3.24

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Module E3: Ratio Test

During the early 1900s, Indian mathematician Srinivasa Ramanujan developed a formula for π using an infinite series:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!\,(1103+26390n)}{(n!)^4 396^{4n}}$$

$(4(n+1))!$ $\quad (4n)!\,(4n+1)(4n+2)\cdots(4n+4)$

$((n+1)!)^4 \cdot \quad n!\cdot(n+1)^4$

Other series for π

More π

a) Prove this infinite series converges.

b) Find the error of this series using only the first two terms. $\quad 8.8818\times 10^{-16}$

c) Plot this series for different values of n alongside the true value of π.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(4(n+1))!\,(1103+26390(n+1))}{((n+1)!)^4 396^{4(n+1)}}\,\frac{(n!)^4 396^{4n}}{(4n)!\,(1103+26390n)}$$

$$= \frac{(4n+1)(4n+2)(4n+3)(4n+4)(1103+26390(n+1))}{(n+1)^4 396^4 (1103+26390n)}$$

$$= \frac{\left(4+\frac{1}{n}\right)\left(4+\frac{2}{n}\right)\left(4+\frac{3}{n}\right)\left(4+\frac{4}{n}\right)\left(\frac{27493}{n}+26390\right)}{\left(1+\frac{1}{n}\right)^4 396^4\left(\frac{1103}{n}+26390\right)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^4}{396^4}$$

Module E4: Manipulating Power Series

Bessel functions comprise a special family of functions which can be applied in many physical scenarios, including heat conduction in a circular plate, the shapes of acoustic membranes, and solutions to Schrödinger equation. The 0th order Bessel's differential equation is given below:

$$x^2y'' + xy' + x^2y = 0, \qquad y(0) = 1, y'(0) = 0$$

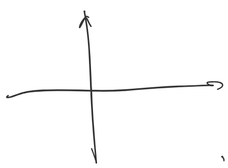
a) Assuming y can be expressed as a power series, find the 0th order Bessel function.

b) Using the Ratio Test, determine at which values of x the power series converges as well as the radius of convergence.

c) Graph the 0th order Bessel function with various order polynomials.

$y(x)$. $\quad x^2y'' + xy' + (x^2-n^2)y = 0$

n-th, Bessel DE. $\quad$ Bessel function



$\sum a_n x^n$

$\sum a_n \cos nx + b_n \sin nx$

$F(a_n, b_n) = 0$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^4$

$A($ $\quad$ vector function

$f, g \quad d(f,g) = \int_{-\infty}^{\infty} f_{(x)} g_{(x)}\,dx$

$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$

funct $\quad$ Function.

Convolution $\quad (f * g)_{(x)} = \int_{-\infty}^{\infty} f(x-y)\, g(y)\,dy$

L_2 norm

$$\boxed{\int_{-\infty}^{\infty} dx \cos nx \cos mx = \delta_{nm} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}}$$

FT

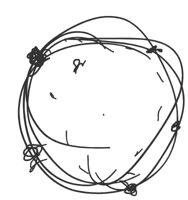
$$\tilde{f}(k) = \frac{1}{2\pi}\int f(x)\, e^{-ikx}\,dx.$$

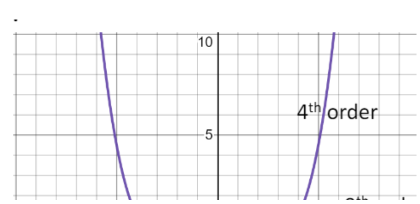
$\tilde{e}_i \cdot \tilde{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$e_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad e_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

L_2

$$E = A\ln r + Br + \sum\left(r^n \begin{matrix} a_n \cos n\theta \\ b_n \sin n\theta \end{matrix}\right) + \left(r^{-n} \begin{matrix} a_n \cdots \\ d_n \cdots \end{matrix}\right)$$





a) Assume $y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Now substitute these expressions into the ODE:

$$x^2\sum_{n=2}^{\infty} n(n-1)\,c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} + x^2\sum_{n=0}^{\infty} c_n x^n = 0$$

a) Assume $y=\sum_{n=0}^{\infty}c_nx^n \Rightarrow y'=\sum_{n=1}^{\infty}nc_nx^{n-1} \Rightarrow y''=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}$. Now substitute these expressions into the ODE:

$$x^2\sum_{n=2}^{\infty}n(n-1)\,c_nx^{n-2}+x\sum_{n=1}^{\infty}nc_nx^{n-1}+x^2\sum_{n=0}^{\infty}c_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}\,x^{n+2}+\left(c_1x+\sum_{n=0}^{\infty}(n+2)c_{n+2}x^{n+2}\right)+\sum_{n=0}^{\infty}c_nx^{n+2}=0$$

$$c_1x+\sum_{n=0}^{\infty}[(n+2)^2c_{n+2}+c_n]\,x^{n+2}=0$$

Since the ODE is true for all values of x, all of its coefficients must be 0. Hence, $c_1=0, c_{n+2}=-\frac{c_n}{(n+2)^2}$. Solving this recurrence relation in terms of c_0 yields $c_{2n}=\left(-\frac{1}{4}\right)^n\left(\frac{1}{n!}\right)^2c_0, c_{2n+1}=0$ for all $n\geq 0$. The solution is given as:

$$y=\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n\left(\frac{1}{n!}\right)^2c_0x^{2n}$$

Substituting the initial conditions yields $c_0=1$.