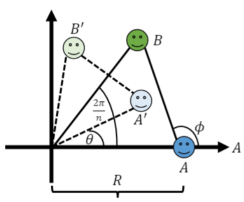
A group of n agents is initially spaced equally apart at a distance R from the origin. Each agent is chasing their neighbour who is positioned counter-clockwise from them. At any given point in time, an agent is facing their target.

1. Determine a polar equation of motion for a given agent. Without loss of generality, you may find the trajectory of an agent initially positioned on the positive x-axis.

2. Determine the total distance travelled by each agent.



1-form

$f: dx, dy, dr, d\theta$

$\int f\,dx = g$

$\int g\,dx\,dy = f \cdot dx$

$\int_C ds = L$

$ds = \sqrt{dx^2 + dy^2}$

$dx \to \ldots\,d\theta$, $dy \to \ldots\,d\theta$

$d\theta = \frac{v}{r} \cdot \cos\phi$

$dr = v \cdot dt \cdot \sin\phi$

$\theta(t)$, $r(t)$, $r(\theta)$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + (f')^2}\,dx$$

$$ds = \sqrt{(r\,d\theta)^2 + dr^2}$$

$$= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$r = Re^{-\theta \tan\left(\frac{\pi}{n}\right)}$$

$$d = \int_0^\infty \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$r(\theta)$

$\int ds = L$

$$\int d\theta = \frac{v\,dt}{r} \cos\phi = \int \frac{v\,dt}{R - vt\sin\phi} \cos\phi$$

$$\boxed{d = R\sec\left(\frac{\pi}{n}\right) \int_0^\infty e^{-\theta \tan\left(\frac{\pi}{n}\right)}\,d\theta}$$

$$\theta = A \cdot \ln(R - vt\sin\phi) + C$$

$$A = \frac{-\cos\phi}{\sin\phi}$$

$$d = R\csc\left(\frac{\pi}{n}\right)$$

$$r = R\,e^{-\theta \tan\phi}$$

Since Agent A is always facing Agent B, its trajectory's slope is equal to the slope of AB. Hence, the following equation is true:

$$\tan(\theta + \phi) = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Using angle addition formulas, we can obtain $\frac{dr}{r} = \cot\phi\,d\theta = -\tan\left(\frac{\pi}{n}\right)d\theta$ (by the

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Using angle addition formulas, we can obtain $\frac{dr}{r} = \cot\phi\, d\theta = -\tan\left(\frac{\pi}{n}\right) d\theta$ (by the expression for ϕ). The initial condition of this 1st order ODE is $r(\theta = 0) = R$, so the final polar equation is given by $r = Re^{-\theta \tan\left(\frac{\pi}{n}\right)}$.

The formula for arc distance in polar coordinates can be applied to the polar equation found in the previous question where the bounds of integration are 0 and ∞ since the agents move towards the origin forever:

$$d = \int_0^\infty \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

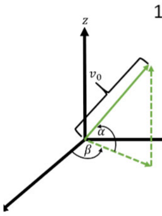
$$d = R\sec\left(\frac{\pi}{n}\right) \int_0^\infty e^{-\theta \tan\left(\frac{\pi}{n}\right)}\, d\theta$$

$$d = R\csc\left(\frac{\pi}{n}\right)$$

Modules F4-F5: Vector Valued Functions and Arc Length and Curvature

On a windy day, you are playing catch with one of your friends. The line connecting you and your friend is along the north-south axis, and the wind is blowing from west to east. The acceleration due to gravity is g, the wind speed is b.

1. Assume your friend throws the ball along the north-south line (y-axis) at a speed v_0, an angle α from the vertical (z-axis), and an angle β from the west-east line (x-axis). Find the position vector of the ball as a function of time.
2. Find the curvature of the ball as a function of time.
3. Assuming a wind speed of 0, what is the total distance travelled by the ball after time t?



1. The left diagram illustrates the initial velocity of the ball given by your friend. By trigonometry, the components are given by $\langle v_o\cos\alpha\cos\beta\,, v_o\cos\alpha\sin\beta\,, v_0\sin\alpha\rangle$. Factoring in the wind in the y direction and gravity in the z direction, the position vector of the ball is given by:

$$\vec{r}(t) = \left\langle (v_0\cos\alpha\cos\beta - b)t, v_0\cos\alpha\sin\beta\, t, h + v_0\sin\alpha\, t - \frac{1}{2}gt^2 \right\rangle$$

2. We may simplify this problem by assuming no wind speed since the curvature since gravity only acts in the z-direction.
   Using the formula for curvature $\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$, we have:

$$\kappa(t) = \frac{v_0 g \cos\alpha}{(v_o^2 - 2gv_0 \sin\alpha\, t + g^2t^2)^{\frac{3}{2}}}$$

   To maximize this function, we set its derivative to 0:

$$\kappa'(t) = \frac{3v_0 g^2 \cos\alpha\,(v_0 \sin\alpha - g^2 t)}{(v_0^2 - 2gv_0 \sin\alpha\, t + g^2t^2)^{\frac{5}{2}}} = 0 \Rightarrow t = \frac{v_0 \sin\alpha}{g}$$

   As expected, this corresponds to the maximum height of the ball.

3. Using $\vec{r}'(t) = \langle v_0 \cos\alpha \cos\beta\,, v_0 \cos\alpha \sin\beta\,, v_0 \sin\alpha - gt\rangle$, we can plug this into the arc length function:

$$s(t) = \int_0^t \sqrt{g^2u^2 - 2gv_0 \sin\alpha\, u + v_0^2}\, du$$

$$s(t) = g \int_0^t \sqrt{\left(u - \frac{v_0 \sin\alpha}{g}\right)^2 + \left(\frac{v_0 \cos\alpha}{g}\right)^2}\, du$$

3. Using a trigonometric substitution of $u - \frac{v_0 \sin\alpha}{g} = \frac{v_0 \cos\alpha}{g} \tan\theta$, the integral becomes:

$$s(t) = \frac{v_0^2 \cos^2\alpha}{g} \left(\int_{-\alpha}^{\tan^{-1}\left(\frac{gt}{v_0 \cos\alpha} - \tan\alpha\right)} \sec^3\theta\, d\theta \right)$$

   The antiderivative of $\sec^3\theta$ is given by $\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|$, so the final expression becomes:

$$s(t) = \frac{v_0^2 \cos^2\alpha}{g} \left(\sec\alpha \tan\alpha - \ln(\sec\alpha - \tan\alpha\,) + \frac{(gt - v_0 \sin\alpha)\sqrt{g^2t^2 - 2v_0 g \sin\alpha\, t + v_0^2}}{v_0^2 \cos^2\alpha} + \ln\left(\frac{\sqrt{g^2t^2 - 2v_0 g \sin\alpha\, t + v_0^2} + gt - v_0 \sin\alpha}{v_0 \cos\alpha} \right) \right)$$