Introduction to Smooth Manifolds

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Chapter 1

Smooth Manifolds

In simple terms, smooth manifolds are spaces that locally look like $\mathbb{R}^n$, and on which we can do calculus. We can visualize them like smooth plane curves like circles and parabolas.

The simplest manifold and topological manifolds, which encode just the properties of what we mean by "locally look like $\mathbb{R}^n$ ". However, to do calculus (volume, curvature, etc.), we need a stronger restriction – the notion of smoothness. Intuitively, we can describe smoothness by having a tangent structure that moves continuously from point to point. For more sophisticated applications we can restrict it to be embedded in some ambient Euclidean vector space. The structure of this ambient space is superfluous that is not guaranteed by the internal structure of the manifold itself.

Also, it is evidently that we cannot define smoothness solely by topological structure. A circle and a square are homeomorphic topological space, but we all agree that square is not smooth but circle is. Therefore, we should think a smooth manifold has two layers of structure: topological manifolds and smoothness.

1.1 Topological Manifolds

Definition 1.1.1: Topological Manifolds

Suppose $(M, \mathcal{T})$ is a topological space, we say that M is a topological manifold of dimension n if it has the following property:

- M is a Hausdorff space. $\forall p \neq q$ in M, there are disjoint open sets $U, V \subseteq M$ such that $p \in U, q \in V$.
- M is second-countable. There exists a countable basis for the topology of M.
- M is locally Euclidean of dimension n : Each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$, in the Euclidean topology. We call the n here the dimension of the topological manifold, denoted $\dim M$.

The last property can be expressed explicitly as: $\forall p \in M, \exists$ open set $U \subseteq M, p \in U$ and $\hat{U} \subseteq \mathbb{R}^n$ such that $U \cong \hat{U}$.

Remark:
We can change the definition to letting U to be homeomorphic to some open balls in $\mathbb{R}^n$. This is equivalent to the original definition.

Proof. If we have a neighborhood that is homeomorphic to a open subspace of $\mathbb{R}^n$, then we have an open ball subspace that would do. □

We also abbreviate M being a topological manifold of dimension n by M^n. It is worth mentioning that we do not consider spaces with mixed dimensions, like a disjoint union of a plane and a line. The dimension here is global to all the point in the space.

Theorem 1.1.1: Topological Invariant of Dimension

A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.

Remark:
The empty set satisfies the definition of a topological manifold of dimension n for every n. But in most circumstances we shall just ignore the trivial case.

A basic example of an n-dimensional topological space is $\mathbb{R}^n$ itself. As every metrizable space is Hausdorff and $\{B(a, r) \mid a \in \mathbb{Q}^n, r \in \mathbb{Q}\}$ is a countable basis.

1.1.1 Coordinate Chart

Definition 1.1.2: Coordinate Chart

Let M be a topological manifold of dimension n, a coordinate chart on M is a pair (U, φ), where U is an open set of M and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.
An *atlas* on M is a collection of coordinate charts $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$.

By the definition of a topological manifold, $\forall p \in M$, we can find a neighborhood where we can define a (U, φ).

- If $\varphi(p) = 0$, we say that the chart is centered at p. (We can always find a chart centered at p by subtracting $\varphi(p)$.

- Given a (U, φ), we say U a coordinate domain. If $\varphi(U)$ is a ball, we say U a coordinate ball.

- φ is called a (local) coordinate map. And the component functions $(x^1, \ldots, x^n)$ of φ are called local coordinates on U. We have $\varphi(p) = (x^1(p), \ldots, x^n(p))$.

1.1.2 Examples

Example: **Graphs of Continuous Functions**

Let $U \subseteq \mathbb{R}^n$ be an open set. And $f : U \to \mathbb{R}^k$ be a continuous function. The graph of f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \wedge y = f(x)\right\} \tag{1.1}$$

with the subspace topology. Let $\pi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection map, and let $\varphi : \Gamma(f) \to U$ be the restriction of π to $\Gamma(f)$.

$$\varphi(x, y) = x, (x, y) \in \Gamma(f)$$

Then $\Gamma(f)$ is a topological manifold of dimension n. $(\Gamma(f), \varphi)$ is a global coordinate chart.

Example: **Spheres**

For each $n \in \mathbb{N}$, the uni sphere $\mathbb{S}^n$ is a subspace of $\mathbb{R}^{n+1}$, and a local part (hemisphere would do) is the graph of a continuous mapping.

Example: **Projective Spaces**

The n-dimensional real projective space $\mathbb{RP}^n$, is defined as $(X, \mathcal{T})$, where

- X is the 1-dimensional linear subspaces of $\mathbb{R}^n$. (The lines that cross the origin)
- $\mathcal{T}$ is the quotient topology.

Example: **Product Manifold**

Suppose $M_1, \ldots, M_k$ are topological manifolds of dimension $n_1, \ldots, n_k$ respectively. Then the product space $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.

Proof. The Hausdorff and second-countable properties follows from the product topology itself. Given any point $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$, we can find a neighborhood U_i of p_i such that $U_i \cong \hat{U}_i \subseteq \mathbb{R}^{n_i}$. Then $U = U_1 \times \ldots \times U_k$ is a neighborhood of p and homeomorphic to $\hat{U}_1 \times \ldots \times \hat{U}_k \subseteq \mathbb{R}^{n_1 + \ldots + n_k}$. $\square$

1.1.3 Topological Properties of Manifolds

We shall see that manifolds have a well-behaved topological structure, thanks to the Hausdorff and second-countable properties.

Lemma 1.1.1: Precompact Coordinate Balls

Every topological manifold has a countable basis of precompact coordinate balls. (Precompact means its closure is compact)

First we shall show that every second countable space is Lindelöf(every open cover has a countable subcover).

Proof. First, let $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\}$ be a countable basis of the topology of M. Given any open cover $\{U_\alpha \mid \alpha \in A\}$ of M, for each B_i, we can find a U_{α_i} such that $B_i \subseteq U_{\alpha_i}$. Then $\{U_{\alpha_i} \mid i \in \mathbb{N}\}$ is a countable subcover of M. □

Now we prove the lemma. For any chart (U, φ), as $\varphi(U)$ is an open subset of $\mathbb{R}^n$, we can find a countable basis of precompact balls $\{B_i \mid i \in \mathbb{N}\}$ of $\varphi(U)$. Then $\{\varphi^{-1}(B_i) \mid i \in \mathbb{N}\}$ is a countable basis of precompact coordinate balls of U. As M is Lindelöf, we can find a countable collection of charts that cover M. The union of the countable bases of precompact coordinate balls of these charts is a countable basis of precompact coordinate balls of M.

Connectedness Topological manifolds also have nice connectedness properties.

Proposition: **Connectedness Properties of Manifolds**

Let M be a topological manifold, then

- M is locally path-connected.
- M is connected iff it is path-connected.
- The components of M are the same as its path components.
- M has countably many components, each is an open subset of M and a topological manifold itself.

Proof. Since each coordinate ball is path-connected, M has a basis of path-connected neighborhoods, so it is locally path-connected. The second and third properties follows from general topology. The openness of components follows from local path-connectedness. The countability of components follows from second-countability and the disjointness of components.(The components are an open cover of M, so we can find a countable subcover. As the components are disjoint, the only subcover is itself.) □

Local Compactness and Paracompactness Topological manifolds are also locally compact and paracompact.

Definition 1.1.3: Exhaustion

Let X be a topological space, an exhaustion of X is a sequence of compact sets $\{K_j\}_{j\in\mathbb{Z}}$ such that

- $K_j \subseteq \operatorname{Int} K_{j+1}$ for all $j \in \mathbb{Z}$.

- $\bigcup_{j\in\mathbb{Z}} K_j = X$.

We say that X is *exhausted* by $\{K_j\}_{j\in\mathbb{Z}}$.

We can see that for a second-countable locally compact Hausdorff space, we can find a countable exhaustion. This is because we can find a countable basis of compact sets, and we can take the union of these compact sets to form an exhaustion.

Proposition: **Local Compactness and Paracompactness of Manifolds**

Let M be a topological manifold, then

- M is locally compact.
- M is paracompact. In fact, given any open cover $\mathcal{X}$ of M and any basis $\mathcal{B}$, there is a countable, locally finite open refinement of $\mathcal{X}$ by elements of $\mathcal{B}$.

Proof. Local compactness follows from the fact that each point has a precompact coordinate ball neighborhood. As second-countable Hausdorff spaces are normal, and every regular Lindelöf space is paracompact, M is paracompact. For a construction, let $\{K_j\}_{j\in\mathbb{Z}}$ be an exhaustion of M by compact sets. For each j, let $V_j = K_{j+1} - \operatorname{Int} K_j$ and $W_j = \operatorname{Int} K_{j+2} - K_{j-1}$. Then □

Fundamental Groups of Manifolds The topological restrictions on manifolds also limit their fundamental groups, which is of great importance when we study covering spaces of manifolds.

Theorem 1.1.2: Fundamental Groups of Manifolds

The fundamental group of a topological manifold is at most countable.

Proof. SORRY, but fairly obvious due to the countability of coordinate balls. □

1.2 Smooth Structures

If we only have the topological structure of a manifold, we cannot do calculus on it. One may try to define derivatives of functions on the manifold by using coordinate charts, but the problem is that this definition is not invariant under homeomorphisms.

For example, the map given by

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \varphi(x, y) = (x^{1/3}, y^{1/3})$$

is a homeomorphism, but it is easy to construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that f is differentiable at 0, but $f \circ \varphi$ is not differentiable at 0.

The smooth structure allows us to formalize the idea of smooth transition between different coordinate charts, so that we can define derivatives of functions on the manifold in an invariant way. Let $U \in \mathbb{R}^n$, and $V \in \mathbb{R}^m$ be two open sets, a map $F : U \to V$ is said to be *smooth* (or C^∞, infinitely differentiable) if all its component functions have continuous partial derivatives of all orders. If F is bijective and both F and F^{-1} are smooth, then F is called a *diffeomorphism*.

Let M be a n-dimensional topological manifold, and for $p \in M$, take a coordinate chart (U, φ) with $p \in U$. We would think that a function $f : U \to \mathbb{R}$ is smooth if $f \circ \varphi^{-1} : \hat{U} \to \mathbb{R}$ is smooth (here $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$). But this would only make sense if this is independent of the choice of the chart (U, φ). Therefore, we need to impose some restrictions on the charts, called *smooth charts*. As this is not preserved by arbitrary homeomorphisms, we should thought this as a new structure on the manifold, called *smooth structure*.

Definition 1.2.1: Transition Map

For an n-dimensional topological manifold M, let (U, φ) and (V, ψ) be two coordinate charts such that $U \cap V \neq \emptyset$. Then the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \tag{1.2}$$

is called a *transition map* from (U, φ) to (V, ψ). It is a composition of homeomorphisms, so it is a homeomorphism itself.
Two coordinate charts (U, φ) and (V, ψ) are said to be *smoothly compatible* if either $U \cap V = \emptyset$, or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. A smooth atlas on M is an atlas whose charts are pairwise smoothly compatible.

However, there may be many different smooth atlases that gave the same set of smooth functions on M. We could define an equivalence relation on the set of smooth atlases, but a more straightforward way is to define a maximal smooth atlas: A smooth atlas $\mathcal{A}$ is said to be *maximal* or *complete* if any coordinate chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.

Definition 1.2.2: Smooth Structure

Let M be a topological manifold. A *smooth structure* on M is a maximal smooth atlas $\mathcal{A}$ on M. A smooth manifold is a pair $(M, \mathcal{A})$.

It is not convenient to work with maximal smooth atlases directly, so we have the following theorem that allows us to work with arbitrary smooth atlases.

Proposition: **Existence of Maximal Smooth Atlas**

Let M be a topological manifold,

- Every smooth atlas $\mathcal{A}$ on M is contained in a unique maximal smooth atlas, called the maximal smooth atlas determined by $\mathcal{A}$.
- Two smooth atlases $\mathcal{A}$ and $\mathcal{A}'$ on M determine the same smooth structure if and only if their union $\mathcal{A} \cup \mathcal{A}'$ is a smooth atlas.

Remark:
Intuitively, this means that we can define an equivalence relation on the set of smooth atlases, where two atlases are equivalent if they can be combined to form a larger smooth atlas. Each equivalence class has a unique maximal element, and all elements in the equivalence class are

just the sub-atlases of this maximal element.

Proof. Let $\mathcal{A}$ be a smooth atlas on M. Let $\overline{\mathcal{A}}$ be the set of all coordinate charts that are smoothly compatible with every chart in $\mathcal{A}$. We claim that $\overline{\mathcal{A}}$ is a maximal smooth atlas containing $\mathcal{A}$.

First, let $(U,\varphi),(V,\psi) \in \overline{\mathcal{A}}$, for $x = \varphi(p) \in \varphi(U \cap V)$, we have some chart $(W,\theta) \in \mathcal{A}$ with $p \in W$. Therefore, we have

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$$

is smooth in a neighborhood of x, so we have $\overline{\mathcal{A}}$ is a smooth atlas. Moreover, every chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ is also smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. Therefore, $\overline{\mathcal{A}}$ is maximal.

For the second part, if $\mathcal{A}$ and $\mathcal{A}'$ determine the same smooth structure, then they are both contained in the same maximal smooth atlas, so their union is a smooth atlas. Conversely, if their union is a smooth atlas, then every chart in $\mathcal{A}'$ is smoothly compatible with every chart in $\mathcal{A}$, so $\mathcal{A}' \subseteq \overline{\mathcal{A}}$. Then both $\mathcal{A}$ and $\mathcal{A}'$ are contained in $\overline{\mathcal{A}}$, so they determine the same smooth structure. □

Remark:
There exists topological manifolds that do not admit any smooth structure. For example, the E8 manifold in dimension 4. The first such example was constructed by Kervaire in 1960. On the other hand, there are also topological manifolds that admit more than one smooth structure. The first such example is the 7-sphere, discovered by Milnor in 1956. In fact, it is known that for every $n \geq 7$, there exist topological manifolds of dimension n that admit more than one smooth structure.
NOTE that different smooth manifold can be diffeomorphic, which we shall justify later.

We can produce various kinds of structures by changing the requirements on the transition maps:

- If we require the transition maps to be homeomorphisms, we get the notion of a topological manifold.
- If we require the transition maps to be diffeomorphisms of class C^k (i.e., having continuous derivatives up to order k), we get the notion of a C^k-manifold.
- If we require the transition maps to be real-analytic (can be expanded as a convergent power series around each point) diffeomorphisms, we get the notion of a real-analytic manifold.
- If we have even dimension, we can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and require the transition maps to be holomorphic (analytic) diffeomorphisms, we get the notion of a complex manifold.

1.2.1 Local Coordinate Representation

If M is a smooth manifold, any chart (U,φ) in the smooth structure is called a smooth chart, and the coordinate map φ is called a smooth coordinate map.

We say a set $B \subseteq M$ is *Regular coordinate ball* if there is a larger coordinate ball $B' \subseteq M$ such that $\overline{B} \subseteq B'$ and a smooth coordinate map $\varphi : B' \to \mathbb{R}^n$ such that for some positive number $r < r'$, we have

$$\varphi(B) = B_r(0), \qquad \varphi(\overline{B}) = \overline{B_r(0)}, \qquad \varphi(B') = B_{r'}(0) \tag{1.3}$$

Therefore, the regular coordinate ball is precompact.

Remark:
This is not true for arbitrary coordinate balls, take $M = \mathbb{R} - \{0\}$, and $B = B_{(1)}(1)$, there is no larger coordinate ball that contains the closure of B, and it is not precompact.

Proposition: **Countabe Basis of Regular Coordinate Balls**

Every smooth manifold has a countable basis of regular coordinate balls.

Proof. This is a slight improvement of lemma 1.1.1. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ be a countable atlas of smooth charts that cover M. For each $\alpha \in A$, as $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$, we can find a countable basis of regular balls $\{B_{\alpha,i} \mid i \in \mathbb{N}\}$ of $\varphi_\alpha(U_\alpha)$. Then $\{\varphi_\alpha^{-1}(B_{\alpha,i}) \mid \alpha \in A, i \in \mathbb{N}\}$ is a countable basis of regular coordinate balls of M. ☐

If we have a chart (U, φ), we can simply identify U with $\varphi(U) \subseteq \mathbb{R}^n$. Therefore, for simplicity we shall say that a point $p \in M$ has coordinates $(x^1(p), \ldots, x^n(p))$ instead of writing $\varphi(p) = (x^1(p), \ldots, x^n(p))$.

A simple example is the polar coordinate on an open set of $\mathbb{R}^2$.

1.3 Examples of Smooth Manifolds

0-dimensional Smooth Manifolds 0-dimensional topological manifolds are just countable discrete spaces. Therefore, the only smooth structure on a 0-dimensional topological manifold is the trivial one, where every chart is a homeomorphism to an open subset of $\mathbb{R}^0 = \{0\}$.

Euclidean Spaces For each $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is a smooth manifold of dimension n with the smooth structure given by the *standard smooth atlas*, which consists of the single global chart $(\mathbb{R}^n, {}_{\mathbb{R}^n})$.

There are other smooth structures on $\mathbb{R}^n$. For example, consider $\psi(x) = x^3$, then $(\mathbb{R}, \psi)$ determines a smooth structure on $\mathbb{R}$ that is different from the standard one. However, it can be shown that there are diffeomorphic.

Finite-Dimensional Vector Spaces Let V be a finite-dimensional real vector space of dimension n. Then V is isomorphic to $\mathbb{R}^n$ as a vector space if we take a basis. It is fairly obvious that all basis give the same smooth structure on V, making it a smooth manifold of dimension n, called the *standard smooth structure* on V.

1.3.1 Einstein Summation Convention

In differential geometry, we often deal with objects that have multiple components, such as vectors and tensors. To simplify the notation, we use the Einstein summation convention, which states that when an index appears twice in a single term, once as a subscript and once as a superscript, it implies summation over all possible values of that index. For example,

$$E(x) = x^i e_i = \sum_{i=1}^{n} x^i e_i$$

To be consistent, we shall use superscripts for components of vectors and subscripts for basis vectors.

Space of Matrices Let $m, n \in \mathbb{N}$, the space of all $m \times n$ real matrices, denoted by $\mathbb{R}^{m\times n}$, is a finite-dimensional vector space of dimension mn. Therefore, it has a standard smooth structure, making it a smooth manifold of dimension mn.

Open Submanifolds Let M be a smooth manifold of dimension n, and let $U \subseteq M$ be an open subset. Then U is a topological manifold of dimension n with the subspace topology. Define a smooth structure on U by

$$\mathcal{A}_U = \{(V, \varphi) \in \mathcal{A}_M, V \subseteq U\} \tag{1.4}$$

Then $\mathcal{A}_U$ is a smooth atlas on U, making it a smooth manifold of dimension n, called an *open submanifold* of M.

Remark:
As $\mathcal{A}$ is maximal, for any chart, its subchart is also in $\mathcal{A}$. Therefore, our requirement is sufficient for a mere inclusion.

The General Linear Group Let $n \in \mathbb{N}$, the general linear group $(n, \mathbb{R})$ is the set of all invertible $n \times n$ real matrices, which is an open subset of $\mathbb{R}^{n\times n}$ (the determinant function is continuous, and $(n, \mathbb{R})$ is the preimage of $\mathbb{R} - \{0\}$). Therefore, it is a smooth manifold of dimension n^2, with the smooth structure induced from the standard smooth structure on $\mathbb{R}^{n\times n}$.

Full Rank Matrices Let $m < n$ be two natural numbers, the set of all $m \times n$ real matrices of rank m, denoted by $M_m(m \times n, \mathbb{R})$, is an open subset of $\mathbb{R}^{m\times n}$ (the map that sends a matrix to the maximum absolute value of its $m \times m$ minors is continuous, and $M_m(m \times n, \mathbb{R})$ is the preimage of $(0, \infty)$). Therefore, it is a smooth manifold of dimension mn, with the smooth structure induced from the standard smooth structure on $\mathbb{R}^{m\times n}$.

For $m = n$, we have $M_n(n \times n, \mathbb{R}) = (n, \mathbb{R})$.

Linear Map Spaces Let V and W be finite-dimensional real vector spaces of dimension m and n respectively. The set of all linear maps from V to W, denoted by $\mathcal{L}(V, W)$, is a finite-dimensional vector space of dimension mn. Therefore, it has a standard smooth structure, making it a smooth manifold of dimension mn.

Graphs of Smooth Functions Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^k$ be a smooth function. The graph of f is a n-dimensional smooth manifold, by the projection map as a global smooth chart.

Example: **Spheres**

For each $n \in \mathbb{N}$, the unit sphere $\mathbb{S}^n$ is a topological n-manifold. Each hemisphere is the graph of a smooth mapping, and it is fairly easy to check that the transition maps are all smooth. Therefore, $\mathbb{S}^n$ is a smooth manifold of dimension n, called the *standard smooth structure* on $\mathbb{S}^n$.

Level Sets of Smooth Functions Suppose $U \in \mathbb{R}^n$ is an open set, and $\Phi : U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set

$$M_c = \Phi^{-1}(c) = \{x \in U \mid \Phi(x) = c\} \tag{1.5}$$

is called a *level set* of Φ. Suppose $M_c \neq \emptyset$, and for every $a \in M_c$, the derivative $D\Phi(a) : \mathbb{R}^n \to \mathbb{R}$ is non zero. Then by the implicit function theorem, take $\partial\Phi/\partial x^i(a) \neq 0$, we can find a neighborhood U_a of a such that $M_c \cap U_a$ is the graph of a smooth function from an open subset of $\mathbb{R}^{n-1}$ to $\mathbb{R}$. Therefore, M_c is a topological manifold of dimension $n-1$. By checking the transition maps, we can see that M_c is a smooth manifold of dimension $n-1$.

Projective Spaces The n-dimensional real projective space $\mathbb{RP}^n$ can be given a smooth structure by using standard charts.

Proposition: **Smooth Product Manifolds**

Suppose $M_1, \ldots, M_k$ are smooth manifolds of dimension $n_1, \ldots, n_k$ respectively. Then the product space $M_1 \times \ldots \times M_k$ is a smooth manifold of dimension $n_1 + \ldots + n_k$, with the smooth structure determined by charts of the form

$$(U_1 \times \ldots \times U_k, \varphi_1 \times \ldots \times \varphi_k)$$

where (U_i, φ_i) is a smooth chart on M_i.

Up to now, we construct smooth manifolds from topological manifolds. By the following lemma, we can construct smooth manifolds directly from smooth atlases.

Lemma 1.3.1: The Smooth Manifold Chart Lemma

Let M be a set, and let $\{U_\alpha\}_{\alpha \in A}$ be a collection of subsets of M and $\varphi_\alpha : U_\alpha \to \hat{U}_\alpha \subseteq \mathbb{R}^n$, such that

- For each $\alpha \in A$, φ_α is a bijection from U_α to an open subset $\hat{U}_\alpha$ of $\mathbb{R}^n$.
- For each $\alpha, \beta \in A$, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$, and the map

  $$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

  is a diffeomorphism.
- Countable many U_α cover M.

- For any two distinct points $p, q \in M$, either there is an $\alpha \in A$ such that $p, q \in U_\alpha$, or there are $\alpha, \beta \in A$ such that $p \in U_\alpha, q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.

Then there is a unique topology and smooth structure on M such that $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ is a smooth atlas on M, making M a smooth manifold of dimension n.

Remark:
The sets U_α gives local properties of every point in M, so we can define a topology on M by declaring open sets of M by inverses of open sets in $\mathbb{R}^n$ via the maps φ_α. The second requirement ensures that the charts are smoothly compatible, so we can define a smooth structure on M. The third requirement ensures that M is second-countable, and the fourth requirement ensures that M is Hausdorff.

Example: **Grassmann Manifolds**

Let V be a finite-dimensional real vector space of dimension n. For each $k \leq n$, the Grassmannian $G_k(V)$ is the set of all k-dimensional linear subspaces of V. We can give $G_k(V)$ a smooth structure.

Proof. SORRY □

1.4 Manifolds with Boundary

Many spaces, like closed balls and half-spaces, are not manifolds in the usual sense, because they have "edges". However, we can generalize the notion of manifolds to include such spaces because they still locally resemble Euclidean spaces, except at the boundary points.

Definition 1.4.1: Manifold with Boundary

An n-dimensional topological manifold with boundary is a Hausdorff, second-countable topological space M such that for every point $p \in M$, there exists a neighborhood U of p that is either homeomorphic to an open subset of $\mathbb{R}^n$ or to an (relative) open subset of the closed half-space $\mathbb{H}^n$, where

$$\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\right\}$$

We call a chart (U, φ) a *boundary chart* if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ with $\varphi(U) \cap \partial\mathbb{H}^n \neq \emptyset$, and an *interior chart* if $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

A point $p \in M$ is called an *interior point* if there is an interior chart (U, φ) with $p \in U$. p is a *boundary point* if there is a boundary chart (U, φ) with $p \in U$ and $\varphi(p) \in \partial\mathbb{H}^n$.

The set of all boundary points of M is called the *boundary* of M, denoted by ∂M. The set of all interior points of M is called the *interior* of M, denoted by $\operatorname{Int} M$.

Remark:
A point must be either a boundary point or an interior point. If p is not a boundary point, then either it is in the domain of an interior chart, or it is in the domain of a boundary chart

but mapped to the interior of $\mathbb{H}^n$. In the latter case, we can shrink the domain to get an interior chart containing p.

The following theorem shows that a point cannot be both a boundary point and an interior point.

Theorem 1.4.1: Topological Invariance of Boundary

Let M be a topological manifold with boundary, then each point $p \in M$ is either a boundary point or an interior point, but not both. Thus

$$M = \partial M \cup \operatorname{Int} M, \qquad \partial M \cap \operatorname{Int} M = \emptyset \tag{1.6}$$

Remark:
NOTE that here the concept of boundary is not the same as the boundary of a subspace in topology. When in confusion, we shall call the former the *manifold boundary* and the latter the *topological boundary*.
Manifold boundary is a local, absolute concept, while topological boundary is a global, relative concept. For example, consider the closed unit disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ as a manifold with boundary. The manifold boundary of D is the unit circle $\mathbb{S}^1$, while the topological boundary of D in $\mathbb{R}^2$ is also $\mathbb{S}^1$. However, if we consider D as a subspace of itself, then its topological boundary is empty, since D has no points outside itself.

Proposition: **Manifold Structure on Interior and Boundary**

Let M be a topological manifold with boundary of dimension n. Then

- The interior $\operatorname{Int} M$ is an n-dimensional topological manifold (without boundary), with the subspace topology.
- The boundary ∂M is an $(n-1)$-dimensional topological manifold (without boundary), with the subspace topology.
- M is a topological manifold (without boundary) iff $\partial M = \emptyset$.
- If $n = 0$, then $\partial M = \emptyset$ and M is a 0-dimensional topological manifold (without boundary).

Proof. SORRY □

Proposition: **Topological Properties of Manifolds with Boundary**

Let M be a topological manifold with boundary, then

- M has countable basis of precompact coordinate balls and half-balls.
- M is locally compact.

- M is paracompact.
- M is locally path-connected.
- M has countablely many components, each is an open subset of M and a connectd topological manifold with boundary itself.
- The fundamental group of M is at most countable.

1.4.1 Smooth Structure on Manifolds with Boundary

First we shall define smooth functions on arbitrary subset of $\mathbb{R}^n$:

Definition 1.4.2: Smooth Maps on subset of $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be an arbitrary subset, a map $f : A \to \mathbb{R}^k$ is said to be *smooth* if for every point $p \in A$, there is an open neighborhood U of p in $\mathbb{R}^n$ and a smooth map $\tilde{f} : U \to \mathbb{R}^k$ such that $\tilde{f}|_{U \cap A} = f|_{U \cap A}$.

The definition of smooth atlases and smooth structures on manifolds with boundary are similar to those on manifolds without boundary, except that we now allow charts to be homeomorphisms to open subsets of $\mathbb{H}^n$, and tweak the definition of smooth compatibility accordingly.

Proposition: **Properties of Smooth Manifolds with Boundary**

Let M be a smooth manifold with boundary of dimension n. Then

- The interior $\operatorname{Int} M$ is an n-dimensional smooth manifold (without boundary), with the subspace topology and the smooth structure induced from M.
- The boundary ∂M is an $(n-1)$-dimensional smooth manifold (without boundary), with the subspace topology and the smooth structure induced from M.
- Every smooth manifold with boundary has a countable basis of regular coordinate balls and half-balls.
- The smooth manifold chart lemma 1.3.1 also holds for smooth manifolds with boundary. Just replace $\mathbb{R}^n$ by $\mathbb{R}^n$ or $\mathbb{H}^n$ accordingly.

As a product of $\mathbb{H}^m$ and $\mathbb{H}^n$ is not a half space, the product of two manifolds with boundary is not a manifold with boundary in general. (It is a smooth manifold with corners, which we shall not discuss here.)

Proposition: **Products of Smooth Manifold with Boundary**

Suppose $M_1, M_2, \ldots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then the product space $M_1 \times M_2 \times \ldots \times M_k \times N$ is a smooth manifold with boundary, with the boundary

$$\partial(M_1 \times M_2 \times \ldots \times M_k \times N) = M_1 \times M_2 \times \ldots \times M_k \times \partial N$$

Chapter 2

Smooth Maps

We shall do calculus on smooth manifolds via smooth maps between them.

2.1 Smooth Functions and Smooth Maps

Although formally maps and functions are the same thing, we shall technically denote functions as maps from a manifold to $\mathbb{R}^n$ and maps as maps between manifolds.

2.1.1 Smooth Functions on Manifolds

Definition 2.1.1: Smooth Functions on Manifolds

Let M be a smooth n-manifold and $k \in \mathbb{N}$. A function $f : M \to \mathbb{R}^k$ is a **smooth function** if for every $p \in M$, there exists a smooth chart (U, φ) containing p the corresponding coordinate representation $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is a smooth function (in the usual sense) on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.
The definition for manifolds with boundary is similar.
We denote all smooth functions from M to $\mathbb{R}^k$ by $C^\infty(M, \mathbb{R}^k)$ or simply $C^\infty(M)$ when $k = 1$. It is a vector space over $\mathbb{R}$.

Remark:
If $M \subseteq \mathbb{R}^n$, the definition coincide with the usual definition of smooth functions on subsets of $\mathbb{R}^n$, obviously.

We shall see that the definition holds for all charts containing p if it holds for one chart containing p, thanks to the smoothness of transition maps.

Proposition: **Smoothness is Chart-Independent**

Let M be a smooth manifold, with or without boundary, and let $f : M \to \mathbb{R}^k$ be a function. Then f is a smooth function if and only if for every $p \in M$ and every smooth chart (U, φ) containing p, the coordinate representation $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is a smooth function (in the usual sense) on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.

Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) on M, the function

$$\hat{f} : \varphi(U) \to \mathbb{R}^k, \quad \hat{f} = f \circ \varphi^{-1} \tag{2.1}$$

is called the **coordinate representation** of f with respect to the chart (U, φ). Then the definition just says that f is smooth if and only if for every $p \in M$, there exists a chart (U, φ) containing p such that the coordinate representation $\hat{f}$ is smooth in the usual sense.

2.1.2 Smooth Maps Between Manifolds

Definition 2.1.2: Smooth Maps Between Manifolds

Let M and N be smooth manifolds. A map $F : M \to N$ is a **smooth map** if for every $p \in M$, there exist smooth charts (U, φ) on M containing p and (V, ψ) on N containing $F(p)$ such that $F(U) \subseteq V$ and the coordinate representation

$$\hat{F} : \varphi(U) \to \psi(V), \quad \hat{F} = \psi \circ F \circ \varphi^{-1} \tag{2.2}$$

is a smooth map (in the usual sense) between the open subsets $\varphi(U) \subseteq \mathbb{R}^m$ and $\psi(V) \subseteq \mathbb{R}^n$.
The definition for manifolds with boundary is similar.
We denote all smooth maps from M to N by $C^\infty(M, N)$.

Our previous definition of smooth functions is a special case of this definition when $N = \mathbb{R}^k$.

Remark:
The requirement that $F(U) \subseteq V$ is crucial, as we need to make F completely in control when we express it in coordinates. So we can identify F with its coordinate representation $\hat{F}$ on U.

Proposition: **Smooth Maps are Continuous**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a map. If F is smooth, then it is continuous.

Proof. As F is smooth, for every $p \in M$, there exist smooth charts (U, φ) on M containing p and (V, ψ) on N containing $F(p)$ such that the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth in the usual sense.

$$F_U = \psi^{-1} \circ \hat{F} \circ \varphi : U \to V$$

is continuous as a composition of continuous maps. Since F agrees with F_U on U, F is continuous at p. As p is arbitrary, F is continuous. $\square$

Proposition: **Characterization of Smooth Maps**

Suppose M and N are smooth manifolds, with or without boundary, and let $F : M \to N$ be a map. Then F is smooth if and only if one of the following equivalent conditions holds:

- For every $p \in M$ there exist smooth charts (U, φ) on M containing p and (V, ψ) on N

containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ is smooth in the usual sense.

- F is continuous and there exists smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ on M and $\{(V_\beta, \psi_\beta)\}$ on N such that for every α and β, the composite map $\psi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to \psi_\beta(V_\beta)$ is smooth in the usual sense.

It is also obvious that smooth maps does not depend on the choice of charts, thanks to the smoothness of transition maps.

Proposition: **Smoothness is Local**

Let M and N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a map. Then F is smooth if and only if for every $p \in M$ and every smooth chart (U, φ) on M containing p and every smooth chart (V, ψ) on N containing $F(p)$ such that $F(U) \subseteq V$, the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth in the usual sense.

Proposition: **Algebra of Smooth Maps**

Let M, N, P be smooth manifolds, with or without boundary.

- The constant map $C : M \to N$ defined by $C(p) = q$ for some fixed $q \in N$ is smooth.
- The identity map $\mathrm{Id}_M : M \to M$ is smooth.
- If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota : U \hookrightarrow M$ is smooth.
- If $F : M \to N$ and $G : N \to P$ are smooth maps, then the composition $G \circ F : M \to P$ is smooth.

Proposition: **Smooth Maps by Components**

Suppose $M_1, \ldots, M_k$ and N are smooth manifolds, with or without boundary (at most one of $M_1, \ldots, M_k$ has nonempty boundary), and let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ be the projection map onto the i-th factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each component map $F_i = \pi_i \circ F : N \to M_i$ is smooth for $i = 1, \ldots, k$.

Example: **Smooth Maps**

- If M is a 0-manifold, then every map $F : M \to N$ is smooth.
- The wrapping map $\epsilon : \mathbb{R} \to S^1$ defined by $\epsilon(t) = \exp(2\pi i t)$ is smooth. So is $\epsilon^n : \mathbb{R}^n \to T^n$ defined by $\epsilon^n(t_1, \ldots, t_n) = (\exp(2\pi i t_1), \ldots, \exp(2\pi i t_n))$.
- The inclusion map $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.
- The quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ defined by $\pi(x) = [x]$ is smooth.

- If $M_1, \dots, M_k$ are smooth manifolds, then each projection map $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is smooth.

2.1.3 Diffeomorphisms

Definition 2.1.3: Diffeomorphisms

A **diffeomorphism** is a smooth map $F : M \to N$ that is a bijection and whose inverse $F^{-1} : N \to M$ is also smooth. If such a map exists, we say that M and N are **diffeomorphic**, denoted by $M \cong N$.

Remark:
Diffeomorphisms are isomorphisms in the category of smooth manifolds, so diffeomorphic manifolds are "the same" from the smooth manifold point of view.

Diffeomorphisms give an equivalence relation on the class of smooth manifolds. And it is fairly interesting to ask whether a given manifold has multiple smooth structures that are not diffeomorphic to each other. As it turns out, for $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure up to diffeomorphism, while for $n = 4$, there are uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$!

2.2 Partitions of Unity

The Gluing lemma in topology states that

> Let X, Y be topological spaces, and if one of the following holds:
>
> - X is the union of finitely many closed subsets $A_1, \dots, A_n$.
> - X is the union of open subsets $\{U_\alpha\}_{\alpha \in A}$.
>
> If we are given continuous maps $f_i : A_i \to Y$ (or $f_\alpha : U_\alpha \to Y$) that agree on the overlaps, then there exists a unique continuous map $f : X \to Y$ such that $f|_{A_i} = f_i$ (or $f|_{U_\alpha} = f_\alpha$).

We can glue smooth maps for the open cover case, but not for the closed cover case. This is fairly obvious, Take $f(x) = |x|$ on $\mathbb{R}$, and cover $\mathbb{R}$ by the two closed sets $(-\infty, 0]$ and $[0, \infty)$. The restrictions $f|_{(-\infty,0]}$ and $f|_{[0,\infty)}$ are both smooth, but f is not smooth at 0.

A slight disadvantage of gluing smooth maps over open covers is that we need to make sure the maps agree on the overlaps. To get around this, we introduce partitions of unity, which allow us to glue local smooth properties into global smooth properties without worrying about the overlaps.

Our discussion is based on the existence of smooth bump functions that are positive in a specific part and vanish outside a slightly larger part. Take the function

$$f(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

on $\mathbb{R}$ for example.

Lemma 2.2.1: Smooth Bump on $\mathbb{R}^n$

Given any $0 < r_1 < r_2$, there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H(x) = 1$ for $||x|| \leq r_1$, $H(x) = 0$ for $||x|| \geq r_2$, and $0 < H(x) < 1$ for $r_1 < ||x|| < r_2$.

Proof. Using f to patch the two regions together would do. □

Definition 2.2.1: Partition of Unity

Suppose M is a topological space and $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ is an open cover of M. A **partition of unity subordinate to $\mathcal{X}$** is a collection of continuous functions $\{\varphi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

- For each $\alpha \in A$, $0 \leq \varphi_\alpha(p) \leq 1$ for all $p \in M$.
- $\operatorname{supp} \varphi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.
- The family of supports $\{\operatorname{supp} \varphi_\alpha\}_{\alpha \in A}$ is locally finite.
- For every $p \in M$, $\sum_{\alpha \in A} \varphi_\alpha(p) = 1$ (only finitely many terms are nonzero by local finiteness).

If each φ_α is smooth, we say it is a **smooth partition of unity**.

Theorem 2.2.1: Existence of Smooth Partitions of Unity

Let M be a smooth manifold, with or without boundary, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be any open cover of M. Then there exists a smooth partition of unity $\{\varphi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ subordinate to $\mathcal{U}$.

Proof. SORRY □

As you can see, we can use smooth partitions of unity to glue local smooth properties into global smooth properties. This is extremely useful in differential geometry.

Definition 2.2.2: Bump functions

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be closed and $A \subseteq U \subseteq M$ for some open set U. A **bump function** for A supported in U is a continuous function $\psi : M \to \mathbb{R}$ such that

- $0 \leq \psi(p) \leq 1$ for all $p \in M$.
- $\psi(p) = 1$ for all $p \in A$.
- $\operatorname{supp} \psi \subseteq U$.

Proposition: **Existence of Smooth Bump Functions**

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be closed and $A \subseteq U \subseteq M$ for some open set U. Then there exists a smooth bump function for A

supported in U.

Proof. Use the existence of smooth partitions of unity. Let $U_0 = U, U_1 = M - A$. □

Now we deal with smooth maps on arbitrary subsets of manifolds. Suppose M, N are smooth manifolds, with or without boundary, and $A \subseteq M$ is arbitrary. A map $F : A \to N$ is **smooth** if for every $p \in A$, there exists an open neighborhood U of p in M and a smooth map $\tilde{F} : U \to N$ such that $\tilde{F}|_{U \cap A} = F|_{U \cap A}$.

Lemma 2.2.2: Extension Lemma for Smooth Functions

Let M be a smooth manifold, with or without boundary, and let $A \subseteq M$ be closed and $f : A \to \mathbb{R}^k$ be a smooth function. Then for any open set U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\operatorname{supp} \tilde{f} \subseteq U$.

Proof. For each $p \in A$, take an open neighborhood $W_p \subseteq U$ and a smooth function $\tilde{f}_p : W_p \to \mathbb{R}^k$ such that $\tilde{f}_p|_{W_p \cap A} = f|_{W_p \cap A}$. Then the family $\{W_p\}_{p \in A} \cup \{M - A\}$ is an open cover of M. Let $\{\varphi_p : M \to \mathbb{R}\}_{p \in A} \cup \{\varphi_0\}$ be a smooth partition of unity subordinate to this cover. Define

$$\tilde{f}(x) = \sum_{p \in A} \varphi_p(x) \tilde{f}_p(x).$$

From local finiteness, the sum is well-defined and smooth. Also, $\operatorname{supp} \tilde{f} \subseteq U$ and for any $x \in A$,

$$\tilde{f}(x) = \sum_{p \in A} \varphi_p(x) \tilde{f}_p(x) = \sum_{p \in A} \varphi_p(x) f(x) = f(x).$$

Remark:
Note that the codomain is $\mathbb{R}^k$ here, this lemma would fail for arbitrary manifolds.

□

Definition 2.2.3: Exhaustion Functions

If M is a topological space, a continuous function $f : M \to \mathbb{R}$ is an **exhaustion function** if for every $c \in \mathbb{R}$, the sublevel set $M_c = f^{-1}((-\infty, c])$ is compact.

Well, as $n \in \mathbb{Z}_+$, the sets M_n forms an exhaustion of M by compact sets, hence the name.

Proposition: **Existence of Smooth Exhaustion Functions**

Every smooth manifold M without boundary admits a smooth positive exhaustion function.

Proof. SORRY □

Theorem 2.2.2: Level Sets of Smooth Functions

Let M be a smooth manifold. If K is a closed subset of M, then there exists a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

Proof. SORRY □

Chapter 3

Tangent Vectors

The basic idea of calculus is linear approximation.

In analysis, we come across the idea of geometric tangent vectors in $\mathbb{R}^n$, which are used for "directional derivatives" of multivariable functions. We shall follow this path initially, and then move to a more abstract definition of tangent vectors as derivations.

3.1 Tangent Vectors

Take $S^{n-1} \subseteq \mathbb{R}^n$ for example. For a point $x \in S^{n-1}$, usually we think it as a location, expressed by coordinates $(x^1, x^2, \ldots, x^n)$. But when doing calculus, we sometimes need to think of it as a vector. Geometrically, we can think of a vector as an arrow which has arbitrary start point. We imagine tangent vectors as arrows starting from the point x. That is to say, they live in a copy of $\mathbb{R}^n$ that is "attached" to the point x.

3.1.1 Geometric Tangent Vectors

Given $a \in \mathbb{R}^n$, define the geometric tangent space to $\mathbb{R}^n$ at a as the vector space

$$\mathbb{R}^n_a = \{a\} \times \mathbb{R}^n = \{(a, v) : v \in \mathbb{R}^n\}, \quad (a, v) + (a, w) = (a, v + w), \quad c(a, v) = (a, cv). \tag{3.1}$$

A geometric tangent vector in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$. We shall denote $v_a = (a, v)$

From this perspective, we can think of the tangent space of S^{n-1} at $a \in S^{n-1}$ as a subspace of $\mathbb{R}^n_a$: As all vectors in $\mathbb{R}^n_a$ that are perpendicular to the radius vector from the origin to a. To do this, we must have an inner product inherited from $\mathbb{R}^n$ via the natural isomorphism between $\mathbb{R}^n_a$ and $\mathbb{R}^n$.

This cannot be generalized to arbitrary manifolds, since there is no ambient Euclidean space to provide such a notion of perpendicularity. We shall use smooth structures to define tangent vectors in a more abstract way.

We turn to directional derivatives to motivate our definition. Every geometric tangent vector $v_a \in \mathbb{R}^n_a$ defines a map

$$D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad D_v|_a f = D_v f(a) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a + tv). \tag{3.2}$$

This map is linear and satisfies the Leibniz product rule:

$$D_v|_a(fg) = f(a) D_v|_a g + g(a) D_v|_a f.$$

If $v_a = v^i e_{i,a}$ in the standard basis, then we have

$$D_v|_a(f) = v^i \frac{\partial f}{\partial x^i}(a).$$

We now reverse the process.

Definition 3.1.1: Derivation

A **derivation** at $a \in \mathbb{R}^n$ is a linear map

$$w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

that satisfies the Leibniz product rule:

$$w(fg) = f(a)w(g) + g(a)w(f)$$

for all $f, g \in C^\infty(\mathbb{R}^n)$. The set of all derivations at a is denoted by $T_a\mathbb{R}^n$. Then $T_a\mathbb{R}^n$ is a vector space under the operations

$$(w_1 + w_2)(f) = w_1(f) + w_2(f), \quad (cw)(f) = cw(f).$$

It is fairly surprising that $T_a\mathbb{R}^n$ is isomorphic to the geometric tangent space $\mathbb{R}^n_a$.

Lemma 3.1.1: Property of Derivations

Suppose $a \in \mathbb{R}^n$ and $w \in T_a\mathbb{R}^n$, $f, g \in C^\infty(\mathbb{R}^n)$.

- If f is constant, then $w(f) = 0$.
- If $f(a) = g(a) = 0$, then $w(fg) = 0$.

Proof. If $f(x) = 1$, then $wf = w(ff) = f(a)wf + f(a)wf = 2wf$, so $wf = 0$. If $f(x) = c$, then $wf = w(cf_1) = cw(f_1) = 0$. □

Proposition: **The Structure of $T_a\mathbb{R}^n$**

Let $a \in \mathbb{R}^n$. Then

- For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a$ defined above is a derivation at a.
- The map $v_a \mapsto D_v|_a$ is a vector space isomorphism from $\mathbb{R}^n_a$ to $T_a\mathbb{R}^n$.

Proof. To prove isomorphism:

- Linearity: we have
$$D_{c_1 v + c_2 w}|_a f = (c_1 v + c_2 w)^i \frac{\partial f}{\partial x^i}(a) = c_1 v^i \frac{\partial f}{\partial x^i}(a) + c_2 w^i \frac{\partial f}{\partial x^i}(a) = c_1 D_v|_a f + c_2 D_w|_a f.$$
- Injectivity: if $D_v|_a = 0$, then for all $f \in C^\infty(\mathbb{R}^n)$, $D_v|_a f = v^i \dfrac{\partial f}{\partial x^i}(a) = 0$. Taking $f(x) = x^j$, we have $v^j = 0$ for all j, so $v = 0$.

- Surjectivity: let $w \in T_a\mathbb{R}^n$. Define $v^i = w(x^i)$, and let $v_a = v^i e_i|_a$. For any $f \in C^\infty(\mathbb{R}^n)$, by Taylor's theorem, we have

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + R(x),$$

$$R(x) = \sum_{i,j=1}^{n} (x^i - a^i)(x^j - a^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))\,\mathrm{d}t.$$

As $R(x)$ is the sum of products of functions vanishing at a, by the previous lemma we have $w(R) = 0$. Thus,

$$w(f) = w\left(f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) w(x^i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) v^i = D_v|_a f.$$

□

We have thus established the equivalence. And this definition can be generalized to arbitrary smooth manifolds.

3.1.2 Tangent Vectors on Manifolds

Definition 3.1.2: Tangent Vectors on Manifolds

Let M be a smooth manifold, with or without boundary, and let $p \in M$. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a **derivation at** p if it satisfies the Leibniz product rule:

$$v(fg) = f(p)v(g) + g(p)v(f), \quad \forall f, g \in C^\infty(M).$$

The set of all derivations at p is denoted by T_pM and called the **tangent space** of M at p. Its elements are called **tangent vectors** to M at p.

Proposition: **Property of Tangent Vectors on Manifolds**

Let M be a smooth manifold, with or without boundary, and let $p \in M$. If $v \in T_pM$ and $f, g \in C^\infty(M)$, then

- If f is constant, then $v(f) = 0$.
- If $f(p) = g(p) = 0$, then $v(fg) = 0$.

3.2 The Differential of a Smooth Map

We talk about the differential in analysis as linear approximations of functions at a given point. In the manifold case, there makes no sense to talk about linear transformations between manifolds, so we do it in terms of tangent spaces.

Definition 3.2.1: Differential on Manifolds

If M, N are smooth manifolds, with or without boundary, and $F : M \to N$ is a smooth map, then for each $p \in M$, we define a map

$$\mathrm{d}F_p : T_pM \to T_{F(p)}N \tag{3.3}$$

to be the **differential** of F at p, defined by

$$(\mathrm{d}F_p(v))(f) = v(f \circ F), \quad \forall f \in C^\infty(N), v \in T_pM. \tag{3.4}$$

Remark:
This is quite natural. To give a geometric intuition, take a curve γ in M with $\gamma(0) = p$ and $\gamma'(0) = v$. Then $F \circ \gamma$ is a curve in N with $(F \circ \gamma)(0) = F(p)$, and the tangent vector of $F \circ \gamma$ at 0 is $\mathrm{d}F_p(v)$.
Here v is a directional derivative operator acting on functions on M. which is given: $v(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(\gamma(t))$ for $g \in C^\infty(M)$. Then $\mathrm{d}F_p(v)$ is also a directional derivative operator acting on functions on N: for $f \in C^\infty(N)$,

$$(\mathrm{d}F_p(v))(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f((F \circ \gamma)(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ F)(\gamma(t)) = v(f \circ F).$$

The operator $\mathrm{d}F_p$ is linear, as for $v, w \in T_pM$, $c \in \mathbb{R}$, we have

$$(\mathrm{d}F_p(cv + w))(f) = (cv + w)(f \circ F) = cv(f \circ F) + w(f \circ F) = c(\mathrm{d}F_p(v))(f) + (\mathrm{d}F_p(w))(f).$$

Is also follows the Leibniz product rule:

$$\begin{aligned}(\mathrm{d}F_p(v))(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))(\mathrm{d}F_p(v))(g) + g(F(p))(\mathrm{d}F_p(v))(f).\end{aligned}$$

Proposition: **Properties of Differential**

Let M, N, P be smooth manifolds, with or without boundary, and let $F : M \to N$ and $G : N \to P$ be smooth maps. Then for each $p \in M$,

- $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is a linear map.
- (Chain Rule) $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$.
- If $\mathrm{id}_M : M \to M$ is the identity map, then $\mathrm{d}(\mathrm{id}_M)_p$ is the identity map on T_pM.
- If F is a diffeomorphism, then $\mathrm{d}F_p$ is an isomorphism, and $(\mathrm{d}F_p)^{-1} = \mathrm{d}(F^{-1})_{F(p)}$.

Our first application of differentials is to relate tangent spaces of manifoldes to those of Euclidean spaces via charts. But first we shall prove that tangent vectors are local-behaved, for charts only give local information.

Proposition: **Locality of Tangent Vectors**

Let M be a smooth manifold, with or without boundary, and let $p \in M$. If $v \in T_pM$ and $f, g \in C^\infty(M)$ agree on an open neighborhood of p, then $v(f) = v(g)$.

Proof. Let $f, g \in C^\infty(M)$ agree on an open neighborhood U of p. Then $h = f - g$ vanishes on U. Let $\psi \in C^\infty(M)$ be a smooth bump function that is 1 on $\operatorname{supp} h$ and $\operatorname{supp} \psi \subseteq M - \{p\}$. Then $h = h\psi$, so by the previous proposition, we have

$$v(h) = v(h\psi) = h(p)v(\psi) + \psi(p)v(h) = 0.$$

Thus, $v(f) = v(g)$. □

Proposition: **Tangent Space to Open Subsets**

Let M be a smooth manifold, with or without boundary, and let $U \subseteq M$ be an open subset. Let $\iota : U \hookrightarrow M$ be the inclusion map. Then for $p \in U$, the differential

$$\mathrm{d}\iota_p : T_pU \to T_pM$$

is an isomorphism.

Proof. Via the extension lemma, every $f \in C^\infty(U)$ can be extended to a function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_U = f$. Thus, with the locality of tangent vectors, we can easily see the result. □

Therefore, it is safe to identify T_pU with T_pM via the inclusion map.

Theorem 3.2.1: Dimension of Tangent Space

Let M be a smooth manifold of dimension n, and let $p \in M$. Then T_pM is an n-dimensional real vector space.

Proof. Take a chart (U, φ) containing p. Then as φ is a diffeomorphism from U to an open subset $\hat{U} \subseteq \mathbb{R}^n$, by the previous proposition, we have an isomorphism

$$T_pM \cong T_pU \cong T_{\varphi(p)}\hat{U} \cong T_{\varphi(p)}\mathbb{R}^n.$$

Thus, by the earlier result on $\mathbb{R}^n$, we have $\dim T_pM = n$. □

Now we address points on the boundary of manifolds with boundary. The situation is similar. First, we shall relate the tangent spaces of $T_a\mathbb{H}^n$ to those of $\mathbb{R}^n$ when $a \in \partial\mathbb{H}^n$. As $\mathbb{H}^n$ is not an open subset of $\mathbb{R}^n$, we cannot use the previous proposition **??** directly. However, we have the following result.

Lemma 3.2.1: Inclusion of $\mathbb{H}^n$

Let $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ be the inclusion map. Then for each $a \in \partial\mathbb{H}^n$, the differential $\mathrm{d}\iota_a : T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ is a linear isomorphism.

Proof. Assume $\mathrm{d}\iota_a(v) = 0$, then for all $f \in C^\infty(\mathbb{H}^n)$, we let $\tilde{f} \in C^\infty(\mathbb{R}^n)$ be an extension of f. Thus, $\tilde{f} \circ \iota = f$, and we have

$$v(f) = v(\tilde{f} \circ \iota) = (\mathrm{d}\iota_a(v))(\tilde{f}) = 0.$$

So $v = 0$ and $\mathrm{d}\iota_a$ is injective.

For surjectivity, let $w \in T_a\mathbb{R}^n$. Define $v : C^\infty(\mathbb{H}^n) \to \mathbb{R}$ by $v(f) = w(\tilde{f})$, where $\tilde{f} \in C^\infty(\mathbb{R}^n)$ is any extension of f. Thus

$$v(f) = w^i \frac{\partial \tilde{f}}{\partial x^i}(a).$$

From continuity, this does not depend on the choice of extension $\tilde{f}$, as we can get the result by limiting process from points in the interior of $\mathbb{H}^n$. So we have $\mathrm{d}\iota_a(v) = w$. □

Therefore, it is safe to identify $T_a\mathbb{H}^n$ with $T_a\mathbb{R}^n$ via the inclusion map, even for $a \in \partial\mathbb{H}^n$.

Proposition: **Dimension of Tangent Space with Boundary**

Let M be a smooth manifold of dimension n with boundary, and let $p \in M$. Then T_pM is an n-dimensional real vector space.

Next, as we know that for a finite-dimensional vector space, there exists a natural smooth structure on it. We shall see that the tangent space to a vector space at any point is naturally isomorphic to the vector space itself.

Proposition: **Tangent Space to a Vector Space**

Let V be a finite-dimensional real vector space with the standard smooth structure, and let $v \in V$. Then there is a natural isomorphism $V \cong T_vV$, defined by

$$v \mapsto D_v|_a, \quad D_v|_a(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a + tv), \quad \forall f \in C^\infty(V).$$

For any liner transformation $T : V \to W$ between finite-dimensional real vector spaces, we have

$$L \cong \mathrm{d}L_a, \qquad \mathrm{d}L_a(D_v|_a) = D_{L(v)}|_{L(a)}. \tag{3.5}$$

Therefore, we can identify T_vV with V itself via the above isomorphism. For example, since $GL(n,\mathbb{R})$ is an open subset of the vector space $M_{n\times n}(\mathbb{R})$, we can identify $T_AGL(n,\mathbb{R})$ with $M_{n\times n}(\mathbb{R})$ for each $A \in GL(n,\mathbb{R})$.

For products, we have the following result.

Theorem 3.2.2: Tangent Space to Product Manifolds

Let $M_1, \ldots, M_k$ be smooth manifolds, at most one have boundary. Let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection map onto the j-th factor. Then for each $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha_p : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k, \quad \alpha_p(v) = (\mathrm{d}\pi_1)_p(v), \ldots, (\mathrm{d}\pi_k)_p(v)$$

Is an isomorphism.

Therefore, we can identify $T_p(M_1 \times \cdots \times M_k)$ with $T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ via the above isomorphism.

3.3 Computation in Coordinates

We shall use charts to compute tangent vectors and differentials in coordinates.

Suppose M is a smooth manifold of dimension n (without boundary for simplicity), and (U, φ) is a chart containing $p \in M$. Then $\varphi : U \to \hat{U} \subseteq \mathbb{R}^n$ is a diffeomorphism, thus $\mathrm{d}\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism.

In $\mathbb{R}^n$, we have the standard basis

$$\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} : f \mapsto \frac{\partial f}{\partial x^i}(\varphi(p)), \quad i = 1, \ldots, n.$$

Therefore, the preimages of these basis vectors under $\mathrm{d}\varphi_p$ form a basis of T_pM, denoted by

$$\left.\frac{\partial}{\partial x^i}\right|_p = (\mathrm{d}\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = \mathrm{d}(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right), \quad i = 1, \ldots, n. \tag{3.6}$$

Acting on $f \in C^\infty(M)$, we have

$$\left.\frac{\partial}{\partial x^i}\right|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}), \quad \hat{f} = f \circ \varphi^{-1}, \quad \hat{p} = \varphi(p).$$

which is the coordinate expression of f and p in $\mathbb{R}^n$. We call $\left\{\left.\frac{\partial}{\partial x^i}\right|_p : i = 1, \ldots, n\right\}$ the **coordinate basis** of T_pM induced by the chart (U, φ).

Remark:
In $\mathbb{R}^n$, the coordinate basis vectors are just the partial derivative operators along the coordinate axes.

For points on the boundary of manifolds with boundary, the situation is similar, just replacing $\mathbb{R}^n$ with $\mathbb{H}^n$, and using the inclusion isomorphism between $T_a\mathbb{H}^n$ and $T_a\mathbb{R}^n$ for $a \in \partial\mathbb{H}^n$.

Theorem 3.3.1: The Coordinate Basis

Let M be a smooth manifold of dimension n, with or without boundary, and let $p \in M$. Then take any chart (U, φ) containing p. Then the coordinate vectors

$$\left.\frac{\partial}{\partial x^i}\right|_p = \mathrm{d}(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right), \quad i = 1, \ldots, n$$

form a basis of T_pM.

This a tangent vector $v \in T_pM$ can be expressed in coordinates as

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p, \quad v^i = v(x^i), \tag{3.7}$$

where $x^i = \pi_i \circ \varphi$ are the coordinate functions on U. The numbers v^i are called the **components** of v with respect to the coordinate basis induced by the chart (U, φ).

3.3.1 The Differential in Coordinates

Now, we shall do computations of differentials of smooth maps in coordinates form. First, for simplicity consider $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets, and let $F : U \to V$ be a smooth map. For $p \in U$, we have $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ being a linear map. In the standard coordinate bases, we have

$$\mathrm{d}F_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) f = \frac{\partial}{\partial x^i}\bigg|_p (f \circ F) = \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) = \left(\frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\bigg|_{F(p)} \right) f.$$

Thus, we have

$$\mathrm{d}F_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\bigg|_{F(p)}. \tag{3.8}$$

Writing in matrix form, we have

$$\mathrm{d}F_p = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \frac{\partial F^m}{\partial x^2}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix} \tag{3.9}$$

which is just the Jacobian matrix of F at p. The same can be said if U is an open subset of $\mathbb{H}^n$, so do V.

For a more general case, let M, N be smooth manifolds of dimension n, m respectively, with or without boundary, and let $F : M \to N$ be a smooth map. Take charts (U, φ) and (V, ψ) containing $p \in M$ and $F(p) \in N$ respectively. Then we have $\mathrm{d}\hat{F}_{\hat{p}} : T_{\hat{p}}\mathbb{R}^n \to T_{\hat{F}(\hat{p})}\mathbb{R}^m$ being the differential of the smooth map $\hat{F} = \psi \circ F \circ \varphi^{-1} : \hat{U} \to \hat{V}$ at $\hat{p} = \varphi(p)$. In the coordinate bases, we have

$$\begin{aligned} \mathrm{d}F_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \mathrm{d}(\psi^{-1})_{\hat{F}(\hat{p})} \circ \mathrm{d}\hat{F}_{\hat{p}} \circ \mathrm{d}\varphi_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \\ &= \mathrm{d}(\psi^{-1})_{\hat{F}(\hat{p})} \left(\mathrm{d}\hat{F}_{\hat{p}} \left(\frac{\partial}{\partial x^i}\bigg|_{\hat{p}} \right) \right) \\ &= \mathrm{d}(\psi^{-1})_{\hat{F}(\hat{p})} \left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\bigg|_{\hat{F}(\hat{p})} \right) \\ &= \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\bigg|_{F(p)}. \end{aligned} \tag{3.10}$$

Which is just the pushforward of the Jacobian matrix of $\hat{F}$ at $\hat{p}$ via the charts.

3.3.2 Change of Coordinates

Suppose (U, φ) and (V, ψ) are two smooth charts on M and $p \in U \cap V$. Denote the coordinate functions of φ by $x^i = \pi_i \circ \varphi$ and those of ψ by $\tilde{x}^i = \pi_i \circ \psi$. Therefore, any tangent vector

$v \in T_pM$ can be expressed in both coordinate bases, and we want to find the relation between the components.

To do it, consider the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$, and we write its coordinate functions by

$$\varphi \circ \psi^{-1}(x) = (\tilde{x}^1(x), \ldots, \tilde{x}^n(x)).$$

We have identified V with $\psi(V) \subseteq \mathbb{R}^n$ via the chart ψ, so we use the same notation $\tilde{x}^i$ for the coordinate functions on V for simplicity. Then we have the differential

$$\mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} : T_{\varphi(p)}\mathbb{R}^n \to T_{\psi(p)}\mathbb{R}^n.$$

by the previous result, we have

$$\mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\psi(p)}. \tag{3.11}$$

So we have pull back to T_pM via the charts:

$$\begin{aligned}
\left.\frac{\partial}{\partial x^i}\right|_p &= \mathrm{d}(\varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) \\
&= \mathrm{d}(\psi^{-1})_{\psi(p)} \circ \mathrm{d}(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) \\
&= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p .
\end{aligned} \tag{3.12}$$

Therefore, the components of v in the two coordinate bases are related by

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) v^i. \tag{3.13}$$

3.4 The Tangent Bundle

Definition 3.4.1: The Tangent Bundle

Let M be a smooth manifold, with or without boundary. The **tangent bundle** of M is the disjoint union

$$TM = \bigsqcup_{p \in M} T_pM = \{(p, v) : p \in M, v \in T_pM\}.$$

The map $\pi : TM \to M$ defined by $\pi(p, v) = p$ is called the **bundle projection**.

For example, the tangent bundle of $\mathbb{R}^n$ is naturally isomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ via the isomorphism

$$\mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n, \quad (a, v) \mapsto (a, D_v|_a).$$

But for general manifolds, we cannot identify TM with $M \times \mathbb{R}^n$ globally because we cannot have a natural way to identify each tangent space T_pM with each other.

Theorem 3.4.1: Structure of the Tangent Bundle

Let M be a smooth manifold of dimension n. Then TM has a natural topology and smooth structure such that TM is a smooth manifold of dimension $2n$. With this structure, the bundle projection $\pi : TM \to M$ is a smooth map.

Proof. The ultimate intuition is to do it locally via charts. For each smooth chart (U, φ) on M, note that $\pi^{-1}(U) = \bigsqcup_{p\in U} T_pM$. Define a map $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{\varphi}(p, v) = \tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right), \tag{3.14}$$

So the image set is $\hat{U} \times \mathbb{R}^n$, being an open subset of $\mathbb{R}^{2n}$. It is also a bijection from $\pi^{-1}(U)$ to $\hat{U} \times \mathbb{R}^n$, because

$$\tilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) = v_i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x^1,\dots,x^n)}.$$

Now suppose we have two smooth charts (U, φ) and (V, ψ) on M and let $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ be the corresponding charts on TM. Then the sets

$$\tilde{\varphi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \varphi(U \cap V) \times \mathbb{R}^n, \quad \tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \psi(U \cap V) \times \mathbb{R}^n$$

are both open subsets of $\mathbb{R}^{2n}$. The transition map is given by

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) = \left(\tilde{x}^1(x), \dots, \tilde{x}^n(x), \frac{\partial \tilde{x}^j}{\partial x^1}(x)v^i, \dots, \frac{\partial \tilde{x}^j}{\partial x^n}(x)v^i\right),$$

which is smooth.

Finally, choose a countable cover of charts $\{(U_\alpha, \varphi_\alpha)\}$ of M, then the corresponding charts $\{(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha)\}$ form an atlas of TM. The conditions of 1.3.1 are easily verified. □

Remark:
For smooth manifolds with boundary, the construction is similar, just replacing $\mathbb{R}^n$ with $\mathbb{H}^n$ in the above proof. We note that the only "half-ness" happens in the base manifold M, while each tangent space T_pM is a full n-dimensional vector space, so no harm is done to the tangent bundle structure.

Proposition: **Single-Chart Tangent Bundle**

If M is a smooth manifold of dimension n (with or without boundary) that can be covered by a single chart (M, φ), then the tangent bundle TM is diffeomorphic to $M \times \mathbb{R}^n$.

Proof. Obvious. □

Remark:
NOTE that although we can locally view TM as $U \times \mathbb{R}^n$ via charts, there is no natural way to identify TM with $M \times \mathbb{R}^n$ globally in general. In fact, this may not be true in many cases.

Putting all pointwise differentials together, we have a map

$$\mathrm{d}F : TM \to TN, \quad \mathrm{d}F(p, v) = (F(p), \mathrm{d}F_p(v)),$$

called the global differential of F.

Theorem 3.4.2: Global Differential is Smooth

Let M, N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. Then the global differential

$$\mathrm{d}F : TM \to TN$$

is a smooth map.

Proof. From the coordinate expression, we have

$$\mathrm{d}F(p, v) = \left(F(p), \frac{\partial F^j}{\partial x^i}(p) v^i \right),$$

which is smooth for F is. □

Proposition: **Properties of Global Differential**

Let M, N, P be smooth manifolds, with or without boundary, and let $F : M \to N$ and $G : N \to P$ be smooth maps. Then for each $(p, v) \in TM$,

- (Chain Rule) $\mathrm{d}(G \circ F) = \mathrm{d}G \circ \mathrm{d}F$.
- If $\mathrm{id}_M : M \to M$ is the identity map, then $\mathrm{d}(\mathrm{id}_M) = \mathrm{id}_{TM}$.
- If F is a diffeomorphism, then $\mathrm{d}F$ is a diffeomorphism, and $(\mathrm{d}F)^{-1} = \mathrm{d}(F^{-1})$.

Just using proposition **??** would do. From now we may denote $\mathrm{d}F^{-1}$ for either $\mathrm{d}(F^{-1})$ or $(\mathrm{d}F)^{-1}$, when F is a diffeomorphism.

3.5 Velocity Vectors of Curves

Definition 3.5.1: Curves

Let M be a manifold, with or without boundary. A **curve** in M is a continuous map $\gamma : J \to M$, where $J \subseteq \mathbb{R}$ is an open interval. Sometimes we may want J to have one or both endpoints, in which case slight modifications are needed.

Definition 3.5.2: Velocity

Let M be a smooth manifold, with or without boundary, and let $\gamma : J \to M$ be a smooth curve. The **velocity** of γ at $t_0 \in J$ is the tangent vector

$$\gamma'(t_0) = \mathrm{d}\gamma_{t_0} \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t_0} \right) \in T_{\gamma(t_0)} M. \tag{3.15}$$

Other notations include

$$\dot{\gamma}(t_0) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t_0) = \left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_{t=t_0}$$

The tangent vector $\gamma'(t_0)$ acts on functions by

$$\gamma'(t_0)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} f(\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (f \circ \gamma)(t).$$

which is the rate of change of f along the curve γ at t_0. For a smooth chart (U, φ) containing $\gamma(t_0)$, we can express the velocity in coordinates as

$$\gamma'(t_0) = \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)} = \left(\frac{\mathrm{d}\gamma^1}{\mathrm{d}t}(t_0), \ldots, \frac{\mathrm{d}\gamma^n}{\mathrm{d}t}(t_0)\right),$$

which is familiar in Euclidean space.

Next, we shall see that every tangent vector can be expressed as the velocity of some curve, which will lead us to an equivalent definition of tangent vectors.

Proposition: **Tangent Vector as Velocity**

Let M be a smooth manifold, with or without boundary, and let $p \in M$. Then for any tangent vector $v \in T_pM$, there exists a smooth curve $\gamma : J \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Proof. First suppose $p \in \operatorname{Int} M$, then let (U, φ) be a smooth chart centering p. Then we write $v = v^i \partial/\partial x^i|_p$. For sufficiently small ϵ, we have a smooth curve $\gamma : (-\epsilon, \epsilon) \to U$ defined by

$$\gamma(t) = \varphi^{-1}\left(tv^1, \ldots, tv^n\right)$$

which is smooth because φ^{-1} is smooth.

Now if $p \in \partial M$, then let (U, φ) be a smooth boundary chart centering p. We can similarly define a smooth curve $\gamma : [0, \epsilon) \to U$ or $(-\epsilon, 0] \to U$ by the same formula for sufficiently small $\epsilon > 0$, depending on the sign of the first component of v. □

For composition, we have the following result.

Proposition: **Velocity under Composition**

Let M, N be smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map. If $\gamma : J \to M$ is a smooth curve, then for each $t_0 \in J$,

$$(F \circ \gamma)'(t_0) = \mathrm{d}F_{\gamma(t_0)}\left(\gamma'(t_0)\right).$$

Proof. Just the chain rule:

$$(F \circ \gamma)'(t_0)(f) = \mathrm{d}(F \circ \gamma)_{t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right)(f) = \mathrm{d}F \circ \mathrm{d}\gamma_{t_0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right)(f) = \mathrm{d}F_{\gamma(t_0)}\left(\gamma'(t_0)\right)(f).$$

□

We can also use curve velocity to compute differentials: Suppose M, N are smooth manifolds, with or without boundary, and let $F : M \to N$ be a smooth map, then to compute $\mathrm{d}F_p(v)$ for $p \in M$ and $v \in T_pM$, we can first find a smooth curve $\gamma : J \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, then we have

$$\mathrm{d}F_p(v) = \mathrm{d}F_p\left(\gamma'(0)\right) = (F \circ \gamma)'(0).$$

3.6 Alternative Definition of Tangent Vectors

3.6.1 Derivations of the Space of Germs

A smooth function element on M is an ordered pair (f, U), where $U \subseteq M$ is an open set and $f \in C^\infty(U)$. Two smooth function elements (f, U) and (g, V) are said to be equivalent at $p \in U \cap V$ if there exists an open neighborhood $W \subseteq U \cap V$ of p such that $f|_W = g|_W$. The equivalence class of (f, U) at p is called the **germ** of f at p, and the set if all germs of smooth functions at p is denoted by $C_p^\infty(M)$.

Remark:
Intuitively, $C_p^\infty(M)$ of a smooth function at p contains all distinguishable smooth functions locally around p.

We notice that $C_p^\infty(M)$ is a real vector space and an associative algebra under operations defined by

- Addition: $[(f, U)] + [(g, V)] = [(f + g, U \cap V)]$.
- Scalar Multiplication: $c[(f, U)] = [(cf, U)]$.
- Multiplication: $[(f, U)] \cdot [(g, V)] = [(fg, U \cap V)]$.

Now we denote the germ of f at p simply by $[f]_p$ when there is no confusion.

A derivation of $C_p^\infty(M)$ is a linear map $v : C_p^\infty(M) \to \mathbb{R}$ such that for all $[f]_p, [g]_p \in C_p^\infty(M)$,

$$v([f]_p \cdot [g]_p) = f(p)v([g]_p) + g(p)v([f]_p). \tag{3.16}$$

The set of all derivations of $C_p^\infty(M)$ is denoted by $\mathcal{D}_p(M)$. And it is simple to verify that $\mathcal{D}_p(M)$ is naturally isomorphic to T_pM.

3.6.2 Equivalent Class of Curves

This definition captures the intuitive idea of tangent vectors as "directions" at a point. Suppose p is a point of M, and consider all smooth curves $\gamma : J \to M$ such that $\gamma(0) = p$. We say two such curves γ_1 and γ_2 are equivalent at p if for any smooth function $f : M \to \mathbb{R}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(f \circ \gamma_1)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(f \circ \gamma_2)(t). \tag{3.17}$$

The equivalence classes are denoted by $[\gamma]$, and all such equivalence classes form a set denoted by $\mathcal{V}_p(M)$, which is naturally isomorphic to T_pM.

3.7 Categories and Functors

A category $\mathcal{C}$ consists of

- A class $\mathrm{Ob}(\mathcal{C})$, whose elements are called objects of $\mathcal{C}$.
- A class $\mathrm{Hom}(\mathcal{C})$, whose elements are called morphisms of $\mathcal{C}$.
- For each morphism $f \in \mathrm{Hom}(\mathcal{C})$, there are two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ called the source and target of f, denoted by $f : X \to Y$.
- For each triplet of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, there is a mapping called composition

$$\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z), \quad (g, f) \mapsto g \circ f.$$

  where $\mathrm{Hom}(A, B)$ is the class of all morphisms from A to B.

The morphisms and objects must satisfy the following axioms:

- (Associativity) For each $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (Identity) For each object $X \in \mathrm{Ob}(\mathcal{C})$, there exists an identity morphism $\mathrm{id}_X : X \to X$ such that for each $f : X \to Y$.

$$\mathrm{id}_Y \circ f = f, \quad f \circ \mathrm{id}_X = f.$$

A morphism $f : X \to Y$ is called an isomorphism if there exists a morphism $g : Y \to X$ such that

$$g \circ f = \mathrm{id}_X, \quad f \circ g = \mathrm{id}_Y.$$