3.8

Monday, March 8, 2021 7:46 PM

1.

A particle is observed to be moving such that its trajectory satisfies the following ODEs: $y'' - 3y' + 2y = \cos 2t$ and $y'' + 4y = e^{2t} + e^t$. What must have been its initial position and velocity at time $t = 0$?

① $\begin{cases} y_1 = y_{c1} + y_{p1} \end{cases}$ ① $\begin{cases} y_{c1} = Ae^{2t} + Be^t \\ y_{p1} = -\frac{1}{20}\cos 2t - \frac{3}{20}\sin 2t \end{cases}$

② $\begin{cases} y_2 = y_{c2} + y_{p2} \end{cases}$ ② $\begin{cases} y_{c2} = F\cos 2t + G\sin 2t \\ y_{p2} = \frac{1}{8}e^{2t} + \frac{1}{5}e^t \end{cases}$

$y_1 = y_2$

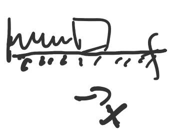
$A = 1/8 \quad B = 1/5 \quad F = -\frac{1}{20} \quad G = -3/20$

$\begin{cases} y(0) = A + B + F = \frac{11}{40} \\ y'(0) = 2A + B + 2G = \frac{3}{20} \end{cases}$

2.

Consider a damped spring-mass system with nonzero mass m, spring constant k, and damping coefficient b which is subjected to an external force $F(t) = F_0 \sin(\omega t)$. Driven force

a) Write a 2nd order ODE for the position x of the mass.
b) Is the characteristic solution always transient?
c) Find the particular solution. Is it transient or steady-state?
d) Convert the above into the form $C\sin(\omega t + \phi)$.
e) Set $m = k = 1$. Graph C as a function of ω for different values of b and comment on the effect of b and ω on the trajectory of the spring-mass system.



$m\ddot{x} = -b\dot{x} - kx + F_0\sin\omega t$ physics

$\boxed{m\ddot{x} + b\dot{x} + kx} = F_0\sin\omega t$ ODE.

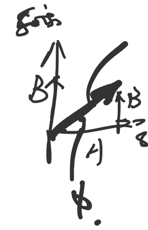
internal outside

$A\cos + B\sin$

$\to C\sin(\omega t + \phi)$

$C = \sqrt{A^2 + B^2}$

$\phi = \tan^{-1}\frac{A}{B}$

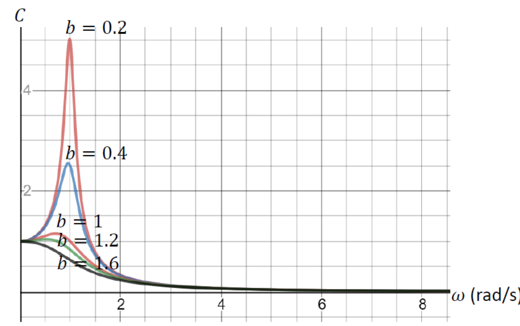


b)

$\begin{cases} m > 0 \\ b > 0 \\ k > 0 \end{cases} \to \lim_{t\to\infty} x \to 0.$

$e^{-2t}(\cancel{A\cos\omega t + B\sin\omega t}) \to 0.$

c) $x_p = A\cos(\omega t) + B\sin(\omega t)$ steady-state



$$\begin{cases} -Am\omega^2 + Bb\omega + Ak = 0 \\ -Bm\omega^2 - Ab\omega + Bk = F_0 \end{cases}$$

cos term

sin term

$$\begin{cases} A(k - m\omega^2) = -Bb\omega \\ A(b\omega) = B(k - m\omega^2) - F_0 \end{cases}$$

$$\begin{cases} A = \dfrac{-F_0 b\omega}{(m\omega^2 - k)^2 + (b\omega)^2} \\ B = \dfrac{-F_0(m\omega^2 - k)}{(m\omega^2 - k)^2 + (b\omega)^2} \end{cases}$$

$$\begin{cases} A \cdot m + nB = 0 \\ A \cdot n = Bm - F \end{cases}$$

$$F \begin{cases} \dfrac{m^2}{m^2+n^2} + \dfrac{n^2}{m^2+n^2} = 1 \\ \dfrac{mn}{m^2+n^2} - \dfrac{mn}{m^2+n^2} = 0 \end{cases}$$