Topology

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Contents

Chapter 1

Topological Spaces

The concept of topological space grew out of the study of the real line and Euclidean space and the study of continuous functions on these spaces.

1.1 Topological Spaces

Definition 1.1.1: Topology

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X (that is, $T \subseteq P(X)$) such that the following conditions hold:

- $\emptyset, X \in \mathcal{T}$.
- The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.
- The intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

Example: **Topologies**

- The *discrete topology* on a set X is the topology $\mathcal{T} = P(X)$, where $P(X)$ is the power set of X. In this topology, every subset of X is open.
- The *indiscrete topology*, or trivial topology, on a set X is the topology $\mathcal{T} = \{\emptyset, X\}$. In this topology, only the empty set and the entire set are open.
- Topologies on $\mathbb{R}$.
  1. $\mathcal{T}_1$ consisting of $\mathbb{R}$, $\emptyset$, and all open intervals (a, b).
  2. $\mathcal{T}_2$ consisting of $\mathbb{R}$, $\emptyset$, and all $[-n, n]$ for $n \in \mathbb{Z}_+$.
- Topologies on $\mathbb{N}$.
  1. The initial segment topology: $\mathcal{T}_1$ consisting of $\mathbb{N}$, $\emptyset$ and the set $\{1, \ldots, n\}$ for $n \in \mathbb{N}$.

2. The final segment topology: $\mathcal{T}_2$ consisting of $\mathbb{N}$, $\emptyset$ and the set $\{n, n+1\ldots\}$ for $n \in \mathbb{N}$.

Definition 1.1.2: Finer Topologies

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is said to be *finer* than $\mathcal{T}_1$. Similarly, $\mathcal{T}_1$ is said to be *coarser* than $\mathcal{T}_2$. If $\mathcal{T}_1$ is neither finer nor coarser than $\mathcal{T}_2$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be *incomparable*.

1.2 Basis for a Topology

Sometimes it is rather hard to explicitly describe all the elements of a given topology. Therefore, we turn to the concept of a basis: "using fewer elements that generates the whole space", just like what we do in linear algebra with linear spaces.

Definition 1.2.1: Basis

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ for subsets of X, i.e. $\mathcal{B} \subseteq P(X)$, such that

- $\forall x \in X, \exists B \in \mathcal{B}, x \in B$.
- If $B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2$, then $\exists B \in \mathcal{B}, x \in B \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ is a basis, we say that $\mathcal{T} = \{B = \bigcup_\alpha B_\alpha, B_\alpha \in \mathcal{B}\}$ is the topology generated by $\mathcal{B}$. (Here the union index α can take $\emptyset$, so that $\emptyset \in \mathcal{T}$.)
A different equivalent to describe $\mathcal{T}$ is that $U \in \mathcal{T}$ iff $\forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U$.

Remark:
We use the concept of "infinite union" closure of topology here. The $\mathcal{T}$ above is the smallest topology that contains the basis $\mathcal{B}$. Also, the second axiom of basis stands for the "finite intersection" closure. (The first is given for the whole space to be open.)

Another more general way to see basis is as follows: Let $\mathcal{T}$ be a topology on X, then $\mathcal{B}$ is a basis of $\mathcal{T}$ iff every $U \in \mathcal{T}$ is the union of some $B_\alpha \in \mathcal{B}$.

Example: **Basis for Topologies**

We shall see that different basis can generate the same topology. For instance, in $\mathbb{R}^2$, Let $\mathcal{B}_1$ be all the open circular regions, and $\mathcal{B}_2$ be all open squares.

The following proof shows that the $\mathcal{T}$ defined above is indeed a topology on X.

Proof.

- First $\emptyset \in \mathcal{T}$. Let $U_x \in \mathcal{B}$ contains x. Then $\bigcup_{x\in X} U_x = X \in \mathcal{T}$.
- If $U_1, U_2 \in \mathcal{T}$, if the intersection is not $\emptyset$, $\forall x \in U_1 \cap U_2$, Take $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1, x \in B_2 \subseteq U_2$, then $\exists B_x \in \mathcal{B}, x \in B_x \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$, then $U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B_x \in \mathcal{T}$.

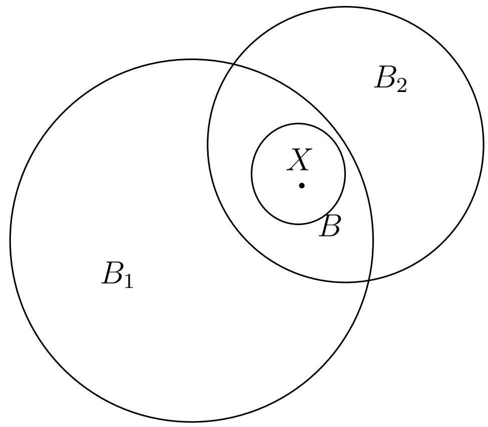

Figure 1.1: Basis of Topology

- The infinite union part is quite easy, just union all the index would do.

□

We can compare topologies via basis.

Lemma 1.2.1: Comparison of Topology by Basis

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the following are equivalent:

- $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e. $\mathcal{T} \subseteq \mathcal{T}'$.
- $\forall x \in X, \forall B \in \mathcal{B}$ such that $x \in B$, $\exists B' \in \mathcal{B}', x \in B' \subseteq B$. (For every $B \in \mathcal{B}, B = \bigcup_\alpha B'_\alpha$ for some $B'_\alpha \in \mathcal{B}'$.)
- $\forall B \in \mathcal{B}, B \in \mathcal{T}'$. (All the basis of $\mathcal{T}$ is open in $\mathcal{T}'$, then all the open sets in $\mathcal{T}$ is also open in $\mathcal{T}'$.)

Example: **Criterion for Comparison**

Using the above criterion we can easily show that the open disks and squares generate the same topology on $\mathbb{R}^2$.

Now we focus on topologies on $\mathbb{R}$.

Definition 1.2.2: Topologies on $\mathbb{R}$

The following are 3 topologies on $\mathbb{R}$:

1. Let $\mathcal{B} = \{(a,b) : a < b\}$ be all the open intervals on $\mathbb{R}$, then the topology generated by $\mathcal{B}$ is the standard (Euclidean) topology on $\mathbb{R}$, denoted by $\mathbb{R}$.
2. The topology generated by $\mathcal{B}' = \{[a,b) : a < b\}$ is called the lower limit topology, denoted $\mathbb{R}_l$.
3. Let $K = \left\{\frac{1}{n} : n \in \mathbb{Z}_+\right\}$, then the topology generated by $\mathcal{B}'' = \mathcal{B} \cup \{(a,b) - K : a < b\}$ is called the K-topology on $\mathbb{R}$, denoted by $\mathbb{R}_K$.

Proposition: **Comparison of $\mathbb{R}$ topologies**

- $\mathbb{R} \subseteq \mathbb{R}_l, \mathbb{R}_K$
- $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Proof.

- For every $(a,b) = \bigcup_{\epsilon>0}[a+\epsilon, b)$. And $\mathcal{B}'' \subseteq \mathcal{B}$.
- First $[2,3) \in \mathbb{R}_l$ but $\notin \mathbb{R}_K$, also $U = (-1,1) - K \in \mathbb{R}_K$ but $\notin \mathbb{R}_l$, because $0 \in U$ but $\forall 0 \in [a,b)$ we have $b > 0$, and $\exists n \in \mathbb{Z}_+, 1/n \in [a,b)$, so there is no $0 \in [a,b) \subseteq U$.

$\square$

To eliminate the need of all intersections, we introduce the concept of subbasis, which is a smaller collection that can also generate the whole topology.

Definition 1.2.3: Subbasis

A subbasis $\mathcal{S}$ for a topology $\mathcal{T}$ on X is also a collection of subsets of X such that $\mathcal{B} = \left\{B = \bigcap_{n=1}^{N} S_n : S_n \in \mathcal{S}, N \in \mathbb{Z}_+\right\}$ is a basis for $\mathcal{T}$.

Remark:
The topology $\mathcal{T}$ is generated by first taking finite intersections of $\mathcal{S}$ and then arbitrary union. The topology induced by $\mathcal{S}$ is also the smallest topology containing $\mathcal{S}$. Therefore, we still have the following criterion.

Theorem 1.2.1: Criterion for Finer Topologies from Subbasis

If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X, and $\mathcal{S}$ is a subbasis of $\mathcal{T}$. If $\mathcal{S} \subseteq \mathcal{T}'$ then $\mathcal{T} \subseteq \mathcal{T}'$.

1.3 The Order Topology

Suppose there is an order relation $<$ on X. The order here is the simple order:

- $\forall x \neq y, x < y \vee y < x$.
- $\forall x \in X, \neg(x < x)$.

- $x < y \wedge y < z \to x < z$.

We define intervals similar to that of $\mathbb{R}$:

- $(a, b) = \{x : a < x < b\}$, $[a, b] = \{x : a \leq x \leq b\}$
- $(a, b]$ and $[a, b)$ are similar.

Definition 1.3.1: Order Topology

Let X be a set with order $<$, and has more than 1 elements. Let $\mathcal{B}$ consists of the following subsets:

- All (a, b) in X.
- All $[a_0, b)$ in X if a_0 is the smallest element (if any) in X, i.e. $\forall x \in X, x \geq a_0$
- All $(a, b_0]$ if b_0 is the largest element (if any) in X.

The topology generated by $\mathcal{B}$ is called the order topology.

It is easy to show that $\mathcal{B}$ is indeed a basis.

Example: **Order topology**

- The standard (Euclidean) topology on $\mathbb{R}$
- **Dictionary order:** On $\mathbb{R}^2$, we say $(a, b)_p < (c, d)_p$ iff $(a < c) \vee (a = c \wedge b < d)$.
- The discrete topology on $\mathbb{Z}_+$.

If X is an ordered set, and a is an element. The following four sets are called rays:

- $(a, \infty) = \{x : x > a\}$
- $(-\infty, a) = \{x : x < a\}$
- $[a, \infty) = \{x : x \geq a\}$
- $(-\infty, a] = \{x : x \leq a\}$

The first two are open rays, while the last two are closed rays. (The openness here is the openness in order topology).

Remark:
The open rays forms a subbasis of the open topology, actually.

1.4 The Product Topology

If X, Y are topological spaces, there is a standard way to define a topology on $X \times Y$.

Definition 1.4.1: Product Topology

If X, Y are topological spaces, with topology $\mathcal{T}_X, \mathcal{T}_Y$. The product topology on $X \times Y$ is generated by a basis $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$.

We should check that $\mathcal{B}$ is indeed a basis.

Proof.

- For $X \times Y \in \mathcal{B}$, all elements are contained in some basis.
- If $(x, y)_p \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, the latter is a basis element.

$\square$

Remark:
We cannot directly define the product topology to be $\mathcal{T} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ because this only contains rectangles, but we want circles to be open as well.

The next result shows that we can directly construct a basis for the product topology by a basis of X and Y, without knowing all open sets of $\mathcal{T}_X, \mathcal{T}_Y$.

Theorem 1.4.1: Basis for the Product Topology

If $\mathcal{B}_X, \mathcal{B}_Y$ are basis for X and Y. Then the collection

$$\mathcal{B} = \{B_x \times B_y, B_x \in \mathcal{B}_X, B_y \in B_Y\}$$

is a basis for the product topology of $X \times Y$.

Remark:
We have a standard topology (The order topology or Euclidean topology) on $\mathbb{R}$, so we have the standard topology on $\mathbb{R}^2$ simply by the product topology, with the basis $(a, b) \times (c, d)$, i.e., all open rectangles.

To describe a product topology with subbasis, we need projection function:

Definition 1.4.2: Projection

The function $\pi_1 : X \times Y \to X$, $\pi_1(x, y)_p = x$ and $\pi_2 : X \times Y \to Y$, $\pi_2(x, y)_p = y$ are projections from $X \times Y$ onto X or Y.

If $U \in \mathcal{T}_X$, then $\pi_1^{-1}(U) = U \times Y$, which is open in the product topology on $X \times Y$. Similarly, $\pi_2^{-1}(V) = X \times V$ is also open. The intersection is the open rectangle $U \times V$.

Theorem 1.4.2: Subbasis by Projection

$$\mathcal{S} = \left\{\pi_1^{-1}(U) : U \in \mathcal{T}_X\right\} \cup \left\{\pi_2^{-1}(V) : V \in \mathcal{T}_Y\right\}$$

is a subbasis for the product topology on $X \times Y$.

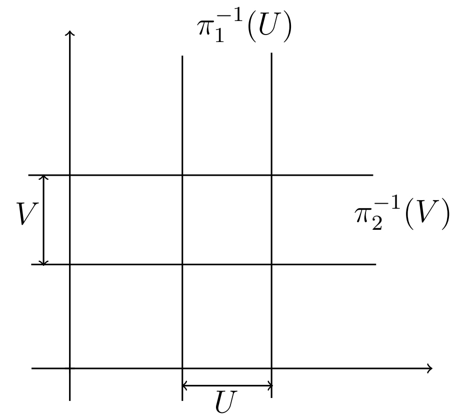

Figure 1.2: Subbasis by Projection

1.5 Subspace Topology

Definition 1.5.1: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space, $Y \subseteq X$ be any subset. The collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y

It is easy to show that $\mathcal{T}_Y$ is a topology, we can use the same method to construct its basis:

Lemma 1.5.1: Basis for Subspace Topology

If $\mathcal{B}$ is a basis for the topology of X then the collection

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for the subspace topology $\mathcal{T}_Y$.

Remark:
Openness of a set does not preserve in $\mathcal{T}$ and $\mathcal{T}_Y$, BE CAREFUL! However, if Y is open in X, them every open set in Y is also open in X.

Lemma 1.5.2: Preserve of Openness in Open Subspaces

If Y is open in X, them every open set in Y is also open in X.

Theorem 1.5.1: Exchange of Subspace and Product Topology

If $A \subseteq X, B \subseteq Y$, then the following is equal:

- The subspace topology of $A \times B$ of the product topology of $X \times Y$.
- The product topology of subspace topology of $A \subseteq X$ and $B \subseteq Y$.

Proof. If $\mathcal{B}_X = \{B_x\}$ and $\mathcal{B}_Y = \{B_y\}$ are basis of X and Y, then $\{(B_x \cap A) \times (B_y \cap B)\} = \{(B_x \times B_y) \cap (A \times B)\}$ are basis for $A \times B$ in both situation. ☐

It is shown that we can exchange the order of subspace topology and product topology, but the following example shows that order topology and subspace topology may not be exchangeable.

Example: **Subspace Topology and Order Topology**

Let $(X, \mathcal{T})$ be a topological space with order topology, and $Y \subseteq X$. The order topology on Y MAY NOT BE the subspace topology.

- Consider $X = \mathbb{R}$ and $Y = [0, 1]$. In this case, the subspace topology and order topology are the same.
- Let $X = \mathbb{R}$ and $Y = [0, 1) \cup \{2\}$. The set $\{2\}$ is open in the subspace topology but not in the oeder topology, which requires the form

$$\{x \in Y : a \in Y, a < x \leq 2\}$$

  (It is also clear that $Y \cong [0, 1]$ in the order sense.)
- Let $X = \mathbb{R}^2$ with the dictionary order, and $Y = [0, 1] \times [0, 1]$. Then $\{1/2\} \cup [0, 1]$ is open in the subspace topology but not in the order topology

The dictionary order in $I^2 = [0, 1] \times [0, 1]$ is called the ordered square, denoted I_o^2.

Remark:
Taking a closer look, the problem occurs when there are "break points" in Y, which introduce us to the concept of convexity.

Definition 1.5.2: Convexity

Given an ordered set X, we say $Y \subseteq X$ is convex iff

$$\forall a, b \in Y, a < b, \{x \in X, a < x < b\} \subseteq Y$$

Be careful that we must use (a, b) in X to verify convexity.

Theorem 1.5.2: Exchange of Ordered Topology and Subspace Topology

Of X is an ordered set and $Y \subseteq X$ is convex in X. Then the order topology on Y is the same as the subspace topology.

Proof.
- Consider in X the ray $(a, +\infty)$, and $U = (a, +\infty) \cap Y$.

  If $a \in Y$, then $U = \{x \in Y : x > a \in Y\}$ is just an open ray in Y. If $a \notin Y$, for Y is convex, a is either a lower bound or an upper bound, $U = Y$ or $U = \emptyset$, both open.

  Then $(-\infty, b) \cap Y$ is also open in order topology. As $(a, +\infty) \cap Y$ and $(-\infty, b) \cap Y$ forms a subbasis of subspace topology, and are open in order topology, we have subspace topology $\subseteq$ order topology.

- For the convexity of Y, the open ray $\{x > a : x \in Y\} = \{x > a : x \in X\} \cap Y$. So any open ray in the order topology is still open in the subspace topology.

□

!!NOTE: To avoid ambiguity, if $Y \subseteq X$, the topology on Y is always the subspace topology unless explicitly stated.

1.6 Closed Sets and Limit Points

1.6.1 Closed Sets

Definition 1.6.1: Closed Sets

A subset $A \subseteq X$ is closed iff $X - A$ is open.

Closed sets have dual status to open sets, which means that we can also fully describe a topological space by closed sets.

Theorem 1.6.1: Basic Properties of Closed Sets

Let X be a topological space.

- $\emptyset$ and X are closed.
- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Just like open sets, the closed sets in the subspace topology is given by intersection of a closed set with the subset.

Theorem 1.6.2: Closed sets in the Subspace Topology

Let Y be a subspace of X, Then $A \subseteq Y$ is closed iff $A = B \cap Y$ for some closed set $B \subseteq X$.

Proof. A is closed iff $Y - A$ is open in Y iff $Y - A = (X - B) \cap Y$ for some $B \subseteq X$ that $X - B$ is open in X. □

The following is analogous to the open set case.

Theorem 1.6.3: Criterion for Closed Sets

Let $Y \subseteq X$ is closed. If A is closed in Y, then A is closed in X.

1.6.2 Closure and Interior

Definition 1.6.2: Closure and Interior

Given a subset $A \subseteq X$.

- The **interior** of A is the union of all open sets contained in A, denoted $\operatorname{Int} A$.

$$\operatorname{Int} A = \bigcup \{U : U \in \mathcal{T}, U \subseteq A\}$$

- The **closure** of A is the intersection of all closed sets containing A, denoted $\overline{A}$.

$$\overline{A} = \bigcap \{U : X - U \in \mathcal{T}, A \subseteq U\}$$

A first observation shows that $\operatorname{Int} A$ is open and $\overline{A}$ is closed. Furthermore,

$$\operatorname{Int} A \subseteq A \subseteq \overline{A}$$

Remark:
The interior is the largest open set contained in A, and the closure is the smallest closed set containing A.
If A is open then $A = \operatorname{Int} A$, also if A is closed, $A = \overline{A}$.

Theorem 1.6.4: Closure in Subspace

Let Y be a subspace of X, let $A \subseteq Y$, $\overline{A}$ is the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.

Proof. Let B be the closure of A in Y. Then because $\overline{A}$ is closed in X, $\overline{A} \cap Y$ is closed in Y, then for $A \subseteq Y \subseteq \overline{A} \cap Y$, we have $B \subseteq \overline{A} \cap Y$.

The other side, for B is closed in Y, then $B = C \cap Y$ for some closed C in X. For $A \subseteq C$, then $\overline{A} \subseteq C$, then $\overline{A} \cap Y \subseteq C \cap Y = B$. $\square$

Notation 1.6.1: Intersect

We say that A intersects B if $A \cap B \neq \emptyset$.

The definition gives us no way to actually find the closure of a set without finding all $A \subseteq U$. A basis is helpful here.

Theorem 1.6.5: Criterion for Closure

Let A be a subspace of X.

- $x \in \overline{A}$ iff $\forall U \in \mathcal{T}(x \in U \to U \cap A \neq \emptyset)$.
- Suppose the topology has a basis $\mathcal{B}$. Then $x \in \overline{A}$ iff $\forall B \in \mathcal{B}(x \in B \to B \cap A \neq \emptyset)$.

Proof.

- $\forall A \subseteq V$ and V is closed, then $X - V$ is open, but $X - V \cap A = \emptyset$, so $x \notin X - V$, so $x \in V$. Therefore, $x \in \overline{A}$.

  The other way, if $x \in \overline{A}$, then if $\exists U \in \mathcal{T}, x \in U, U \cap A = \emptyset$, then $X - U$ is closed and $A \subseteq X - U$, contradicts.

- For $U \in \mathcal{T}, U = \bigcup_\alpha B_\alpha$ for some $B_\alpha \in \mathcal{B}$ would do.

□

Definition 1.6.3: Neighborhood

We say U is a neighborhood of x when $U \in \mathcal{T}, x \in U$.
(Some would define neighborhood as: U is a neighborhood of x when $\exists V \in \mathcal{T}, x \in V \subseteq U$, which is looser)

Using the criterion, we can restate the previous criterion as:

If $A \subseteq X$, then $x \in \overline{A}$ iff every neighborhood of x intersects A.

1.6.3 Limit Points

Limit point are another way to describe the closure of a set.

Intuitively, a limit point is what we can reach by the limit process. In $\mathbb{R}$ the subset $(0, 1]$ has the limit point set $[0, 1]$. The 0 we can reach by taking all $1/n$ as $n \to \infty$.

Definition 1.6.4: Limit Points

If $A \subseteq X$, $x \in X$, then x is a limit point of A iff every neighborhood of x intersects A in some point other than x itself. Formally, if

$$\forall U \in \mathcal{T}(x \in U \to \exists y \in A \cap U \land y \neq x)$$

then x is a limit point of A.

Remark:
We cay already see that every limit points are in the closure. But NOT vice versa, "Single" points that are "out the group" may be in A but some open set containing it only intersets A with it. For instance, in $\mathbb{R}$ the subset $[0, 1] \cup \{2\}$, the point 2 is not a limit point but still in the closure.

Theorem 1.6.6: Closure by Limit Points

Let $A \subseteq X$, let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A' \tag{1.1}$$

This is quite natural actually, just adding all "single points" would do.

Proof.

- If $x \in \overline{A}$: when $x \in A$, it is trivial, when $x \notin A$, $\forall U \in \mathcal{T}(x \in U \to U \cap A \notin \emptyset)$, then $\exists y \neq x, y \in U \cap A$, so $x \in A'$.
- $A, A' \subseteq \overline{A}$ would do.

□

Corollary 1.6.1: Closed Sets and Limit Points

A subset $A \subseteq X$ is closed iff it contains all its limit points.

1.6.4 Hausdorff Spaces

In more abstract spaces, our intuition of $\mathbb{R}^n$ of openness and closeness would fail. Consider the topology $\{\emptyset, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ on the 3-point space $\{a,b,c\}$. Then the one-point set $\{b\}$ is not closed.

Definition 1.6.5: Convergence

If $(X, \mathcal{T})$ is a topological space. We say a sequence $\{x_1, x_2, \ldots\}$ converge to x iff $\forall U \in \mathcal{T}, x \in U, \exists N \in \mathbb{Z}_+(\forall n > N, x_n \in U)$.

In the $\mathbb{R}^n$ case, this is exactly a restatement of the ϵ-N language without the need of a metric, but lead to the same result.

However, in arbitrary topological spaces, the same sequence may converge to multiple points, for example, the sequence $\{x_n\} : x_n = b$ converge to a, b, c.

These examples are pretty strange. Then main problem is that we cannot explicit separate two points by open sets. Then open sets of the two points are all jagged together, making the limit process by open sets not sufficient to tell then apart.

Definition 1.6.6: Hausdorff Space

A topological space X is called a Hausdorff space if $\forall x_1, x_2 \in X, x_1 \neq x_2$, there exists neighborhood U_1, U_2 of x_1, x_2 such that $U_1 \cap U_2 = \emptyset$. Formally,

$$\forall x_1, x_2 \in X, x_1 \neq x_2, \exists U_1, U_2 \in \mathcal{T}, x_1 \in U_1 \wedge x_2 \in U_2 \wedge U_1 \cap U_2 = \emptyset \tag{1.2}$$

Theorem 1.6.7: Finite Closure in Hausdorff Space

Every finite subset in a Hausdorff space is closed.

Proof. Let $x_0 \in X$, $\forall x \in X, x \neq x_0, \exists x \in U_x \in \mathcal{T}, x_0 \notin U_x$, then $X - \{x_0\} = \bigcup_{x \neq x_0} U_x \in T$, so $\{x_0\}$ is closed. So all finite subset is closed. □

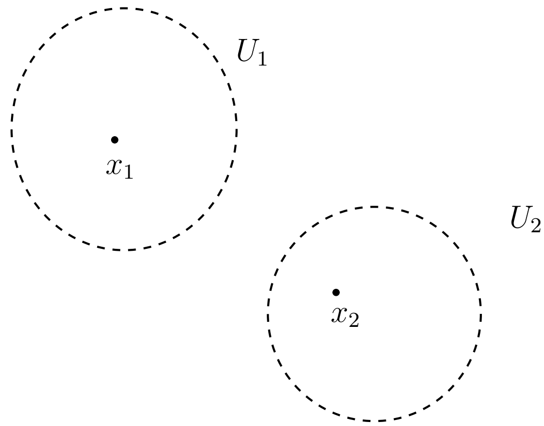

Figure 1.3: Hausdorff Space

Remark:
Note that we do not use the two disjoint neighborhoods simultaneously. This implies that *the condition "Every Single subset is Closed" is weaker than the Hausdorff space condition.*

We call the condition "Every Single subset is Closed" the T_1-axiom. For instance, $\mathbb{R}$ with the finite complement topology is not a Hausdorff space but is a T_1-space. (very easy to prove)

Definition 1.6.7: T_1-axiom

If $(X, \mathcal{T})$ is a topological space, and $\forall x \in X, \{x\}$ is closed, then X satisfies the T_1-axiom.

Theorem 1.6.8: Limit Points of T_1-spaces

Let X satisfies T_1-axiom, and $A \subseteq X$, then $x \in X$ is a limit point of A iff every neighborhood of x contains infinitely many points of A.

Proof.

- $\rightarrow$ part: If some neighborhood U of X intersects A with only finite points, then $U \cap (A - \{x\}) = \{x_1, \ldots, x_n\}$ (n could be 0). Then $U \cap (X - \{x_1, \ldots, x_n\})$ is an open subset that intersects A with at most x not at all.

- $\leftarrow$ part: Obvious.

□

However, T_1-axioms are too loose to have the following properties.

Theorem 1.6.9: Uniqueness of Limit Points of Hausdorff Spaces

If X is a Hausdorff space, then a sequence in X converge to at most onr point in X. Denoted $\lim_{n\to\infty} x_n = x$.

Proof. If $\{x_n\}$ converge to both x, y with $x \neq y$. Then let $x \in U_1 \in \mathcal{T}, y \in U_2 \in \mathcal{T}, U_1 \cap U_2 = \emptyset$. Then $X - U_1$ contains finite points in x_n, contradicts. □

Theorem 1.6.10: Properties of Hausdorff Spaces

- Every topological space with order topology is a Hausdorff space. (simple order)
- The product of two Hausdorff spaces is a Hausdorff space.
- A subspace of a Hausdorff space is a Hausdorff space.

Proof.

- For $x < y$, if $\exists z : x < z < y$, take $U_1 = (-\infty, z), U_2 = (z, +\infty)$ would do. Otherwise, $\forall z \in X, z \leq x \vee z \geq y$, taking $U_1 = (-\infty, y), U_2 = (x, +\infty)$ would do.
- For $(x_1, y_1), (x_2, y_2) \in X \times Y$, we have $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$, also $y_1 \in V_1, y_2 \in V_2, V_1 \cap V_2 = \emptyset$, then we have $(x_1, y_1) \in U_1 \times V_1, (x_2, y_2) \in U_2 \times V_2$.
- If $S \subseteq X$, then for $x \neq y$ in S, we take $U_1 \cap S$ and $U_2 \cap S$ would do.

□

1.7 Continuous Functions

In common sense, a continuous function preserve locality between range and domain, so it is quite natural to define continuity by the following way.

Definition 1.7.1: Continuity of a Function

Let X, Y be topological spaces. A function $f : X \to Y$ is continuous if for each open $V \subseteq Y$, we have $f^{-1}(V) \subseteq X$ is open.
(Note that the continuity of a function depend on the topology defined on X and Y.)

Remark:
If Y is given a basis $\mathcal{B}$, then if $f^{-1}(B)$ is open for $\forall B \in \mathcal{B}$, then f is continuous.
Also, given a subbasis $\mathcal{S}$, then $f^{-1}(S)$ is open for $\forall S \in \mathcal{S}$ implies f being continuous.
The above to statements is given by:

$$f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$$

$$f^{-1}(\bigcap_{i=1}^n S_i) = \bigcap_{i=1}^n f^{-1}(S_i)$$

Example: **Continuity of $\mathbb{R} \to \mathbb{R}$**

In analysis the continuity of a function $f : \mathbb{R} \to \mathbb{R}$ is given by ϵ-δ language: if $\forall \epsilon > 0 \exists \delta > 0 \forall x(|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon)$, then f is continuous at x_0.
This definition is compatible with the topological definition above in the standard topology.

Proof. If f follows the ϵ-δ continuity, let $V = (a, b)$ be an element of the basis of $\mathbb{R}$, and $U = f^{-1}(V)$, then $\forall x \in U, f(x) \in (a, b)$, then let $\epsilon > 0$ be such that $(f(x)-\epsilon, f(x)+\epsilon) \subseteq (a, b)$, then $\exists \delta > 0$ such that $U_x = (x - \delta, x + \delta), f(U_x) \subseteq V$, then $U_x \subseteq f^{-1}(V)$. As $U = \bigcup_{x \in U} U_x$, we have U being open.
For the other way, if f follows the topological continuity, given $x_0 \in \mathbb{R}$, the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is open, then $f^{-1}(V)$ is open. As $x_0 \in f^{-1}(V)$, then there is basis element $x_0 \in (a, b) \subseteq f^{-1}(V)$, letting δ be smaller would do. □

Example: **Continuity and Topology**

Let $\mathbb{R}$ denote the standard topology, and $\mathbb{R}_l$ the lower limit topology.

- Let $f : \mathbb{R} \to \mathbb{R}_l$ be the identity function, then f is not continuous, for $f^{-1}([a, b)) = [a, b)$ which is not open in $\mathbb{R}$.
- However, $f^{-1} : \mathbb{R}_l \to \mathbb{R}$ is continuous.

Theorem 1.7.1: Equivalent Definitions for Continuity

Let X, Y to topological spaces, and $f : X \to Y$. Then the following are equivalent.

1. f is continuous. $\forall A \subseteq Y, A \in \mathcal{T}_Y, f^{-1}(A) \in \mathcal{T}_X$.
2. $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$.
3. $\forall B \subseteq Y$ which is closed, $f^{-1}(B)$ is closed.
4. $\forall x \in X$ and $\forall$ neighborhood V of $f(x)$, $\exists$ a neighborhood U of x such that $f(U) \subseteq V$.

Proof.

- $1 \to 2 :$ We shall show that $x \in \overline{A} \to f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x, so $\exists y \in f^{-1}(V) \cap A$. So $f(y) \in f(A) \cap V$, so $f(x) \in \overline{f(A)}$.
- $2 \to 3 :$ Let $A = f^{-1}(B)$, then $f(A) \subseteq B$, so $f(\overline{A}) \subseteq \overline{f(A)} \subseteq B$, so $\overline{A} \subseteq f^{-1}(B) = A$, and $A \subseteq \overline{A}$ implies $A = \overline{A}$, meaning A is closed.
- $3 \to 1 :$ $f^{-1}(B)$ and $f^{-1}(Y - B)$ part X would do.
- $1 \to 4 :$ Letting $U = f^{-1}(V)$ would do.
- $4 \to 1$: Similar of example 1.7.

□

All the above illustrate our intuition of continuity: preserving locality.

1.7.1 Homeomorphisms

Homeomorphisms in topology is just isomorphisms between topological spaces. We need to preserve openness. As continuity show preservation of openness in one direction, we can simplify sayings by it.

Definition 1.7.2: Homeomorphism

Let X, Y be topological spaces, $f : X \to Y$ be a bijection. If both f and f^{-1} are continuous, then f is a homeomorphism.

Homeomorphisms preserve all topological properties of the two spaces. If there is a homeomorphism between X and Y, then we can treat then the same in topology.

Sometimes X is homeomorphic to a subspace of Y, which is called an imbedding.

Definition 1.7.3: Topological Imbedding

Suppose $f : X \to Y$ is an injective continuous map. Let $Z = f(X)$, then $f' : X \to Z$ is a bijection. If f' is a homeomorphism of X and Z (Z with the subspace topology), then we say f is a topological imbedding of X in Y.

Example: **Continuity and Homeomorphisms**

Let $S^1 \subseteq \mathbb{R}^2$ denote the unit circle. Let

$$F : [0, 1) \to S^1, F(t) = (\cos 2\pi t, \sin 2\pi t)$$

Then f is indeed continuous and bijective, but not an isomorphism, for $f([0, \frac{1}{2}))$ is not open in S^1.
The function $g : [0, 1) \to \mathbb{R}^2, g(x) = f(x)$ is continuous but not an imbedding.

1.7.2 Constructing Continuous Functions

Sometimes it is rather hard to directly verify the continuity of a function. We use construction rules in analysis: compound functions and operations etc. Some can be generalized here.

Theorem 1.7.2: Construction Rules for Continuous Functions

Let X, Y, Z be topological spaces. All topologies in subspace are subspace topologies.

1. (Constant Function) If $f : X \to Y, x \mapsto y_0$, a constant point in Y, then f is continuous.
2. (Inclusion) If $A \subseteq X$, the inclusion function $j : A \to X, x \mapsto x$ is continuous.
3. (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.
4. (Restricting the Domain) If $f : X \to Y$ is continuous, then for $\forall A \subseteq X$, $f|_A : A \to Y$ is continuous.
5. (Expanding/Restricting the Range) If $f : X \to Y$ be continuous, and $f(X) \subseteq Z \subseteq Y$ or $f(X) \subseteq Y \subseteq Z$, then $f : X \to Z$ is continuous.
6. (Local formulation of Continuity): $f : X \to Y$ is continuous if $X = \bigcup_\alpha U_\alpha$ such that U_α is open and $f|_{U_\alpha}$ is continuous.

Most of the above is obvious, just using the definition would do.

Theorem 1.7.3: The Pasting Lemma

Let $X = A \cup B$, where A, B are both closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous, and $\forall x \in A \cap B, f(x) = g(x)$. And let $h : X \to Y$, where $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$. Then h is continuous.
This theorem also holds for A, B being open, which is just a special case for the local formulation of continuity.

Proof. Let C be a closed subset of Y, we have

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Since f, g are continuous, so $f^{-1}(C)$ is closed in A, and A is closed in X, so $f^{-1}(C)$ is closed in X, so is $g^{-1}(C)$. □

Remark:
Both closeness is quite necessary for the theorem to hold. For instance,

$$h(x) = \begin{cases} x - 2, & \text{if } x \leq 0 \\ x + 2, & \text{if } x > 0 \end{cases}$$

is not continuous in $\mathbb{R}$, but is continuous in $\mathbb{R}_{\leq 0}$ and $\mathbb{R}_{>0}$ respectively.

Theorem 1.7.4: Maps into Products

Let $f : A \to X \times Y$ be given by

$$f(a) = (f_1(a), f_2(a))$$

Then f is continuous iff f_1, f_2 are both continuous.
f_1, f_2 are called coordinate functions of f.

Proof. Taking the subbasis $\mathcal{S} = \{U_1 \times Y : U_1 \in \mathcal{T}_X\} \cup \{X \times U_2 : U_2 \in \mathcal{T}_Y\}$. If f_1, f_2 are continuous, then $f^{-1}(U_1 \times Y) = f_1^{-1}(U_1) \times A$ which is open, so does $f^{-1}(Y \times U_2)$.

If f is continuous, we also have the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous. So the composite $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$ are continuous. □

This result is just the continuity of parametrized curve (or vector field) in calculus, with $a \in A$ as the parameter.

1.8 The Product Topology

Previous discussion shows how we can impose a topology on finite Cartesian products $X_1 \times \cdots \times X_n$, which is just a generalization for the two-space case $X \times Y$. Here we take a closer look on the infinite number of Cartesian products.

Consider the products

$$X_1 \times \cdots \times X_n \text{ and } X_1 \times X_2 \times \cdots$$

We give a more formal definition of Cartesian products for infinite products:

Definition 1.8.1: Tuples

Let J be an index set. Given a set X, define a J-tuple of elements of X to be a function $x : J \to X$, we also denote $x(\alpha)$ by x_α, called the α-th coordinate of x. We also denote the function x by $(x_\alpha)_{\alpha \in J}$.
X^J denotes all J-tuples of X.

Definition 1.8.2: Cartesian Products

Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets. Let $X = \bigcup_{\alpha \in J} A_\alpha$. The Cartesian product of $\{A_\alpha\}_{\alpha \in J}$ is denoted by

$$\prod_{\alpha \in J} A_\alpha = \{(x_\alpha)_{\alpha \in J} : \forall \alpha \in J, x_\alpha \in A_\alpha\}$$

Which is all the J-tuples that $x_\alpha \in A_\alpha$, being a subset of X^J.

To generate a basis, we have two ways, generalized by the two previous ways to describe the topology on $X \times Y$.

- *The Box Topology*: Taking all $\prod_\alpha U_\alpha$ where $U_\alpha \in \mathcal{T}_\alpha$ as a basis.
- *The Product Topology*: Taking all $\pi_\alpha^{-1}(U_\alpha)$ as a subbasis.

Previous discussion has shown that these two ways result in the same topology in finite case.

Definition 1.8.3: The Box Topology

Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed set family of topological spaces. Let

$$X = \prod_{\alpha\in J} X_\alpha$$

be the product space. Define a topology with the basis $\mathcal{B}$ consisting of all the following form

$$\prod_{\alpha\in J} U_\alpha, \text{ where } \forall\alpha\in J, U_\alpha \text{ is open in } X_\alpha$$

the topology is called the box topology of X.

To see that the box topology is indeed a topology, we have $X = \prod X_\alpha$ is itself a basis element, also

$$\left(\prod_{\alpha\in J} U_\alpha\right) \cap \left(\prod_{\alpha\in J} V_\alpha\right) = \prod_{\alpha\in J} (U_\alpha \cap V_\alpha)$$

The intersection itself is another basis element.

To generalize the subbasis formulation, define

$$\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta \tag{1.3}$$

$$\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta$$

is called the projection mapping associated with the index β.

Definition 1.8.4: The Product Topology (Subbasis)

Let $\mathcal{S}_\beta$ denote the following collection:

$$\mathcal{S}_\beta = \left\{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\right\}$$

And let

$$\mathcal{S} = \bigcup_{\beta\in J} \mathcal{S}_\beta$$

The topology generated by $\mathcal{S}$ as a subbasis is called the product topology. In this topology the product $\prod_{\beta\in J} X_\beta$ is called the product space.

To see the basis generated by $\mathcal{S}$, we take finite intersections.

First, intersecting elements inside $\mathcal{S}_\beta$ would not give us anything new, for

$$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta)$$

Intersection of elements among different $\mathcal{S}_{\beta_i}$ is what we want. Let $\beta_1, \ldots, \beta_n$ be distinct element in J and U_{β_i} be open in X_{β_i}, then

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1} \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

is a typical element of the basis $\mathcal{B}$.

Theorem 1.8.1: Comparison of the box and product topologies

- The box topology on $\prod X_\alpha$ has a basis of the form $\prod U_\alpha$ where U_α is open in X_α.
- The product topology on $\prod X_\alpha$ has a basis of the form $\prod U_\alpha$, where $U_\alpha = X_\alpha$ except for finite many values of α, and all $U_\alpha \subseteq X_\alpha$ are open.

For finite Cartesian products the two topologies are the same. In general case the box topology is finer than the product topology.

The product topology plays a more significant role in topology than the box topology, for it is more natural to consider the product of topological spaces. The box topology is too fine to be useful in most cases.

NOTE: Whenever we consider the product $\prod X_\alpha$, we will always assume the product topology unless otherwise specified.

Theorem 1.8.2: Product Topology by Basis

Suppose the topology on each X_α is given by a basis $\mathcal{B}_\alpha$. Then the box topology on $\prod X_\alpha$ has a basis consisting of all sets of the form

$$\prod_{\alpha \in J} B_\alpha, \text{ where } B_\alpha \in \mathcal{B}_\alpha \text{ for all } \alpha \in J$$

Similarly, the product topology on $\prod X_\alpha$ has a basis consisting of all sets of the form

$$\prod_{\alpha \in J} B_\alpha, \text{ where } B_\alpha \in \mathcal{B}_\alpha \text{ for finitely many } \alpha \in J, \text{ and } B_\alpha = X_\alpha \text{ for all other } \alpha \in J$$

Proof. By closure under arbitrary unions, it is obvious. □

Theorem 1.8.3: Subspace Topology of Products

$\forall \alpha \in J$, let A_α be a subspace of X_α. Then the topology $\prod_\alpha A_\alpha$ is the subspace topology on $\prod_\alpha X_\alpha$, either by the box topology or the product topology.

Proof. It is just a generalization of theorem 1.5.1. Using the fact that

$$\prod_{\alpha \in J} (B_\alpha \cap A_\alpha) = \left(\prod_{\alpha \in J} B_\alpha \right) \cap \left(\prod_{\alpha \in J} A_\alpha \right)$$

□

Theorem 1.8.4: Products of Hausdorff Spaces

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is a Hausdorff space, in both the box topology and the product topology.

Theorem 1.8.5: Products of Closure

Let $\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces, and $A_\alpha \subseteq X_\alpha$ for each $\alpha \in J$. Then

$$\overline{\prod_{\alpha\in J} A_\alpha} = \prod_{\alpha\in J} \overline{A_\alpha}$$

in both the box topology and the product topology.

Proof. Let $x = (x_\alpha) \in \prod \overline{A_\alpha}$, we show that $x \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element of $\prod X_\alpha$ that $x \in U$. Then as $x_\alpha \in \overline{A_\alpha}$, then $\exists y_\alpha \in U_\alpha \cap A_\alpha$, so $y \in \prod U_\alpha \cap A_\alpha = \prod U_\alpha \cap \prod A_\alpha$. So $U \cap \prod A_\alpha \neq \emptyset$, so $x \in \overline{\prod A_\alpha}$.

Conversely, suppose $x \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element of $\prod X_\alpha$ that $x \in U$. Then $\exists y \in U \cap \prod A_\alpha$. So $y_\alpha \in U_\alpha \cap A_\alpha$, so $x_\alpha \in \overline{A_\alpha}$ for all $\alpha \in J$. So $x \in \prod \overline{A_\alpha}$. □

Up till now, all the results holds the same for the box topology and the product topology. A first difference arises in the study of continuity.

Theorem 1.8.6: Continuity of Products

Let $f : A \to \prod_{\alpha\in J} X_\alpha$ be given by

$$f(a) = (f_\alpha(a))_{\alpha\in J}$$

where $f_\alpha : A \to X_\alpha$ for each $\alpha \in J$. Then f is continuous in the product topology iff f_α is continuous for each $\alpha \in J$.

Proof.

- $\Rightarrow$ part: If f is continuous, then for each $\alpha \in J$, we have $\pi_\alpha \circ f = f_\alpha$ is continuous
- $\Leftarrow$ part: If f_α is continuous for each $\alpha \in J$, then for each open set $U_\alpha \subseteq X_\alpha$, we have $\pi_\alpha^{-1}(U_\alpha)$ is open in the product topology, and $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha)$ is open in A, so f is continuous (all the preimages of a subbasis is open).

□

Note that the $\Leftarrow$ part uses the finite intersection of the subbasis elements $\pi_\alpha^{-1}(U_\alpha)$ to generate the basis elements of the product topology. While the box topology allows infinite intersection, which would bring problems

Example: **Conterexamples of Continuity of Box Topology**

Consider $\mathbb{R}^{\mathbb{Z}_+}$. Let $f : \mathbb{R} \to \mathbb{R}^{\mathbb{Z}_+}$ to be

$$f(t) = (t, t, \ldots)$$

Then f is continuous in the product topology. In box topology, however, let

$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$$

We have $f^{-1}(B) = \{0\}$ which is not open in $\mathbb{R}$.

1.9 The Metric Topology

Most of our intuition of open/closed sets and neighborhoods and locality and continuous functions are based on the metric space, mostly $\mathbb{R}^n$. Indeed, if given a metric, we can naturally define a topology based on it.

Definition 1.9.1: Metric

A metric on a set X is a function $d : X \times X \to \mathbb{R}$:

- $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is called the distance between x and y. Also, if given $\epsilon > 0$, the set

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$$

is called the ϵ-ball centered at x.

Definition 1.9.2: Metric Topology

If d is a metric on X, then the collection of all ϵ-balls forms a basis

$$\mathcal{B} = \{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$$

The topology generated by $\mathcal{B}$ is called the metric topology on X induced by d.

The following proposition is widely used in analysis.

Proposition: **Openness of a Ball**

If $y \in B(x, \epsilon)$, then $\exists \delta > 0, y \in B(y, \delta) \subseteq B(x, \epsilon)$.

Proof. For we have $d(x, y) < \epsilon$, take $\delta = \epsilon - d(x, y) > 0$, then for $z \in B(y, \delta)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + \epsilon - d(x, y) = \epsilon$$

So $z \in B(x, \epsilon)$, so $B(y, \delta) \subseteq B(x, \epsilon)$. $\square$

Next we prove that $\mathcal{B}$ is indeed a basis for a topology on X.

Proof.

- The first condition is trivial for $\forall x \in X, x \in B(x, \epsilon)$.
- Let B_1, B_2 be two basis element, if $y \in B_1 \cap B_2$, we can choose $\delta_1, \delta_2 > 0$ such that $y \in B(y, \delta_1) \subseteq B_1$ and $y \in B(y, \delta_2) \subseteq B_2$. Let $\delta = \min\{\delta_1, \delta_2\}$, then $B(y, \delta) \subseteq B_1 \cap B_2$, so $B(y, \delta)$ is a basis element containing y.

$\square$

The following result returns us to the original definition of openness in analysis:

Proposition: **Openness of Sets in Metric Topology**

A set U is open in the metric space (X, d) iff

$$\forall x \in U, \exists \delta > 0, B_d(x, \delta) \subseteq U \tag{1.4}$$

Proof. Clearly the condition implies U being open. Conversely, if U is open, $\forall x \in U$, there is $x \in B(y, \epsilon) \subseteq U$, then we can take δ that $B_d(x, \delta) \subseteq B(y, \epsilon) \subseteq U$. □

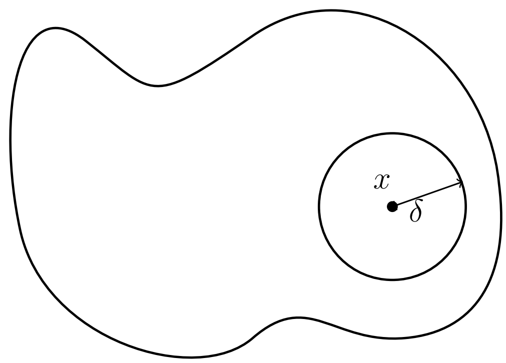

Figure 1.4: Openness in Metric Spaces

Proposition: **Continuity of Metric**

Let (X, d) be a metric space. Then the metric $d : X \times X \to \mathbb{R}$ is continuous, where $X \times X$ has the product topology and $\mathbb{R}$ has the standard topology.
Also, $x \mapsto d(x, y)$ is continuous for fixed $y \in X$.

Proof. Let $(x_0, y_0) \in X \times X$, and $\epsilon > 0$. Let $\delta = \frac{\epsilon}{2}$. Then for $(x, y) \in B(x_0, \delta) \times B(y_0, \delta)$, we have

$$|d(x, y) - d(x_0, y_0)| \leq |d(x, y) - d(x_0, y)| + |d(x_0, y) - d(x_0, y_0)| \leq d(x, x_0) + d(y, y_0) < 2\delta = \epsilon$$

So d is continuous.

The second part follows from the Restricting of Domain rule of continuous functions. □

Example: **Metric Topology**

- The standard metric on $\mathbb{R}$ is $d(x, y) = |x - y|$, the metric topology is the standard topology on $\mathbb{R}$.

- On $\mathbb{R}^n$, we can define the p-metric as

$$d(x - y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \text{ for } p \geq 1$$

  When $p = 2$, it is the Euclidean metric, the balls are spheres.

  When $p = \infty$, it is the maximum metric (or square metric), defined as

$$d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

  the balls are cubes.

Definition 1.9.3: Metrizable

A topological space X is called metrizable if there exists a metric d on X such that the topology induced by d is the same as the topology of X.

Remark:
Metrizable spaces are valuable but rare in general. A metric gives powerful tools to study the topology, such as compactness, connectedness, and convergence, just like what we did in analysis.
It is worth noting that not all topological spaces are metrizable. And it is of fundamental importance to find conditions under which a topological space is metrizable: they are expressed in the Urysohn Metrization Theorem, which will be discussed later.
We shall also note that a metric is not a topological property, and properties of a metric space may not be preserved under homeomorphisms.

Definition 1.9.4: Boundedness

Let (X, d) be a metric space. A subset $A \subseteq X$ is bounded if there exists $M > 0$ such that $\forall x, y \in A, d(x, y) < M$.
If A is bounded and nonempty, we define the diameter of A to be

$$\operatorname{diam} A = \sup \{d(x, y) : x, y \in A\}$$

We define another metric $\tilde{d} : X \times X \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \min \{d(x, y), 1\}$$

then $\tilde{d}$ is called the standard bounded metric corresponding to d. We can verify that $\tilde{d}$ is indeed a metric, and the topology induced by $\tilde{d}$ is the same as the topology induced by d.

The open balls induced by $\tilde{d}$ are comprised of two parts:

- All the balls $B_{\tilde{d}}(x, \epsilon)$ for $\epsilon < 1$ are the same as those induced by d.
- The whole space

Remark:
Different metrics may induce the same topology, such as the Euclidean metric and the square metric on $\mathbb{R}^n$.

Lemma 1.9.1: Comparing Topologies via Metrics

Let d_1, d_2 be two metrics on a set X. Let $\mathcal{T}_1, \mathcal{T}_2$ be the topologies induced by d_1, d_2 respectively. Then $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, ($\mathcal{T}_1 \subseteq \mathcal{T}_2$), iff

$$\forall x \in X, \forall \epsilon > 0, \exists \delta > 0, B_{d_1}(x, \delta) \subseteq B_{d_2}(x, \epsilon)$$

This is just restatement of proving the basis for $\mathcal{T}_2$ is all open in $\mathcal{T}_1$.
Using the lemma, it is quite easy to see that all the p-metric on $\mathbb{R}^n$ gives the same topology.
Now we consider metrics on the infinite Cartesian product $\mathbb{R}^{\mathbb{Z}_+}$, for which we can just generalize the p-metric.

- For finite p, we can define
$$d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}}$$
This does not always make sense, for the infinite sum may not converge. However, if $X \subseteq \mathbb{R}^{\mathbb{Z}_+}$ consists of all sequences x such that $\sum_{i=1}^{\infty} x_i^p$ converges, then the above metric is well-defined and induces a topology on X.
- For $p = \infty$, we can define the metric
$$d(x, y) = \sup_{i \in \mathbb{Z}_+} |x_i - y_i|$$
This also does not always make sense, but if we replace d with $\tilde{d}$ of $\mathbb{R}$, this is well-defined then, called the uniform metric on $\mathbb{R}^{\mathbb{Z}_+}$.

Definition 1.9.5: Uniform Metric

Given an index set J, and $x, y \in \mathbb{R}^J$, define a metric $\tilde{\rho}$ on $\mathbb{R}^J$ by

$$\tilde{\rho}(x, y) = \sup \left\{ \tilde{d}(x_\alpha, y_\alpha) : \alpha \in J \right\} \tag{1.5}$$

where $\tilde{d}$ is the standard bounded metric on $\mathbb{R}$.
Then $\tilde{\rho}$ is called the uniform metric on $\mathbb{R}^J$. And the topology it induces is called the uniform topology on $\mathbb{R}^J$.

The basis of the uniform topology is given by all the rectangles whose edges has length < 1 or the whole $\mathbb{R}$. So it is quite easy to compare it to the product topology and box topology on $\mathbb{R}^J$.

Theorem 1.9.1: Comparison of Uniform Topology

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology.
The three topologies are the same if and only if J is finite.

Proof. Using the fact that $\mathcal{B}_{\text{product}} \subseteq \mathcal{B}_{\text{uniform}} \subseteq \mathcal{B}_{\text{box}}$ would do.

In finite case, it is easy to see that all three topologies are the same. For the infinite case,

- The cube $(0, \frac{1}{2})^J$ is not open in the product topology, but is open in the uniform topology.
- Consider the set
$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$$
For a countable part of J, and the whole $\mathbb{R}$ for the rest of J.

  Then B is open in the box topology, but not in the uniform topology, for $0 \in B$, and $\forall \epsilon > 0, B(0, \epsilon)$ has element outside $(-\frac{1}{n}, \frac{1}{n})$ for sufficiently large n.

□

When J is infinite, we have not determined whether $\mathbb{R}^J$ is metrizable in box or product topology. We shall further see that the only case that it is metrically is when J is countable, and in the product topology.

Theorem 1.9.2: Metrization of $\mathbb{R}^{\mathbb{Z}_+}$

Let $\tilde{d} = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. $\forall x, y \in \mathbb{R}^{\mathbb{Z}_+}$, we define

$$D(x, y) = \sup_{i \in \mathbb{Z}_+} \left\{ \frac{\tilde{d}(x_i, y_i)}{i} \right\}$$

Then D is a metric that induced the product topology on $\mathbb{R}^{\mathbb{Z}_+}$.

We notice that the D given above gives a set of balls with first finite elements are some bounded intervals, and the rest are all $\mathbb{R}$. For sufficiently large i, $\forall x_i, y_i \in \mathbb{R}$, we have $\tilde{d}(x_i, y_i)/i < \epsilon$. This is just what we want for the product topology.

Proof. The properties of a metric are easy to verify: For the triangle inequality, because

$$\forall i \in \mathbb{Z}_+, \frac{\tilde{d}(x_i, z_i)}{i} \leq \frac{\tilde{d}(x_i, y_i)}{i} + \frac{\tilde{d}(y_i, z_i)}{i} \leq D(x, y) + D(y, z)$$

So $D(x, y) \leq D(x, z) + D(z, y)$.

Next we prove that D gives the product topology. Let U be open in the metric topology, and $x \in U$, then $\exists \epsilon > 0$ such that $B_D(x, \epsilon) \subseteq U$. So for $N > 1/\epsilon$, the coordinates of $\mathcal{B}_D(x, \epsilon)$ range over $\mathbb{R}$. So we take

$$V = \prod_{i=1}^{N} B_{\tilde{d}}(x_i, \epsilon) \times \prod_{i > N} \mathbb{R}$$

Then V is open in the product topology, and $x \in V \subseteq B_D(x, \epsilon) \subseteq U$. Therefore, U is open in the product topology.

Conversely, consider a basis element

$$U=\prod_{i=1}^{N} U_i \times \prod_{i>N} \mathbb{R}$$

of the product topology, where U_i is open in $\mathbb{R}$ for $i \leq N$. For $x \in U$, we take ϵ_i that $B_{\tilde{d}}(x, \epsilon_i) \subseteq U_i$. Define

$$\epsilon=\min\left\{\epsilon_i / i: 1 \leq i \leq N\right\}$$

Then $x \in B_D(x, \epsilon) \subseteq U$. □

Now we discuss the relation of metric spaces to other interesting properties.

Proposition: **Properties of Metric Spaces**

- Subspace: If A is a subspace of a metric space (X, d), then the restriction of d on to $A \times A \rightarrow \mathbb{R}$ is a metric of A that induces the subspace topology on A.
- The Hausdorff Axiom: Every metric space is Hausdorff.
- The product topology: Countable products of metric spaces are metrizable in the product topology.

Proof.

- The subspace topology is generated by the basis elements of the form $B_d(x, \epsilon) \cap A$, which is just the open balls in the restricted metric.
- Let $x, y \in X$ with $x \neq y$, then $d(x, y)>0$. Let $\epsilon=\frac{d(x, y)}{2}$, then $B_d(x, \epsilon) \cap B_d(y, \epsilon)=\emptyset$, so X is Hausdorff.
- This proof is similar (more to say, exactly the same) to that of theorem 1.9.2. Let d_i be the metric on each X_i, then we can define a metric on $\prod_{i=1}^{\infty} X_i$ by

$$D(x, y)=\sup_{i \in \mathbb{Z}_{+}}\left\{\frac{\tilde{d}_i(x_i, y_i)}{i}\right\}$$

where $\tilde{d}_i$ is the standard bounded metric on X_i. Then D is a metric that induces the product topology on $\prod_{i=1}^{\infty} X_i$.

□

About continuous functions there is more to say, as in analysis.

First, we shall see the familiar ϵ-δ language can be carried out to describe arbitrary metric spaces, and so does the convergence-of-sequences language.

Next, we shall also construct continuous functions via the limit of a uniformly convergent sequence of continuous functions, which we took much care in analysis.

Theorem 1.9.3: The ϵ-δ Language for Metric Spaces

Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f: X \rightarrow Y$ is continuous iff

$$\forall x \in X, \forall \epsilon>0, \exists \delta>0, \forall y \in X,\left(d_X(x, y)<\delta \rightarrow d_Y(f(x), f(y))<\epsilon\right) \tag{1.6}$$

Proof. Quite similar to the $\mathbb{R} \to \mathbb{R}$ case.

- Suppose f is continuous, Given x and ϵ, consider
$$f^{-1}(B(f(x), \epsilon))$$
is open in X and contains x. Then $\exists \delta > 0, B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$, as desired.

- Conversely, suppose the ϵ-δ condition holds. Let V be open in Y. Let $x \in f^{-1}(V)$, then $\exists \epsilon > 0, B(f(x), \epsilon) \subseteq V$, then $\exists \delta > 0, B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)) \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is open in X, so f is continuous.

□

From our intuition in analysis, if x lies in the closure of $A \subseteq X$, then there is a sequence of points in A converging to x. This is not true in general, but is in metric spaces.

Lemma 1.9.2: The Sequence Lemma

Let X be a topological space. Let $A \subseteq X$. If there is a sequence of points of A converging to $x \in X$, then $x \in \overline{A}$.
The converse holds if X is a metric space.

Proof.
- The first part is just the definition of closure, for if x is a limit point of A, then $\forall U$ open containing x, $U \cap A \neq \emptyset$, so $x \in \overline{A}$.

- Conversely, if X is a metric space, and $x \in \overline{A}$, then take $x_n \in B(x, 1/n) \cap A$ for all $n \in \mathbb{Z}_+$. We assert that $x_n \to x$. For every open U containing x has a $x \in B(x, \epsilon) \subseteq U$, then we take $N > 1/\epsilon$, then for all $n > N$, we have $x_n \in B(x, 1/n) \subseteq B(x, \epsilon) \subseteq U$. So $x_n \to x$.

□

Theorem 1.9.4: Continuity and Convergence

Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.
The converse holds if X is a metric space. (The metrizability of Y is not required.)

Proof.
- If f continuous and $x_n \to x$, then let V be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x. So $\exists N, \forall n > N, x_n \in f^{-1}(V), f(x_n) \in V$.

- To prove the converse, we need to show that $f(\overline{A}) \subseteq \overline{f(A)}$. If $x \in \overline{A}$, then by lemma 1.9.2, there is a sequence $x_n \in A$ converging to x. By the assumptions, we have $f(x_n) \to f(x)$, so $f(x) \in \overline{f(A)}$.

□

Remark:
We notice that we do not need the full strength of metrizability of X in the proof of lemma 1.9.2 and theorem 1.9.4. The only place we use the metric is to show that there is a good-behaved basis $B(x, 1/n)$, so that every neighborhood of x contains one of the basis. This fact leads us to a weaker condition, called first-countability.

Theorem 1.9.5: The First Countability Axiom

A topological space X is first-countable (have a countable basis at every point x) if $\forall x \in X$, there is a countable collection $\{U_n\}_{n\in\mathbb{Z}_+}$ of neighborhoods of x, such that $\forall$ neighborhood U of x, $\exists n \in \mathbb{Z}_+$ such that $U_n \subseteq U$.

Theorem 1.9.6: First Countability and Continuity

If X, Y are topological spaces, and X is first countable. Let $f : X \to Y$. Then f is continuous iff for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.

Proof. Similar to that of lemma 1.9.2 and theorem 1.9.4. Just replace the basis $B(x, 1/n)$ with the countable collection $\{U_n\}_{n\in\mathbb{Z}_+}$ of neighborhoods of x. ☐

Next, we consider additional ways to construct continuous functions: by operations, just like in analysis.

Theorem 1.9.7: Operations of Continuous Functions

If X is a topological space, and $f, g : X \to \mathbb{R}$ are continuous, then $f + g, f - g, f \cdot g, f/g$ (if $\forall x \in X, g(x) \neq 0$) are continuous.

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by

$$h(x) = (f(x), g(x))$$

is continuous according to 1.8.6. Then addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, so $f + g = + \circ h$ is continuous. The others are similar. ☐

Now we come to uniformly convergence.

Definition 1.9.6: Uniformly Convergence

Let X, Y be topological spaces and Y has a metric d. Let $f_n : X \to Y$ be a sequence of functions. We say that the sequence f_n converges uniformly to $f : X \to Y$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_+, \forall n > N, \forall x \in X, d(f_n(x), f(x)) < \epsilon$$

Theorem 1.9.8: Uniform Limit Theorem

Let X, Y be topological spaces and Y has a metric d. Let $f_n : X \to Y$ be a sequence of continuous functions. If f_n converges uniformly to $f : X \to Y$, then f is continuous.

Proof. This is similar of that in analysis. Let V be open in Y and $x_0 \in f^{-1}(V)$, We want to find a neighborhood U of x_0 such that $f(U) \subseteq V$.

Take $\epsilon > 0$ that $B(f(x_0), \epsilon) \subseteq V$, then $\exists N \in \mathbb{Z}_+, \forall n > N, \forall x \in X$ we have

$$d(f_n(x), f(x)) < \epsilon/3$$

Then use the continuity of f_N, we take $U = f_N^{-1}(B(f_N(x_0), \frac{\epsilon}{3}))$, i.e. $f_N(U) \subseteq B(f_N(x_0), \frac{\epsilon}{3})$. We claim that $f(U) \subseteq V$. $\forall x \in U$, we have

$$\begin{aligned} d(f(x), f_N(x)) &< \epsilon/3 && \text{by uniform convergence.} \\ d(f_N(x), f_N(x_0)) &< \epsilon/3 && \text{by continuity of } f_N. \\ d(f_N(x_0), f(x_0)) &< \epsilon/3 && \text{by uniform convergence.} \end{aligned}$$

Through the triangle inequality, we have $d(f(x), f(x_0)) < \epsilon$, so $f(x) \in B(f(x_0), \epsilon) \subseteq V$. So $f(U) \subseteq V$, and f is continuous. ☐

Remark:
The concept of uniform convergence is relevant to the uniform metric $\tilde{\rho}$. Consider the space $\mathbb{R}^X$ of all functions $f : X \to \mathbb{R}$ with the uniform metric $\tilde{\rho}$, where $\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|$. Then f_n converges uniformly to f iff f_n converge to f in the metric space $(\mathbb{R}^X, \tilde{\rho})$.

Proof. This is just the definition of uniform convergence. ☐

Now we give some examples of spaces that are not metrizable.

Example: **Spaces that are not Metrizable**

- $\mathbb{R}^{\mathbb{Z}_+}$ in the box topology is not metrizable.

  Proof. We show that the sequence lemma 1.9.2 does not hold here. Let $A \subseteq \mathbb{R}^{\mathbb{Z}_+}$ consist of points whose coordinates are all positive:

  $$A = \{(x_1, x_2, \ldots) : \forall i \in \mathbb{Z}_+, x_i > 0\}$$

  ☐

  We have $0 \in \overline{A}$, obviously. However, there is no sequence that converge to 0 in A. Let $a_n = (x_{1n}, x_{2n}, \ldots)$ be a sequence in A, that is $\forall i, n, x_{in} > 0$. We let

  $$B = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times (-x_{33}, x_{33}) \times \cdots$$

  Then B is open in the box topology, and $0 \in B$, but $a_n \notin B$ for all n. So there is no sequence in A converging to 0.

- Let J be a uncountable set. The $\mathbb{R}^J$ is not metrizable in both the box and the product topology.

  Proof. The box topology case follows the same as the previous one, just pick a countable $j_1, j_2, \ldots$ from J would do. For any sequence a_n, consider the basis element B whose coordinates are given:

  $$B_{j_n} = (-(a_n)_{j_n}, (a_n)_{j_n}) \text{ for } i \in \mathbb{Z}_+, \text{ and the rest of the coordinates are all } \mathbb{R}$$

  For the product topology, consider $A \subseteq \mathbb{R}^J$ consisting of all points (x_α) such that $x_\alpha = 1$ for all but finite many values of α.

We assert that $0 \in \overline{A}$. Let $0 \in \prod U_\alpha$ be a basis element, all but finite many $U_\alpha = \mathbb{R}$, say, except for $\alpha_1, \ldots, \alpha_n$. Let $x_\alpha = 0$ for $\alpha = \alpha_1, \ldots, \alpha_n$ and $x_\alpha = 1$ for the rest. Then $(x_\alpha) \in A \cap \prod U_\alpha$.

Next we prove that there is no sequence in A that converge to 0. Let a_n be a sequence of points in A. As in each a_n, there are only finite many coordinates that are not 1, so there are only countable many coordinates that are not 1 in the sequence. So $\exists \beta \in J$, such that $\forall n \in \mathbb{Z}_+, (a_n)_\beta = 1$.

Now let $U = \pi_\beta^{-1}((-1, 1))$, then U is a neighborhood of 0 in the product topology. But $\forall n \in \mathbb{Z}_+, a_n \notin U$. $\square$

NOTE: It is sometimes hard to visulize the uncountable-infinite $\mathbb{R}^J$. Think of it as all the functions $\mathbb{R} \to \mathbb{R}$ if it helps. The product topology basis elements are mostly $\mathbb{R}^2$ except for finite vertical lines, and the box topology basis elements are vertical intervals aligned on x-axis.

1.10 The Quotient Topology

The motivation of quotient topology comes in two ways: Geometrically, it is used for the "cut and paste" operation, such as gluing two points together, or identifying a point with a set of points. Algebraically, it is used to construct a new space from an equivalence relation on a given space, meaning that two points we see them as the same point.

A torus can be constructed by taking a square and gluing the opposite edges together.

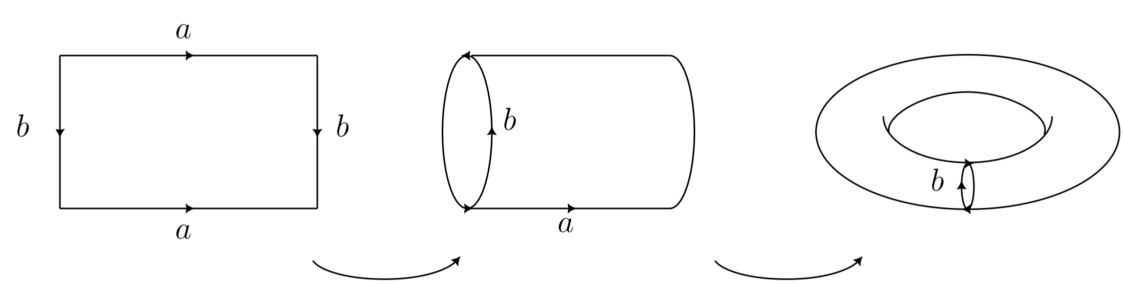

Figure 1.5: Construction of a torus

Definition 1.10.1: Quotient Map

Let X, Y be topological spaces. Let $p : X \to Y$ be a surjective map. Then p is a quotient map iff

$$\forall U \subseteq Y, U \in \mathcal{T}_Y \Leftrightarrow p^{-1}(U) \in \mathcal{T}_X$$

The condition of a quotient map is stronger than continuity. An equivalent restatement via closeness is obvious:

$$\forall V \subseteq Y, V \text{ is closed in } Y \Leftrightarrow p^{-1}(V) \text{ is closed in } X$$

If p is bijective, then a quotient map p is just a homeomorphism.

We also give a restatement via saturated set. Any subjective map provides an equivalent relation on X by

$$x \sim y \Leftrightarrow p(x) = p(y)$$

A saturated set is a set containing some equivalent classes of the relation.

Definition 1.10.2: Saturated sets

A subset $A \subseteq X$ is saturated with respect to a surjective map $p : X \to Y$ iff $\exists B \subseteq Y, A = p^{-1}(B)$.
The saturation of a set A with respect to p is defined as $p^{-1}(p(A))$.

To say that p is a quotient map is equivalent to say that p is continuous and p maps all open saturated sets to open sets. Algebraically, quotient topologies are topologies constructed on the set of equivalent classes.

An important collection of quotient maps are continuous open maps/closed maps.

Definition 1.10.3: Open and Closed Maps

A map $p : X \to Y$ is called an open map if it maps open sets to open sets, i.e. $\forall U \in \mathcal{T}_X, p(U) \in \mathcal{T}_Y$.
A map $p : X \to Y$ is called a closed map if it maps closed sets to closed sets.

Remark:
To verify an open map we only need to check that the image of a basis element is open in the target space. For the fact that

$$p(\bigcup_{\alpha \in J} B_\alpha) = \bigcup_{\alpha \in J} p(B_\alpha)$$

It is easy to see that continuous open maps and closed maps are quotient maps, but the converse is not true in general.

Example: **Quotient Maps**

- Let $X = [0,1] \cup [2,3]$, and $Y = [0,2]$, let

$$p(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ x-1, & \text{if } x \in [2,3] \end{cases}$$

then p is surjective, continuous and closed, thus is a quotient map. However, it is not open, for $p([0,1]) = [0,1]$ is not open in Y.

If we let $A = [0,1) \cup [2,3]$, then restricting p to $q : A \to Y$ is not a quotient map. For $q^{-1}([1,2]) = [2,3]$ is open in A.

- $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and surjective, and it is also open. For $U \times V$ is an nonempty basis element, then $\pi_1(U \times V) = U$ is open in $\mathbb{R}$. Thus π_1 is a quotient map. However, π_1 is not a closed map, for

$$C = \{(x,y)_p : xy = 1\}$$

is closed in $\mathbb{R}^2$, but $\pi_1(C) = \mathbb{R} - \{0\}$ is not closed.

Restricting π_1 to $A = C \cup \{0\}$ is not a quotient map. For $\{0\} = \pi_1^{-1}(\{0\})$ is open in $\mathbb{R}$.

Now we show that quotient maps can induce a topology on the target space, called the quotient topology.

Definition 1.10.4: Quotient Topology

If X is a topological space and A is a set and $p : X \to A$ is a surjective map, then there exists exactly one topology on A that makes p a quotient map, called the quotient topology induced by p.
The topology $\mathcal{T}$ is given by: $U \in \mathcal{T} \Leftrightarrow p^{-1}(U) \in \mathcal{T}_X$.

It is easy to see that $\mathcal{T}$ is indeed a topology. Using the fact $p^{-1}(\emptyset) = \emptyset$, $p^{-1}(A) = X$, and

$$p^{-1}(\bigcup_{\alpha \in J} U_\alpha) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha)$$

$$p^{-1}(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} p^{-1}(U_i)$$

Example: **Quotient Topologies**

- Let $p : \mathbb{R} \to A$, where $A = \{a, b, c\}$, given by

$$p(x) = \begin{cases} a, & \text{if } x > 0 \\ b, & \text{if } x < 0 \\ c, & \text{if } x = 0 \end{cases}$$

then the quotient topology is given by $\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$.

The set A in the definition above is actually the set of equivalent classes of the equivalence relation induced by p. We can also define the quotient topology on a set of equivalent classes directly.

Definition 1.10.5: Quotient Spaces

Let X be a topological space, and X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that $p(x)$ is the element $U \in X^*, x \in U$. In the quotient topology induced by p, the space X^* is called the quotient space.
If the partition is given by an equivalent relation $\sim$, we can denote $X^* = X/\sim$.

Example: **Quotient Spaces**

- Let $X = \{(x, y)_p : x^2 + y^2 \leq 1\}$ be the unit closed disk in $\mathbb{R}^2$, then let $\sim$ be

$$x \sim y \Leftrightarrow x = y \vee \|x\| = \|y\| = 1$$

  Then we can show that $X/\sim$ is homeomorphic to S^2.

- Let $X = [0, 1] \times [0, 1]$, and define $\sim$ be:

$$a \sim b \Leftrightarrow a = b \vee (a = (0, y)_p \wedge b = (1, y))_p \vee (a = (x, 0)_p \wedge b = (x, 1)_p) \text{ for some } x, y$$

  Then $X/\sim$ is homeomorphic to the torus T^2.

Now we consider the relation of quotient spaces with the previous topological properties.

We've noticed that subspaces do not behave well in the quotient topology. If $p : X \to Y$ is a quotient map, and $A \subseteq X$, then the restriction map $q : A \to p(A)$ may not be a quotient map. However, we have the following theorem.

Theorem 1.10.1: Subspaces of Quotient Spaces

Let $p : X \to Y$ be a quotient map, and $A \subseteq Y$ be saturated with respect to p. Let $q : A \to p(A)$ be the restriction of p to A.

- If A is either open or closed, the q is a quotient map.
- If p is either an open map or a closed map, then q is a quotient map.

Proof.

- (Saturation) For either case, we have

$$q^{-1}(V) = p^{-1}(V), \text{ if } V \subseteq p(A)$$
$$p(U \cap A) = p(U) \cap p(A), \text{ if } U \subseteq X$$

  These lines come form saturation of A. (We have $p(U \cap A) \subseteq p(U) \cap p(A)$ for any U, A, and if $y = p(u) = p(a)$, then as A is saturated, $u \in p^{-1}(p(a)) \subseteq A$, so $u \in U \cap A$.)

- (Open A or p) Given $A \subseteq p(A)$, assume $q^{-1}(V)$ is open in A:
  - If A is open, then $q^{-1}(V) = q^{-1}(V)$ is open in X, so V is open in Y, so $V = V \cap A$ is open in A.

  - If p is open, then as $q^{-1}(V) = p^{-1}(V)$ is open in A, we have $p^{-1}(V) = U \cap A$ for some open $U \subseteq X$. For p is surjective, we have

$$V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$$

    Then as $p(U)$ is open in Y, we have V is open in $p(A)$.

- For closed A or p it is similar, just replace open with closed.

□

The products of quotient spaces do not behave well. The Cartesian product of two quotient maps may not be a quotient map. To make it work, we need to use additional conditions, such as local compactness, or when p, q are both open maps. (The latter is easy to verify, as the product of two open maps is an open map.)

The product of two maps is defined: If $p : X_1 \to Y_1, q : X_2 \to Y_2$, then $p \times q : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $p \times q(x_1, x_2) = (p(x_1), q(x_2))$.

The Hausdorff condition is also not preserved in the quotient topology. The product of two Hausdorff spaces is Hausdorff, but the quotient of a Hausdorff space may not be Hausdorff.

The interesting part took place in the study of continuous functions on the quotient space. We've studies whether a map $f : Z \to \prod X_\alpha$ is continuous, and the counterpart is to determine the continuity of $f : X^* \to Z$ out of a quotient space.

Theorem 1.10.2: Continuity on the Quotient Space

Let $p : X \to Y$ be a quotient map, and Z be a space, $g : X \to Z$ be a map that is constant for each $p^{-1}(\{y\}), \forall y \in Y$. (In this case, all the element in an equivalent class have the same target). Then $\exists f : Y \to Z$ that $g = f \circ p$.

- f is continuous iff g is continuous.
- f is a quotient map iff g is a quotient map.

Proof.

- Using the fact
$$g^{-1}(V) = p^{-1}(f^{-1}(V))$$
and p being a quotient map would do.

- If f is a quotient map, then $g = f \circ p$ is also a quotient map. Conversely, if g is a quotient map, then f is surjective, Let $V \subseteq Z$, if $f^{-1}(V)$ is open, then $p^{-1}(f^{-1}(V)) = g^{-1}(V)$ is open, so V is open.

□

Remark:
If we take $Y = X^* = \{g^{-1}(\{z\}) : z \in Z\}$, then X^* is just the equivalent classes by g, and f becomes a homeomorphism.

Example: **The Product of two quotient map may not be a quotient map**

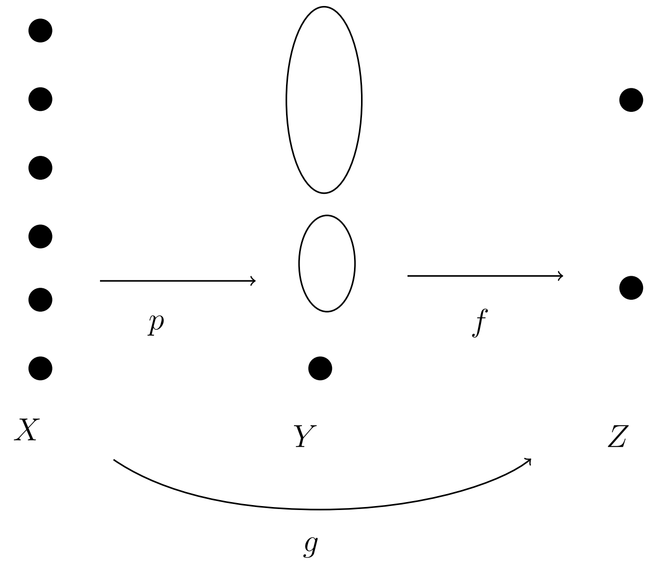

Figure 1.6: Continuity on Quotient Spaces

Let $X = \mathbb{R}$ and $\sim$: $x \sim y \Leftrightarrow x = y \vee x, y \in \mathbb{Z}_+$. The equivalent class of all $\mathbb{Z}_+$ is denoted b. Let $p : X \to X^*$ be the quotient map, and $i : \mathbb{Q} \to \mathbb{Q}$ be the identity. We show that

$$p \times i : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$$

is not a quotient map.

Proof. Consider U be the following region 1.7: where $c_n = \sqrt{2}/n$, and the width of each strip is $1/2$. Then U contains each $\{n\} \times \mathbb{Q}$ for c_n is not rational.
Then U is open in $X \times \mathbb{Q}$, it is also saturated for it contains all $\mathbb{Z}_+ \times \mathbb{Q}$. If $p \times i$ is a quotient map, then $U' = p \times i(U)$ is open in $X^* \times \mathbb{Q}$.
Consider $(b, 0)_p \in p \times i(U) \subseteq X^* \times \mathbb{Q}$. Then U' contains some $W \times I_\delta$. Where W is some neighborhood of b in X^* and $I_\delta = \{y \in \mathbb{Q} : |y| < \delta\}$. Choose N sufficiently large so that $c_N < \delta$. As $p^{-1}(W)$ is open in X and contains $\mathbb{Z}_+$, then $(n, 0)_p \in p^{-1}(W)$. We can choose $\epsilon < \frac{1}{4}$ that $V = (N - \epsilon, N + \epsilon) \times I_\delta \subseteq p^{-1}(W) \subseteq U$. But the figure shows that there are points that do not lie in V, which contradicts. $\square$

1.11 A Note on Topological Groups

So far, we have seen that the quotient spaces behave similar to the quotient groups in algebra. In fact, the quotient topology gets its name from the quotient of a topological group by a subgroup.

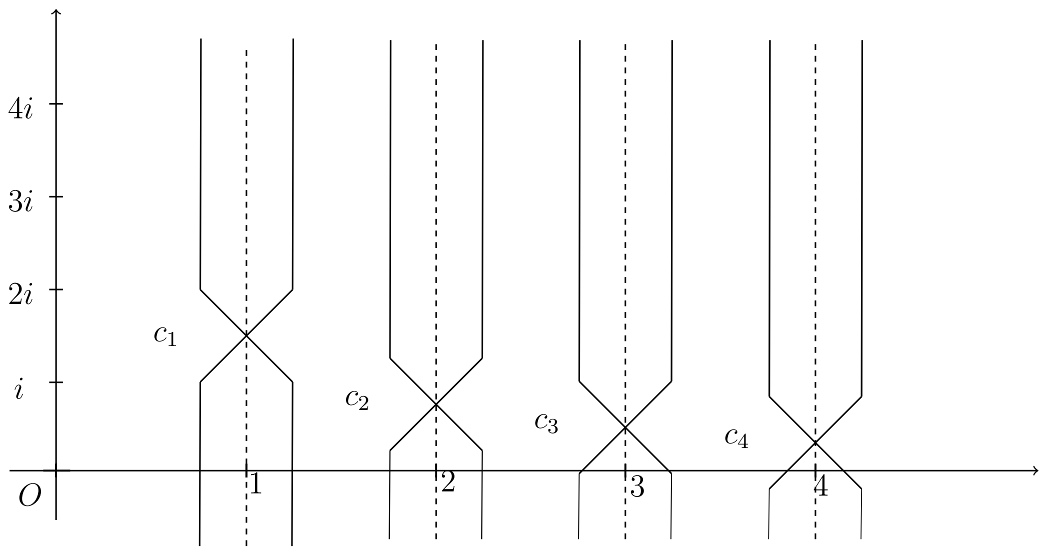

Figure 1.7: The Region U

Definition 1.11.1: Topological Groups

A topological group is a group G with a topology $\mathcal{T}$ such that the following holds:

- G satisfies the T_1-axiom.
- The group operation $\mu : G \times G \to G, (x, y)_p \mapsto x \cdot y$ is continuous.
- The inverse map $\iota : G \to G, x \mapsto x^{-1}$ is continuous.

Chapter 2

Connectedness and Compactness

In analysis, three important properties of continuous functions are listed below:

- **Intermediate Value Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous and $r \in \mathbb{R}$ is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = r$.

- **Maximum Value Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in [a, b]$ such that $\forall x \in [a, b], f(x) \leq f(c)$.

- **Uniform Continuity Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous, then for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [a, b]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

These theorems play a crucial role in analysis, upon which the inverse function theorem, implicit function theorem, and theorems on differentiability and integratiability are built.

These theorems not only depend on the continuity of the function, but also on the topological properties of the domain $[a, b]$.

- The property of the space $[a, b]$ on which the intermediate value theorem holds is called **connectedness**.

- The property of the space $[a, b]$ on which the maximum value theorem and uniform continuity theorem hold is called **compactness**.

2.1 Connected Spaces

When we say a space is not connected, we mean that it can separate into two disjoint parts that do not interfere with each other.

Definition 2.1.1: Separation and Connectedness

Let X be a topological space. A separation of X is a pair of disjoint nonempty open sets U and V such that $X = U \cup V$.
X is said to be **connected** if it cannot be separated into two disjoint nonempty open sets. In other words, the only sets that are both open and closed in X is $\emptyset$ and X.

Lemma 2.1.1: Connectedness of Subspaces

If $Y \subseteq X$, then nonempty sebsets A, B are a separation of Y iff $A \cap B = \emptyset, A \cup B = Y$, and neither of which contains a limit point of the other.

Proof.
- Suppose that A, B is a separation of Y, then A is both open and closed in Y. The closure of A in Y is $\overline{A} \cap Y$. Then we have $A = \overline{A} \cap Y$, as $B \subseteq Y, A \cap B = \emptyset$, then $\overline{A} \cap B = \emptyset$.
- Conversely, if A, B are nonempty disjoint subsets of Y, and neither contains a limit point of the other, then $\overline{A} \cap B = \emptyset, A \cap \overline{B} = \emptyset$. Thus, $A \subseteq \overline{A} \cap Y \subseteq Y - B$, and $A = Y - B$, so $A = \overline{A} \cap Y$ so A is closed, and so is B.

□

Example: **Connected and Disconnected Spaces**

- In discrete topology, every subset is open and closed, so every space is disconnected. (This is what we mean by "discrete" actually.)
- $Y = [-1, 0) \cup (0, 1] \subseteq X$, then Y is disconnected, as it can be separated into $[-1, 0)$ and $(0, 1]$.
- $X = [-1, 1]$ is connected, and we shall prove it later.
- $\mathbb{Q}$ is not connected. For any irrational number a, we have $\mathbb{Q} = (-\infty, a) \cup (a, +\infty)$, which are both open.
- In $\mathbb{R}^2$, the graph of x-axis and $y = 1/x$ is disconnected, for neither contains a limit point of the other.

We can see that proving a space to be connected is not easy. For disconnected spaces we only need to construct a separation.

Now we introduce some theorems that can help us construct connected spaces.

Lemma 2.1.2: Connected Subspace in Separation

If C, D forms a separation of X, and $Y \subseteq X$ is connected. Then $Y \subseteq C$ or $Y \subseteq D$.

Proof. As $C \cap Y$ and $D \cap Y$ are open in Y, so one of them is $\emptyset$. □

Theorem 2.1.1: Union of Connected Subspaces

The union of a collection of connected subspaces of X that has a point in common is connected, i.e. If $A_\alpha \subseteq X$ are connected, then

$$\bigcap_\alpha A_\alpha \neq \emptyset \to \bigcup_\alpha A_\alpha \text{ is connected.}$$

Proof. Let $p \in \bigcap U_\alpha$, and $Y = \bigcup U_\alpha$. Suppose $Y = C \cup D$ is a separation, without loss of generality, we can assume $p \in C$. Then from lemma 2.1.1 we know that $\forall \alpha, U_\alpha \subseteq C$, thus $Y = C$, contradiction. □

Theorem 2.1.2: Inside Boundary of Connected Subspaces

Let $A \subseteq X$ be connected, if $A \subseteq B \subseteq \overline{A}$, then B is connected.

Proof. If $B = C \cup D$ a separation, then $A \subseteq C$ or D, let $A \subseteq C$. As C is closed, so $\overline{A} \subseteq \overline{C}$ so $B \subseteq \overline{C}$. As $\overline{C} \cap D = \emptyset$, we have $D = \emptyset$. □

Theorem 2.1.3: Image of a Connected Space

The image of a connected space under a continuous function is connected. That is, if $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Proof. Restricting Y to $f(X)$, then $f : X \to f(X)$ is continuous. Let $f(X) = C \cup D$ be a separation, then $f^{-1}(C)$ and $f^{-1}(D)$ are open in X, and they are a separation of X. Thus, X is disconnected, contradiction. □

Theorem 2.1.4: Products of Finite Connected Spaces

A finite Cartesian product of connected spaces is connected. That is, if $X_1, X_2, \ldots, X_n$ are connected spaces, then $X_1 \times X_2 \times \cdots \times X_n$ is connected.

Proof. Proving for two spaces X, Y would suffice. Consider the horizontal and vertical lines $X \times \{b\} \cong X$, and $\{a\} \times Y \cong Y$, so both are connected. So the cross

$$T_{a,b} = X \times \{b\} \cup \{a\} \times Y$$

is connected in $X \times Y$, according to theorem 2.1.1. Then we have

$$X \times Y = \bigcup_{b \in Y} T_{a,b}$$

is connected. □

When we ask if arbitrary products of connected spaces are connected, the answer depends on what topology we use.

Theorem 2.1.5: Products of Connected Spaces

If X_α is connected for all $\alpha \in J$, then the product space $\prod_{\alpha \in J} X_\alpha$ is connected in the product topology.

Proof. Let $X = \prod_{\alpha \in J} X_\alpha$, $z \in X$ be a given point. Let $\sigma \subseteq J$ be a finite set, and $V_\sigma \subseteq X$ be defined as:

$$V_\sigma = \{x \in X : \forall \alpha \notin \sigma, x_\alpha = z_\alpha\} \cong \prod_{\alpha \in \sigma} X_\alpha$$

Then V_σ, being the space of all $x \in X$ that different from z only in the index in σ, is a finite product of connected spaces, so it is connected by theorem 2.1.4.

Next we define

$$V = \bigcup_{\sigma \subseteq J, |\sigma| < \infty} V_\sigma$$

Then as $\forall \sigma, z \in V_\sigma$, then V is connected from the union of connected spaces theorem 2.1.1.

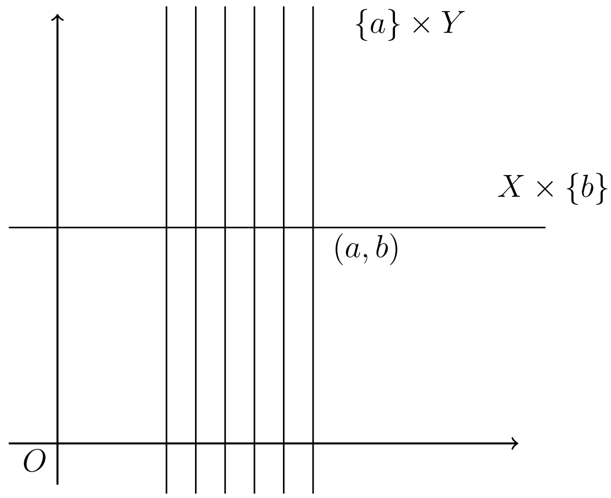

Figure 2.1: Products of Connected Spaces

Now we prove that $\overline{V} = X$: For all $x \in X$, and all $U = \prod U_\alpha$ open in X and contains x, let $\sigma = \{\alpha \in J : U_\alpha \neq X_\alpha\}$, then σ is finite by the definition of product topology, let

$$y = (y_\alpha)_{\alpha \in J} \in V_\sigma, \text{ such that } \forall \alpha \in \sigma, y_\alpha = x_\alpha$$

Then $y \in U \cap V$. Thus, $\overline{V} = X$.

Using theorem 2.1.2, we know that X is connected. □

In box topology, however, the result is not necessarily true.

Example: **Products of Connected Spaces**

Consider $\mathbb{R}^{\mathbb{Z}_+}$ in the box topology. Let A be the set of all bounded sequence and B all unbounded sequences. Then A and B form a separation of $\mathbb{R}^{\mathbb{Z}_+}$.

Proof. If $a = (a_i)_{i \in \mathbb{Z}_+} \in \mathbb{R}^{\mathbb{Z}_+}$, then let

$$U = \prod_{i=1}^{\infty} (a_i - 1, a_i + 1)$$

be an open set, then $U \subseteq A$ if $a \in A$, and $U \subseteq B$ if $a \in B$. □

In product topology, however, we prove that $\mathbb{R}^{\mathbb{Z}_+}$ is indeed connected.

2.2 Connected Subspaces for $\mathbb{R}$

We have proved many ways to construct connected spaces by now, and most of our examples and conterexamples comes from spaces constructed by $\mathbb{R}$ and its subspaces. Now we turn to that.

First we tend to the connectedness of $\mathbb{R}$ itself, and its intervals and rays. It turns out that the connectedness of $\mathbb{R}$ do not depend on its algebraic structure, but only on its order properties. Sets having the enough order properties to imply connectedness are called **linear continuum**.

Definition 2.2.1: Linear Continuum

A simply ordered set L has more than one element is called a **linear continuum** if the following holds.

- L has the least upper bound property.
- For any $a, b \in L$, if $a < b$, then there exists $c \in L$ such that $a < c < b$.

We know that $\mathbb{R}$ is the only ordered field that is a linear continuum, up to isomorphism, so linear continuum is just a generalization of $\mathbb{R}$.

Example: **Linear Continuum**

- The Ordered square $I \times I$ is a linear continuum. It is easy to see that for a given $A \subseteq I \times I$, let $b = \pi_1(A)$. If A intersects $\pi_1^{-1}(\{b\})$, then taking the supremum of the intersection would do, if it doesn't intersect, then $(b, 0)_p$ is the supremum.
- If X is a well-ordered set, i.e. Every nonempty subset of X has a least element, then $X \times [0, 1)$ is a linear continuum in the dictionary order.

Theorem 2.2.1: Connectedness of Linear Continuum

If L is a linear continuum with the order topology, then L is connected, and so are all its intervals and rays. That is, convex subsets of L are connected.

Proof. Suppose $Y \subseteq L$ is convex, and has a separation $A \cup B$. Without loss of generality, we let $a \in A, b \in B, a < b$. Then $[a, b] \subseteq Y$, so $[a, b]$ is the union of disjoint subsets

$$A_0 = A \cap [a, b] \text{ and } B_0 = B \cap [a, b]$$

A_0, B_0 are open in $[a, b]$, in the subspace topology, and from theorem 1.5.2, the order topology is just the subspace topology, also A_0, B_0 are not empty, as $a \in A_0$ and $b \in B_0$. So A_0, B_0 is a separation of $[a, b]$.

Now we let $c = \sup A_0$, as b is an upper bound, and $a \in A_0$, we have $c \in [a, b]$. Now we prove that $c \notin A_0, c \notin B_0$:

- Suppose $c \in B_0$, then $c = b$ or $a < c < b$. For B_0 is open, $\exists d, (d, c] \subseteq B_0$. According to the linear continuum property, $\exists z, d < z < c$, then z is a smaller upper bound of A_0, contradicts.
- If $c \in A_0$, similarly, there is $[c, d) \subseteq A_0$, and $c < z < d$, and $z \in A_0$, so c is not an upper bound of A_0, contradicts.

□

So $\mathbb{R}$ is connected, and so are all its intervals and rays.

Theorem 2.2.2: Intermediate Value Theorem

Let X be connected and Y an ordered set with order topology. Let $f : X \to Y$ be continuous. If $a, b \in X$ and r is between $f(a)$ and $f(b)$, then there exists $c \in X$ such that $f(c) = r$.

Proof. Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, +\infty)$. If there are no point c in $[a, b]$ such that $f(c) = r$, then $f(a) \in A$ and $f(b) \in B$, and $A \cup B = f(X)$, so A, B is a separation of X, contradicting to the fact that the continuous image of a connected space is connected. □

The connectedness of intervals in $\mathbb{R}$ gives a more useful and intuitive form of connectedness.

Definition 2.2.2: Path Connected

Given $x, y \in X$, a path in X from x to y is a continuous function $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. If such a path exists for every pair x, y, we say that X is **path connected**.

Proposition: **Properties of Path-Connected Space**

- A path connected space is connected.

  Proof. Suppose $X = A \cup B$ is a separation. Let $f : [a, b] \to X$ be path in X that $f(a) \in A, f(b) \in B$, then $f([a, b])$ is connected, which is a contradiction as $f([a, b]) \subseteq A$ or B. □

  The converse does not hold.

- The continuous image of a path connected space is path connected.

  Proof. By the composite of continuous functions. □

Example: **Path Connected and Connected**

- The unit ball in $\mathbb{R}^n$:
  $$B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$
  is path connected, as we can connect any two points by a straight line, and the straight line is continuous. Also every open and closed ball in $\mathbb{R}^n$ is path connected.

- The unit sphere in $\mathbb{R}^n$:
  $$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$$
  is path connected. For the map $g : \mathbb{R} - \{0\} \to S^{n-1}$ by $g(x) = x/\|x\|$ is continuous and surjective.

- The ordered square I_o^2 is connected but not path-connected.

Proof. Being a linear continuum, I_o^2 is connected. Let $p = (0,0)_p, q = (1,1)_p$, Suppose there is a path $f : [a,b] \to I_o^2$ that joins p, q, and by the immediate value theorem, $f([a,b]) = I_o^2$.

Then $\forall x \in I$, the preimage of a vertical segment

$$U_x = f^{-1}(\{x\} \times (0,1))$$

is a nonempty open subset of $[a,b]$. $\forall x \in I$, choose $q_x \in U_x$ and $q_x \in \mathbb{Q}$. Then the function $x \mapsto q_x$ is injective, contradicting that I is uncountable.

(An interval of $\mathbb{R}$ cannot be disjoint union of uncountable open subsets.) □

- Let a set $S \subseteq \mathbb{R}^2$ be the graph of the function $y = \sin \frac{1}{x}$, i.e.

$$S = \left\{ (x,y)_p : y = \sin \frac{1}{x}, x \in (0,1] \right\}$$

For S is a continuous image of $(0,1]$, it is connected. Then $\overline{S}$ is also connected by theorem 2.1.2. The set

$$\overline{S} = S \cup \{(0,y)_p : y \in [-1,1]\}$$

is called the **topologist's sine curve**. Suppose there is a path $f : [a,c] \to \overline{S}$ that $f(a) = (0,0)_p, f(c) \in S$, then the set

$$\{t : f(t) \in \{0\} \times [-1,1]\} = f^{-1}\left(\{0\} \times [-1,1]\right)$$

is closed in $[a,c]$, so it has a maximum b. Then $f : [b,c] \to \overline{S}$ is a path that $f(b) \in \{0\} \times [-1,1]$ and the others to be in S. Replace $[b,c]$ to $[0,1]$ for convenience.

Let $f(t) = (x(t), y(t))_p$, then $x(0) = 0$ and $\forall t > 0, x(t) > 0, y(t) = \sin \frac{1}{x(t)}$, but there is a sequence $t_n \to 0$ that $y(t_n) = (-1)^n$, contradicting to continuity of f.

2.3 Components and Local Connectedness

Given an arbitrary space, how do we break it into connected (or path-connected) pieces? We consider the process now.

Definition 2.3.1: Components

X is a topological space. Define an equivalent relation $\sim$, that

$$x \sim y \Leftrightarrow \exists \text{ a connected } U \subseteq X, x \in U, y \in U$$

The equivalent classes are called components.
A restatement is that the components of X are disjoint subspaces of X whose union is X and each nonempty connected subspace of X intersects only one of them.

Proof. The symmetry and reflexivity is obvious. For transitivity, if $x \sim y$ and $y \sim z$, then there exists connected sets U, V such that $x, y \in U$ and $y, z \in V$. Then $U \cup V$ is connected, as they intersect at y. ☐

Remark:
Note that although the definition of disconnected requires the split into two disjoint open sets, the components are not necessarily open. When you continuously divide open sets for infinite times, you may end up with a set that is not open, but still connected. (Openness is only closed under finite intersection)

Path connectedness also has a similar definition about path components.

Definition 2.3.2: Path Components

X is a topological space. Define an equivalent relation $\sim$, that

$$x \sim y \Leftrightarrow \exists \text{ a path } f : [a, b] \to X, f(a) = x, f(b) = y$$

The equivalent classes are called path components.
A restatement is that the path components of X are disjoint subspaces of X whose union is X and each nonempty path connected subspace of X intersects only one of them.

Proof. The symmetry and reflexively is obvious. For transitivity, if $x \sim y$ and $y \sim z$, then there exists paths $f : [a, b] \to X$ and $g : [b, c] \to X$ such that $f(a) = x, f(b) = y$ and $g(b) = y, g(c) = z$. Then the pasting path $h : [a, c] \to X$ defined by

$$h(t) = \begin{cases} f(t) & t \in [a, b] \\ g(t) & t \in [b, c] \end{cases}$$

is a path from x to z, by the pasting lemma 1.7.3. So $x \sim z$. ☐

Remark:
NOTE: each component in a space X is closed. For if A is a component, $\overline{A}$ is also one. Then as $A \subseteq \overline{A}$, we have $A = \overline{A}$. If there are only finite components, then eevery component is also open.
For path components, we can say less, for they can be neither open or closed.

Example: **Components**

- In $\mathbb{Q}$ the components are the singletons, saying $\mathbb{Q}$ is totally disconnected.
- The topologist's sine curve $\overline{S}$ has one component and two path components, one is S, and the other is the vertical segment $\{(0, y)_p : y \in [-1, 1]\}$. One is open and one closed.

Sometimes it is more important to study the close neighborhood of a point in a space, rather than the whole space. We define the local connectedness of a space.

Definition 2.3.3: Local Connectedness

A space X is locally connected at x if $\forall$ neighborhood U of x, $\exists$ connected neighborhood V of x such that $V \subseteq U$. If X is locally connected at every point, we say that X is **locally connected**.
Similarly, a space X is locally path connected at x if $\forall$ neighborhood U of x, $\exists$ path connected neighborhood V of x such that $V \subseteq U$. If X is locally path connected at every point, we say that X is **locally path connected**.

Neither local connectedness nor connectedness implies the other.

Example: **Local Connectedness and Connectedness**

- Each interval and ray in $\mathbb{R}$ is both locally connected and connected.
- $[-1, 0) \cup (0, 1]$ is locally connected but not connected, as it can be separated into two disjoint open sets $[-1, 0)$ and $(0, 1]$.
- The topologist's sine curve $\overline{S}$ is connected but not locally connected. Taking a point P on the vertical line $(0, y)_p$ for $y \in [-1, 1]$, then any neighborhood of $P = (0, y)_p$ intersects both S and the vertical line, so it cannot be locally connected.
- $\mathbb{Q}$ is neither locally connected nor connected.

Theorem 2.3.1: Criterion for Local Connectedness

A space X is locally connected iff $\forall U \in \mathcal{T}_X$, each component of U is open in X.

Proof.
- Suppose X is locally connected, and U is open in X. C is a component of U, Let $x \in C$, then there is a connected neighborhood V_x, $x \in V_x \subseteq U$, then $V_x \subseteq C$. As $C = \bigcup V_x$ then C is open.
- If components of open sets of X are open, then $\forall x \in X, \forall U \in \mathcal{T}_X, x \in U$, let C be the component of U containing x, then C is an open and connected set.

□

Theorem 2.3.2: Criterion for Local Path-Connectedness

A set X is locally path-connected iff $\forall U \in \mathcal{T}_X$, each path component of U is open in X.

Proof. Similar to the previous one. □

The following theorem shows the relationship of connectedness and path-connectedness.

Theorem 2.3.3: Relation of two Components

If X is a topological space, then each component is the union of some path-components.
If X is locally connected, then the components and path-components are the same.

Proof. Let C be a component of X, and $x \in C$, and P be the path component of X containing x, then P is connected, so $P \subseteq C$. Suppose $P \subsetneq C$, then as X is locally path connected, then

C are the union of many path components, which are open, these forms a separation of C. So $P = C$. □

2.4 Compact Spaces

Compactness is another important property of spaces, which is closely related to the maximum value theorem and uniform continuity theorem. It is not so easy to formulate compared to connectedness. SOME property of the closed interval on $\mathbb{R}$ guaratees these theorems, and it is once thought to be the existence of limit points of infinite subsets, but it turns out that it needs a stronger formulation, in terms of open coverings.

Definition 2.4.1: Cover

A collection $\mathcal{A}$ of subsets of X is said to be a cover of X iff $X = \bigcup \mathcal{A}$.
If all elements in $\mathcal{A}$ is open, then it is an open covering.

Definition 2.4.2: Compactness

A space X is compact iff for every open covering $\mathcal{A}$ of X, there exists a finite subcovering $\mathcal{A}' \subseteq \mathcal{A}$ such that $X = \bigcup \mathcal{A}'$.

Example: **Compactness**

- Let
$$X = \{0\} \cap \left\{ \frac{1}{n} : n \in \mathbb{Z}_+ \right\}$$
Then X is compact. Given an open covering $\mathcal{A}$ if X, $\exists U \in \mathcal{A}, 0 \in U$, then there are only finite many $1/n$ outside U.
- The interval $(0, 1]$ is not compact, as the open covering $\mathcal{A} = \{(1/n, 1] : n \in \mathbb{Z}_+\}$ does not have a finite subcovering.

First let's consider how to generate compact spaces from known ones.

Lemma 2.4.1: Compactness of Subspaces

Let $Y \subseteq X$, then Y is compact iff every covering of Y by open sets in X has a finite subcovering.

Proof. Obvious. □

Remark:
This shows that compactness is a "local" property, it does not depend on whether the set is itself or a subset of another space.

Theorem 2.4.1: Closed Subpaces of a Compact Space

All closed subspaces of a compact space are compact.

Proof. Let $Y \subseteq X$ be closed, the for every covering $\mathcal{A}$ of Y by open sets in X, extend it by adding $X - Y$ to form an open covering of X, then it has a finite subcover due to the compactness of X. $\square$

Theorem 2.4.2: Compact Subspaces of Hausdorff Space

Let X be a Hausdorff space, then every compact subspace of X is closed.

Proof. Let $Y \subseteq X$ be compact, and $x_0 \in X - Y$ which is open. Then $\forall y \in Y$, there exists disjoint neighborhoods U_y, V_y of x_0, y. Then $\bigcup_{y \in Y} V_y$ is an open covering of Y. It has a finite subcover $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$, then $Y \subseteq \bigcup_{i=1}^n V_{y_i}$, and $U = \bigcap_{i=1}^n U_{y_i}$ is a neighborhood of x_0 that is disjoint from Y. Thus, Y is closed. $\square$

The proof above shows the following statement:

Lemma 2.4.2: Compact Subspaces of Hausdorff Space

If Y is a compact subspace of a Hausdorff space X, and $x_0 \notin Y$, then there exists disjoint open sets of X containing x_0 and Y respectively.

About continuity:

Theorem 2.4.3: Continuous Image of Compact Spaces

Let $f : X \to Y$ be a continuous function, if X is compact, then $f(X)$ is compact.

Proof. Let $\mathcal{A}$ be an open covering of $f(X)$, then $\{f^{-1}(U) : U \in \mathcal{A}\}$ is an open covering of X. As X is compact, there exists a finite subcovering $\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)\}$, then $\{U_1, U_2, \ldots, U_n\}$ is a finite subcovering of $f(X)$. $\square$

We can use this feature to verify homeomorphisms.

Theorem 2.4.4: Homeomorphism by Compactness

Let $f : X \to Y$ be a continuous bijection, if X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. If A is closed in X, then A is compact, so $f(A)$ is compact in Y, and as Y is Hausdorff, $f(A)$ is closed in Y. Thus, f^{-1} is continuous. $\square$

About products:

Theorem 2.4.5: Finite Products of Compact Spaces

Let $X_1, X_2, \ldots, X_n$ be compact spaces, then the product space $X_1 \times X_2 \times \cdots \times X_n$ is compact in the product topology.

Proof. We only need two prove $n = 2$ case.

- Suppose we have X, Y with Y compact, and $x_0 \in X$, N is an open set in $X \times Y$ containing $\{x_0\} \times Y$. We shall prove:

  > There is a neighborhood W of x_0 in X that $W \times Y \subseteq N$. $W \times Y$ is called a tube around $\{x_0\} \times Y$.

  First we have $\forall (x_0, y)_p$, taking a basis element $(x_0, y)_p \in U_y \times V_y \subseteq N$, then all the sets forms a covering of $\{x_0\} \times Y$. For the compactness of $\{x_0\} \times Y$, it has a subcover:

  $$U_1 \times V_1, \ldots, U_n \times V_n$$

  Define $W = U_1 \cap \cdots \cap U_n$, then W is a neighborhood of x_0, and $W \times Y \subseteq N$.

- Now er prove the theorem. Let X, Y be compact spaces, and $\mathcal{A}$ be an open covering of $X \times Y$, then $\forall x \in X$, there is finite $A_1, \ldots, A_m \in \mathcal{A}$ that covers $\{x\} \times Y$, as Y is compact, Let $N = A_1 \cup \cdots \cup A_m$ be an open set containing $\{x\} \times Y$, then N contains a tube $W_x \times Y$ around $\{x\} \times Y$, which is covered by $A_1, \ldots, A_m$. Now we have a covering of X by W_x, and as X is compact, there exists a finite subcovering $W_1, \ldots, W_n$, then $\{W_i \times A_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a finite subcovering of $\mathcal{A}$, which covers $X \times Y$.

$\square$

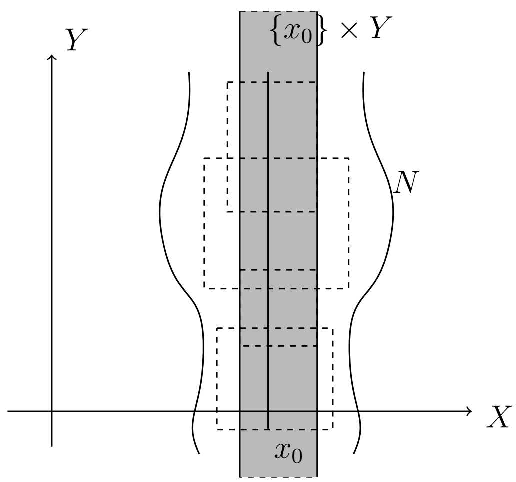

Figure 2.2: Construction of Tube

In the proof, we came across the tube lemma:

Lemma 2.4.3: The Tube Lemma

If X, Y are topological spaces and Y is compact, let $x_0 \in X$, and N is an open set of $X \times Y$ containing $\{x_0\} \times Y$. Then there exists a neighborhood W of x_0 in X such that $W \times Y \subseteq N$.

Remark:
The tube lemma shows some properties of closed sets in analysis. It implies how well the finite subcovering property is related to it.
It may not be true for not compact spaces. For example, let Y be the y-axis in $\mathbb{R}^2$, and

$$N = \left\{ (x, y)_p : |x| < \frac{1}{y^2 + 1} \right\}$$

has no tube around $\{0\} \times Y$.

It is natural to ask if arbitrary products of compact spaces are compact. The answer is yes, and the theorem is called Tychonoff's theorem. There is no way passing to the infinite case to finite case, as we've done to the connected space. We need to find a new route.

The following theorem is another criterion for compactness, in terms of closed sets.

Definition 2.4.3: Finite Intersection Property

A collection $\mathcal{C}$ of subsets of X has the finite intersection property if for every finite subcollection $\mathcal{C}' \subseteq \mathcal{C}$, $\bigcap \mathcal{C}' \neq \emptyset$.

Theorem 2.4.6: Finite Intersection Property and Compactness

Let X be a topological space, then X is compact iff every collection $\mathcal{C}$ of closed sets in X that has the finite intersection property, the intersection $\bigcap \mathcal{C} \neq \emptyset$.

Proof.

- The $\Rightarrow$ part: If $\bigcap \mathcal{C} = \emptyset$, then the collection $\{X - C : C \in \mathcal{C}\}$ is an open covering of X, so it has a subcovering, $X - C_1, \ldots, X - C_n$, then $\bigcap_{i=1}^n C_i = \emptyset$, contradicting to the finite intersection property.

- The $\Leftarrow$ part: If X is not compact, then there exists an open covering $\mathcal{A}$ of X that has no finite subcovering. Let $\mathcal{C} = \{X - A : A \in \mathcal{A}\}$, then we get a collection of closed sets that has the finite intersection property, but $\bigcap \mathcal{C} = \emptyset$, contradicting to the assumption.

$\square$

A corollary of the theorem is the nested closed sets:

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

Then $\{V_i\}$ satisfies the finite intersection property, so $\bigcap_{i=1}^\infty V_i \neq \emptyset$.

This theorem is just a restatement of the definition of compactness, but it is useful in the proof of Tychonoff's theorem.

2.5 Compact Subspaces of $\mathbb{R}$

We shall prove every closed interval in $\mathbb{R}$ is compact, to do this we only need to use the least upper bound property of $\mathbb{R}$.

Theorem 2.5.1: Compactness of Closed Intervals

Let X be a simply ordered set having the least upper bound property, then every closed interval $[a, b]$ in X is compact in the order topology.

Proof. **Step 1**: Given $a < b$, let $\mathcal{A}$ be an open covering of $[a, b]$ open sets in the subspace topology. (which is also order topology)

First we prove the following

> If $x \in [a, b], x \neq b$, then $\exists x < y \leq b$ that $[x, y]$ can be covered by one or two elements of $\mathcal{A}$.

Proof.
- If x has an immediate successor y, then $[x, y]$ has two elements.
- If x has no immediate successor, then let $A \in \mathcal{A}, x \in A$, then $\exists [x, c) \subseteq A$, taking $y \in (x, c)$ would do.

□

Step 2: Let $C = \{y \in [a, b] : y > a \wedge [a, y] \text{ have a finite subcover of } \mathcal{A}\}$. The previous step shows that C is not empty. We let $c = \sup C$. Now we show $c \in C$. Choose $A \in \mathcal{A}, c \in A$, then $\exists (d, c] \subseteq A$. If $c \notin C$, then $\exists z \in C, z \in (c, d)$, otherwise d is a smaller upper bound of C. As $[a, z]$ can be covered by finite subcollection of $\mathcal{A}$, adding A would cover $[a, c]$, contradicts.

Step 3: As $c \in C$, if $c \neq b$, then $\exists c < d \leq b$ that $[c, d]$ can be covered by one or two elements of $\mathcal{A}$. So $d \in C$, contradicts.

Therefore, $c = b$. □

Corollary 2.5.1: Closed Intervals in $\mathbb{R}$

Every closed interval $[a, b]$ in $\mathbb{R}$ is compact in the order topology.

Theorem 2.5.2: Compact Subspaces of $\mathbb{R}^n$

A subspace $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded in the p-metric. (Usually consider square metric ρ)

Proof. Suppose A is compact, as $\mathbb{R}^n$ is Hausdorff, A is closed. Consider the sets $\{B_\rho(0, m) : m \in \mathbb{Z}_+\}$, some finite subcollection contains A, so A is bounded.

Conversely, Suppose A is closed and bounded under ρ, and $\rho(x, y) \leq N, \forall x, y \in A$. Let $x_0 \in A, \rho(x_0, 0) = b$, then $\rho(x, 0) \leq N + b, \forall x \in A$. Let $P = N + b$, then $A \subseteq [-P, P]^n$ which is compact. As A is closed, A is compact. □

Theorem 2.5.3: Extreme Value Theorem

Let $f : X \to Y$ be continuous, Y is an ordered set in the order topology. If X is compact, then $\exists c, d \in X, \forall x \in X, f(c) \leq f(x) \leq f(d)$.

Proof. As $A = f(X)$ is compact, if A has no largest element, then the collection

$$\{(-\infty, a) : a \in A\}$$

forms an open covering of A. It has a finite subcover

$$\{(-\infty, a_i) : i = 1, \ldots, n\}$$

let $a = \max a_i$, then a is the largest element, contradicts. $\square$

To prove the uniform continuity theorem of calculus, we introduce the notion of Lebesgue number for an open covering of a metric space.

Definition 2.5.1: Distance of a point to set

Let (X, d) be a metric space, and $A \subseteq X, A \neq \emptyset$, then for $x \in X$ we have

$$d(x, A) = \inf_{a \in A} d(x, a)$$

It is easy to show that for a fixed A, the function $x \mapsto d(x, A)$ is continuous.

Proof. $\forall a \in A, x, y \in X$, we have $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$, so $d(x, A) \leq d(x, y) + d(y, A)$. Using ϵ-δ language would do, for

$$|d(x, A) - d(y, A)| \leq d(x, y)$$

$\square$

Lemma 2.5.1: Lebesgue Number Lemma

Let $\mathcal{A}$ be an open covering of the metric space (X, d), and if X is compact, there is $\delta > 0$ such that

$$\forall U \subseteq X, (\operatorname{diam} U < \delta \rightarrow \exists A \in \mathcal{A}, U \subseteq A)$$

A sufficiently small set is contained in some open set in the covering.
The number δ is called a **Lebesgue number** of the covering $\mathcal{A}$. Note that all smaller numbers are also Lebesgue numbers.

Proof. If $X \subseteq \mathcal{A}$, it is done, we assume $X \subsetneq A$. Choose a finite subcollection $\{A_1, \ldots, A_n\}$ that covers X. Let $C_i = X - A_i$ and let $f : x \rightarrow \mathbb{R}$ be the average of the distance to the closed sets C_i:

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

$\forall x \in X$, let $x \in A_i$, then $\exists x \in B(x, \epsilon) \subseteq A_i$, so $d(x, C_i) \geq \epsilon$, $f(x) > 0$.

Since f is continuous, it has a minimal value δ. Let $\operatorname{diam} B < \delta$ and $x_0 \in B$, then

$$\delta \leq f(x_0) \leq d(x_0, C_m) = \max_{1 \leq i \leq n} d(x_0, C_i)$$

Then the $B \subseteq B(x_0, \delta) \subseteq A_m$. $\square$

Definition 2.5.2: Uniform Continuity

X, Y are metric spaces. A function $f : X \rightarrow Y$ is uniformly continuous if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in X, (d_X(x, y) < \delta \rightarrow d_Y(f(x), f(y)) < \epsilon)$$

Theorem 2.5.4: Uniform Continuity Theorem

Let X, Y be metric spaces and X compact. Then if $f : X \rightarrow Y$ is continuous, then f is uniformly continuous.

Proof. Given $\epsilon > 0$, taking an open cover of Y by balls $B(y, \epsilon/2)$, and $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2), y \in Y\}$. Choose δ to be a Lebesgue number of $\mathcal{A}$, then we finish. □

A interesting application of compactness is that of the uncountability of some Hausdorff spaces, including $\mathbb{R}$.

Definition 2.5.3: Isolated Points

If $x \in X$, that $\{x\}$ is open, then x is an **isolated point** of X.

Theorem 2.5.5: Uncountable Hausdorff Spaces

Let X be a nonempty Hausdorff space, if X has no isolated points, then X is uncountable.

Proof.

- Lemma:

  Let U be a nonempty open set in X, and $x \in X$, then there is an nonempty open set $V \subseteq U$ and $x \notin \overline{V}$.

  It is possible to take a $y \in U, y \neq x$, and separate x, y with W_1, W_2, then take $W_2 \cap U$ would do.

- Now we show $f : \mathbb{Z}_+ \to X$ is not surjective. Let $x_n = f(n)$, using the lemma, choose $V_1 \subseteq X, x_1 \notin \overline{V_1}$. For each $n \in \mathbb{Z}_+$, given V_{n-1} open and nonempty, choose open V_n that $V_n \subseteq V_{n-1}$ and $x_n \notin \overline{V_n}$. Then for the nested closed sets

$$\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$$

  satisfying the finite intersection property, we have $\bigcap_{n=1}^{\infty} \overline{V_n} \neq \emptyset$. So there exists $x \in X$ that $x \neq x_n$ for all $n \in \mathbb{Z}_+$. Thus, f is not surjective, so X is uncountable.

□

So we have every closed interval in $\mathbb{R}$ is uncountable.

2.6 Limit Point Compactness

Definition 2.6.1: Limit Point Compactness

A space X is limit point compact if every infinite subset of X has a limit point in X.

This is the original definition of compactness, and it is looser than the definition in terms of open coverings.

Theorem 2.6.1: Compactness implies Limit Point Compactness

Compactness implies limit point compactness, but not vice versa.

Proof. Let X be compact, and $A \subseteq X$ has no limit points. We have A is closed. And $\forall a \in A, \exists U_a$ that $U_a \cap A = \{a\}$. As $\{U_a\}$ is an open cover of A, so it has a finite subcover, so A is finite. □

Example: **Limit Point Compactness**

Let $Y = \{a, b\}$ and $\mathcal{T}_Y$ is the indiscrete topology, then $X = \mathbb{Z}_+ \times Y$ is limit point compact, but not compact.

Another version of compactness is sequential compactness.

Definition 2.6.2: Sequential Compactness

Let X be a topological space, then X is sequentially compact if every sequence in X has a convergent subsequence.

These three definitions of compactness are equivalent in metric spaces, but not in general topological spaces.

Theorem 2.6.2: Equivalence of Compactness in Metric Spaces

Let X be metrizable. Then the following are equivalent:

- X is compact.
- X is limit point compact.
- X is sequentially compact.

Proof. SORRY □

2.7 Local Compactness

Definition 2.7.1: Local Compactness

A space X is locally compact at x if $\exists U \in \mathcal{T}_X$ such that $x \in U \subseteq C$ and C is compact. If X is locally compact at every point, we say that X is **locally compact**.
Or to say, for every point $x \in X$, there exists a neighborhood U of x such that $\overline{U}$ is compact.

NOTE that a compact space is locally compact, as it is itself a neighborhood of each point.

Example: **Locally Compact Spaces**

- $\mathbb{R}$ is locally compact, as every point has a neighborhood that is contained in a closed interval. So is $\mathbb{R}^n$.
- Every simply ordered set with the least upper bound property is locally compact, as every point has a neighborhood that is contained in a closed interval.

Metrizable spaces and compact Hausdorff spaces are well-behaved. If a given space is neither, we may hope to find some property of it is a subspace of one of these spaces. The subspace of a metrizable space is metrizable, but the subspace of a compact Hausdorff space is not necessarily compact Hausdorff. What spaces are homeomorphic to subspaces of compact Hausdorff spaces?

Theorem 2.7.1: Criterion of a Locally Compact Hausdorff Space

A space X is locally compact Hausdorff iff there is an Y satisfying the following:

1. $X \subseteq Y$.
2. $Y - X$ consists of a single point.
3. Y is a compact Hausdorff space.

If Y, Y' both satisfies the condition, then there is a homeomorphism $Y \to Y'$ that is an identity function on X.

Proof.

- First we verify uniqueness. Let Y, Y' be two spaces satisfying the condition, then Let $h : Y \to Y'$ that is the identity function on X, and map the single point p in $Y - X$ to the single point q in $Y' - X$. Then h is continuous (need some verification), and as Y is compact, h is a homeomorphism.

  The continuity of h : let $U \subseteq X$, if $p \notin U$, we're done, else $C = Y - U$ is closed in Y, so it is a compact subspace of X, then $h(U) = Y' - U$ is also closed compact.

- Now suppose X is locally compact Hausdorff and we construct Y. Take a point ∞ not in X (use the symbol for convenience). $Y = X \cup \{\infty\}$. Define $\mathcal{T}_Y$ to contain:

  - All open sets in X.
  - $Y - C$ where C is each compact subspace of X.

  It leads on to some labor to check $\mathcal{T}_Y$ is indeed a topology.

  To show that Y is compact, take $\mathcal{A}$ covering Y, then there is some $Y - C \in \mathcal{A}$ to cover ∞. So as C is compact, we finish.

  To show Y is Hausdorff, for $x, y = \infty$, we find a compact C contain a neighborhood of x, that will do.

- Conversely, X is naturally Hausdorff. $\forall x \in X$, choose U, V separating x, ∞, then $C = Y - V$ is closed compact.

$\square$

Remark:
$\mathbb{R}$ is not compact, but adding ∞ would become compact, for an open covering containing ∞ must contain some $\{x : x < a \vee x > b\}$, which take cares of the infinite region. It is homeomorphic to a circle.
The one point compactification of $\mathbb{C}$ is the Riemann sphere, or extended complex plane $\mathbb{C}^\infty$.

Definition 2.7.2: Campactification

If Y is compact Hausdorff and $X \subsetneq Y, \overline{X} = Y$, then Y is called a **compactification** of X.
If $Y - X$ is a single point, then Y is called a **one-point compactification** of X.

Usually "local" means true in every neighborhood "arbitrary small". Now we give an equivalent definition of local compactness in a Hausdorff space.

Theorem 2.7.2: Criterion of Local Compactness

Let X be a Hausdorff space. Then X is local compact iff $\forall x \in X, \forall$ neighborhood U of x, there exists a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

Proof. Then $\Leftarrow$ part is obvious. For the $\Rightarrow$ part, let $x \in X$ and U be a neighborhood of x. Taking the one-point compactification Y of X, let $C = Y - U$, then C is closed and compact in Y. According to lemma 2.4.2, we choose separation V, W of x, C, then $\overline{V}$ would do. □

Corollary 2.7.1: Subspaces of Locally Compact Hausdorff Spaces

Let X be locally compact Hausdorff. And $A \subseteq X$ be open or closed, then A is locally compact Hausdorff.

Corollary 2.7.2: Homeomorphism to Subspace of Compact Hausdorff Spaces

A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.

Proof. Follows from 2.7.1 and 2.7.2. □

2.8 Nets

Chapter 3

Countability and Separation Axioms

The intuition of countability and separation axioms do not arise from the study of analysis like compactness and connectedness, but rather from the study of topology. The separation axioms are a set of conditions that describe how distinct points and sets can be separated by neighborhoods.

3.1 The Countability Axioms

Recall the definition earlier for the first countable space.

Definition 3.1.1: First Countable

A space X has a countable basis at $x \in X$ iff there is a countable collection $\mathcal{B}$ of neighborhoods of x such that for every neighborhood U of x, there exists a $B \in \mathcal{B}$ such that $B \subseteq U$.
A space X is **first countable** if every point $x \in X$ has a countable basis at x. This is called the **first countability axiom**.

We've seen that every metric space is first countable, with balls as a countable basis at each point. In a first countable space, convergent sequences is adequate to detect limit points and continuity.

Theorem 3.1.1: Sequence Detection of First Countable Spaces

Let X be a topological space.

- Let $A \subseteq X$, then if there is a sequence of points converging to x, then $x \in \overline{A}$. The reverse holds if X is first countable.
- Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y. The reverse holds if X is first countable.

Proof. The same as lemma 1.9.2 and theorem 1.9.4. □

Definition 3.1.2: Second Countable

A topological space X is **second countable** if it has a countable basis. This is called the **second countability axiom**.

Obviously, every second countable space is first countable, but the reverse is not true. Second countability is so strong that not all metric spaces are second countable.

Example: **Second Countable Spaces**

- $\mathbb{R}$ is second countable, with the collection of open intervals with rational endpoints as a countable basis.
- $\mathbb{R}^{\mathbb{Z}_+}$ with the product topology is second countable, with the collection of open sets of the form $\prod U_n$, with finitely many U_n are rational intevals and the rest are $\mathbb{R}$, as a countable basis.
- In the uniform topology, $\mathbb{R}^{\mathbb{Z}_+}$ is first countable (being metrizable) but not second countable. We choose $A \subseteq R^{\mathbb{Z}_+}$ to be all sequences consisting of 0 and 1, then A is discrete but uncountable, so it cannot be covered by a countable basis.

Countable axioms are well-behaved when taking subspaces and countable products.

Theorem 3.1.2: Subspaces and Products of Countability Axioms

- A subspace of a first countable space is first countable.
- A subspace of a second countable space is second countable.
- A countable product of first countable spaces is first countable.
- A countable product of second countable spaces is second countable.

Proof. Consider the second countable case. If $\mathcal{B}$ is a countable basis of X, then $\{B \cap A, B \in \mathcal{B}\}$ is a countable basis of $A \subseteq X$. If $\mathcal{B}_i$ is a countable basis of X_i for each $i \in \mathbb{Z}_+$, then $\{\prod U_i\}$, where $U_i \in \mathcal{B}_i$ for finitely many i, is a countable basis of $\prod X_i$. The first countable case is similar, using countable bases at each point. $\square$

Definition 3.1.3: Denseness

A subset A of a topological space X is **dense** in X if $\overline{A} = X$. This means that every point in X is either in A or is a limit point of A.

Theorem 3.1.3: Second Countability and Denseness

Suppose X is second countable.

- Every open covering of X has a countable subcovering.
- There is a countable dense subset of X.

Proof. Let $\mathcal{B}$ be a countable basis of X.

- Let $\mathcal{A}$ be an open covering of X, then for every $n \in \mathbb{Z}_+$, choose $A_n \in \mathcal{A}$ that $B_n \subseteq A_n$. Then A_n is a countable subcovering of $\mathcal{B}$.

- For every $n \in \mathbb{Z}_+$, choose $x_n \in B_n$, then $D = \{x_n\}$ is dense. $\forall x \in X$, every basis element containing x intersects D, so $x \in \overline{D}$.

□

Remark:

- A space for which every open cover has a countable subcover is called a **Lindelöf space**.
- A space that has a countable dense subset is called a **separable space**.

Each of the axioms is equivalent to the second countability axiom in metric spaces.

Proof.

- Let $\mathcal{B}_n = \{B(x, 1/n) : x \in X\}$, then it has a countable subcovering $\mathcal{B}'_n$, we let

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}'_n.$$

  which is also countable.

- Let D be a countable dense subset. Let $\mathcal{B} = \{B(x, r) : x \in D, r \in \mathbb{Q}_+\}$, then $\mathcal{B}$ is a countable basis.

□

Example: **The Countability of $\mathbb{R}_l$**

The space $\mathbb{R}_l$ satisfies all countability axioms except second countability.

Example: **Sorgenfrey plane**

The Sorgenfrey plane is the product topology of $\mathbb{R}_l \times \mathbb{R}_l$. It is not Lindelöf.

3.2 The Separation Axioms

The separation axioms are a set of conditions that describe how distinct points and sets can be separated by neighborhoods.

Definition 3.2.1: Separation Axioms

- A space X is **T0** (Kolmogorov) if for every pair of distinct points $x, y \in X$, there exists an open set containing one but not the other.
- A space X is **T1** (Frechet) if for every pair of distinct points $x, y \in X$, there exists an open set containing x but not y and vice versa.
- A space X is **T2** (Hausdorff) if for every pair of distinct points $x, y \in X$, there exist disjoint open sets containing x and y respectively.
- A space X is **T3** (Regular) if it is T1 and for every closed set F and point $x \notin F$, there exists disjoint open sets containing x and F respectively.
- A space X is **T4** (Normal) if it is T1 and for every pair of disjoint closed sets, there exist disjoint open sets containing them.
- A space X is **T5** (completely normal) if it is T1 and for every pair of separated sets, there exist disjoint open sets containing them. Two sets A, B are **separated** if $A \cap \overline{B} = \emptyset$ and $B \cap \overline{A} = \emptyset$.

We can write it in logical form:

- T0: $\forall x, y \in X, x \neq y \implies \exists U \in \mathcal{T}, (x \in U, y \notin U) \vee (y \in U, x \notin U)$.
- T1: $\forall x, y \in X, x \neq y \implies \exists U, V \in \mathcal{T}, x \in U, y \notin U, y \in V, x \notin V$.
- T2: $\forall x, y \in X, x \neq y \implies \exists U, V \in \mathcal{T}, x \in U, y \in V, U \cap V = \emptyset$.
- T3: T1 and $\forall F$ closed, $\forall x \notin F \implies \exists U, V \in \mathcal{T}, x \in U, F \subseteq V, U \cap V = \emptyset$.
- T4: T1 and $\forall F, G$ closed, $F \cap G = \emptyset \implies \exists U, V \in \mathcal{T}, F \subseteq U, G \subseteq V, U \cap V = \emptyset$.
- T5: T1 and $\forall A, B, A \cap \overline{B} = \emptyset, B \cap \overline{A} = \emptyset \implies \exists U, V \in \mathcal{T}, A \subseteq U, B \subseteq V, U \cap V = \emptyset$.

Another way to formulate T_1 is that every singleton set $\{x\}$ is closed. Using this property it is easy to see that

$$T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0.$$

There are other ways to formulate the separation axioms.

Lemma 3.2.1: Another Formulation of T3 and T4

Let X be a T_1 space.

- X is regular iff $\forall x \in X, \forall U$ as neighborhood of x, there exists a neighborhood V of x such that $\overline{V} \subseteq U$.
- X is normal iff $\forall F$ closed, $\forall U \in \mathcal{T}, F \subseteq U$, there exists an open set V such that $F \subseteq V$ and $\overline{V} \subseteq U$.

Proof. Using $X - U$ as another closed set will do. $\square$

Theorem 3.2.1: Subspaces and Products of T2 and T3

- A subspace of a Hausdorff space is Hausdorff. The product of Hausdorff spaces is Hausdorff.
- A subspace of a regular space is regular. The product of regular spaces is regular.

Proof. Let $Y \subseteq X$.

- If U, V are disjoint, so are $U \cap Y$ and $V \cap Y$.

  Let $\{X_\alpha\}$ be a family of Hausdorff spaces, and $x, y \in \prod X_\alpha$ be distinct points. There exists β that $x_\beta \neq y_\beta$. Let $U, V \subseteq X_\beta$ be disjoint open sets containing x_β and y_β respectively. Taking $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$, we have disjoint open sets containing x and y respectively.

- First Y is T_1 for every singleton in Y is closed. Let x, B be the disjoint point and closed set. Then $B = \overline{B} \cap Y$, so $x \notin \overline{B}$. Using the regularity of X, we can find disjoint open sets U, V such that $x \in U$ and $\overline{B} \subseteq V$. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets containing x and B respectively.

  Let $\{X_\alpha\}$ be a family of regular spaces, and $X = \prod X_\alpha$, the previous statement shows that X is Hausdorff so is T_1. Let $x = (x_\alpha) \in X$, U be its neighborhood. Choose a basis element $x \in \prod U_\alpha \subseteq U$. For the finite collection of α that U_α is not the whole space, we can find $x_\alpha \in V_\alpha$ and $\overline{V_\alpha} \subseteq U_\alpha$ for open V_α. Then $\prod V_\alpha$ is a neighborhood of x such that $\overline{\prod V_\alpha} \subseteq U$.

□

There is NO analogous statement for T_4 spaces. The subspace and product of normal spaces is not normal in general.

Example: **Separation Axioms and Real Spaces**

- The space $\mathbb{R}_K$ is Hausdorff but not regular.
- The space $\mathbb{R}_l$ is normal.
- The Sorgenfrey plane $\mathbb{R}_l^2$ is regular but not normal.

3.3 Normal Spaces

Theorem 3.3.1: Regular Second Countable Implies Normal

Let X be a second countable regular space. Then X is normal.

Proof. Let $\mathcal{B}$ be a countable basis, and A, B are disjoint closed set in X. $\forall x \in A$, there is a neighborhood U_x that does not intersect B. Using regularity, let V_x be a neighborhood that $\overline{V_x} \subseteq U$, and $B_x \in \mathcal{B}, x \in B_x \subseteq V_x$. So we construct a countable cover of A by B_x, denoted $U = \bigcup_{n=1}^\infty U_n$. (The element U_n are the countable B_x that cover A.)

Similarly, we construct a countable cover $V = \bigcup_{n=1}^\infty V_n$ of B. Now we construct U', V' that are disjoint.

Define

$$U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i}, \qquad V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}.$$

Then U_n' and V_n' are open. Let

$$U' = \bigcup_{n=1}^{\infty} U_n', \qquad V' = \bigcup_{n=1}^{\infty} V_n'.$$

Then if $x \in U' \cap V'$, then $x \in U_j' \cap V_k'$ for some $j, k \in \mathbb{Z}_+$. If $j \leq k$, then contradicts. So U' and V' are disjoint open sets containing A and B respectively. Thus X is normal. $\square$

B_x is a basis satisfying $x \in B_x, \overline{B_x} \subseteq U_x \subseteq X - B$

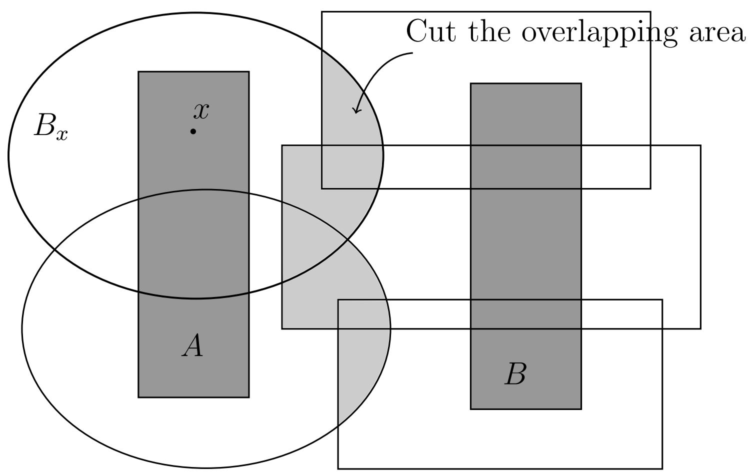

Figure 3.1: Regular Second Countable and Normal

Theorem 3.3.2: Metrizable Implies Normal

Let X be a metrizable space. Then X is normal.

Proof. Let d be a metric on X. Let A, B be disjoint closed sets in X. For every $a \in A$, let $B(a, \epsilon_a) \cap B = \emptyset$, and for every $b \in B$, let $B(b, \epsilon_b) \cap A = \emptyset$. Then let

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2), \qquad V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

We shall see that U and V are disjoint open sets containing A and B respectively. If $x \in U \cap V$, then there exists $a \in A$ and $b \in B$ such that $d(x, a) < \epsilon_a/2$ and $d(x, b) < \epsilon_b/2$. Then the triangle inequality gives us

$$d(a, b) \leq d(a, x) + d(x, b) < \epsilon_a/2 + \epsilon_b/2 \leq \max\{\epsilon_a, \epsilon_b\}.$$

which is a contradiction. $\square$

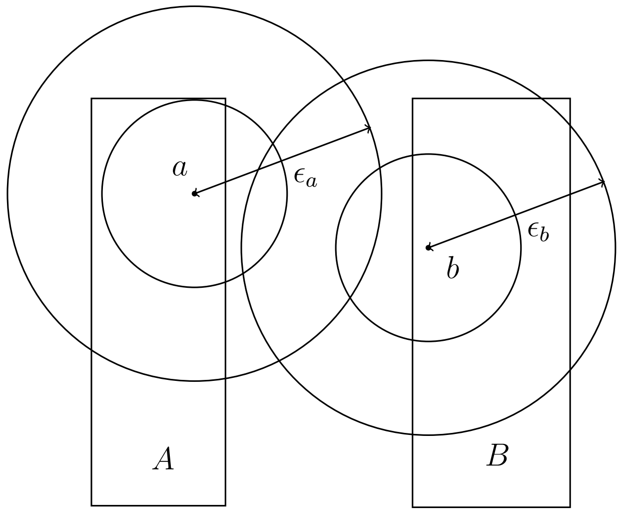

Figure 3.2: Metrizable and Normal

Theorem 3.3.3: Compact Hausdorff Implies Normal

Let X be a compact Hausdorff space. Then X is normal.

Proof. First we prove X is regular. For all F closed and $x \notin F$, since X is Hausdorff, for all $y \in F$, there exist disjoint open sets U_y, V_y such that $x \in U_y$ and $y \in V_y$. There are finite y that $\bigcup V_y$ covers F, taking the intersection of all U_y would do. (Finiteness)

Similar argument applies to show that X is normal. □

Theorem 3.3.4: Order Topology is Normal

Let X be a simply ordered set with the order topology. Then X is normal.

We can prove a weaker statement:

Every well-ordered set is normal in the order topology.

Example: **Spaces that are not Normal**

- If J is uncountable, then $\mathbb{R}^J$ is not normal.
- The product space $S_\Omega \times \overline{S_\Omega}$ is not normal, where S_Ω is the minimal uncountable well-ordered set.

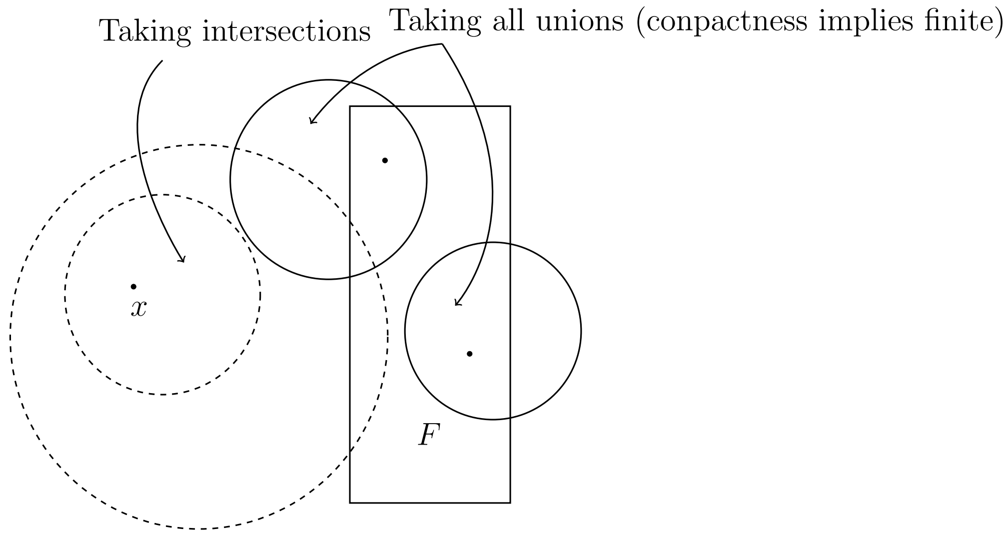

Figure 3.3: Compact Hausdorff and Regularity

3.4 The Urysohn Lemma

Theorem 3.4.1: Urysohn Lemma

Let X be a normal space, and A, B be disjoint closed sets in X. Let $[a, b] \subseteq \mathbb{R}$, then there exists a continuous function $f : X \to [a, b]$ such that $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$.

Proof. We shall only consider $[a, b] = [0, 1]$.

- **Step 1:** Let $P = [0, 1] \cap \mathbb{Q}$. $\forall p \in P$, we can find an open set U_p such that $\forall p, q \in P, p < q \to \overline{U_p} \subseteq U_q$.

  For P being countable, we can use induction to define the U_p. Let $P = \{p_n\}_{n=1}^{\infty}$. Assume $p_1 = 1, p_2 = 0$. Let $P_n = \{p_1, \ldots, p_n\}$.

  Define $U_1 = X - B$, choose U_0 such that $A \subseteq U_0, \overline{U_0} \subseteq U_1$. (this is possible because X is normal). Suppose $\forall p \in P_n$, we have U_p defined such that $\forall p, q \in P_n, p < q \to \overline{U_p} \subseteq U_q$. Let $r = p_{n+1}$, and we need to define U_r.

  In $P_{n+1} = P_n \cup \{r\}$, it is a finite simply ordered set in $[0, 1]$. Let p be the immediate predecessor of r in P_{n+1}, and q the immediate successor. Then we define U_r to be an open set

  $$\overline{U_p} \subseteq U_r, \qquad \overline{U_r} \subseteq U_q.$$

  Then we say that the condition holds for P_{n+1}.

- **Step 2:** We extend the definition of U_p to $p \in \mathbb{Q}$.

$$U_p = \begin{cases} \emptyset, & \text{if } p < 0 \\ U_p \text{ defined above }, & \text{if } 0 \le p \le 1 \,, \qquad p \in \mathbb{Q}. \\ X, & \text{if } p > 1 \end{cases}$$

  We still have $\forall p, q \in \mathbb{Q}, p < q \to \overline{U_p} \subseteq U_q$.

- **Step 3:** Define

$$Q(x) = \{p \in \mathbb{Q}, x \in U_p\} \,.$$

  It is clear that any $p < 0$ is not in $Q(x)$, then $Q(x)$ is bounded below. Define $f(x) = \inf Q(x)$. Then we have $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

- **Step 4:** We show that f is continuous. First we have two results:

  - $x \in \overline{U_r} \to f(x) \le r$.
  - $x \notin U_r \to f(x) \ge r$.

  These two results are obvious. Next $\forall x_0 \in X, \forall (c, d) \subseteq \mathbb{R}, f(x_0) \in (c, d)$, we need to find a neighborhood U of x_0 such that $f(U) \subseteq (c, d)$. Choose rational p, q such that

$$c < p < f(x_0) < q < d$$

  Then we find $U = U_q - \overline{U_p}$ to be the desired neighborhood.

□

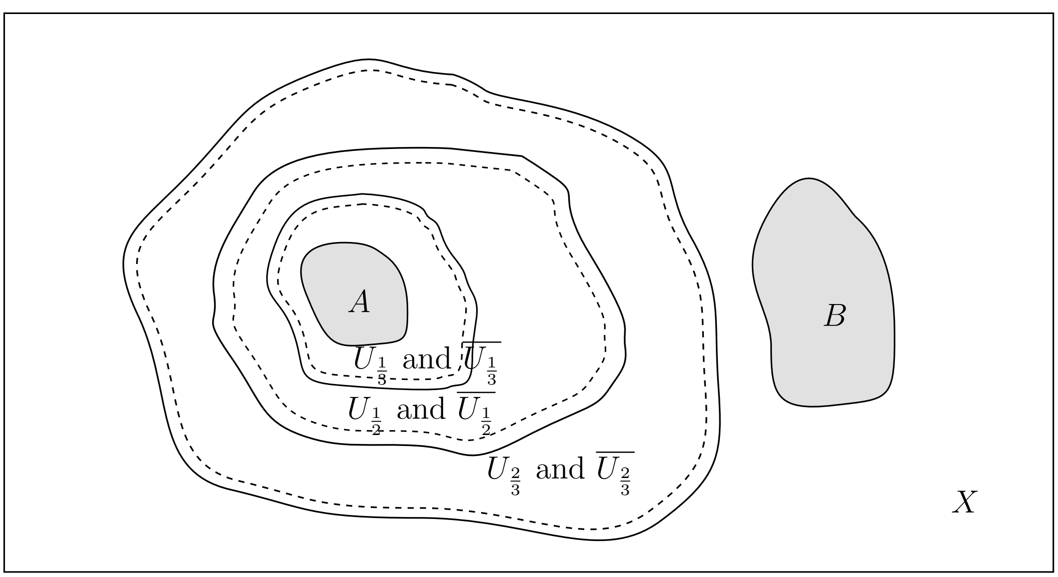

Figure 3.4: Urysohn Lemma

Definition 3.4.1: Separation by Continuous Functions

If $A, B \subseteq X$, and $\exists$ a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B can be **separated by continuous functions**.

In this way, the Urysohn lemma can be stated:

> If every pair of disjoint closed sets can be separated by disjoint open sets, then every such pair can be separated by continuous functions.

Remark:
The regularity is not strong enough for the separation of a point and a closed set by a continuous function. (Similar proof fails at step 1, when we want to find a U_r between U_p and U_q.

Definition 3.4.2: Completely Regular

A space X is completely regular if it is T1 and $\forall$ closed set A and $\forall x_0 \notin A$, there is a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = 0$.

Sometimes this is called T3.5. By Urysohn lemma, we have

$$T_4 \implies T_{3.5} \implies T_3$$

Completely regular spaces are well-behaved under subspaces and products.

Theorem 3.4.2: Subspaces and Products of Completely Regular Spaces

A subspace of a completely regular space is completely regular, and a product of a completely regular space is completely regular.

Proof. Let X be completely regular. Let $Y \subseteq X$, and A is closed under Y, $x_0 \in Y - A$. We choose $f : X \to [0,1]$ that $f(x_0) = 1$ and $f(\overline{A}) = \{0\}$. The restriction of f onto Y is what we want.

Let $X = \prod X_\alpha$ be a product of completely regular spaces, and $b \in X$, A be a closed set in X disjoint from b. Let $\prod U_\alpha$ be a neighborhood of b that does not intersect A. Then $U_\alpha \neq X_\alpha$ for finite many $\alpha_1, \ldots, \alpha_n$. Choose continuous function:

$$f_i : X_{\alpha_i} \to [0,1], \qquad f_i(b_{\alpha_i}) = 1, f_i(X - U_{\alpha_i}) = 0.$$

and $\phi_i(x) = f_i(\pi_{\alpha_i}(x))$. Then

$$f(x) = \phi_1(x) \cdots \phi_n(x)$$

is the desired continuous function. It is clear that $f(b) = 1$ and $f(A) = 0$. $\square$

Example: **Completely regular and not Normal**

The sets $\mathbb{R}_l^2$ and $S_\Omega \times \overline{S_\Omega}$ are completely regular but not normal.

3.5 The Urysohn Metrization Theorem

We use two versions of proof here, both has generalizations.

Theorem 3.5.1: Urysohn Metrization Theorem

Let X be a second countable regular space. Then X is metrizable.

Proof. From theorem 3.3.1, we have X being second countable normal. We shall prove that X is metrizable be imbedding it into a known metrizable space Y.

- **Step 1:** Lemma: *There exists a countable collection of continuous function* $f_n : X \to [0,1]$ *that* $\forall x_0 \in X$ *and* $\forall$ *neighborhood* U *of* x_0*, there exists* n *such that* $f_n(x_0) > 0$ *and* $f_n(X - U) = \{0\}$*.* (Like a "bump" function that lumps at x_0 and vanishes outside U.)

  By normality of X and Urysohn lemma, we have the required function f exists for each pair (x, U). Our goal is to limit the quantity. Let $\{B_n\}$ be a countable basis of X, for each $n, m \in \mathbb{Z}_+, \overline{B_n} \subseteq B_m$ (from normality this pair exists and has infinite), from Urysohn lemma choose c continuous function $g_{n,m} : X \to [0,1]$ that $g_{n,m}(\overline{B_n}) = \{1\}, g_{n,m}(X - B_m) = \{0\}$. Then the collection $g_{n,m}$ is countable and satisfies our condition. (Choose $x_0 \in B_m \in U$ and $x_0 \in B_n, \overline{B_n} \in B_m$).

- **Step 2: First Version of Proof** Given the function in step 1 and take $\mathbb{R}^{\mathbb{Z}_+}$ in the product topology. Define $F : X \to \mathbb{R}^{\mathbb{Z}_+}$ that

  $$F(x) = (f_1(x), f_2(x), \ldots) \tag{3.1}$$

  Then we conclude that F is an imbedding.

  First F is continuous: from 1.8.6 and f_n is continuous. F is injective: $\forall x \neq y$, we have have and n that $f_n(x) > 0$ and $f_n(y) = 0$. So $F(x) \neq F(y)$.

  Finally, F is a homeomorphism $X \to Z = F(X)$. Let $U \in \mathcal{T}_X$, let $z_0 \in F(U), z_0 = F(x_0)$. Take N that $f_N(x_0) > 0$ and $f_N(X - U) = \{0\}$. Define

  $$V = \pi_N^{-1}((0, +\infty))$$

  in $\mathbb{R}^{\mathbb{Z}_+}$, and $W = V \cap Z$. Then W is open in Z. We assert that $z_0 \in W \subseteq F(U)$: First $z_0 \in W$ for $z_N = f_N(x_0) > 0$. Then $W \subseteq F(U)$ because if $z \in W$, then $z = F(x)$ for some $x \in X$, and $\pi_N(z) \in (0, \infty)$. Since $\pi_N(z) = f_N(x)$ so we have $x \in U$, so $z \in F(U)$.

  So we've proved that F is an imbedding of X in $\mathbb{R}^{\mathbb{Z}_+}$.

- **Step 3: Second Version of Proof** In this version, we still use $\mathbb{R}^{\mathbb{Z}_+}$ but do not use the product topology, instead, we use the uniform topology induced by the metric space $(\mathbb{R}^{\mathbb{Z}_+}, \overline{\rho})$. Actually, we imbed it on $[0,1]^{\mathbb{Z}_+}$, on which $\overline{\rho}$ is defined

  $$\overline{\rho}(x, y) = \rho(x, y) = \sup\{|x_i - y_i|\}$$

  Let f_n be constructed by step 1, assume $f_n(x) < 1/n$ for all $x \in X$, (dividing n would do) Define $F : X \to [0,1]^{\mathbb{Z}_+}$ by

  $$F(x) = (f_1(x), f_2(x), \ldots)$$

First, F is injective from step 2, and F maps open sets to open sets in product topology so does in uniform topology, which is finer.

To prove F being continuous, (we cannot use theorem 1.8.6 because we're not using product topology now). Let $x_0 \in X$ and $\epsilon > 0$, and we find a neighborhood U of x_0 that

$$x \in U \to \rho(F(x), F(x_0)) < \epsilon$$

choose N that $1/N < \epsilon/2$. For each $n \in \{1, \dots, N\}$ choose a neighborhood U_n of x_0 that

$$\forall x \in U_n, |f_n(x) - f_n(x_0)| \leq \frac{\epsilon}{2}$$

Denote $U = \bigcap_{n=1}^{N} U_n$ would do. For $x \in U$, if $n < N$ we have $|f_n(x) - f_n(x_0)| \leq \frac{\epsilon}{2}$ and if $n \geq N$, we have $|f_n(x) - f_n(x_0)| \leq \frac{1}{N} \leq \frac{\epsilon}{2}$. So we have

$$\rho(F(x), F(x_0)) = \sup_{n \in \mathbb{Z}_+} |f_n(x) - f_n(x_0)| \leq \frac{\epsilon}{2} < \epsilon$$

Remark:
The condition means that F is only prominently different from zero at low indices.

□

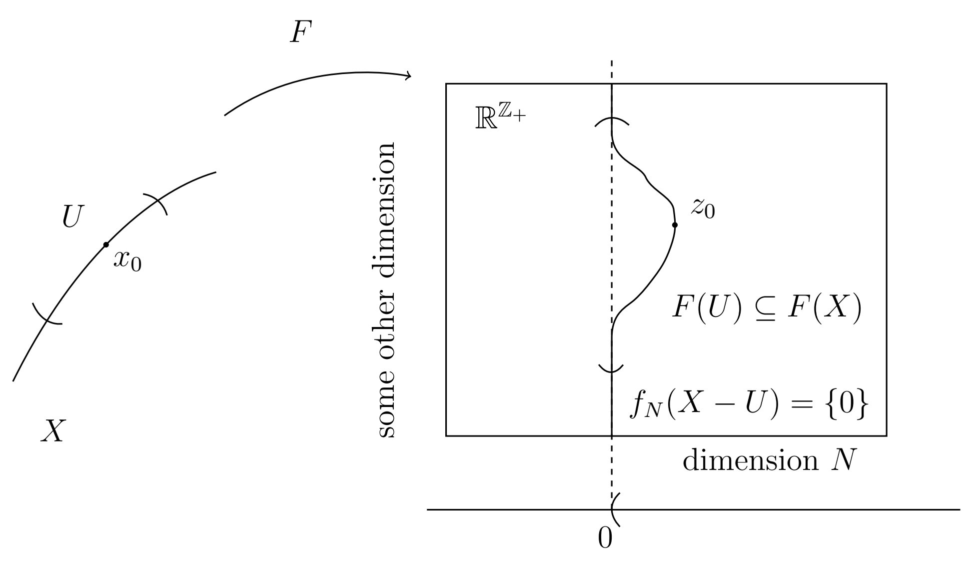

Figure 3.5: Urysohn Metrization Theorem

The step 2 from the proof of Urysohn Metrization Theorem also proved a stronger result:

Theorem 3.5.2: Imbedding Theorem

Let X be a T1 space. Suppose $\{f_\alpha\}_{\alpha\in J}$ is a family of continuous functions from X to $\mathbb{R}$ such that

$$\forall x_0 \in X, \forall \text{ neighborhood } U \text{ of } x_0,\ \exists \alpha \in J, f_\alpha(x_0) > 0 \wedge f_\alpha(X - U) = \{0\}\,.$$

Then the function $F : X \to \mathbb{R}^J, x \mapsto (f_\alpha(x))_{\alpha\in J}$ is an imbedding of X is $\mathbb{R}^J$. If f_α maps X to $[0,1]$, then F imbeds X in $[0,1]^J$. (All in the product topology)

A family of continuous functions that satisfies this condition is said to **separate points from closed sets in** X. And a completely regular space (T3.5) has exactly this property. Reversely, the space $[0,1]^J$ is also completely regular due to theorem 3.4.2, so we have

Theorem 3.5.3: Characterization of Completely Regular Space

A space X is completely regular iff it is homeomorphic to some subspace of $[0,1]^J$ for some J.

3.6 The Tietze Extension Theorem

The Tietze extension theorem is a direct consequence of the Urysohn lemma. It extends a continuous function defined on a subspace of X to the whole space X.

Theorem 3.6.1: Tietze Extension Theorem

Let X be a normal space. Let $A \subseteq X$ be closed.

- Let $f : A \to [a,b]$ be any continuous function. Then there exists a continuous function $\overline{f} : X \to [a,b]$ such that $\overline{f}|_A = f$.
- Let $f : A \to \mathbb{R}$ be any continuous function. Then there exists a continuous function $\overline{f} : X \to \mathbb{R}$ such that $\overline{f}|_A = f$.

Proof. The idea would be to construct a uniformly convergent sequence s_n of continuous function.

- **Step 1:** Consider $f : A \to [-r, r]$. We construct a continuous $g : X \to \mathbb{R}$ that
  - $|g(x)| \leq \frac{1}{3}r$ for all $x \in X$.
  - $|f(x) - g(x)| \leq \frac{2}{3}r$ for all $x \in A$.

□

3.7 Imbedding of Manifolds

We've already know that every second countable regular space is metrizable can can be imbedded in $\mathbb{R}^{\mathbb{Z}_+}$. But for manifolds, we can do better to induce then in a finite dimensional Euclidean space.

Definition 3.7.1: Manifold

An m-manifold is a Hausdorff second countable space X such that every point $x \in X$ has a neighborhood homeomorphic to an open set in $\mathbb{R}^m$.

We shall prove that a compact manifold (thus normal from 3.3.3) can be imbedded in a finite dimensional Euclidean space. (This is also true for non-compact manifolds, but the proof is more complicated.)

Theorem 3.7.1: Support

Let X be a topological space, and $f : X \to \mathbb{R}$ be a continuous function. The **support** of f is the closure of the set $\{x \in X, f(x) \neq 0\}$, denoted $\operatorname{supp}(f)$. i.e

$$\operatorname{supp} f = \overline{f^{-1}(\mathbb{R} - \{0\})} \tag{3.2}$$

Thus if $x \notin \operatorname{supp} f$ then there is a neighborhood U of x such that $f(U) = \{0\}$.

Definition 3.7.2: Partition of Unity

Let $\{U_1, \dots, U_n\}$ be a finite collection of open sets that cover X. A family of continuous functions

$$\phi_i : X \to [0, 1], \qquad i = 1, \dots, n \tag{3.3}$$

is a **partition of unity** dominated by $\{U_i\}$ if

- $\forall i, \operatorname{supp}(\phi_i) \subseteq U_i$.
- $\forall x \in X, \sum_{i=1}^n \phi_i(x) = 1$.

Remark:
A support is like the nonzero bump of a function, and a partition of unity is like that every position is covered by some bump, and the sum of all bumps at each position is 1.

Theorem 3.7.2: Existence of Partition of Unity

Let X be a normal space, and $\{U_1, \dots, U_n\}$ be a finite open cover of X. Then there exists a partition of unity dominated by $\{U_i\}$.

Proof.

- **Step 1:** We shall prove that we can shrink $\{U)i\}$ a little to an open covering $\{V_i\}$ that $\overline{V_i} \subseteq U_i$ for all i. Proceed by induction.

  The set $A = X - (U_2 \cup \cdots \cup U_n)$ is closed, and $A \subseteq U_1$. Using normality, take open V_1 that $A \subseteq V_1, \overline{V_1} \subseteq U_1$. Then $V_1, U_2, \dots, U_n$ covers X. The following proceed in an obvious way.

- **Step 2:** Let $\{V_i\}$ be as in step 1, and also choose an open covering $\{W_i\}$ that $\overline{W_i} \subseteq V_i$. By Urysohn lemma, choose continuous functions

$$\psi_i : X \to [0, 1], \qquad \psi_i(\overline{W_i}) = \{1\}, \psi_i(X - V_i) = \{0\}.$$

Then $\operatorname{supp}\psi_i \subseteq \overline{V_i} \subseteq U_i$. As $\{W_i\}$ covers X, $\forall x \in X, \exists i$ such that $x \in W_i$, so $\psi_i(x) = 1$. Thus $\Psi(x) = \sum_{i=1}^{n} \psi_i(x) \geq 1$ for all $x \in X$. Denote

$$\phi_i(x) = \frac{\psi_i(x)}{\Psi(x)}$$

would do.

□

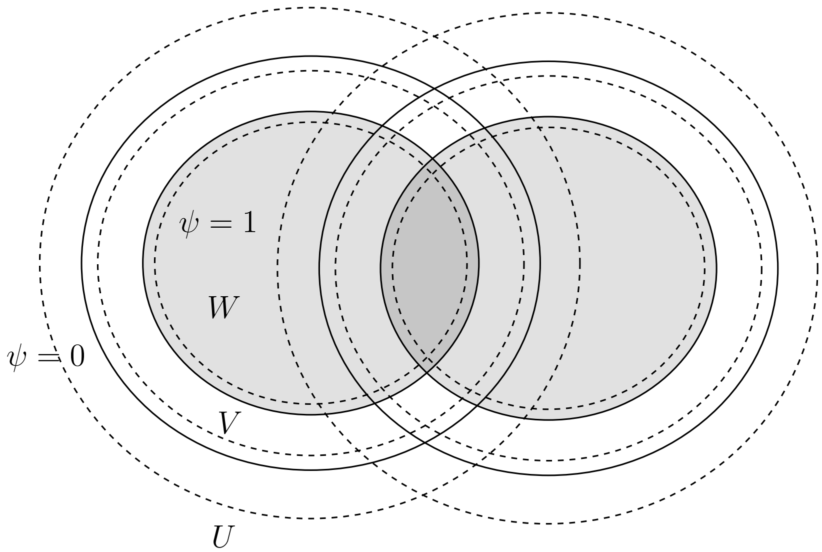

Figure 3.6: Partition of Unity

Theorem 3.7.3: Imbedding of Compact Manifolds

Let X be a compact m-manifold. Then there exists an imbedding of X in $\mathbb{R}^N$ for some $N \in \mathbb{Z}_+$.

Proof. Cover X my finite many open sets $\{U_1, \ldots, U_n\}$, each homeomorphic to an open set in $\mathbb{R}^m$ (this is possible form compactness: every point has a neighborhood homeomorphic to an open set in $\mathbb{R}^m$, and we can extract a finite subcover). Let $g_i : U_i \to \mathbb{R}^m$ be imbeddings.

Being compact Hausdorff, X is normal from theorem 3.3.3, so we can find a partition of unity $\{\phi_i\}$ dominated by $\{U_i\}$. Let $A_i = \operatorname{supp}\phi_i$, define $h_i : X \to \mathbb{R}^m$ by

$$h_i(x) = \begin{cases} \phi_i(x)g_i(x), & \text{if } x \in U_i \\ 0, & \text{if } x \notin A_i \end{cases}$$

Remark:
Here h_i is the stretching of image of g_i by ϕ_i.

Then h_i is well-defined and continuous. Define

$$F : X \to \mathbb{R}^n \times (\mathbb{R}^m)^n \cong \mathbb{R}^{n(m+1)}, \qquad F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$$

So F is continuous. Suppose $F(x) = F(y)$, then $\phi_i(x) = \phi_i(y)$ for all i. Since $\sum \phi_i(x) = 1$, there exists i such that $\phi_i(x) > 0$, so $\phi_i(y) > 0$ and $x, y \in U_i$. Then $h_i(x) = h_i(y)$ gives us $g_i(x) = g_i(y)$, so $x = y$. Thus F is injective.

We have F being an imbedding from theorem 2.4.4, as X is compact and $\mathbb{R}^{n(m+1)}$ is Hausdorff, so F is a homeomorphism of X onto $F(X)$. $\square$

Chapter 4

The Tychonoff Theorem

4.1 The Tychonoff Theorem

In previous chapters, we discussed the compactness of the product of two compact spaces $X \times Y$, by covering sliced $x \times Y$ by finite subcovers and then using these to cover $X \times Y$. However, it is rather tricky to extend this argument to an arbitrary product of compact spaces, one must well-order the index set and use transfinite induction. Another way is to tackle the problem is to use the closed set definition of compactness, using Zorn's lemma.

For simplicity, consider the product of two compact spaces $X_1 \times X_2$. Let $\mathcal{A}$ be a collection of closed subsets of $X_1 \times X_2$ with the finite intersection property, that is, any finite intersection of elements of $\mathcal{A}$ is non-empty. We want to show that the intersection of all elements of $\mathcal{A}$ is non-empty.

- The projection collection $\{\pi_1(A) : A \in \mathcal{A}\}$ is a collection of closed subsets of X with the finite intersection property, so are their closure $\{\overline{\pi_1(A)} : A \in \mathcal{A}\}$. Since X is compact, $\bigcap_{A\in\mathcal{A}} \overline{\pi_1(A)}$ is non-empty. The result holds for π_2 also.

- Take $x_1 \in \bigcap_{A\in\mathcal{A}} \overline{\pi_1(A)}$ and $x_2 \in \bigcap_{A\in\mathcal{A}} \overline{\pi_2(A)}$. We wish that $x_1 \times x_2 \in \bigcap_{A\in\mathcal{A}} A$. However, this is not necessarily true. Consider the following counterexample: $X_1 = X_2 = [0, 1]$ and $p = (1/3, 1/3)_p, q = (1/2, 2/3)_p$, then $\mathcal{A}$ being all elliptical closed sets with p, q as focii has the finite intersection property, but we show that $(1/2, 1/2)_p \notin A$ for any $A \in \mathcal{A}$. To get a valid point, we need to choose on the segment pq.

  So we need a better choice of x_1, x_2, the current choice is to free.

- We can expand the collection $\mathcal{A}$ to $\mathcal{D}$, retaining the finite intersection property. For example, we can add all all circles with p as center. This forces us to choose $x_1 = 1/3$ and $x_2 = 1/3$. The new collection still has the finite intersection property.

- To determine the correct expansion, considering choosing $\mathcal{D}$ "as large as possible", so that no larger collection containing $\mathcal{D}$ has the finite intersection property. This is where Zorn's lemma comes in.

Remark:
Note that we do not use the closedness of elements of $\mathcal{A}$ until the very end, when we need to show the closed set formulation of compactness. We may as well begin with an arbitrary

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collection of subsets of X with the finite intersection property, and expand it to a maximal such collection.

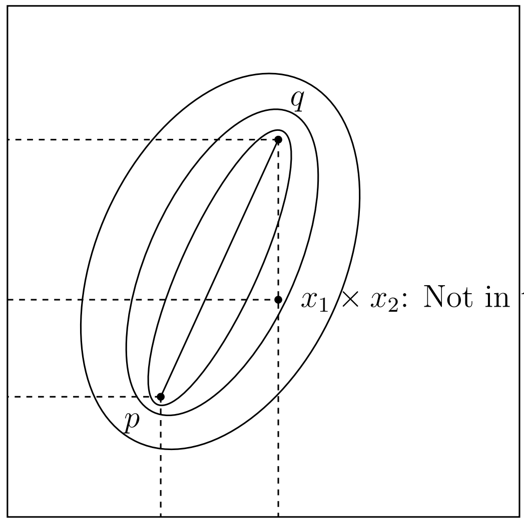

Figure 4.1: Conterexample for Product Common Point

Lemma 4.1.1: Maximal Collection with Finite Intersection Property

Let X be any set, and $\mathcal{A} \subseteq P(X)$ having the finite intersection property. Then there exists a collection $\mathcal{D} \subseteq P(X)$ such that $\mathcal{A} \subseteq \mathcal{D}$, $\mathcal{D}$ has the finite intersection property, and any collection of subsets of X properly containing $\mathcal{D}$ does not have the finite intersection property. The collection $\mathcal{D}$ is called a **maximal collection with the finite intersection property** containing $\mathcal{A}$. Then

- Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.
- If $A \subseteq X$ such that $A \cap D \neq \emptyset$ for all $D \in \mathcal{D}$, then $A \in \mathcal{D}$.

Proof. Let the set

$$\mathscr{A} = \{\mathcal{C} \subseteq P(X) : \mathcal{A} \subseteq \mathcal{C}, \mathcal{C} \text{ has the finite intersection property}\}.$$

The partial order is given by set inclusion. For any chain $\mathscr{B} \subseteq \mathscr{A}$, let

$$\mathcal{E} = \bigcup_{\mathcal{C} \in \mathscr{B}} \mathcal{C}.$$

We shall show that $\mathcal{E} \in \mathscr{A}$, so that $\mathcal{E}$ is an upper bound of $\mathscr{B}$. Clearly, $\mathcal{A} \subseteq \mathcal{E}$. To show that $\mathcal{E}$ has the finite intersection property, let $E_1, E_2, \ldots, E_n \in \mathcal{E}$. Then for each i, there exists $\mathcal{C}_i \in \mathscr{B}$

such that $E_i \in \mathcal{C}_i$. Since $\mathscr{B}$ is a chain, there exists some $\mathcal{C} \in \mathscr{B}$ such that $\mathcal{C}_i \subseteq \mathcal{C}$ for all i. Thus $E_i \in \mathcal{C}$ for all i, and since $\mathcal{C}$ has the finite intersection property, we have $\bigcap_{i=1}^{n} E_i \neq \emptyset$. By Zorn's lemma, $\mathscr{A}$ has a maximal element $\mathcal{D}$, which is the desired collection.

The two properties follows smoothly from the maximally of $\mathcal{D}$. For the first property, if not, joining the finite intersection to $\mathcal{D}$ gives a larger collection with the finite intersection property, contradicting maximality. For the second property, if not, joining A to $\mathcal{D}$ gives a larger collection with the finite intersection property, contradicting maximality. $\square$

Remark:
The maximal collection $\mathcal{D}$ is not unique. For example, if $X = \{1, 2\}$ and $\mathcal{A} = \{X\}$, then both $\mathcal{D}_1 = \{\{1\}, X\}$ and $\mathcal{D}_2 = \{\{2\}, X\}$ are maximal collections with the finite intersection property containing $\mathcal{A}$.

Theorem 4.1.1: Tychonoff Theorem

Let $\{X_\alpha : \alpha \in J\}$ be an indexed family of compact topological spaces. Then the product space $X = \prod_{\alpha \in J} X_\alpha$ is compact in the product topology.

Proof. Let $\mathcal{A}$ be a collection of subsets of X with the finite intersection property. We prove the collection $\bigcap_{A \in \mathcal{A}} \overline{A}$ is non-empty, where the closure is taken in X. By Lemma 4.1.1, choose $\mathcal{D}$ a maximal collection with the finite intersection property containing $\mathcal{A}$. For each $\alpha \in J$, let π_α be the projection map from X to X_α, and let

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)},$$

the intersection is not empty since $\{\overline{\pi_\alpha(D)} : D \in \mathcal{D}\}$ is a collection of closed subsets of the compact space X_α with the finite intersection property. Let $x = (x_\alpha)_{\alpha \in J} \in X$. We show that $x \in \overline{D}$ for all $D \in \mathcal{D}$, which implies that $x \in \overline{A}$ for all $A \in \mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{D}$.

From the second property of Lemma 4.1.1, we know that all subbasis $\pi_\beta^{-1}(U_\beta)$ containing x intersects D, because $x_\beta \in \overline{\pi_\beta(D)}$ implies $\pi_\beta(D) \cap U_\beta \neq \emptyset$. Since finite intersections of subbasis elements form a basis, any basis element containing x intersects D. Thus $x \in \overline{D}$ for all $D \in \mathcal{D}$. $\square$

4.2 The Stone-Čech Compactification

We have already studied one-point compactification, which is some sense the minimal compactification. The Stone-Čech compactification is the maximal compactification.

Definition 4.2.1: Compactification

A compactification of a space X is a compact Hausdorff space Y that $\overline{X} = Y$. Two compactification Y_1, Y_2 of X are said to be equivalent if there exists a homeomorphism $f : Y_1 \to Y_2$ such that $f(x) = x$ for all $x \in X$.

If X has a compactification, then X is completely regular because Y is completely regular. Conversely, if X is completely regular, then X has a compactification because X can be embedded in some $[0, 1]^J$.

Lemma 4.2.1: Compactification by Imbedding

Let X be a space, and $h : X \to Z$ be an imbedding of X into a compact Hausdorff space Z. Then there is a compactification Y of X that there is an imbedding $H : Y \to Z$ that $H|_X = h$. Y is uniquely determined up to equivalence.
We call Y the compactification of X induced by the imbedding h.

Proof. Let $X_0 = h(X) \subseteq Z$, and $Y_0 = \overline{X_0} \subseteq Z$. Then Y_0 is compact Hausdorff so is a compactification of X_0.

Now we construct Y: let Y be the set $X \cup (Y_0 - X_0)$, and define $H : Y \to Y_0$ by

$$H(y) = \begin{cases} h(y) & y \in X \\ y & y \in Y_0 - X_0 \end{cases}.$$

Then H is a bijection. We give Y the topology such that H is a homeomorphism. Then Y is a compactification of X, and $H|_X = h$. ☐

Proposition: **Criterion for Compactifiablility**

A space X has a compactification if and only if X is completely regular.

Usually there are various ways to compactify a given space.

Example: **Compactification**

Compactify $(0, 1) \subseteq \mathbb{R}$:

- Take $h : (0, 1) \to S^1$ by $h(x) = (\cos 2\pi x, \sin 2\pi x)$, then the compactification induced by h is S^1, which is just the one-point compactification.
- Take $h : (0, 1) \to [0, 1]$ by $h(x) = x$, then the compactification induced by h is $[0, 1]$.
- Take $h : (0, 1) \to [0, 1]^2$ by $h(x) = (x, \sin 1/x)_p$, then the compactification induced by h is the topologist's sine curve.

For many occasions, we wish that a continuous real function on X can be extended to a continuous real function on the compactification. First, f must be bounded, since the image $f(Y)$ is compact in $\mathbb{R}$ thus is bounded. But this is not sufficient, taking $f(x) = \sin 1/x$ on $(0, 1)$ for example.

Remark:
In the third case, f can be extended to the compactification: Take the composite function $\pi_2 \circ H$, that $x \to x \times \sin 1/x \to \sin 1/x$.
But in the first and second case, it can't.
Actually, this gives us an idea, that to extend a whole collection of bounded continuous functions on X, we can use them as component functions for an imbedding on $\mathbb{R}^J$ for some J, and thereby get a compactification for which every function in the collection can be extended.

This gives us the intuition behind the Stone-Čech compactification.

Theorem 4.2.1: Stone-Čech Compactification

Let X be a completely regular space. There exists a compactification of X that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map $\overline{f} : Y \to \mathbb{R}$.

Proof. Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous functions from X to $\mathbb{R}$. For each $\alpha \in J$, let I_α be a closed bounded interval containing the image of f_α. Specifically, choose

$$I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)].$$

Then define

$$h : X \to \prod_{\alpha \in J} I_\alpha, \qquad h(x) = (f_\alpha(x))_{\alpha \in J}.$$

By the Tychonoff theorem, $\prod_{\alpha \in J} I_\alpha$ is compact Hausdorff. Since X is completely regular, h separates points from closed sets, so h is an imbedding. Let Y be the compactification of X induced by h. Then there is a unique imbedding $H : Y \to \prod_{\alpha \in J} I_\alpha$ such that $H|_X = h$. Take $\overline{f} = \pi_\alpha \circ H$, where π_α is the projection map from $\prod_{\alpha \in J} I_\alpha$ to I_α. Then $\overline{f}$ is the desired extension of f. □

The uniqueness follows from the next lemma.

Lemma 4.2.2: Uniqueness of Extended Functions

Let $A \subseteq X$, and $f : A \to Z$ be a continuous map, where Z is Hausdorff. Then there exists at most one continuous map $\overline{f} : \overline{A} \to Z$ such that $\overline{f}|_A = f$.

Proof. Suppose that there are two such maps $\overline{f}_1, \overline{f}_2$. Let $x \in \overline{A}$, such that $\overline{f}_1(x) \neq \overline{f}_2(x)$. Since Z is Hausdorff, there exist disjoint open neighborhoods U_1, U_2 of $\overline{f}_1(x), \overline{f}_2(x)$ respectively. Then $\overline{f}_1^{-1}(U_1)$ and $\overline{f}_2^{-1}(U_2)$ are disjoint open neighborhoods of x in $\overline{A}$. Since $x \in \overline{A}$, there exists some $a \in A$ such that $a \in \overline{f}_1^{-1}(U_1) \cap \overline{f}_2^{-1}(U_2)$. But then $\overline{f}_1(a) = f(a) = \overline{f}_2(a)$, contradicting the disjointness of U_1 and U_2. Thus $\overline{f}_1 = \overline{f}_2$. □

Theorem 4.2.2: Universal Property of Stone-Čech Compactification

Let X be a completely regular space, and Y be the Stone-Čech compactification of X. Given any continuous map $f : X \to C$, where C is a compact Hausdorff space, there exists a unique continuous map $\overline{f} : Y \to C$ such that $\overline{f}|_X = f$.

Proof. Note that C is completely regular so can be imbeded into $[0,1]^J$ form some J. So assume $C \subseteq [0,1]^J$, the for each bounded real f_α, there is a unique extension $g_\alpha : Y \to \mathbb{R}$, and $g : Y \to \mathbb{R}^J$ where $g(y) = (g_\alpha(y))_{\alpha \in J}$. Then $g|_X = f$, and $g(Y) \subseteq C$ since g is continuous and

$$g(Y) = g(\overline{X}) \subseteq \overline{g(X)} = \overline{f(X)} \subseteq \overline{C} = C.$$

thereby giving the desired extension $\overline{f} = g$. The uniqueness follows from the uniqueness of each g_α. □

Theorem 4.2.3: Uniqueness of Stone-Čech Compactification

Let X be a completely regular space, and Y_1, Y_2 be two Stone-Čech compactifications of X. Then Y_1 and Y_2 are equivalent.

Proof. Consider the inclusion $j_2 : X \to Y_2$, which is a continuous map, from theorem 4.2.2, we may extend j_2 to $f_2 : Y_1 \to Y_2$ being a continous map. Similarly, we may extend the inclusion $j_1 : X \to Y_1$ to $f_1 : Y_2 \to Y_1$ being a continuous map. Then $f_1 \circ f_2 : Y_1 \to Y_1$ is a continuous map such that $(f_1 \circ f_2)|_X = id_X$. By the uniqueness of extension, $f_1 \circ f_2 = id_{Y_1}$. Similarly, $f_2 \circ f_1 = id_{Y_2}$. Thus f_1, f_2 are homeomorphisms, and Y_1, Y_2 are equivalent. $\square$

Chapter 5

Metrization Theorems and Paracompactness

The Urysohn Metrization Theorem states that a topological space is metrizable if it is regular and has a countable basis. And we hope to strengthen the condition to give an equivalent condition.

We know that the regularity is necessary, but the second countability is not. The eventual conclusion involves a new notion called locally finiteness.

Another way to say a basis $\mathcal{B}$ is countable is that $\mathcal{B}$ can be expressed in

$$\mathcal{B} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{B}_n, \qquad \text{where each } \mathcal{B}_n \text{ is finite.}$$

It is rather aukward, but it gives us a hint to weaken the second countability condition: The Nagata-Smirnov condition, by means of local finiteness.

5.1 Local Finiteness

Definition 5.1.1: Local Finiteness

Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be **locally finite** if for every $x \in X$, there exists an open neighborhood U of x such that U intersects only finitely many elements of $\mathcal{A}$.

Example: **Local Finiteness**

- The collection $\mathcal{A} = \{(n, n+2) : n \in \mathbb{Z}\}$ is locally finite in $\mathbb{R}$.
- The collection $\mathcal{B} = \{(1, 1/n) : n \in \mathbb{Z}^+\}$ is locally finite in $(0, 1)$ but not in $\mathbb{R}$.

Proposition: **Properties of Local Finiteness**

Let X be a topological space and let $\mathcal{A}$ be a locally finite collection of subsets of X. Then

- Every subcollection of $\mathcal{A}$ is locally finite.
- The collection $\mathcal{B} = \left\{\overline{A} : A \in \mathcal{A}\right\}$ is locally finite.

- $\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \overline{A}$.

Proof. First one is trivial. For the second one, note that any open set intersecting $\overline{A}$ needs to intersect A. For the last one, we let

$$Y = \bigcup_{A\in\mathcal{A}} A.$$

In general, we have $\overline{Y} \supseteq \bigcup_{A\in\mathcal{A}} \overline{A}$. For the other direction, let $x \in \overline{Y}$ and let U be an open neighborhood of x intersecting Y. Since $\mathcal{A}$ is locally finite, there exists an open neighborhood V of x intersecting only finitely many elements of $\mathcal{A}$, say $A_1, A_2, \ldots, A_n$. Then x is in one of $\overline{A_i}$: of not, then $U - \bigcup_{i=1}^{n} \overline{A_i}$ is an open neighborhood of x not intersecting Y, a contradiction. □

Remark:
An analogous concept is a **local finite index family** $\{A_\alpha\}_{\alpha\in J}$, where J is an index set. (This allows repeat sets). It is obvious that a local finite index family is a local finite collection if it is local finite as a collection of sets and there are at most finite multiple repeat non-empty set in the family.

Definition 5.1.2: Countably Local Finite

A collection $\mathcal{A}$ of subsets of a topological space X is said to be **countably locally finite** if $\mathcal{A}$ can be expressed in

$$\mathcal{A} = \bigcup_{n\in\mathbb{Z}_+} \mathcal{A}_n, \qquad \text{where each } \mathcal{A}_n \text{ is locally finite.}$$

This is also called σ-locally finite.

Both a countable collection and a locally finite collection are countably locally finite.

Definition 5.1.3: Refinement

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a set X. We say that $\mathcal{B}$ is a **refinement** of $\mathcal{A}$ if for every $B \in \mathcal{B}$, there exists an $A \in \mathcal{A}$ such that $B \subseteq A$. (This means that $\mathcal{B}$ is finer than $\mathcal{A}$.) If the elements in $\mathcal{B}$ are open sets in X, then we say that $\mathcal{B}$ is an **open refinement** of $\mathcal{A}$. If they are closed sets, then we say that $\mathcal{B}$ is a **closed refinement** of $\mathcal{A}$.

Lemma 5.1.1: Existence of Locally Finite Open Refinement

Let X be a metrizable space, and $\mathcal{A}$ be an open cover of X. Then there is an open covering $\mathcal{E}$ of X which is a locally finite open refinement of $\mathcal{A}$.

Proof. SORRY □

5.2 The Nagata-Smirnov Metrization Theorem

Now we give an equivalent condition for metrizability.

Definition 5.2.1: F_σ and G_δ Sets

Let X be a topological space. A subset A of X is called an **F_σ-set** if A can be expressed as a countable union of closed sets in X. A subset B of X is called a **G_δ-set** if B can be expressed as a countable intersection of open sets in X.

Lemma 5.2.1: Countably Locally Finite Basis in a Regular Space

Let X be a regular space with a countably locally finite basis $\mathcal{B}$, then X is normal and every closed set in X is a G_δ-set.

Proof. SORRY □

Lemma 5.2.2: Closed G_δ Sets

Let X be normal, and let A be a closed G_δ-set in X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) > 0$ for all $x \notin A$.

Proof. Write $A = \bigcap_{n=1}^{\infty} U_n$, where each U_n is open in X. Since X is normal, for each n there is a continuous function $f_n : X \to [0, 1]$ such that $f_n(x) = 0$ for all $x \in A$ and $f_n(x) = 1$ for all $x \in X - U_n$ (Urysohn's Lemma). Now define

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}.$$

Then f is continuous due to uniform convergence, and $f(x) = 0$ for all $x \in A$ and $f(x) > 0$ for all $x \notin A$, because there is an n such that $x \notin U_n$, so $f_n(x) = 1$. □

Theorem 5.2.1: The Nagata-Smirnov Metrization Theorem

For a topological space X, then X is metrizable if and only if X is regular and has a σ-locally finite basis.

Proof. SORRY □

5.3 Paracompactness

This is another generalization of compactness: Recall that a space is compact if every open cover has a finite subcover. We can reformulate it as:

> A space is compact if every open cover has a finite open refinement.

So we can generalized it:

Definition 5.3.1: Paracompactness

A topological space X is said to be **paracompact** if every open cover of X has a locally finite open refinement.

Example: **Paracompact Space**

$\mathbb{R}^n$ is paracompact: Let $X = \mathbb{R}^n$, and let $\mathcal{A}$ be an open cover of X. Let B_m be the radius $m \in \mathbb{N}$ open ball centered at the origin ($B_0 = \emptyset$). Then for m, choose finite subcover $\mathcal{A}_m$ of $\mathcal{A}$ covering $\overline{B_m}$. Denote

$$\mathcal{C}_m = \left\{ A \cap (X - \overline{B_{m-1}}) : A \in \mathcal{A}_m \right\}.$$

Then easily $\mathcal{C} = \bigcup_{m=1}^{\infty} \mathcal{C}_m$ is a locally finite open refinement of $\mathcal{A}$.

Proposition: **Properties of Paracompact Spaces**

- Every compact space is paracompact.
- The subspace of a paracompact space is NOT NECESSARILY paracompact.
- A closed subspace of a paracompact space is paracompact.

Theorem 5.3.1: Paracompact Hausdorff Spaces are Normal

Every paracompact Hausdorff space X is normal.

Proof. This is rather similar to the proof of theorem 3.3.3, we first prove that X is regular. Let $a \in X$ and B be a closed set not containing a. For each $b \in B$, choose $U_b \in \mathcal{T}, a \notin \overline{U_b}$. Cover X with all U_b and $X - B$, and choose a locally finite open refinement $\mathcal{C}$. Let $\mathcal{D} \subseteq \mathcal{C}$ be all sets that intersects B, then $\mathcal{D}$ covers B.

Moreover, if $D \in \mathcal{D}$, then D lies in some U_b, so $a \notin \overline{D}$. Let

$$V = \bigcup_{D \in \mathcal{D}} D, \Rightarrow \overline{V} = \bigcup_{D \in \mathcal{D}} \overline{D} \Rightarrow a \notin \overline{V}.$$

then regularity is proved. For normality, repeat the process would do. □

Theorem 5.3.2: Closed Subspaces of Paracompact Spaces

Every closed subspace of a paracompact space is paracompact.

Proof. Let X be a paracompact space, and let Y be a closed subspace of X. Let $\mathcal{A}$ be an open cover of Y with open sets in X. Cover X with $\mathcal{A}$ and $X - Y$, and choose a locally finite open refinement $\mathcal{C}$, then intersecting elements of $\mathcal{C}$ with Y would do. □

Remark:

- A paracompact subspace of a Hausdorff space is NOT NECESSARILY closed. Well, $(0, 1)$ is paracompact, homeomorphic to $\mathbb{R}$, but not closed in $\mathbb{R}$.
- A subspace of a paracompact space is NOT NECESSARILY paracompact.
- The product of two paracompact spaces is NOT NECESSARILY paracompact. Like $\mathbb{R}_l^2$.

- The space $\mathbb{R}^{\mathbb{Z}_+}$ is paracompact in both product and uniform topology. It is not known that whether the box topology is paracompact.
- $\mathbb{R}^J$ is not paracompact for uncountable J, for $\mathbb{R}^J$ is Hausdorff but not normal.

We shall see that every metrizable space is paracompact.

Lemma 5.3.1: Michael's Lemma

Let X be regular, then the following are equivalent:
Every open cover of X has a refinement that is:

- An open cover that is σ-locally finite.
- A covering that is locally finite.
- A closed cover that is locally finite.
- An open cover that is locally finite.

Proof. SORRY □

Theorem 5.3.3: Every Metrizable Space is Paracompact

Every metrizable space is paracompact.

Proof. Let X be a metrizable space, and from lemma 5.1.1, every open cover of X has a σ-locally finite open refinement. Since X is regular, by Michael's lemma, every open cover of X has a locally finite open refinement, so X is paracompact. □

Theorem 5.3.4: Every Regular Lindelöf Space is Paracompact

Every regular Lindelöf space is paracompact. (A Lindelöf space is a space in which every open cover has a countable subcover.)

Proof. A countable subcover is obviously a σ-local finite refinement, using Michael's lemma would do. □

One more intersecting features of paracompact spaces is the existence of partitions of unity.

Definition 5.3.2: Partitions of Unity

Let $\{U_\alpha\}_{\alpha\in J}$ be an indexed open covering of X, then an indexed family of continuous functions

$$\phi_\alpha : X \to [0,1], \quad \alpha \in J$$

is called a **partition of unity** on X dominated by $\{U_\alpha\}_{\alpha\in J}$ if

- $\forall \alpha \in J, \operatorname{supp}\phi_\alpha \subseteq U_\alpha$.
- The indexed family $\{\operatorname{supp}\phi_\alpha\}$ is locally finite.
- $\forall x \in X, \sum_{\alpha\in J}\phi_\alpha(x) = 1$.

(The sum is well-defined due to local finiteness: there are only finitely many non-zero terms.)

Now, we construct partitions of unity in paracompact Hausdorff spaces.

Lemma 5.3.2: Shrinking Lemma

Let X be a paracompact Hausdorff space, and let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of X. Then there is a local finite indexed family open cover $\{V_\alpha\}_{\alpha\in J}$ of X such that $\overline{V_\alpha} \subseteq U_\alpha$ for each $\alpha \in J$.

Proof. SORRY □

Remark:
Sometimes we say that $\{V_\alpha\}_{\alpha\in J}$ is a **precise refinement** of $\{U_\alpha\}_{\alpha\in J}$ if $\overline{V_\alpha} \subseteq U_\alpha$ for each $\alpha \in J$.

Theorem 5.3.5: Existence of Partitions of Unity

Let X be a paracompact Hausdorff space, and let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of X. Then there is a partition of unity on X dominated by $\{U_\alpha\}_{\alpha\in J}$.

Proof. SORRY □

Partitions of unity are very useful in patching local constructed functions to a global one.

Theorem 5.3.6: Patch Theorem

Let X be a paracompact Hausdorff space, and let $\mathcal{C}$ be a collection of subsets of X. For each $C \in \mathcal{C}$, let $\epsilon_C > 0$. If $\mathcal{C}$ is locally finite, then there is a continuous function $f : X \to \mathbb{R}_+$ such that for each $C \in \mathcal{C}$, $f(x) \le \epsilon_C$ for all $x \in C$.

Proof. SORRY □

5.4 The Smirnov Metrization Theorem

Definition 5.4.1: Locally Metrizable

A topological space X is said to be **locally metrizable** if for each $x \in X$, there is a neighborhood U of x that is metrizable.

Theorem 5.4.1: The Smirnov Metrization Theorem

A topological space X is metrizable if and only if X is paracompact, Hausdorff, and locally metrizable.

Chapter 6

Complete Metric Spaces and Function Spaces

Completeness is a fundamental concept in analysis, referring to a property of metric spaces that, while metric in nature, underlies many important topological theorems. The most familiar example is Euclidean space, but another key example is the space $C(X, Y)$ of all continuous functions from a space X to a metric space Y, which is complete in the uniform metric if Y is complete. This chapter examines such examples, demonstrates the completeness of $C(X, Y)$ in the uniform metric, and constructs the Peano space-filling curve as an application. It also explores the relationship between compactness and completeness, leading to Ascoli's theorem about compact subsets of function spaces. Finally, the chapter discusses alternative topologies on $C(X, Y)$ and proves a general version of Ascoli's theorem.

6.1 Complete Metric Spaces

Definition 6.1.1: Cauchy Sequence and Completeness

Let (X, d) be a metric space, and a sequence (x_n) in X is a Cauchy sequence in (X, d) if it satisfies:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N, d(x_n, x_m) < \epsilon. \tag{6.1}$$

A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit in X.

Remark:
Of course every convergent sequence is a Cauchy sequence, but the converse is not true in general. For example, the space of rational numbers $\mathbb{Q}$ with the usual metric is not complete.

Theorem 6.1.1: Closed Subspace of Complete Spaces

A closed subspace of a complete metric space is complete. (in the restricted metric)

Proof. Well, a Cauchy sequence in the closed subspace is also a Cauchy sequence in the complete space, so it converges to some point in the complete space. Since the subspace is closed, the limit point is also in the subspace. □

Proposition: **Standard Bounded Metric and Completeness**

X is complete under d if and only if X is complete under the standard bounded metric $\overline{d}(x,y) = \min\{d(x,y), 1\}$.

This is quite obvious as a sequence is Cauchy under d if and only if it is Cauchy under $\overline{d}$, and a sequence converges under d if and only if it converges under $\overline{d}$.

Lemma 6.1.1: Subsequence Criterion for Completeness

A metric space (X, d) is complete if and only if every Cauchy sequence in X has a convergent subsequence.

Proof. The $\Rightarrow$ side obvious, taking the original sequence would do.

For the $\Leftarrow$ side, let (x_n) be a Cauchy sequence in X. Let (x_{n_i}) is a convergent subsequence of (x_n), and let $x = \lim_{i\to\infty} x_{n_i}$. For any $\epsilon > 0$, there exists N that

$$d(x_n, x_m) < \epsilon/2, \forall m, n > N.$$

Also, there exists I that $n_I > N$ and

$$d(x_{n_i}, x) < \epsilon/2, \forall i > I.$$

So for any $n > N$, we have

$$d(x_n, x) \leq d(x_n, x_{n_I}) + d(x_{n_I}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem 6.1.2: Completeness of $\mathbb{R}^k$

$\mathbb{R}^k$ is complete under the usual metric $d(x,y) = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$ and the square metric $\rho(x,y) = \max_{1\leq i\leq k} |x_i - y_i|$.

Proof. **Proof of** $(\mathbb{R}^k, \rho)$, let (x_n) be a Cauchy sequence in $(\mathbb{R}^k, \rho)$, it is easy to see that x_n is bounded in some cube $[-M, M]^k$. Since $[-M, M]^k$ is compact, it is sequentially compact from theorem 2.6.2. So (x_n) has a convergent subsequence (x_{n_i}) that converges to some $x \in [-M, M]^k$. From lemma 6.1.1, (x_n) converges to x.

Proof of $(\mathbb{R}^k, d)$, equivalently. □

Theorem 6.1.3: Compact Metric Spaces are Complete

Every compact metric space is complete.

Proof. From theorem 2.6.2 and lemma 6.1.1. □

Now we deal with the product space $\mathbb{R}^{\mathbb{Z}_+}$.

Lemma 6.1.2: Convergence in Product Space

Let $X = \prod X_\alpha$ be a product space, and (x_n) be a sequence in X. Then (x_n) converges to $x \in X$ if and only if for each α, the sequence $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$ in X_α.

Proof. The $\Rightarrow$ side is obvious as continuity preserve convergence.

For the $\Leftarrow$ side, let $U = \prod U_\alpha$ be a basis containing x, for $U_\alpha \neq X_\alpha$, choose N_α that $\pi_\alpha(x_n) \in U_\alpha$ for all $n > N_\alpha$. Let $N = \max\{N_\alpha\}$, then for all $n > N$, $x_n \in U$. □

Theorem 6.1.4: Compactness of $\mathbb{R}^{\mathbb{Z}_+}$

There is a metric for $\mathbb{R}^{\mathbb{Z}_+}$ that makes $\mathbb{R}^{\mathbb{Z}_+}$ a complete metric space.

Proof. Let $\overline{d}$ be the standard bounded metric on $\mathbb{R}$, and define a metric D on $\mathbb{R}^{\mathbb{Z}_+}$ by

$$D(x,y) = \sup\left\{\frac{\overline{d}(x_n,y_n)}{n} : n \in \mathbb{Z}_+\right\}. \tag{6.2}$$

Then D induces the product topology on $\mathbb{R}^{\mathbb{Z}_+}$, from theorem 1.9.2.

Now we prove that $\mathbb{R}^{\mathbb{Z}_+}$ is complete with D: Take x_n be a Cauchy sequence, and

$$\overline{d}(\pi_i(x),\pi_i(y)) \leq iD(x,y).$$

So for a fixed i, $(\pi_i(x_n))$ is a Cauchy sequence in $(\mathbb{R},\overline{d})$. From lemma 6.1.2 we have that (x_n) converges. □

Example: **Non-complete Spaces**

- The space $\mathbb{Q}$ of rational numbers with the usual metric is not complete. For example, the sequence
$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$
is a Cauchy sequence in $\mathbb{Q}$ that does not converge to a rational number.
- The space $(-1,1)$ is not complete under the usual metric. For example, the sequence
$$0, 0.9, 0.99, 0.999, 0.9999, \ldots$$
is a Cauchy sequence in $(-1,1)$ that does not converge to a point in $(-1,1)$.

We now turn to $\mathbb{R}^J$ when J is not countable. Well actually, this is not even metrizable with the product topology, see example 1.9. We turn to the topology induced by the uniform metric.

From definition 1.9.5, we know that if (Y,d) is a topological space, then $\overline{\rho}$ is a metric defined by

$$\overline{\rho}(x,y) = \sup\{\overline{d}(x_\alpha,y_\alpha) : \alpha \in J\}, \quad \overline{d} = \min\{d,1\}. \tag{6.3}$$

Or written in function form, Y^J is the set of all functions from J to Y, and

$$\overline{\rho}(f,g) = \sup\{\overline{d}(f(\alpha),g(\alpha)) : \alpha \in J\}, \quad \overline{d} = \min\{d,1\}. \tag{6.4}$$

Theorem 6.1.5: Completeness in Uncountable Products

If (Y,d) is a complete metric space, then $(Y^J,\overline{\rho})$ is a complete metric space, for any set J.

Proof. If (Y,d) is complete, so is $(Y,\overline{d})$. Suppose f_n is a Cauchy sequence in $(Y^J,\overline{\rho})$. For each $\alpha \in J$, $f_n(\alpha)$ is a Cauchy sequence in $(Y,\overline{d})$, so it converges to some $f(\alpha) \in Y$. Let f be the function defined by $f(\alpha) = \lim_{n\to\infty} f_n(\alpha)$ for each $\alpha \in J$. We show that f_n converges to f in $(Y^J,\overline{\rho})$:

For any $\epsilon > 0$, there exists N such that for all $m,n > N$, $\overline{\rho}(f_n, f_m) < \epsilon/2$. We have

$$\overline{d}(f_n(\alpha), f(\alpha)) = \lim_{m\to\infty} \overline{d}(f_n(\alpha), f_m(\alpha)) \le \epsilon/2, \forall n > N, \forall \alpha \in J.$$

(follows from the continuity of metric proposition 1.9 and theorem 1.9.4)

So we have $\overline{\rho}(f_n, f) \le \epsilon/2 < \epsilon$ for all $n > N$. □

Now to be more specific, we may consider the function space Y^X where X is a topological space.

Theorem 6.1.6: Continuous and Bounded Functions

Let X be a topological space and (Y,d) be a metric space. Let $\mathscr{C}(X,Y)$ be the set of all continuous functions from X to Y, and $\mathscr{B}(X,Y)$ be the set of all bounded functions from X to Y.

Then both $\mathscr{C}(X,Y)$ and $\mathscr{B}(X,Y)$ are closed subsets of $(Y^X,\overline{\rho})$. In particular, if (Y,d) is complete, then both $\mathscr{C}(X,Y)$ and $\mathscr{B}(X,Y)$ are complete metric space under the uniform metric.

Proof. For $\mathscr{C}(X,Y)$ it is just the uniform limit throrem 1.9.8, for $\mathscr{B}(X,Y)$ it is also obvious. □

Definition 6.1.2: Sup Metric

Let X be a topological space and (Y,d) be a metric space. The sup metric on $\mathscr{B}(X,Y)$ is defined by

$$\rho(f,g) = \sup\{d(f(x),g(x)) : x \in X\}. \tag{6.5}$$

In fact, in $\mathscr{B}(X,Y)$, ρ and $\overline{\rho}$ are equivalent metrics. And $\overline{\rho}$ is just the standard bounded metric of ρ.

If X is compact, then every continuous function from X to Y is bounded, so $\mathscr{C}(X,Y) \subset \mathscr{B}(X,Y)$. And if Y is complete, then $\mathscr{C}(X,Y)$ is complete under the sup metric.

We now show that every metric space can be isometrically embedded in a complete metric space. Meaning that it is just some part of a complete metric space just like $\mathbb{Q} \subseteq \mathbb{R}$.

Theorem 6.1.7: Completion of Metric Spaces

Let (X,d) be a metric space. There exists a complete metric space (X',d') and an isometric embedding $f : X \to X'$.

The subspace $\overline{f(X)}$ in Y is a complete metric space containing $f(X)$ as a dense subset and is called the completion of X.

Proof. Let $\mathscr{B}(X,\mathbb{R})$ be the set of all bounded functions from X to $\mathbb{R}$, let $x_0 \in X$, and given $a \in X$. Define $\phi_a : X \to \mathbb{R}$ by

$$\phi_a(x) = d(a,x) - d(x,x_0).$$

The triangle inequality shows that $|\phi_a(x)| \le d(a.x_0)$ which is bounded.

Define $\Phi : X \to \mathscr{B}(X, \mathbb{R}), \Phi(a) = \phi_a$. We prove that Φ is an isometric embedding:

$$\rho(\phi_a, \phi_b) = \sup\{|\phi_a(x) - \phi_b(x)| : x \in X\} = \sup\{|d(a,x) - d(b,x)| : x \in X\} = d(a,b).$$

$\square$

Theorem 6.1.8: Uniqueness of Completion

Let X be a metric space. If $h : X \to Y$ and $h' : X \to Y'$ are isometric embeddings of X into complete metric spaces Y and Y' such that both $h(X)$ and $h'(X)$ are dense in Y and Y', respectively, then there exists an isometry $g : Y \to Y'$ such that $g \circ h = h'$.

6.2 Compactness in Metric Spaces

We've already shown in theorem 2.6.2 that in metric spaces, compactness, limit point compactness and sequential compactness are equivalent.

Following the equivalence, every compact metric space is sequentially compact, and thus complete. But the converse is not true, for example, $\mathbb{R}$ is complete but not compact.

Definition 6.2.1: Totally Bounded

A metric space (X, d) is totally bounded if for every $\epsilon > 0$, there exists a finite cover of X by open balls of radius ϵ.

Proposition: **Properties of Totally Boundedness**

- Totally boundedness implies boundedness, but not conversely. For example, $\mathbb{R}$ is bounded in $\overline{d}(x,y) = \min\{|x-y|, 1\}$ but not totally bounded.
- Under $d(x,y) = |x-y|$. $\mathbb{R}$ is complete but not totally bounded. The subspace $(-1,1)$ is totally bounded but not complete. The subspace $[0,1]$ is both complete and totally bounded.

Theorem 6.2.1: Compactness of Metric Spaces

A metric space is compact if and only if it is complete and totally bounded.

Proof.
- The $\Rightarrow$ side is from theorem 6.1.3 and the definition of compactness.
- Let X be complete and totally bounded, and prove that X is sequentially compact. Let (x_n) be a sequence in X. Since X is totally bounded, there exists a finite cover of X by open balls of radius 1. So some ball contains infinitely many terms of the sequence. Call this ball B_1, and let $(x_{1,n})$ be the subsequence of (x_n) consisting of all terms that lie in B_1.

  Next, since X is totally bounded, there exists a finite cover of X by open balls of radius $1/2$. So some ball contains infinitely many terms of the sequence $(x_{1,n})$. Call this ball B_2, and let $(x_{2,n})$ be the subsequence of $(x_{1,n})$ consisting of all terms that lie in B_2.

  Continuing in this way, we obtain a sequence of nested balls B_k of radius $1/k$, each containing infinitely many terms of the previous subsequence. Let $(x_{k,n})$ be the subsequence of $(x_{k-1,n})$

consisting of all terms that lie in B_k. Now define a new sequence (y_k) by letting $y_k = x_{k,k}$. Then (y_k) is a subsequence of (x_n), and for $m, n > N$, both y_m and y_n lie in the ball B_N of radius $1/N$. So

$$d(y_m, y_n) < 2/N.$$

This shows that (y_k) is a Cauchy sequence in X, and since X is complete, (y_k) converges to some point in X. Thus every sequence in X has a convergent subsequence, and so X is sequentially compact.

$\square$