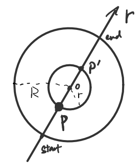
Gauss's law

You are designing a gravity train that will travel along a diametric path through the centre of the earth. Assume the mass and radius of earth are M and R, respectively. The force acting on the train with mass m at position r from the centre of the earth is given by $F_r = -\frac{GMm}{R^3} r$ where G is the gravitational constant.



if the flux $\sim r^{-2}$

a) Find a differential equation for the position of the train with respect to the centre of earth.

b) Assume the train starts at the surface of the earth with downwards speed v_0. Determine the position and velocity of the train as functions of time.

c) How far can the gravity train reach from the centre of the earth? What is it's maximum speed?

d) Assume the gravity train starts at rest. How long will it take to reach the other side of the earth?

a) $F_r = -\frac{GMm}{R^3} r = m\ddot{r}$

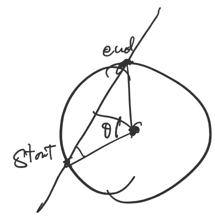
ODE: $\ddot{r} + \frac{GM}{R^3} r = 0$

b) $\dot{r}\big|_{r=0} = v_0$, $v_0 > 0$

say $\frac{GM}{R^3} = k^2$.

$r(t) = -R\cos(kt) + \frac{v_0}{k}\sin(kt)$

potential $\sim r^{-1}$



$\theta\uparrow \quad \sin\theta\downarrow \quad F_r\downarrow$

c) $r'(t) = Rk\sin(kt) + v_0\cos(kt) = C\sin(\omega t + \phi)$

$= \sqrt{R^2k^2 + v_0^2}\,\sin\left(kt + \tan^{-1}\frac{v_0}{Rk}\right)$

$\sqrt{R^2k^2+v_0^2} = \dot{r}_{max}$

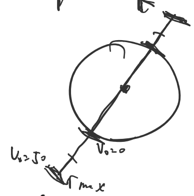
$r(t) = \sqrt{R^2 + \left(\frac{v_0}{k}\right)^2}\,\sin\left(kt + \tan^{-1}\frac{v_0}{Rk}\right)$

$\sqrt{R^2 + \frac{v_0^2}{k^2}} = r_{max}$

d) $r(t)\big|_{v_0=0} = R\sin(kt)$

freq.

$T = \frac{\pi}{k} = 42.25$ mins



According to Einstein's theory of special relativity, the kinetic energy of an object with rest mass m and speed v is given by $\boxed{K} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$ where c is the speed of light.

$\boxed{\gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}}}$

time dialation

length contraction

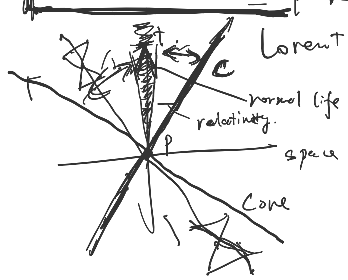
force $\sim r^{-2}$

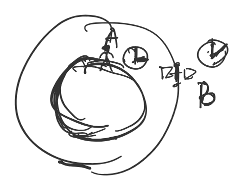
a) Find a second order Maclaurin polynomial for K as a function of $\frac{v^2}{c^2}$ for the kinetic energy.

b) Derive the classical form of kinetic energy by only considering first order terms.

c) Assume $v \le v_1$ for some v_1. Find an upper bound for the error in the second order Maclaurin polynomial.

d) Graph the first order, second order, and exact forms of K.





$K = mc^2(\gamma(v) - 1)$ SR

$K = \frac{1}{2}mv^2$ Newton special case of $K(SR)$ when $v \ll c$, $v/c \ll 1$

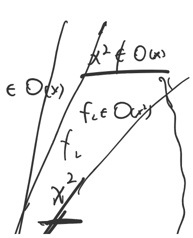
$K = \boxed{mc^2\left[1 + \frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} - 1\right]} + O\left(\left(\frac{v}{c}\right)^6\right)$

$\boxed{f \in O(g) \quad \exists c.\ f \le c \cdot g}$

$\approx mc^2\left[\frac{v^2}{2c^2}\right]$

$O(v^n)$ small when $\frac{v}{c}$ small

$= \frac{1}{2}mv^2$



$\approx mc^2 \left[\frac{v^2}{2c^2} \right]$

$= \frac{1}{2} m v^2$

$O(v^n)$ small when v/c small

$F = 2(v/c)^4 + \left[c_5 (v/c)^5 + c_6 (v/c)^6 + \cdots \right]$ when $v/c < 1$

$\exists\ C = 10$

$F < 10 \cdot (v/c)^4$

so $F \in O((v/c)^4)$