Module D1: Homogeneous 2nd Order ODEs

1. Consider a 2nd order ODE with nonzero constant coefficients: $ay'' + by' + cy = 0$. Under what conditions on a, b, c will the nontrivial solution $y(x)$ satisfy $\lim_{x\to\infty} y(x) = 0$?

2. Suppose we have a string that is constrained on two static ends over a length L. An equation that describes the standing wave amplitude $y(x)$ of the string is given by the boundary equation $y'' + \lambda y = 0$.

Dirichlet bdy. cond.
$\Rightarrow f(0), f(L) = 0.$

   a) Formulate this problem as a boundary value problem (i.e., add constraints for y).
   b) Under what conditions on λ does this equation give a nontrivial solution?
   c) Draw plots of possible solutions. (These correspond to the "normal modes" of the string.)

Q1. in r $\quad ar^2 + br + c = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \boxed{\frac{-b}{2a}} \pm \frac{b}{2a}\sqrt{1 - \frac{4ac}{b^2}}$$

want $\mathrm{Re}(r)$ to be negative. $+ \quad - \quad >1$

① if $\frac{b}{2a} > 0$ then want $\sqrt{\cdots} < 1 \Rightarrow ac > 0$
since $a, c \neq 0$

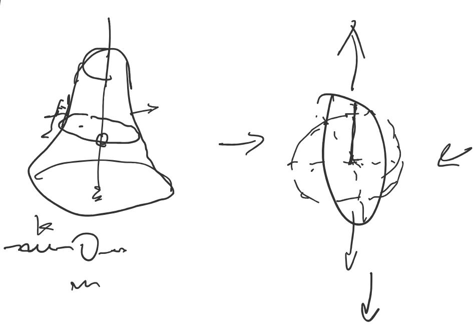
② if $\frac{b}{2a} < 0$ X $\Rightarrow$ a, b can't different sign.

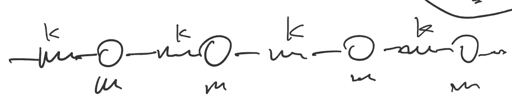
① $\Rightarrow$ ($ab > 0$ and $ac > 0$) $\Rightarrow$ a, b, c have same sign.

$\wedge$ and
$\vee$ or
$\Rightarrow$ implies $\neg$ not $\quad \neg p \wedge p \Rightarrow \perp$ contradiction
$\neg p \vee p \Leftrightarrow \top$

Q2. $y'' + \lambda y = 0$ holds for small y.

model the string as elastic spring chain



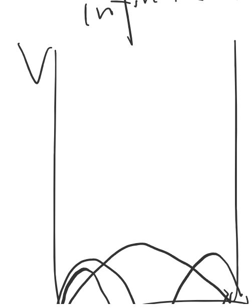


$y(0) = 0 \quad y(L) = 0$

b) $y = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$

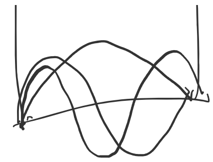
$y(0) = 0 \quad y(L) = 0 \Rightarrow a = 0, \quad \lambda = \left(\frac{n\pi}{L}\right)^2 \in \mathbb{N}$

Schrödinger's eq.
infinite well potential



c). $y = b\sin\left(\frac{n\pi}{L}x\right)$

c). $y = b \cdot \sin\left(\frac{n\pi}{L} x\right)$



Dirichlet bdy.