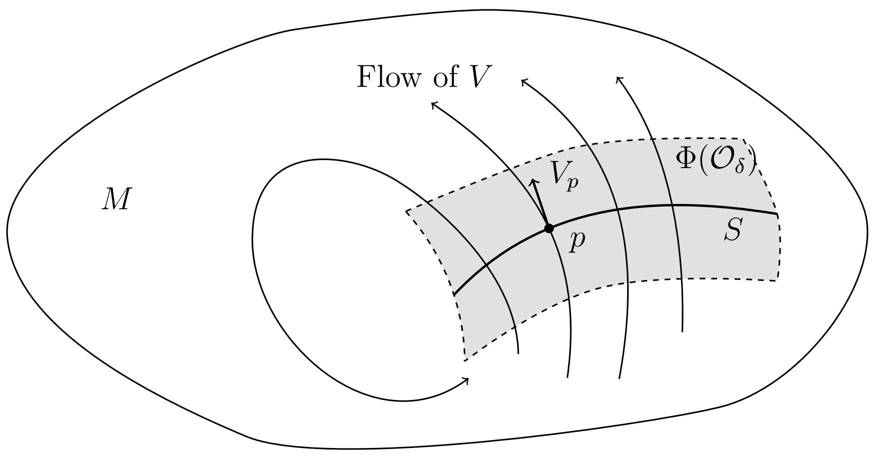

Figure 9.1: Flowout

Proof. SORRY ☐

9.3.1 Regular Points and Singular Points

Definition 9.3.1: Regular Point and Singular Point

Let M be a smooth manifold, and let V be a vector field on M. A point $p \in M$ is called a **regular point** of V if $V_p \neq 0$. Otherwise, it is called a **singular point** of V.

Proposition: **Regular and Singular Points**

Let V be a smooth vector field on a smooth manifold M. Let $\theta : \mathcal{D} \to M$ be the maximal flow of V. Then if $p \in M$ is a singular point of V, then $\mathcal{D}^{(p)} = \mathbb{R}$, and $\theta^{(p)}(t) = p$ for all $t \in \mathbb{R}$ is a constant curve. If p is a regular point of V, then $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is a smooth immersion.

This shows that equilibrium points (The points where $\theta^{(p)}$ is constant) of a flow are exactly the singular points of its infinitesimal generator.

Now we give a complete local structure of a vector field around a regular point.

Theorem 9.3.2: Canonical Form near a Regular Point

Let M be a smooth manifold, and let V be a smooth vector field on M. If $p \in M$ is a regular point of V, then there exists a smooth chart (U, φ) near p, with coordinates $(s^1, \dots, s^n)$, such that on U,

$$V = \frac{\partial}{\partial s^1}.$$

If $S \subseteq M$ is any embedded hypersurface (codimension 1 submanifold) with $p \in S$ and $V_p \notin T_pS$, then the chart (U, φ) can be chosen so that s^1 is the local defining function of S.

9.4 Flows and Flowouts on Manifolds with Boundary

Well from definition point we see that vector fields on manifolds with boundary need not generate flows when the point is on the boundary, which only permits half-open intervals as domains of integral curves. However, there are some flowout results that are important.

Theorem 9.4.1: Boundary Flowout Theorem

Let M be a smooth manifold with nonempty boundary, let N be a smooth vector field on M that is inward points on every $p \in \partial M$. There exists a smooth function $\delta : \partial M \to \mathbb{R}^+$ and a smooth embedding $\Phi : \mathcal{P}_\delta \to M$, where $\mathcal{P}_\delta = \{(t,p) : p \in \partial M, 0 \le t < \delta(p)\} \subseteq \mathbb{R} \times \partial M$, such that $\Phi(\mathcal{P}_\delta)$ is an open neighborhood of ∂M in M, and for each $p \in \partial M$, the curve $\Phi^{(p)} : [0, \delta(p)) \to M$ defined by $\Phi^{(p)}(t) = \Phi(t,p)$ is the integral curve of N starting at p.

Lemma 9.4.1: Existence of Inward Vector Fields

Let M be a smooth manifold with nonempty boundary. There exists a smooth global vector field on M that inward points at every point of ∂M.

Theorem 9.4.2: Collar Neighborhood Theorem

Let M be a smooth manifold with nonempty boundary. A neighborhood of ∂M is called a **collar neighborhood** if it is the image of a smooth embedding $[0,1) \times \partial M \to M$ that restricts to the identity on $\{0\} \times \partial M$.
Then every smooth manifold with nonempty boundary has a collar neighborhood.

Theorem 9.4.3: Homotopy to Interior

Let M be a smooth manifold with nonempty boundary. And let $\iota : \operatorname{Int} M \to M$ be the inclusion map. Then there exists a proper smooth embedding $R : M \to \operatorname{Int} M$ such that both $\iota \circ R : M \to M$ and $R \circ \iota : \operatorname{Int} M \to \operatorname{Int} M$ are smoothly homotopic to the respective identity maps. Therefore, ι is a homotopy equivalence between $\operatorname{Int} M$ and M.

Proof. Quite natural, just shrinking the boundary along the collar neighborhood a little bit to get R would do. □

Theorem 9.4.4: Whitney Approximation Theorem for Manifolds with Boundary

Let M, N be smooth manifolds with boundary, then every continuous map $F : M \to N$ is homotopic to a smooth map.

We next generalize theorem 6.3.5, we have

Theorem 9.4.5: Homotopy and Smooth Homotopy for Manifolds with Boundary

Let M, N be smooth manifolds with boundary, and let $F, G : M \to N$ be smooth maps that are homotopic. Then F and G are smoothly homotopic.

The following theorem shows how to attach manifolds along their boundaries.

Theorem 9.4.6: Attaching Manifolds along their Boundaries

Let M, N be smooth n-manifolds with nonempty boundaries, and suppose $h : \partial N \to \partial M$ is a diffeomorphism. Let

$$M \cup_h N = (M \sqcup N)/\sim, \qquad \text{where } x \sim h(x) \text{ for all } x \in \partial N.$$

Then $M \cup_h N$ is a topological manifold without boundary, and it admits a smooth structure such that there are regular domains $M', N' \subseteq M \cup_h N$ that are diffeomorphic to M and N, respectively, and satisfies

$$M' \cup N' = M \cup_h N, \quad M' \cap N' = \partial M' = \partial N'. \tag{9.3}$$

If M, N are both compact, then $M \cup_h N$ is also compact. If they are both connected then $M \cup_h N$ is also connected.

Proof. SORRY □

Example: **Connected Sums**

Let M_1, M_2 be connected smooth n-manifolds, for $i = 1, 2$ let U_i denote a regular coordinate ball centered at some point $p_i \in M_i$, and let $M_i' = M_i \setminus U_i$. Then M_i' is a smooth manifold with boundary diffeomorphic to $\mathbb{S}^{n-1}$. A smooth **connected sum** of M_1 and M_2, denoted $M_1 \# M_2$, is the smooth manifold obtained by attaching M_1' and M_2' along their boundaries via some diffeomorphism $h : \partial M_2' \to \partial M_1'$. The resulting manifold is independent of the choice of coordinate balls and diffeomorphism, up to diffeomorphism.

Theorem 9.4.5 shows a way of embedding a smooth manifold with boundary into another one without boundary, namely just $\operatorname{Int} M$. We can also have another way.

Example: **The Double of a Manifold with Boundary**

Let M be a smooth manifold with nonempty boundary. The **double** of M, denoted $D(M)$, is the smooth manifold without boundary obtained by attaching two copies of M along their boundaries via the identity map of ∂M, namely, $M \cup_{\mathrm{id}} M$.
Then $D(M)$ is compact if and only if M is compact, connected if and only if M is connected.

Although vector fields on manifolds with boundary may not generate flows, there are circumstances that they do.

Lemma 9.4.2: Tangent Fields Generate Flows

Let M be a smooth manifold, and $D \subseteq M$ is a regular domain. Let V be a smooth vector field on M that is tangent to ∂D at every point of ∂D. Then every integral curve of V that starts in D remains in D for all time.

Theorem 9.4.7: Flows on Manifolds with Boundary

Let M be a smooth manifold with nonempty boundary, and let V be a smooth vector field on M that is tangent to ∂M at every point of ∂M. Then the fundamental theorem of flows 9.2.1 holds for V on M.

Proof. Consider M to be the regular domain in its double $D(M)$ would do. □

Theorem 9.4.8: Canonical Form near a Regular Point on Manifolds with Boundary

Let M be a smooth manifold with boundary, and let V be a smooth vector field on M that is tangent to ∂M. If $p \in \partial M$ is a regular point of V, then there exists a smooth chart (U, φ) near p, with coordinates $(s^1, \ldots, s^n)$, such that on U, $V = \partial/\partial s^1$.

9.5 Lie Derivatives

We know how to interpret a directional derivative of a real-valued function on a manifold. Indeed a tangent vector $v \in T_pM$ can be viewed as this. So what about the directional derivative of a vector field? In Euclidean space $\mathbb{R}^n$, the directional derivative of a smooth vector field W in the direction of a vector $v \in T_p\mathbb{R}^n$ is defined as

$$D_vW(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} W(p + tv) = \lim_{t\to 0} \frac{W(p+tv) - W(p)}{t}.$$

however, this definition depends on the vector space structure of $\mathbb{R}^n$ because we need to do substraction of vectors at different points.

There is no natural way to define the directional derivative of a vector field on a general manifold. However, if we have a vector field V instead of a single tangent vector v, then we can use the flow to push forward and pull back vectors.

Definition 9.5.1: Lie Derivative

Suppose M is a smooth manifold, V is a smooth vector field on M with maximal flow θ. Let W be another smooth vector field on M. The **Lie derivative** of W with respect to V is the rough vector field defined by

$$(\mathcal{L}_V W)_p = \lim_{t\to 0} \frac{(\mathrm{d}\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{d}\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}). \tag{9.4}$$

If M is a smooth manifold with boundary, and V is tangent to ∂M, then the above definition still makes sense using the flow theorem for manifolds with boundary 9.4.7.

Lemma 9.5.1: Smoothness of Lie Derivative

Let M be a smooth manifold, with or without boundary. Let $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \emptyset$, assume that V is tangent to ∂M. Then $\mathcal{L}_V W$ exists and is a smooth vector field on M.

Proof. Let θ be the flow of V. For any $p \in M$, let $(U, (x^i))$ be a smooth chart about p. Choose a small open interval J_0 containing 0 and an open subset $U_0 \subseteq U$ containing p such that θ maps $J_0 \times U_0$ into U. Write $\theta(t, x) = (\theta^1(t, x), \ldots, \theta^n(t, x))$ when $(t, x) \in J_0 \times U_0$. Then for any $(t, x) \in J_0 \times U_0$, the matrix of $(\mathrm{d}\theta_{-t})_{\theta_t(x)}$ with respect to the basis $\{\partial/\partial x^i|_{\theta_t(x)}\}$ is given by

$$\left(\frac{\partial \theta^i}{\partial x^j}(-t, \theta_t(x))\right)_{i,j=1}^n .$$

So we have

$$\mathrm{d}(\theta_{-t})_{\theta_t(x)}(W_{\theta_t(x)}) = \frac{\partial \theta^i}{\partial x^j}(-t, \theta(t, x)) W^j(\theta_t(x)) \left.\frac{\partial}{\partial x^i}\right|_x ,$$

is smooth in both t and x. Thus, taking the derivative with respect to t at $t = 0$ gives a smooth vector field on U_0 that agrees with $\mathcal{L}_V W$ on U_0. Since p was arbitrary, $\mathcal{L}_V W$ is smooth on M. □

The definition is not quite computation friendly. We have the following more useful property that links Lie derivatives with Lie brackets.

Theorem 9.5.1: Structure of Lie Derivative

Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$. then

$$\mathcal{L}_V W = [V, W]. \tag{9.5}$$

Proof. Let $\mathcal{R}(V) \subseteq M$ be the set of regular points of V (points where $V_p \neq 0$). Since $\mathcal{R}(V)$ is open by continuity, its closure is the support of V. Take $p \in M$.

- Case 1, $p \in \mathcal{R}(V)$: By the Canonical Form near a Regular Point 9.3.2, there exists a smooth chart $(U, (u^1, \ldots, u^n))$ about p such that on U, $V = \partial/\partial u^1$. Then the flow of V is $\theta_t(u) = (u^1 + t, u^2, \ldots, u^n)$. For each t, then $\mathrm{d}(\theta_{-t})_{\theta_t(x)}$ is the identity map on $T_{\theta_t(x)}M$. Thus for any $u \in U$, we have

$$\mathrm{d}(\theta_{-t})_{\theta_t(u)}(W_{\theta_t(u)}) = W^j(u^1 + t, u^2, \ldots, u^n)\frac{\partial}{\partial u^j}\bigg|_u,$$

So we have

$$(\mathcal{L}_V W)_u = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} W^j(u^1 + t, u^2, \ldots, u^n)\frac{\partial}{\partial u^j}\bigg|_u = \frac{\partial W^j}{\partial u^1}(u)\frac{\partial}{\partial u^j}\bigg|_u.$$

On the other hand,

$$[V, W]_u = \left[\frac{\partial}{\partial u^1}, W^j\frac{\partial}{\partial u^j}\right]_u = \frac{\partial W^j}{\partial u^1}(u)\frac{\partial}{\partial u^j}\bigg|_u.$$

- Case 2, $p \in \operatorname{supp} V$. By continuity.
- Case 3, $p \notin \operatorname{supp} V$. Then $V = 0$ on some open neighborhood U of p. Then θ_t is the identity map on U for all t sufficiently small. Thus, for any $u \in U$,

$$\mathrm{d}(\theta_{-t})_{\theta_t(u)}(W_{\theta_t(u)}) = W_u,$$

so $(\mathcal{L}_V W)_u = 0$. On the other hand, since $V = 0$ on U, we have $[V, W]_u = 0$ as well.

□

Remark:
This is the geometric interpretation of Lie brackets we mentioned in the chapter on Lie groups. Lie brackets measure the change of one vector field along the flow generated by another vector field.

Proposition: **Properties of Lie Derivatives**

Suppose M is a smooth manifold, with or without boundary. Let $V, W, X \in \mathfrak{X}(M)$,

- $\mathcal{L}_V W = -\mathcal{L}_W V$.
- $\mathcal{L}_V[W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$. (The Jacobi Identity)
- $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$. (Again the Jacobi Identity)
- $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$, for any $g \in C^\infty(M)$. (Note this gives a geometric interpretation of the product rule for Lie brackets.)
- If $F : M \to N$ is a diffeomorphism, then $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_* V}(F_* W)$.

To compute derivative at other time than 0, we have

Proposition: **Derivatives at Other Times**

Let M be a smooth manifold, with or without boundary. Let $V, W \in \mathfrak{X}(M)$, and if $\partial M \neq \emptyset$, assume that V is tangent to ∂M. Let θ be the flow of V. Then for any $(t_0, p) \in \mathcal{D}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \mathrm{d}(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = \mathrm{d}(\theta_{-t_0})_{\theta_{t_0}(p)}((\mathcal{L}_V W)_{\theta_{t_0}(p)}). \tag{9.6}$$

This can be seen as conputing the Lie derivative at time t_0 AT the position $t = 0$, it pulls back the Lie derivative at time t_0 and all t back to time 0 for comparison. So the answer is just the pullback of the Lie derivative at time t_0.

9.6 Commuting Vector Fields

Let M be a smooth manifold, and let $V, W \in \mathfrak{X}(M)$. We say that V, W commute if their Lie bracket vanishes, i.e. $[V, W] = 0$. A vector field W is said to be invariant under some flow θ if W is θ_t-related to itself for all t in the flow domain. (More precisely, $W|_{M_t}$ is θ_t-related to $W|_{M_{-t}}$ for all $t \in \mathbb{R}$.) This means that $\mathrm{d}(\theta_t)_p(W_p) = W_{\theta_t(p)}$ for all $(t, p) \in \mathcal{D}$.

Theorem 9.6.1: Structure of Invariant Fields under Flows

Let M be a smooth manifold, and let $V, W \in \mathfrak{X}(M)$. Then the following are equivalent:

- V and W commute, i.e. $[V, W] = 0$;
- W is invariant under the flow of V;
- V is invariant under the flow of W.

Specially, every smooth vector field is invariant under its own flow.

Proof. If W is invariant under the flow of V, then for any $(t, p) \in \mathcal{D}$, $W_{\theta_t(p)} = \mathrm{d}(\theta_t)_p(W_p)$. Thus,

$$\mathrm{d}(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = \mathrm{d}(\theta_{-t})_{\theta_t(p)} \circ \mathrm{d}(\theta_t)_p(W_p) = W_p.$$

So $\mathcal{L}_V W = 0$, i.e. $[V, W] = 0$.

Conversely, if $[V, W] = 0$, then reverse the argument above would do. $\square$

Remark:
We can use $[V, V] = 0$ to see that every smooth vector field is invariant under its own flow. But this is quite obvious from the naturality of flows proposition 9.2.1, since V is always θ_t-related to itself, because θ_t action gives the same flow.

As infinitesimal generators of flows, we can visualize commuting vector fields as infinitesimal symmetries of each other. Moving along V a little bit and then moving along W a little bit is the same as moving along W first and then moving along V, which is exactly why we need W not to change too much along the flow of V (and vice versa). We may naturally think that this is also equivalent to the commutivity of the flows generated by V and W.

To be more precise,

Definition 9.6.1: Commuting Flows

Suppose M is a smooth manifold, and let θ, ψ be two flows on M, then we say that θ and ψ **commute** if

$$\begin{aligned}&\forall p \in M, \forall J, K \subseteq \mathbb{R} \text{ open intervals containing } 0\\&[(\forall (s,t) \in J \times K, \theta_s \circ \psi_t \text{ is defined }) \vee (\forall (s,t) \in J \times K, \psi_t \circ \theta_s \text{ is defined })\\&\to (\forall (s,t) \in J \times K, \theta_s \circ \psi_t(p) = \psi_t \circ \theta_s(p)) \text{ are both defined }].\end{aligned} \tag{9.7}$$

For global flows, this means that $\theta_s \circ \psi_t = \psi_t \circ \theta_s$ for all $s, t \in \mathbb{R}$.

Theorem 9.6.2: Equivalence of Field Commuting and Flow Commuting

Smooth vector fields commute if and only if their flows commute.

Proof. SORRY □

Remark:
Note the condition of commutativity of flows is a bit complicated due to the possible non-completeness of the vector fields. It is easily mistaken to be

$$\begin{aligned}&\forall p \in M, \forall J, K \subseteq \mathbb{R} \text{ open intervals containing } 0\\&\forall (s,t) \in J \times K, [\theta_s \circ \psi_t \text{ is defined } \wedge \psi_t \circ \theta_s \text{ is defined } \to \theta_s \circ \psi_t(p) = \psi_t \circ \theta_s(p)].\end{aligned}$$

But this is not correct. There exists certain commuting vector fields and for some certain s, t that both $\theta_s \circ \psi_t$ and $\psi_t \circ \theta_s$ are defined, but they are not equal.

Proof. SORRY □

9.6.1 Commuting Frames

Definition 9.6.2: Commuting Frame

Let M be a smooth n-manifold. A smooth local frame $(U, (E_1, \ldots, E_n))$ on M is called a **commuting frame** if $[E_i, E_j] = 0$ for all $1 \le i, j \le n$.

Example: **Commuting Frames**

- The coordinate frame $(U, (\partial/\partial x^1, \ldots, \partial/\partial x^n))$ associated to any smooth chart $(U, (x^1, \ldots, x^n))$ is a commuting frame.
- On $\mathbb{R} \setminus \{0\}$, define
$$E_1 = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}, \quad E_2 = -\frac{y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}, \quad r = \sqrt{x^2 + y^2}.$$
Then we have
$$[E_1, E_2] = \frac{y}{r^2}\frac{\partial}{\partial x} - \frac{x}{r^2}\frac{\partial}{\partial y}0.$$

So it it not a commuting frame. To see why, in angular coordinates (r, θ), we have

$$E_1 = \frac{\partial}{\partial r}, \quad E_2 = \frac{1}{r}\frac{\partial}{\partial \theta},$$

which has a normalizing factor $1/r$ that varies along the flow of E_1.

Therefore, this frame cannot be expressed as a coordinate frame of any smooth chart on $\mathbb{R}^2 \setminus \{0\}$.

Next, we shall show that commuting is also a sufficient condition for a local frame to be a coordinate frame.

Theorem 9.6.3: Canonical Form of Commuting Vector Fields

Let M be a smooth n-manifold, and let $(V_1, \dots, V_k)$ be a collection of linearly independent commuting smooth vector fields on an open set $W \subseteq M$, then for each point $p \in W$, there exists a smooth chart $(U, (s^i))$ about p such that $V_i = \partial/\partial s^i$ on U for $i = 1, \dots, k$.
If $p \in S \subseteq W$ is an embedded submanifold of M with codimension k such that T_pS is complementary to the span of $\{V_1|_p, \dots, V_k|_p\}$ in T_pM, then the coordinates $(s^1, \dots, s^n)$ can be chosen so that $S \cap U$ is the slice by $s^1 = \cdots = s^k = 0$.

This gives a way to define the local coordinates by commuting vector fields.

- First start with an $n-k$ dimensional submanifold S that is complementary to the span of the k commuting vector fields at a point p;

- For any point p, take it as origin, and trace along the flows of the k vector fields to get k coordinates; As the vector fields commute, the order of tracing does not matter;

Example: **Constructing Local Frames from Commuting Fields**

Take $\mathbb{R}^2$, define

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

We have $[V, W] = 0$. So they commute. The flow for V and W are

$$\theta_t(x, y) = (x\cos t - y\sin t, x\sin t + y\cos t), \quad \eta_t(x, y) = (e^t x, e^t y).$$

For the new coordinate (s, t), start at point $(1, 0)$, trace along the flow of V for time s then along the flow of W for time t, we get

$$\Phi(s, t) = \eta_t(\theta_s(1, 0)) = (e^t \cos s, e^t \sin s).$$

So we have

$$(s, t) = (\tan^{-1}(y/x), \log\sqrt{x^2 + y^2}).$$

for a local chart around $(1, 0)$.

9.7 Time-Dependent Vector Fields

In many physical applications, the vector fields involved depend on time explicitly. We shall generalize our discussion to cover these cases.

Definition 9.7.1: Time-Dependent Vector Field

Let M be a smooth manifold. A **time-dependent vector field** on M is a continuous map $V : J \times M \to TM$, where $J \subseteq \mathbb{R}$ is an interval. For each $t \in J$, the map $V_t : M \to TM$ defined by $V_t(p) = V(t,p)$ is a vector field on M.
An integral curve of V is a differentiable curve $\gamma : J_0 \to M$, where $J_0 \subseteq J$ is an interval, such that

$$\gamma'(t) = V(t, \gamma(t)) \quad \forall t \in J_0. \tag{9.8}$$

Note that time-dependent vector fields may not generate flows in the usual sense, since the integral curves starting at the same point at different times may not agree.

Theorem 9.7.1: Fundamental Theorem on Time-Dependent Flows

Let M be a smooth manifold, and let $J \subseteq \mathbb{R}$ be an open interval. Let $V : J \times M \to TM$ be a time-dependent smooth vector field on M. There exists an open subset $\mathcal{E} \subseteq J \times J \times M$ and a smooth map $\psi : \mathcal{E} \to M$ called the **time-dependent flow** of V, such that

- For each $(t_0, p) \in J \times M$, the set $\mathcal{E}^{(t_0,p)} = \{t \in J : (t_0, t, p) \in \mathcal{E}\}$ is an open interval containing t_0, and the smooth curve $\psi^{(t_0,p)} : \mathcal{E}^{(t_0,p)} \to M$ defined by $\psi^{(t_0,p)}(t) = \psi(t, t_0, p)$ is the unique maximal integral curve of V satisfying $\psi^{(t_0,p)}(t_0) = p$.

- If $t_1 \in \mathcal{E}^{(t_0,p)}$, $q = \psi^{(t_0,p)}(t_1)$, then $\mathcal{E}^{(t_1,q)} = \mathcal{E}^{(t_0,p)}$ and $\psi^{(t_1,q)} = \psi^{(t_0,p)}$.

- For any $(t_1, t_0) \in J \times J$, the set $M_{(t_1,t_0)} = \{p \in M : (t_0, t_1, p) \in \mathcal{E}\}$ is an open subset of M, and the map $\psi_{(t_1,t_0)} : M_{(t_1,t_0)} \to M$ defined by $\psi_{(t_1,t_0)}(p) = \psi(t_1, t_0, p)$ is a diffeomorphism from $M_{(t_1,t_0)}$ onto $M_{(t_0,t_1)}$, with inverse $\psi_{(t_0,t_1)}$.

- If $p \in M_{t_1,t_0}$ and $\psi_{(t_1,t_0)}(p) \in M_{t_2,t_1}$, then $p \in M_{t_2,t_0}$ and

$$\psi_{(t_2,t_0)}(p) = \psi_{(t_2,t_1)}(\psi_{(t_1,t_0)}(p)).$$

We can reduce time-dependent vector fields to time-independent ones by considering an augmented manifold $J \times M$. Consider the vector field $\tilde{V}$ on $J \times M$ defined by

$$\widetilde{V}_{(s,p)} = \left(\left.\frac{\partial}{\partial t}\right|_s, V(s,p) \right) \in T_s J \oplus T_p M \cong T_{(s,p)}(J \times M). \tag{9.9}$$

where s is the standard coordinate on J. Let $\tilde{\theta} : \tilde{\mathcal{D}} \to J \times M$ be the flow of $\tilde{V}$. Then we can write

$$\tilde{\theta}(t, (s,p)) = (\alpha(t, (s,p)), \beta(t, (s,p))),$$

Then $\alpha : \tilde{\mathcal{D}} \to J$ and $\beta : \tilde{\mathcal{D}} \to M$ have

$$\begin{aligned} \frac{\partial \alpha}{\partial t}(t, (s,p)) &= 1, & \alpha(0, (s,p)) &= s; \\ \frac{\partial \beta}{\partial t}(t, (s,p)) &= V(\alpha(t, (s,p)), \beta(t, (s,p))), & \beta(0, (s,p)) &= p. \end{aligned}$$

So we have $\alpha(t,(s,p)) = t+s$, so

$$\frac{\partial \beta}{\partial t}(t,(s,p)) = V(t+s, \beta(t,(s,p))).$$

Let $\mathcal{E} \subseteq \mathbb{R} \times J \times M$ defined by

$$\mathcal{E} = \{(t, t_0, p) : (t - t_0, (t_0, p)) \in \tilde{\mathcal{D}}\}.$$

If $(t, t_0, p) \in \mathcal{E}$, then $t = t - t_0 + t_0 = \alpha(t - t_0, (t_0, p)) \in J$, so $\mathcal{E} \subseteq J \times J \times M$.

Now $\psi : \mathcal{E} \to M$ defined by

$$\psi(t, t_0, p) = \beta(t - t_0, (t_0, p))$$

also corresponds to the time-dependent flow of V. All the above results can be obtained from the properties of $\tilde{\theta}$.

9.8 First-Order PDEs

In coordinated, any first order PDE for a single unknown function $u : \mathbb{R}^n \to \mathbb{R}$ can be written as

$$F(x^1, \ldots, x^n, u(x), \frac{\partial u}{\partial x^1}(x), \ldots, \frac{\partial u}{\partial x^n}(x)) = 0, \tag{9.10}$$

where F is a smooth function of $2n+1$ variables. We also add a initial / boundary condition: given a smooth hypersurface $S \subseteq \mathbb{R}^n$ and a smooth function $\varphi : S \to \mathbb{R}$, we require that

$$u|_S = \varphi. \tag{9.11}$$

The problem of finding a solution to the PDE in a neighborhood of S that satisfies the initial condition is called the **Cauchy problem**.

We point out that not all Cauchy problems have solutions, we need to impose some conditions between F and S to ensure the existence of solutions, called some **non-characteristic** conditions.

9.8.1 Linear Equations

$$a^1(x)\frac{\partial u}{\partial x^1} + \cdots + a^n(x)\frac{\partial u}{\partial x^n} + b(x)u(x) = f(x), \tag{9.12}$$

where a^i, b, f are smooth functions on some $\Omega \subseteq \mathbb{R}^n$. Take a smooth vector field $A \in \mathfrak{X}(\Omega)$ defined by

$$A_x = a^i(x) \left.\frac{\partial}{\partial x^i}\right|_x.$$

then the PDE can be written as

$$Au + bu = f, \tag{9.13}$$

with initial hypersurface S being non-characteristic if A is nowhere tangent to S at every point of S.

Proposition: **The Linear First-Order Cauchy Problem**

Let M be a smooth n-manifold, let $S \subseteq M$ be an embedded hypersurface, let $A \in \mathfrak{X}(M)$ be a smooth vector field that is nowhere tangent to S, and let $b, f \in C^\infty(M)$, and $\varphi \in C^\infty(S)$. Then for some neighborhood U of S in M, there exists a unique smooth function $u \in C^\infty(U)$ such that

$$Au + bu = f \qquad u|_S = \varphi. \tag{9.14}$$

Chapter 10

Vector Bundles

10.1 Vector Bundles

We have encountered the tangent bundle TM of a smooth manifold M. Locally it looks like a product $U \times \mathbb{R}^n$, but globally it may be twisted in a nontrivial way. This motivates the following definition.

Definition 10.1.1: Vector Bundles

Let M be a topological space. A (real) **vector bundle** of rank k over M is a topological space E together with a surjective continuous map $\pi : E \to M$ such that

- For each $p \in M$, the fiber $E_p := \pi^{-1}(p)$ is equipped with the structure of a real vector space of dimension k.
- For each $p \in M$, there exists an open neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ called a **local trivialization** such that
  - $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection onto the first factor.
  - For each $q \in U$, the restriction $\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a vector space isomorphism.

If M, E are smooth manifolds, with or without boundary, and π is a smooth map, and the local trivializations Φ are diffeomorphisms, then we say that E is a **smooth vector bundle** over M.
The space E is called the **total space** of the vector bundle, M is called the **base space**, and the map π is called the **projection map**.

If there exists a local trivialization of E over all of M, then we say that it is a **global trivialization**, and E is called a **trivial vector bundle**. In this case, E is homeomorphic (or diffeomorphic, in the smooth case) to the product manifold $M \times \mathbb{R}^k$.

Example: **Vector Bundles**

- Product bundles: For any topological space (or smooth manifold) M and integer $k \geq 0$, the product space $M \times \mathbb{R}^k$ with the projection map $\pi : M \times \mathbb{R}^k \to M$ defined by

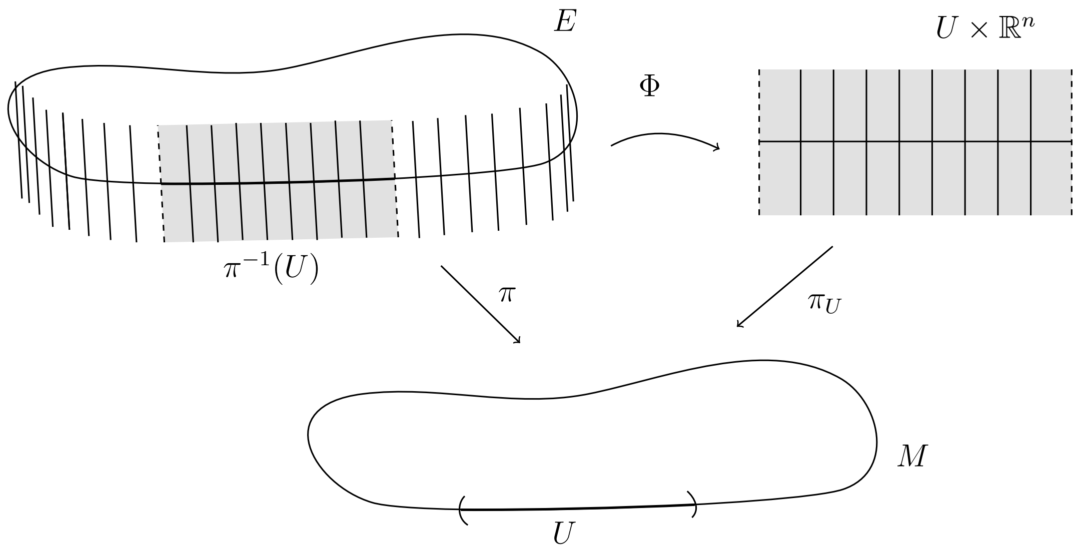

Figure 10.1: Vector Bundles

$\pi(p, v) = p$ is a trivial vector bundle of rank k over M.

- The Möbius Bundle: Define an equivalence relation on $\mathbb{R}^2$ generated by

$$(x, y) \sim (x + 1, -y).$$

Let $E = \mathbb{R}^2/\sim$ be the quotient space, and let $q : \mathbb{R}^2 \to E$ be the quotient map. For any $[-r, r] \subseteq \mathbb{R}$, the image of the strip $\mathbb{R} \times [-r, r]$ under q is called the **Möbius band**. It is a compact manifold with boundary. It is easy to see that $E \to S^1$ is a smooth vector bundle of rank 1 over the circle, called the **Möbius bundle**, which is nontrivial.

Proposition: **Tangent Bundle as Vector Bundle**

Let M be a smooth manifold of dimension n, with or without boundary, and let TM be its tangent bundle. With the projection map $\pi : TM \to M$ defined by $\pi(v) = p$ for $v \in T_pM$, TM is a smooth vector bundle of rank n over M.

Next, we show how two local trivializations of a vector bundle are related on their overlap.

Lemma 10.1.1: Local Trivialization Overlap

Let $\pi : E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two local trivializations of E that $U \cap V \neq \emptyset$. Then there exists a smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ is given by

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v) \tag{10.1}$$

Here τ is called the **transition function** between the two local trivializations.

This means that the change of coordinates between two local trivializations is a smoothly varying linear isomorphism along the manifold points.

Lemma 10.1.2: Vector Bundle Chart Lemma

Let M be a smooth manifold, with or without boundary, and suppose for each $p \in M$, we have a vector space E_p of dimension k. Let $E = \bigsqcup_{p\in M} E_p$ and $\pi : E \to M$ be the natural projection map $E_p \mapsto p$. Suppose we are given the following data:

- An open cover $\{U_\alpha\}$ of M.
- For each α, a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism onto $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$.
- For each α, β that $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that for all $p \in U_\alpha \cap U_\beta$ and $v \in \mathbb{R}^k$,

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v). \tag{10.2}$$

Then E has a unique topology and smooth structure such that it is a smooth manifold with or without boundary, and also a smooth vector bundle of rank k over M with smooth local trivializations $\{U_\alpha, \Phi_\alpha\}$.

Example: **Whitney Sums**

Given a smooth manifold M and two smooth vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ of ranks k_1 and k_2 respectively, we can construct a new vector bundle called the **Whitney sum** of E_1 and E_2,:

- The total space is $E = E_1 \oplus E_2 = \bigsqcup_{p\in M} (E_1)_p \oplus (E_2)_p$.
- The projection map $\pi : E \to M$ is defined by $(E_1)_p \oplus (E_2)_p \to p$.
- For a neighborhood U of $p \in M$ with local trivializations $\Phi_1 : \pi_1^{-1}(U) \to U \times \mathbb{R}^{k_1}$ and $\Phi_2 : \pi_2^{-1}(U) \to U \times \mathbb{R}^{k_2}$, we define a local trivialization for E by

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{k_1+k_2}, \quad \Phi(v_1, v_2) = (p, (\pi_{\mathbb{R}^{k_1}} \circ \Phi_1(v_1), \pi_{\mathbb{R}^{k_2}} \circ \Phi_2(v_2))),$$

Example: **Restricting a Vector Bundle**

Suppose $\pi : E \to M$ is a vector bundle of rank k and $S \subseteq M$ is any subset. Then the restriction of E to S is a vector bundle. The total space is $E|_S = \bigsqcup_{p\in S} E_p$, and the projection map is the restriction $\pi|_{E|_S} : E|_S \to S$. If Φ is a local trivialization of E over an open set U in M, then its restriction is $\Phi|_S : \pi^{-1}(U \cap S) \to (U \cap S) \times \mathbb{R}^k$ is a local trivialization of $E|_S$ over $U \cap S$.

If E is a smooth vector bundle and $S \subseteq M$ is an immersed or embedded submanifold, then $E|_S$ is also a smooth vector bundle over S.
For the tangent bundle, the restriction $TM|_S$ is called the **ambient tangent bundle** of S in M. (Note it is NOT the same as the tangent bundle TS of S because it still maintains full dimension of M in each fiber.)

10.2 Local and Global Sections

Similar to previous definitions about sections.

Definition 10.2.1: Sections of Vector Bundles

Let $\pi : E \to M$ be a vector bundle. A (global) section of E is a section of the projection map π, i.e., a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. For every $p \in M$, we have $\sigma(p) \in E_p$.
A local section of E over an open subset $U \subseteq M$ is a continuous map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$.
If M is a smooth manifold, with or without boundary, and E is a smooth vector bundle over M, then a smooth (global or local) section of E is a smooth map σ as above. A not necessarily continuous section is called a **rough section**. The zero section is a global section $\zeta : M \to E$ defined by $\zeta(p) = 0 \in E_p$ for all $p \in M$. The support of a section σ is defined as

$$\operatorname{supp}(\sigma) = \overline{\{p \in M : \sigma(p) \neq 0\}}.$$

Example: **Sections of Vector Bundles**

Suppose M is a smooth manifold, with or without boundary.

- Sections of TM are vector fields on M.
- Given an immersed submanifold $S \subseteq M$, sections of the ambient tangent bundle $TM|_S$ are called vector fields along S. It is different from a vector field on S because the vectors lie in the tangent spaces of M instead of S.
- If $E = M \times \mathbb{R}^k$ is a trivial vector bundle, then sections of E are exactly (one-to-one correspondence) continuous functions from M to $\mathbb{R}^k$:

  $$F : M \to \mathbb{R}^k, \Leftrightarrow \tilde{F} : M \to M \times \mathbb{R}^k, \quad \tilde{F}(p) = (p, F(p)).$$

  So $C^\infty(M)$ can be identified as the space of smooth sections of the trivial line bundle $M \times \mathbb{R}$.

If $E \to M$ is a smooth vector bundle, then the set of all smooth global sections of E is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication, denoted by $\Gamma(E)$.

$$(c_1\sigma_1 + c_2\sigma_2)(p) = c_1\sigma_1(p) + c_2\sigma_2(p), \quad \forall p \in M. \tag{10.3}$$

Then we have $\mathfrak{X}(M) = \Gamma(TM)$.

We can also multiply a section by a smooth function. If $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, then we define a new section $f\sigma \in \Gamma(E)$ by

$$(f\sigma)(p) = f(p)\sigma(p), \quad \forall p \in M.$$

Lemma 10.2.1: Extension Lemma for Vector Bundles

Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M, with or without boundary. If $A \subseteq M$ is closed and $\sigma : A \to E$ is a section of $E|_A$ that is smooth in a neighborhood of each point of A. For each open neighborhood U of A in M, there exists a smooth global section $\tilde{\sigma} \in \Gamma(E)$ such that $\tilde{\sigma}|_A = \sigma$ and $\operatorname{supp}(\tilde{\sigma}) \subseteq U$.

10.2.1 Local and Global Frames

We can view sections as generalizations of vector fields. Similarly, we generalize the concept of frames.

Definition 10.2.2: Frames of Vector Bundles

Let $\pi : E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is linearly independent if for each $p \in U$, the set $\{\sigma_1(p), \dots, \sigma_k(p)\}$ is linearly independent in the fiber E_p. It is a local frame of E over U if for each $p \in U$, the set $\{\sigma_1(p), \dots, \sigma_k(p)\}$ is a basis of the fiber E_p. It is called a **global frame** of E if it is defined over all of M.

Proposition: **Completion of Local Frames for Vector Bundles**

Let $\pi : E \to M$ be a smooth vector bundle of rank k.

- If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subseteq M$, with $m < k$, then for each $p \in U$, there exists an open neighborhood $V \subseteq U$ of p and smooth local sections $\sigma_{m+1}, \dots, \sigma_k$ of E over V such that $(\sigma_1, \dots, \sigma_k)$ is a local frame of E over V.
- If $(v_1, \dots, v_m)$ is a linearly independent m-tuple of vectors in the fiber E_p over some point $p \in M$, with $m \leq k$, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ of E over an open neighborhood U of p such that $\sigma_i(p) = v_i$ for $1 \leq i \leq m$.
- If $A \subseteq M$ is a closed set and $(\tau_1, \dots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ that is smooth in a neighborhood of each point of A, then there exists a smooth global frame $(\sigma_1, \dots, \sigma_k)$ of E over some open neighborhood U of A such that $\sigma_i|_A = \tau_i$ for $1 \leq i \leq k$.

Intuitively, local frames of a vector bundle are intimately related to local trivializations. This is indeed the case.

Proposition: **Local Frames and Local trivializations**

Let $\pi : E \to M$ be a smooth vector bundle. If $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization of E over an open subset $U \subseteq M$, then we can construct a local frame $(\sigma_1, \ldots, \sigma_k)$ of E over U by defining

$$\sigma_i(p) = \Phi^{-1}(p, e_i), \quad \forall p \in U,$$

Conversely, if $(\sigma_1, \ldots, \sigma_k)$ is a smooth local frame of E over an open subset $U \subseteq M$, then we can construct a smooth local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over U by defining

$$\Phi(v) = (p, (v^1, \ldots, v^k)), \quad \text{where } v = v^i \sigma_i(p) \in E_p, \; p \in U.$$

Therefore, a smooth vector bundle is smoothly trivial if and only if it admits a smooth global frame.

In a local chart (V, φ) of M with coordinates (x^i), and suppose there is a local frame (σ_i) of E over V. The define $\tilde{\varphi} : \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ by

$$\tilde{\varphi}(v^i \sigma_i(p)) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^k).$$

Then $(\pi^{-1}(V), \tilde{\varphi})$ is a smooth chart of the total space E.

Proposition: **Local Frame Criterion for Smoothness**

Let $\pi : E \to M$ be a smooth vector bundle, and let $\tau : M \to E$ be a rough section of E. If (σ_i) is a smooth local frame of E over an open subset $U \subseteq M$, then τ is smooth on U if and only if the component functions $\tau^i : U \to \mathbb{R}$ defined by

$$\tau(p) = \tau^i(p) \sigma_i(p), \quad \forall p \in U$$

are smooth.

Proposition: **Uniqueness of Smooth Structure on TM**

Let M be a smooth n-manifold, with or without boundary. The topology and smooth structure on the tangent bundle TM defined before are the unique ones such that $\pi : TM \to M$ is a smooth vector bundle with the given vector space structure on each fiber by derivations, and such that every coordinate vector field on M is a smooth section of TM.

Proof. Suppose there is another topology and smooth structure on TM satisfying the same conditions. If (U, φ) is a smooth chart of M, then the corresponding frame $\partial/\partial x^i$ of TM over U is a smooth local frame over U. So there is a smooth local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ associated to this local frame. But this is exactly the same as the local trivialization defined before. $\square$

10.3 Bundle Homomorphisms

Definition 10.3.1: Bundle Homomorphisms

Let $\pi : E \to M$ and $\pi' : E' \to M'$ be vector bundles. A continuous map $F : E \to E'$ is a **bundle homomorphism** from E to E' if there exists a map $f : M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

with the property that for each $p \in M$, the restriction $F|_{E_p} : E_p \to E'_{f(p)}$ is a linear map between vector spaces. We say that F covers f.
We can deduce that f is actually continuous and if the bundles and F are smooth, then f is also smooth.
A bijection bundle homomorphism whose inverse is also a bundle homomorphism is called a **bundle isomorphism**. If there exists a bundle isomorphism between two vector bundles, then they are said to be **isomorphic**.
If $M' = M$ is the same space, we say that F is a bundle homomorphism over M with base map id_M.

Example: **Bundle Homomorphisms**

- If $F : M \to N$ is a smooth map between smooth manifolds, then its differential $dF : TM \to TN$ is a smooth bundle homomorphism covering F.
- If $\pi : E \to M$ is a smooth vector bundle, and $S \subseteq M$ is an immersed submanifold, with or without boundary, then the inclusion map $i : E|_S \to E$ is a smooth bundle homomorphism covering the inclusion map of S into M.

Definition 10.3.2: Linear over $C^\infty(M)$

Let $\pi : E \to M$ and $\pi' : E' \to M$ be smooth vector bundles over the same smooth manifold M, with or without boundary. A map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is said to be **linear over** $C^\infty(M)$ if for all $f, g \in C^\infty(M)$ and $\sigma, \tau \in \Gamma(E)$, we have

$$\mathcal{F}(f\sigma + g\tau) = f\mathcal{F}(\sigma) + g\mathcal{F}(\tau).$$

Now, if $E \to M$ and $E' \to M$ are smooth vector bundles over a smooth manifold M, with or without boundary, then if $F : E \to E'$ is a smooth bundle homomorphism over M, then F induces a map $\tilde{F} : \Gamma(E) \to \Gamma(E')$ defined by

$$\tilde{F}(\sigma)(p) = F(\sigma(p)), \quad \forall p \in M. \tag{10.4}$$

It is easily verified that $\tilde{F}$ is linear over $C^\infty(M)$.

Lemma 10.3.1: Bundle Homomorphism Characterization Lemma

Let $\pi : E \to M$ and $\pi' : E' \to M$ be smooth vector bundles over the same smooth manifold M, with or without boundary. A map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is linear over $C^\infty(M)$ if and only if there exists a unique smooth bundle homomorphism $F : E \to E'$ over M such that $\mathcal{F}(\sigma) = F \circ \sigma$ for all $\sigma \in \Gamma(E)$.

Proof. SORRY □

Example: **Bundle Homomorphism over Manifolds**

- Let M be a smooth manifold, and $f \in C^\infty(M)$. Then $\mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $X \mapsto fX$ is linear over $C^\infty(M)$, and thus defines a smooth bundle homomorphism $TM \to TM$ over M.
- If Z is a smooth vector field in $\mathbb{R}^3$, then the cross product with Z: $X \mapsto X \times Z$ is linear over $C^\infty(\mathbb{R}^3)$, and thus defines a smooth bundle homomorphism $T\mathbb{R}^3 \to T\mathbb{R}^3$ over $\mathbb{R}^3$.
- If Z is a smooth vector field on $\mathbb{R}^n$, then the Euclidean inner product with Z: $X \mapsto \langle X, Z\rangle$ is linear over $C^\infty(\mathbb{R}^n)$, and thus defines a smooth bundle homomorphism $T\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ over $\mathbb{R}^n$.

Note that many maps the involve differentials are NOT bundle homomorphisms because they do not satisfy the right linearity conditions. For example, the Lie bracket $[X, Y]$ of two vector fields is not linear over $C^\infty(M)$ in either argument.

10.4 Subbundles

Definition 10.4.1: Subbundles

Given a vector bundle $\pi_E : E \to M$, a **subbundle** of E is a vector bundle $\pi_D : D \to M$, where D is a topological subspace of E, and π_D is the restriction of π_E to D, such that for each $p \in M$, the fiber $D_p = D \cap E_p$ is a vector subspace of E_p.
If $E \to M$ is a smooth vector bundle, then a **smooth subbundle** of E is a subbundle $D \to M$ such that D is a smooth vector bundle and an embedded submanifold of E, with or without boundary.

The following proposition gives a useful criterion for determining when a union of subspace $D = \bigsqcup_{p\in M} D_p, D_p \subseteq E_p$ is a smooth subbundle of E.

Lemma 10.4.1: Local Frame Criterion for Subbundles

Let $\pi : E \to M$ be a smooth vector bundle, and for each $p \in M$, we are given an m-dimensional linear subspace $D_p \subseteq E_p$. Let $D = \bigsqcup_{p\in M} D_p \subseteq E$, then D is a smooth subbundle of E if and only if:
Each point $p \in M$ has an open neighborhood $U \subseteq M$ that there exists a smooth local sectrions $\sigma_1, \ldots, \sigma_m : U \to E$ of E over U such that for each $q \in U$, the set $\{\sigma_1(q), \ldots, \sigma_m(q)\}$ is a basis of the subspace $D_q \subseteq E_q$.

Example: **Subbundles**

- If M is a smooth manifold, and V is a nowhere vanishing smooth vector field on M, then $D \subseteq TM$ defined by $D_p = \operatorname{span}\{V(p)\}$ is a smooth subbundle of TM of rank 1.
- Suppose $E \to M$ is any trivial bundle, and let $E_1, \ldots, E_k$ be a smooth global frame of E. For any $0 \leq m \leq k$, the subspace $D_p = \operatorname{span}\{E_1(p), \ldots, E_m(p)\} \subseteq E_p$ defines a smooth subbundle D of E of rank m.
- Suppose M is a smooth manifold, with or without boundary, and $S \subseteq M$ is an immersed k-dimensional submanifold, with or without boundary. Then TS is a smooth subbundle of the ambient tangent bundle $TM|_S$ of rank k.

Definition 10.4.2: Constant Rank Bundle Homomorphisms

Let $\pi : E \to M$ and $\pi' : E' \to M$ be smooth vector bundles over the same smooth manifold M, with or without boundary. A smooth bundle homomorphism $F : E \to E'$ over M is said to have **constant rank** if for each $p \in M$, the linear map $F_p : E_p \to E'_p$ defined by the restriction of F to the fiber E_p has the same rank.

Theorem 10.4.1: Smooth Subbundles from Constant Rank

Let E, E' be smooth vector bundles over the same smooth manifold M, and let $F : E \to E'$ be a smooth bundle homomorphism over M. Define subsets

$$\ker F = \bigsqcup_{p \in M} \ker F_p \subseteq E, \quad \operatorname{image} F = \bigsqcup_{p \in M} \operatorname{image} F_p \subseteq E'. \tag{10.5}$$

Then both $\ker F$ and $\operatorname{image} F$ are smooth subbundles of E and E' respectively if and only if F has constant rank.

Proof. Quite obvious. □

Lemma 10.4.2: Orthogonal Complement Bundles

Let M be an immersed submanifold of $\mathbb{R}^n$, with or without boundary. Let D be a smooth rank-k subbundle of $T\mathbb{R}^n|_M$. For each $p \in M$, define $D_p^\perp \subseteq T_p\mathbb{R}^n$ to be the orthogonal complement of D_p in $T_p\mathbb{R}^n \cong \mathbb{R}^n$ with respect to the standard Euclidean inner product. Then $D^\perp = \bigsqcup_{p \in M} D_p^\perp$ is a smooth subbundle of $T\mathbb{R}^n|_M$ of rank $n - k$.
If we take $D = TM$ to be the tangent bundle of M, then $D^\perp$ is called the **normal bundle** of M in $\mathbb{R}^n$.

10.5 Fiber Bundles

A vector bundle is a special case of a more general concept called a fiber bundle, where the fibers are not necessarily vector spaces.

Definition 10.5.1: Fiber Bundles

A **fiber bundle** is a triple (E, M, π), where E and M are topological spaces (or smooth manifolds, with or without boundary), and $\pi : E \to M$ is a continuous surjection (or smooth map) satisfying the following conditions:

- For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is homeomorphic (or diffeomorphic) to a fixed topological space (or smooth manifold) F, called the **fiber**.
- For each $p \in M$, there exists an open neighborhood U of p in M and a homeomorphism (or diffeomorphism) $\Phi : \pi^{-1}(U) \to U \times F$ called a **local trivialization** such that $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times F \to U$ is the projection onto the first factor.

Example: **Fiber Bundles**

- Every product space $M \times F$ with the projection map $\pi : M \times F \to M$ defined by $\pi(p, x) = p$ is a trivial fiber bundle over M with fiber F.
- Every rank-k vector bundle is a fiber bundle with fiber $\mathbb{R}^k$.
- The Möbius strip is a fiber bundle over S^1 with fiber $[-1, 1]$. It is not a trivial fiber bundle.
- Every covering map $\pi : E \to M$ is a fiber bundle with discrete fiber.

Chapter 11

The Cotangent Bundle

11.1 Covectors

As we know from linear algebra, given a finite-dimensional vector space V, we can form its dual space V^*, the space of all linear functionals from V to $\mathbb{R}$. Elements of V^* are called **covectors**. Now let M be a smooth manifold, with or without boundary. At each point $p \in M$, we have the tangent space T_pM, which is a finite-dimensional vector space. We can then consider the dual space of the tangent space at each point, denoted by $T_p^*M = (T_pM)^*$. Elements of T_p^*M are called **cotangent vectors** or **covectors** at the point p.

Given smooth local coordinates $(x^1, x^2, \dots, x^n)$ in $U \subseteq M$, the coordinate basis $\partial/\partial x^i|_p$ of T_pM induces a dual basis of T_p^*M, for now denoted by $\lambda^i|_p$. Any $\omega \in T_p^*M$ can be expressed uniquely as $\omega = \omega_i \lambda^i|_p$, where

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right).$$

Now if $(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n)$ is another smooth coordinate whose domain $\tilde{U}$ contains p, then we have

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p,$$

and thus we can write $\omega = \omega_i \lambda^i|_p = \tilde{\omega}_j \tilde{\lambda}^j|_p$, where

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \tilde{\omega}_j \frac{\partial \tilde{x}^j}{\partial x^i}(p).$$

Remark:
In the early days of differential geometry, tangent vectors were often thought of a n-tuple of reals assigned to a point p in a coordinate chart. Note this definition DEPENDS on the choice of the coordinate chart. A real coordinate-free tangent vector can be thought of as an equivalence class of such n-tuples under the transformation law:

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(p) v^i.$$

for any coordinate change. Similarly, a covector can be thought of as an equivalence class of

n-tuples of reals assigned to p under the transformation law:

$$\omega_i = \tilde{\omega}_j \frac{\partial \tilde{x}^j}{\partial x^i}(p).$$

It has been a custom calling tangent covectors **covariant vectors** and tangent vectors **contravariant vectors** because of the way their components transform under coordinate changes. However, this terminology is somewhat outdated and can be confusing, so it is generally avoided in modern texts.

11.1.1 Covector Fields

Definition 11.1.1: Cotangent Bundle

Let M be a smooth manifold, with or without boundary. The **cotangent bundle** of M is the disjoint union of the cotangent spaces at each point in M:

$$T^*M = \bigsqcup_{p \in M} T_p^*M = \{(p, \omega) : p \in M, \omega \in T_p^*M\}.$$

There is a natural projection map $\pi : T^*M \to M$ defined by $\pi(p, \omega) = p$. For any $U \subseteq M$ with local coordinates $(x^1, x^2, \ldots, x^n)$, the coordinate covectors $\lambda^i|_p$ defines n maps $\lambda^i : U \to T^*M$ called coordinate covector fields.

Proposition: **Cotangent Bundle as Vector Bundle**

Let M be a smooth n-dimensional manifold, with or without boundary. Then with the projection and the natural vector space structure on each fiber, the cotangent bundle T^*M can be uniquely made into a smooth vector bundle of rank n over M, for which all coordinate covector fields λ^i are smooth local sections.

Proof. It is quite easy to find local trivializations, just take any coordinate chart $(U, (x^1, x^2, \ldots, x^n))$ on M, and define

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n, \quad \Phi(\xi_i \lambda^i|_p) = (p, (\xi_1, \xi_2, \ldots, \xi_n)).$$

Next follows straight by vector bundle chart lemma 10.1.2. □

This process can be generalized to any vector bundle. Suppose $E \to M$ is a smooth vector bundle over M. Then we can define the dual bundle $E^* \to M$ by taking the dual space of each fiber.

The projection map produce natural coordinate chart for T^*M. For any chart (U, x^i) on M, the map

$$\pi^{-1}(U) \to \mathbb{R}^{2n}, \quad \xi_i \lambda^i|_p \mapsto (x^1(p), x^2(p), \ldots, x^n(p), \xi_1, \xi_2, \ldots, \xi_n)$$

is a smooth local coordinate chart for T^*M. We call (x^i, ξ_i) the **natural coordinates** on T^*M associated to the coordinate chart (U, x^i) on M.

A (local or global) section of T^*M is called a **covector field** or a **differential 1-form**. For any covector field ω, we denote its value at p by $\omega|_p$ or simply ω_p. If ω is a rough covector field and X is a rough vector field, then we can define a rough real-valued function $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)(p) = \omega|_p(X|_p).$$

In component form, if we write $\omega = \omega_i \lambda^i$ and $X = X^i \partial/\partial x^i$, then

$$\omega(X) = \omega_i X^i.$$

The set of smooth covector fields on M is denoted by $\mathfrak{X}^*(M)$.

Proposition: **Smoothness Criterion for Covector Fields**

Let M be a smooth manifold, with or without boundary, and let $\omega : M \to T^*M$ be a rough covector field. The following are equivalent:

- ω is smooth.
- For every smooth chart, the component functions of ω are smooth.
- Each point $p \in M$ has a smooth chart $(U, (x^1, x^2, \dots, x^n))$ such that the component functions of ω with respect to this chart are smooth on U.
- For every smooth vector field X on M, the function $\omega(X) : M \to \mathbb{R}$ is smooth.
- For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X) : U \to \mathbb{R}$ is smooth on U.

11.1.2 Coframes

Definition 11.1.2: Coframe

Let M be a smooth manifold, with or without boundary. Take open subset $U \subseteq M$, a **local coframe** is a collection of n covector fields $(\omega^1, \omega^2, \dots, \omega^n)$ on U such that at each point $p \in U$, the set $(\omega^1|_p, \omega^2|_p, \dots, \omega^n|_p)$ forms a basis for the cotangent space T_p^*M. If $U = M$, then we call $(\omega^1, \omega^2, \dots, \omega^n)$ a **global coframe** on M.

Example: **Coframes**

- The coordinate covector fields $(\lambda^1, \lambda^2, \dots, \lambda^n)$ associated to any smooth chart $(U, (x^1, x^2, \dots, x^n))$ form a local coframe on U.
- For any local frame $(E_1, E_2, \dots, E_n)$ for TM over U, there is a unique local coframe $(\epsilon^1, \epsilon^2, \dots, \epsilon^n)$ for T^*M over U such that $\epsilon^i(E_j) = \delta^i_j$, called the **dual coframe** to $(E_1, E_2, \dots, E_n)$. Conversely, given any local coframe, there is a unique dual local frame. And E_i is smooth if and only if ϵ^i is smooth.

Given any local coframe (ϵ^i) over U, any covector field ω over U can be uniquely expressed as $\omega = \omega_i \epsilon^i$ for some rough functions $\omega_i : U \to \mathbb{R}$. If the coframe and ω are smooth, then so are the component functions ω_i.

Proposition: **Coframe Criterion for Smoothness**

Let M be a smooth manifold, with or without boundary, and let ω be a rough covector field on an open subset $U \subseteq M$. Let $(\epsilon^1, \epsilon^2, \ldots, \epsilon^n)$ be a local coframe on U, and write $\omega = \omega_i \epsilon^i$ for some rough functions $\omega_i : U \to \mathbb{R}$. Then ω is smooth if and only if each component function ω_i is smooth.

11.2 The Differential of a Smooth Function

In elementary analysis, the gradient of a smooth real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as the vector field whose components are the partial derivatives of f. As:

$$f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}. \tag{11.1}$$

Unfortunately, this definition does not generalize to the coordinate free setting here (the index convention is a hint). However, we find that the partial derivatives $\partial f / \partial x^i$ behave like the components of a covector field rather than a vector field.

Definition 11.2.1: Differential of a Smooth Function

Let M be a smooth manifold, with or without boundary, and let $f : M \to \mathbb{R}$ be a smooth function. The **differential** of f is the covector field $\mathrm{d}f \in \mathfrak{X}^*(M)$ defined by

$$\mathrm{d}f_p(v) = v(f) \tag{11.2}$$

for all $p \in M$ and $v \in T_pM$.

Proposition: **Differential is Smooth Covector Field**

The differential of any smooth function $f : M \to \mathbb{R}$ is a smooth covector field on M.

In coordinate representation, if $(U, (x^1, x^2, \ldots, x^n))$ is a smooth chart on M, then we can write $\mathrm{d}f_p = A_i(p)\lambda^i|_p$ for some functions $A_i : U \to \mathbb{R}$. By definition, for any $p \in U$,

$$A_i(p) = \mathrm{d}f_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = \left.\frac{\partial}{\partial x^i}\right|_p (f) = \frac{\partial f}{\partial x^i}(p).$$

So we have

$$\mathrm{d}f_p = \frac{\partial f}{\partial x^i}(p)\lambda^i|_p, \tag{11.3}$$

If $f = x^j$, then $\mathrm{d}x^j|_p = \lambda^j|_p$. So the coordinate covector fields can be written as differentials of the coordinate functions! This gives

$$\mathrm{d}f_p = \frac{\partial f}{\partial x^i}(p)\mathrm{d}x^i|_p, \qquad \mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i. \tag{11.4}$$

which is just the classical expression for the differential of f in multivariable calculus.

Proposition: **Properties of Differential**

Let M be a smooth manifold, with or without boundary. For any smooth functions $f, g \in C^\infty(M)$ and any real number $a \in \mathbb{R}$, the following hold:

- Linearity: $\mathrm{d}(af + bg) = a\mathrm{d}f + b\mathrm{d}g$.
- Product Rule: $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$.
- Quotient Rule: $\mathrm{d}\left(\frac{f}{g}\right) = \frac{g\mathrm{d}f - f\mathrm{d}g}{g^2}$, provided g is nowhere zero.
- If $J \subseteq \mathbb{R}$ is an open interval containing the image of f, and $h : J \to \mathbb{R}$ is a smooth function, then $\mathrm{d}(h \circ f) = (h' \circ f)\mathrm{d}f$.
- If f is constant, then $\mathrm{d}f = 0$.

Proposition: **Vanishing Differential**

Let M be a smooth manifold, with or without boundary, and let $f : M \to \mathbb{R}$ be a smooth function. Then $\mathrm{d}f = 0$ if and only if f is constant on each connected component of M.

Proof. If M is connected and $\mathrm{d}f = 0$, then let $p \in M$, let $\mathcal{C} = \{q \in M : f(q) = f(p)\}$. For any $q \in \mathcal{C}$, take a smooth coordinate ball (or half-ball if $q \in \partial M$) U centered at q. From elementary calculus, f is constant on U, so $U \subseteq \mathcal{C}$. This shows that $\mathcal{C}$ is open. On the other hand, it is closed by continuity of f. Since M is connected, we must have $\mathcal{C} = M$, so f is constant. □

Remark:
In elementary calculus, the differential $\mathrm{d}f_p$ at a point p is often thought of as the approximation of the change in f for a small change of x^i. Here the intuition is also valid, just take a local coordinate chart, identify the tangent space to $T_p\mathbb{R}^n$ and the same goes for Taylor's theorem. For any tangent vector $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, we have

$$\Delta f = f(p + v) - f(p) \approx \frac{\partial f}{\partial x^i}(p)v^i = \mathrm{d}f_p(v).$$

Proposition: **Derivative along a Curve**

Let M be a smooth manifold, with or without boundary, and let $\gamma : J \to M$ be a smooth curve, and let $f : M \to \mathbb{R}$ be a smooth function. Then the derivative of the composition $f \circ \gamma : J \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = \mathrm{d}f_{\gamma(t)}(\gamma'(t))$$

Remark:
Indeed, there are two ways to interpret $(f \circ \gamma)'(t)$. The first is seeing $f \circ \gamma$ as a smooth function from J to $\mathbb{R}$, and taking its derivative at t. The second is seeing it as a smooth

curve in $\mathbb{R}$, and taking the tangent vector at t. One is an element of $\mathbb{R}$, the other is an element of $T_{f(\gamma(t))}\mathbb{R}$. Obviously, they can be identified via the canonical isomorphism.

11.3 Pullback of Covector Fields

A smooth map yields a linear map on tangent spaces called the differential. Dually, it also yields a linear map on cotangent spaces called the pullback.

Definition 11.3.1: Pullback

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. The differential $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ induces a dual map

$$\mathrm{d}F_p^* : T_{F(p)}^*N \to T_p^*M, \qquad \mathrm{d}F_p^*(\omega)(v) = \omega(\mathrm{d}F_p(v)) \tag{11.5}$$

called the pointwise pullback map.

When we discuss pushforwards of vector fields, we have to impose some conditions on F to ensure the existence of the pushforward vector field: Only when F is a diffeomorphism or a Lie group homomorphism can we guarantee the existence of the pushforward vector field. However, for pullbacks of covector fields, no such conditions are needed. Given a smooth map $F : M \to N$ and a covector field ω on N, we can define a rough covector field $F^*\omega$ on M by

$$(F^*\omega)_p = \mathrm{d}F_p^*(\omega_{F(p)}), \qquad (F^*\omega)_p(v) = \omega_{F(p)}(\mathrm{d}F_p(v)) \tag{11.6}$$

It does not have any problem because the pointwise pullback map is defined for all points in M. (This can be seen as a consequence of the "direction" of the smooth map).

Proposition: **Function Multiplication under Pullback**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. Let u be a continuous real-valued function on N, and let ω be a rough covector field on N. Then

$$F^*(u\omega) = (u \circ F)F^*\omega. \tag{11.7}$$

If u is smooth, then

$$F^*\mathrm{d}u = \mathrm{d}(u \circ F). \tag{11.8}$$

Proof. From computation

$$(F^*(u\omega))_p(v) = (u\omega)_{F(p)}(\mathrm{d}F_p(v)) = u(F(p))\omega_{F(p)}(\mathrm{d}F_p(v)) = ((u \circ F)F^*\omega)_p(v).$$

$$(F^*\mathrm{d}u)_p(v) = \mathrm{d}u_{F(p)}(\mathrm{d}F_p(v)) = \mathrm{d}(u \circ F)_p(v).$$

would do. ☐

Proposition: **Smoothness of Pullback**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. If ω is a smooth covector field on N, then the pullback $F^*\omega$ is a smooth covector field on M.

Proof. Take any $p \in M$, and choose smooth coordinates (y^i) on an open neighborhood V of $F(p)$ in N, and $U = F^{-1}(V)$. Then let $\omega = \omega_j \mathrm{d}y^j$ on V. So we have

$$F^*\omega = F^*(\omega_j \mathrm{d}y^j) = (\omega_j \circ F)F^*\mathrm{d}y^j = (\omega_j \circ F)\mathrm{d}(y^j \circ F) = (\omega_j \circ F)\mathrm{d}F^j,$$

This also gives a simple expression to compute the components of $F^*\omega$ in local coordinates. This is exactly how we do it in multivariable calculus. □

11.3.1 Restricting Covector Fields to Submanifolds

Suppose M is a smooth manifold, with or without boundary, and $S \subseteq M$ is an immersed submanifold with inclusion map $\iota : S \hookrightarrow M$. Given any smooth covector field ω on M, we can define a smooth covector field on S by restricting ω to S via the pullback $\iota^*\omega$. To see it,

$$\iota^*\omega|_p(v) = \omega_{\iota(p)}(\mathrm{d}\iota_p(v)) = \omega_p(v)$$

which is just the restriction of ω to S. However, ω may vanish on S even if it is nonvanishing on M, when ω annihilates all vectors in T_pS for each $p \in S$.

Remark:
Here is something to clarify. We say ω vanishes along S if $\omega_p = 0$ for all $p \in S$ in the context of being a covector field on M. We say the pullback ω to S vanishes if $\iota^*\omega|_p = 0$ for all $p \in S$ in the context of being a covector field on S. The former implies the latter, but not vice versa.

11.4 Line Integrals

Using covector fields, we can define line integrals on smooth manifolds coordinnate-freely.

We begin with $\mathbb{R}$. Let $[a, b]$ be a closed interval, and ω be a smooth covector field on $[a, b]$ (meaning it is smooth on some open neighborhood of $[a, b]$ in $\mathbb{R}$). Let t be the standard coordinate on $\mathbb{R}$, then we can write $\omega = f(t)\mathrm{d}t$ for some smooth function $f : [a, b] \to \mathbb{R}$. We DEFINE the line integral of ω over $[a, b]$ by

$$\int_{[a,b]} \omega = \int_a^b f(t)\mathrm{d}t. \tag{11.9}$$

Well, the left hand side does not seem to depend on the choice of coordinate t. We shall see that this is indeed the case.

Proposition: **Diffeomorphism Invariance of Line Integrals**

Let ω be a smooth covector field on $[a, b] \subseteq \mathbb{R}$, and $\varphi : [c, d] \to [a, b]$ be an increasing diffeomorphism. Then

$$\int_{[a,b]} \omega = \int_{[c,d]} \varphi^* \omega. \tag{11.10}$$

If φ is decreasing, then

$$\int_{[a,b]} \omega = -\int_{[c,d]} \varphi^* \omega. \tag{11.11}$$

Proof. By computation, we have $(\varphi^*\omega)_s = \omega_{\varphi(s)}(\varphi'(s)\partial/\partial t|_{\varphi(s)}) = f(\varphi(s))\varphi'(s)\mathrm{d}s$. Thus,

$$\int_{[c,d]} \varphi^*\omega = \int_c^d f(\varphi(s))\varphi'(s)\mathrm{d}s = \int_a^b f(t)\mathrm{d}t = \int_{[a,b]} \omega.$$

□

Now let M be a smooth manifold, with or without boundary. A curve segment in M is a continuous curve $\gamma : [a,b] \to M$.

- A smooth curve segment is a smooth map $\gamma : [a,b] \to M$, where $[a,b]$ is regarded as a smooth manifold with boundary.
- A piecewise smooth curve segment if there is a finite partition $a = t_0 < t_1 < \cdots < t_k = b$ such that each restriction $\gamma|_{[t_{i-1},t_i]}$ is a smooth curve segment.

Proposition: **Connecting Points via Curves**

If M is a connected smooth manifold, with or without boundary, then for any two points $p, q \in M$, there exists a piecewise smooth curve segment $\gamma : [a,b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

Proof. Take $p \in M$, let $\mathcal{C} = \{q \in M : \text{there is a piecewise smooth curve segment from } p \text{ to } q\}$. For any $q \in \mathcal{C}$, take a smooth coordinate ball (or half-ball if $q \in \partial M$) U centered at q. For any $r \in U$, we can connect q to r via a straight line in the coordinate chart, so there is a piecewise smooth curve segment from p to r. This shows that $\mathcal{C}$ is open. On the other hand, if $q \in \partial\mathcal{C}$, then any smooth coordinate ball (or half-ball) U centered at q must contain some point $r \in \mathcal{C}$. Again, we can connect r to q via a straight line in the coordinate chart, so there is a piecewise smooth curve segment from p to q. This shows that $\mathcal{C}$ is closed. Since M is connected, we must have $\mathcal{C} = M$, so any two points in M can be connected via a piecewise smooth curve segment. □

If $\gamma : [a,b] \to M$ is a smooth curve segment and ω is a smooth covector field on M, we define the line integral of ω along γ by

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega. \tag{11.12}$$

if γ is piecewise smooth with partition $a = t_0 < t_1 < \cdots < t_k = b$, we define

$$\int_\gamma \omega = \sum_{i=1}^k \int_{\gamma|_{[t_{i-1},t_i]}} \omega. \tag{11.13}$$

Proposition: **Properties of Line Integrals**

Let M be a smooth manifold, with or without boundary, and let $\gamma : [a, b] \to M$ be a piecewise smooth curve segment. For any smooth covector fields $\omega, \omega_1, \omega_2$ on M,

- $\forall c_1, c_2 \in \mathbb{R}, \quad \displaystyle\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$
- If γ is a constant curve, then $\displaystyle\int_\gamma \omega = 0.$
- If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ for some $c \in (a, b)$, then $\displaystyle\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$
- If $F : M \to N$ is a smooth map between smooth manifolds, with or without boundary, and η is a smooth covector field on N, then $\displaystyle\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$

Line integrals are independent of reparameterization up to sign. If $\gamma : [a, b] \to M$ and $\tilde{\gamma} : [c, d] \to M$ are two piecewise smooth curve segments, we say that they are **reparameterizations** of each other if there is a diffeomorphism $\varphi : [c, d] \to [a, b]$ such that $\tilde{\gamma} = \gamma \circ \varphi$. If φ is increasing, then it is said to be an **forward reparameterization**; if φ is decreasing, then it is said to be a **backward reparameterization**.

Proposition: **Parameter Independence of Line Integrals**

Let M be a smooth manifold, with or without boundary, and let $\gamma : [a, b] \to M$ and $\tilde{\gamma} : [c, d] \to M$ be two piecewise smooth curve segments that are reparameterizations of each other. For any smooth covector field ω on M,

$$\int_{\tilde{\gamma}} \omega = \begin{cases} \displaystyle\int_\gamma \omega, & \text{if } \tilde{\gamma} \text{ is a forward reparameterization of } \gamma; \\ \displaystyle -\int_\gamma \omega, & \text{if } \tilde{\gamma} \text{ is a backward reparameterization of } \gamma. \end{cases}$$

Proof. Using proposition **??**, we have, for forward reparameterization,

$$\int_{\tilde{\gamma}} \omega = \int_{[c,d]} \tilde{\gamma}^*\omega = \int_{[c,d]} (\gamma \circ \varphi)^*\omega = \int_{[c,d]} \varphi^*(\gamma^*\omega) = \int_{[a,b]} \gamma^*\omega = \int_\gamma \omega,$$

Same goes for backward reparameterization. □

Also, from the definition, we have (using local coordinates):

$$\gamma^*\omega = \omega_{\gamma(t)}(\gamma'(t))\mathrm{d}t,$$

So we can write the line integral in the classical form:

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\mathrm{d}t. \tag{11.14}$$

Theorem 11.4.1: Fundamental Theorem of Line Integrals

Let M be a smooth manifold, with or without boundary, and let $\gamma : [a, b] \to M$ be a piecewise smooth curve segment. For any smooth function $f : M \to \mathbb{R}$,

$$\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a)).$$

Proof. Suppose γ is smooth (then the piecewise smooth case follows by additivity). We have

$$\int_\gamma \mathrm{d}f = \int_a^b (\mathrm{d}f)_{\gamma(t)}(\gamma'(t))\mathrm{d}t = \int_a^b (f \circ \gamma)'(t)\mathrm{d}t = f(\gamma(b)) - f(\gamma(a)).$$

□

11.5 Conservative Covector Fields

Definition 11.5.1: Exact, Conservative and Closed Covector Fields

Let M be a smooth manifold, with or without boundary. A smooth covector field ω on M is said to be **exact** if there exists a smooth function $f : M \to \mathbb{R}$ such that $\omega = \mathrm{d}f$. f is called a **potential function** for ω.
We say that a smooth covector field ω on M is **conservative** if for every piecewise smooth closed curve segment γ in M,

$$\int_\gamma \omega = 0.$$

We say that a smooth covector field ω on M is **closed** if for every smooth chart $(U, (x^1, x^2, \dots, x^n))$ on M, writing $\omega = \omega_i \mathrm{d}x^i$ on U, we have

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$

Theorem 11.5.1: Conservative Field Theorem

Let M be a smooth manifold, with or without boundary. A smooth covector field ω on M is conservative if and only if it is exact.

Proof. SORRY □

As we did in calculus, there is an easy necessary condition for a covector field to be conservative from the exchange of mixed partial derivatives.

Proposition: **Exact Covector Field is Closed**

Every exact (hence every conservative) covector field on a smooth manifold, with or without boundary, is closed.

Proof. Just take a local chart and use the equality of mixed partial derivatives. □

Well, actually, we do not need to check every chart to see if a covector field is closed.

Proposition: **Criterion for Closed Covector Fields**

Let M be a smooth manifold, with or without boundary, and let ω be a smooth covector field on M. Then the following are equivalent:

- ω is closed.
- For every point $p \in M$, there is a smooth chart $(U, (x^1, x^2, \dots, x^n))$ containing p such that writing $\omega = \omega_i \mathrm{d}x^i$ on U, we have
$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$
- For any open subset $U \subseteq M$ and smooth vector fields $X, Y \in \mathfrak{X}(U)$,
$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y]). \tag{11.15}$$

We also know pullbacks of local diffeomorphisms preserve these properties, because it is just like an isomorphism.

Corollary 11.5.1: Pullback by Local Diffeomorphisms

Suppose $F : M \to N$ is a local diffeomorphism between smooth manifolds, with or without boundary. Then the pullback $F^* : \mathfrak{X}^*(N) \to \mathfrak{X}^*(M)$ takes exact (conservative) covector fields on N to exact (conservative) covector fields on M. It also takes closed covector fields on N to closed covector fields on M.

Proof. Exactness follows directly from
$$F^* \mathrm{d}u = \mathrm{d}(u \circ F).$$
Closedness follows from local coordinate representation and change of variables. □

The question of whether a closed covector field is exact is of great importance in differential geometry, leading to the development of de Rham cohomology. For now, we just state the result for a specific type of manifold:

A **Star-shaped domain** in $\mathbb{R}^n$ is a subset $U \subseteq \mathbb{R}^n$ such that there exists a point $c \in U$ such that for every point $x \in U$, the line segment connecting c to x lies entirely in U.

Theorem 11.5.2: Poincaré Lemma for Covector Fields

Let U be a star-shaped open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. Then every closed covector field on U is exact.

Proof. Assume $c = 0$ by some diffeomorphism. For any $x \in U$, define $\gamma_x : [0, 1] \to U$ by $\gamma_x(t) = tx$. Define a function $f : U \to \mathbb{R}$ by
$$f(x) = \int_{\gamma_x} \omega = \int_0^1 \omega_{\gamma_x(t)}(\gamma_x'(t))\mathrm{d}t = \int_0^1 \omega_i(tx)x^i \mathrm{d}t.$$

As the integrand is smooth in x, f is smooth. Now we have

$$\begin{aligned}\frac{\partial f}{\partial x^j}(x) &= \int_0^1 \left(t\frac{\partial \omega_i}{\partial x^j}(tx)x^i + \omega_j(tx)\right) \mathrm{d}t \\ &= \int_0^1 \left(t\frac{\partial \omega_j}{\partial x^i}(tx)x^i + \omega_j(tx)\right) \mathrm{d}t \\ &= \int_0^1 \frac{\partial}{\partial t}\left(t\omega_j(tx)\right) \mathrm{d}t = \omega_j(x).\end{aligned}$$

from the closedness of ω. Thus, $\mathrm{d}f = \omega$. □

Corollary 11.5.2: Local Exactness of Closed Covector Fields

Let M be a smooth manifold, with or without boundary, $and let \omega$ be a closed covector field on M. Then for every point $p \in M$, there exists an open neighborhood U of p such that the restriction $\omega|_U$ is exact.

Chapter 12

Tensors

We new generalize the application of linear algebra in differential geometry by introducing tensor fields on smooth manifolds. Tensor fields are multilinear maps that take in multiple vector and covector fields and output smooth functions. They play a crucial role in various areas of differential geometry, including the study of Riemannian metrics, differential forms, and curvature.

12.1 Multilinear Algebra

We have already encountered the concept of multilinear maps in the context of vector spaces in linear algebra. A multilinear map is $F : V_1 \times V_2 \times \cdots \times V_k \to W$ that is linear in each argument when the others are held fixed.

Example: **Multilinear Map**

- The dot product on $\mathbb{R}^n$ is a bilinear map.
- The cross product on $\mathbb{R}^3$ is a bilinear map.
- The determinant function on $n \times n$ matrices is a multilinear map.
- The bracket operation in a Lie algebra is a bilinear map.

Example: **Tensor Product of Covectors**

Let V be a vector space and let $\omega, \eta \in V^*$ be covectors. Define a bilinear map $\omega \otimes \eta : V \times V \to \mathbb{R}$ by

$$\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2). \tag{12.1}$$

This can also be generalized to define the tensor product of multilinear maps: if $F : V_1 \times \cdots \times V_k \to \mathbb{R}$ and $G : W_1 \times \cdots \times W_m \to \mathbb{R}$ are multilinear maps, then their tensor product $F \otimes G : (V_1 \times \cdots \times V_k) \times (W_1 \times \cdots \times W_m) \to \mathbb{R}$ is defined by $(F \otimes G)(v_1, \ldots, v_k, w_1, \ldots, w_m) = F(v_1, \ldots, v_k)G(w_1, \ldots, w_m)$.
It is easily verified that the tensor product operation acting on multilinear maps is itself multilinear and also associative.

12.1.1 Tensors Products of Vector Spaces

The vector space of multilinear functions $L(V_1, \ldots, V_k; \mathbb{R})$ can be viewed as all linear combinations of objects of the form $\omega_1 \otimes \cdots \otimes \omega_k$ where each $\omega_i \in V_i^*$. This motivates the following contruction of tensor products of vector spaces.

The intuition is clear: given finite-dimensional vector spaces $V_1, \ldots, V_k$, we want to construct a new vector space $V_1 \otimes \cdots \otimes V_k$ which formally consists of all linear combinations of objects of the form $v_1 \otimes \cdots \otimes v_k$ where each $v_i \in V_i$, depending multilinearly on each v_i.

Definition 12.1.1: Formal Linear Combinations

Let S be a set. A **formal linear combination** of elements of S is a function $f : S \to \mathbb{R}$ such that $f(s) \neq 0$ for only finitely many $s \in S$. The set of all formal linear combinations of elements of S is called the **free vector space** on S, denoted by $\mathcal{F}(S)$.
If we identify $x \subseteq S$ with the formal linear combination δ_x, we can say $S \subseteq \mathcal{F}(S)$ and every element of $\mathcal{F}(S)$ can be written as a finite linear combination of elements of S by $f = \sum_{i=1}^m a_i x_i$, where $a_i = f(x_i)$. Therefore, S is a basis for $\mathcal{F}(S)$.

We can directly see this as construction from a set as basis to form a vector space. So $\mathcal{F}(S)$ is finite-dimensional if and only if S is finite. However, usually the elements of S can be transformed to each other by some linear relations, which we need to quotient out.

Also, as S is thought of as a basis, for any vector space W, any function $A : S \to W$ extends uniquely to a linear map $\tilde{A} : \mathcal{F}(S) \to W$ by defining $\tilde{A}(\sum_{i=1}^m a_i x_i) = \sum_{i=1}^m a_i A(x_i)$.

Now, we need to rigorously form the tensor product of vector spaces.

- We start from the intuitive formal linear combinations of elements of the form $v_1 \otimes \cdots \otimes v_k$ where each $v_i \in V_i$. So we consider the free vector space $\mathcal{F}(V_1 \times \cdots \times V_k)$.
- Next, we need to identify elements that are related by multilinearity. We define an equivalence relation. Let $\mathcal{R}$ be the vector space spanned by all elements of the following forms:
  - $(v_1, \ldots, v_i + v_i', \ldots, v_k) - (v_1, \ldots, v_i, \ldots, v_k) - (v_1, \ldots, v_i', \ldots, v_k)$ for all $v_i, v_i' \in V_i$,
  - $(v_1, \ldots, av_i, \ldots, v_k) - a(v_1, \ldots, v_i, \ldots, v_k)$ for all $v_i \in V_i$ and $a \in \mathbb{R}$.

  for all $1 \leq i \leq k$. By quotienting out $\mathcal{R}$, we identify elements that are related by multilinearity.

Definition 12.1.2: Tensor Product of Vector Spaces

Let $V_1, \ldots, V_k$ be vector spaces. The **tensor product** of $V_1, \ldots, V_k$, denoted by $V_1 \otimes \cdots \otimes V_k$, is defined as the quotient vector space

$$V_1 \otimes \cdots \otimes V_k = \mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R}, \tag{12.2}$$

where $\mathcal{R}$ is the subspace defined above.
The natural projection map is $\Pi : \mathcal{F}(V_1 \times \cdots \times V_k) \to V_1 \otimes \cdots \otimes V_k$. Then for $v_i \in V_i$, we define

$$v_1 \otimes \cdots \otimes v_k = \Pi((v_1, \ldots, v_k)). \tag{12.3}$$

called the tensor product of $v_1, \ldots, v_k$.

From the definition we have multilinearity built in:

$$v_1 \otimes \cdots \otimes (v_i + v_i') \otimes \cdots \otimes v_k = v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k + v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_k,$$
$$v_1 \otimes \cdots \otimes (av_i) \otimes \cdots \otimes v_k = a(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k).$$

Proposition: **Characterization Property of Tensor Product Space**

Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces and let $A : V_1 \times \cdots \times V_k \to X$ be a multilinear map to a vector space X. Then there exists a unique linear map $\tilde{A} : V_1 \otimes \cdots \otimes V_k \to X$ such that the following diagram commutes:

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "A"] \arrow[d, "\pi"'] & X \\ V_1 \otimes \cdots \otimes V_k \arrow[ur, "\tilde{A}"'] & \end{tikzcd}$$

Proof. Note first that every A extends uniquely to a linear map $\overline{A} : \mathcal{F}(V_1 \times \cdots \times V_k) \to X$ by defining $\overline{A}(\sum_{i=1}^m a_i(v_{1,i}, \ldots, v_{k,i})) = \sum_{i=1}^m a_i A(v_{1,i}, \ldots, v_{k,i})$. So the subspace $\mathcal{R}$ is contained in the kernel of $\overline{A}$ by the multilinearity of A. Therefore, by the first isomorphism theorem, there exists a unique linear map $\tilde{A} : V_1 \otimes \cdots \otimes V_k \to X$ such that $\tilde{A} \circ \Pi = \overline{A}$. Since $\pi = \Pi \circ i$ where $i : V_1 \times \cdots \times V_k \to \mathcal{F}(V_1 \times \cdots \times V_k)$ is the inclusion map, so we have $\tilde{A} \circ \pi = \tilde{A} \circ \Pi \circ i = \overline{A} \circ i = A$.

Uniqueness follows from requiring $\tilde{A}(v_1 \otimes \cdots \otimes v_k) = A(v_1, \ldots, v_k)$ for all $v_i \in V_i$ and linearity. □

We can find a basis for the tensor product space easily.

Proposition: **Basis of Tensor Product Space**

Let $V_1, \ldots, V_k$ be finite-dimensional vector spaces with dimensions $n_1, \ldots, n_k$ respectively. Let $\{E_1^{(j)}, \ldots, E_{n_j}^{(j)}\}$ be a basis for V_j for each $1 \le j \le k$. Then the set

$$\mathcal{C} = \{E_{i_1}^{(1)} \otimes E_{i_2}^{(2)} \otimes \cdots \otimes E_{i_k}^{(k)} \mid 1 \le i_j \le n_j, 1 \le j \le k\}$$

is a basis for $V_1 \otimes \cdots \otimes V_k$. So the dimension of $V_1 \otimes \cdots \otimes V_k$ is $n_1 n_2 \cdots n_k$.

Proposition: **Associativity of Tensor Product**

Let V_1, V_2, V_3 be finite-dimensional vector spaces. Then there is a natural isomorphism

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \cong V_1 \otimes V_2 \otimes V_3.$$

Proposition: **Concrete Tensor Products**

If $V_1, \ldots, V_k$ are finite-dimensional vector spaces, there is a canonical isomorphism

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R}) \tag{12.4}$$

under which the tensor product $\omega_1 \otimes \cdots \otimes \omega_k$ defined abstractly corresponds to the multilinear

map defined earlier.
Similarly, there is a canonical isomorphism

$$V_1 \otimes \cdots \otimes V_k \cong L(V_1^*, \ldots, V_k^*; \mathbb{R}) \tag{12.5}$$

from the identification $V \cong V^{**}$.

Proof. Define a multilinear map $\Phi : V_1^* \times \cdots \times V_k^* \to L(V_1, \ldots, V_k; \mathbb{R})$ by $\Phi(\omega_1, \ldots, \omega_k)(v_1, \ldots, v_k) = \omega_1(v_1) \cdots \omega_k(v_k)$. By the characterization property of tensor product space, Φ induces a linear map $\tilde{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \to L(V_1, \ldots, V_k; \mathbb{R})$ such that $\tilde{\Phi}(\omega_1 \otimes \cdots \otimes \omega_k) = \Phi(\omega_1, \ldots, \omega_k)$.

Now $\tilde{\Phi}$ is an isomorphism. We can use a basis above to prove it, but itself is canonical. □

12.1.2 Covariant and Contravariant Tensors

Definition 12.1.3: Covariant and Contravariant Tensors

Let V be a finite-dimensional vector space, $k \in \mathbb{N}$.

- A **covariant k-tensor** on V is an element of $(V^*)^{\otimes k} = V^* \otimes \cdots \otimes V^*$ (k times).
- A **contravariant k-tensor** on V is an element of $V^{\otimes k} = V \otimes \cdots \otimes V$ (k times).

We also denote the vector space of all covariant k-tensors on V by $T^k(V^*)$. Similarly, we denote the vector space of all contravariant k-tensors on V by $T^k(V)$.

- A 1-covariant tensor is just a covector, $T^1(V^*) = V^*$.
- A covariant 2-tensor is an element of $V^* \otimes V^*$, which can be thought of as a bilinear form on V.
- The determinant is a covariant n-tensor on $\mathbb{R}^n$.

Remark:
Even if we can identify $V^{\otimes k}$ with $L((V^*)^k; \mathbb{R})$, we usually just think of contravariant tensors as formal objects instead of multilinear maps on covectors.

More generally, we can mix covariant and contravariant tensors.

Definition 12.1.4: Mixed Tensors

Let V be a finite-dimensional vector space, $k, l \in \mathbb{N}$. A **mixed (k, l)-tensor** on V is an element of $V^{\otimes k} \otimes (V^*)^{\otimes l}$. We denote the vector space of all mixed (k, l)-tensors on V by $T^{(k,l)}(V)$.

12.2 Symmetric and Alternating Tensors