Module D1: Homogeneous 2nd Order ODEs

1. Consider a 2nd order ODE with nonzero constant coefficients: $ay'' + by' + cy = 0$. Under what conditions on a, b, c will the nontrivial solution $y(x)$ satisfy $\lim_{x\to\infty} y(x) = 0$?

2. Suppose we have a string that is constrained on two static ends over a length L. An equation that describes the standing wave amplitude $y(x)$ of the string is given by the boundary equation $y'' + \lambda y = 0$.

   a) Formulate this problem as a boundary value problem (i.e., add constraints for y).
   b) Under what conditions on λ does this equation give a nontrivial solution?
   c) Draw plots of possible solutions. (These correspond to the "normal modes" of the string.)

Modeling

Dirichlet bdy. cond. $f(y_{(x)}) = 0$

Newmann bdy. cond. $g(y') = 0$

$h(y, y') = 0$ Cauchy bdy.

Suppose ... k m k m k m k

T T T T

$y'' = -\lambda y$ in case $y \ll 1$ $F = kx$

0 L k m

$y : [0, L] \to \mathbb{R}$.

Now we have our model $y'' + \lambda y = 0$. for a string.

$f(t) = \cos\omega t$

$\begin{cases} y(0) = 0 \\ y(L) = 0 \end{cases}$ fixed ends at $x=0$, $x=L$ to $y=0$

$\{ y(0) = 0$ free end at $x=L$. fixed.. at $x=0$ $y=0$

$\lambda > 0$ $\begin{cases} y = a\sin\sqrt{\lambda}x + b\cos(\sqrt{\lambda}x) \\ y(0) = 0 \\ y(L) = 0 \end{cases}$

$\Rightarrow b = 0$

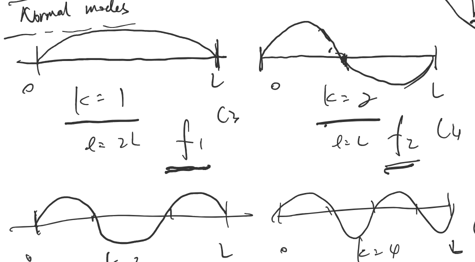
$\Rightarrow \sin\sqrt{\lambda}L = 0$

$\sqrt{\lambda}L = 2\pi k$ $k \in \mathbb{Z}$

$\boxed{\lambda = \left(\frac{2\pi k}{L}\right)^2} = \left(\frac{2\pi}{L}\right)^2 \cdot k^2$

base freq.

Normal modes



0 L $k=1$ $\ell = 2L$ f_1 c_3

0 L $k=2$ $\ell = L$ f_2 c_4

0 L $k=3$ $\ell = \frac{2}{3}L$ f_3

0 L $k=4$ $\ell = \frac{2L}{k}$ f_4 ... c_5 ...

$\{\sin(\sqrt{\lambda}x), \sqrt{\lambda} \in \mathbb{R}\}$

$\lambda = \left(\frac{2\pi}{L}\right)^2 \cdot k^2$; $k \in \mathbb{Z}$

period ℓ.

$\lambda \Leftrightarrow$ freq.

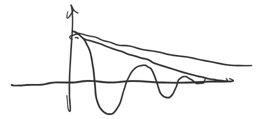
$\Rightarrow f = \sum_{i=0}^{\infty} A_i f_i$

Combination of normal modes.

$(i\uparrow$, $a_i\downarrow)$

amplitude normal modes

Q1. $y = Ce^{\lambda t}$



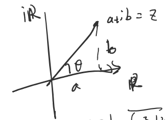
$a = 0$

$\lambda = a + ib$, $a, b \in \mathbb{R}$.

$\lambda = ib$

$y = C \cdot \boxed{e^{at}}\, e^{ibt} \to$

$\cos bt + i\sin bt$



$i\mathbb{R}$ $a+ib = z$ b θ a $\mathbb{R}$

$|z| = \sqrt{a^2+b^2}$

$\theta = bt$

$e^{i\theta} = \cos\theta + i\sin\theta \Leftrightarrow |e^{i\theta}| = 1$

$r=1$ θ

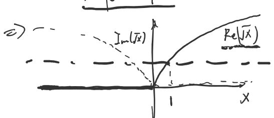
$a\lambda^2 + b\lambda + c = 0$

$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \to \frac{-b}{2a} \pm \frac{b}{2a}\sqrt{1 - \frac{4ac}{b^2}}$

$\sqrt{-36} = \sqrt{(-1)36} = i \cdot \sqrt{36} = i \cdot 6$

want $Re(\lambda) < 0$.

if $ab > 0$, $\frac{-b}{2a} < 0$ $Re\left|\sqrt{\cdots}\right| < 1$



$\Rightarrow$ -- $Im(\sqrt{x})$ $Re(\sqrt{x})$ 1 1 x

$Re\left(\sqrt{1 - \frac{4ac}{b}}\right) < 1 \Leftrightarrow 1 - \frac{4ac}{b^2} < 1$

$\Rightarrow ac > 0$

thus, $ab > 0 \Rightarrow ac > 0 \Rightarrow$ a, b, c same sign.

if $abc < 0$, $-\frac{b}{2a} > 0$. $Re\left(\left(\frac{-b}{2a}\right) + \left(-\frac{b}{2a}\right)\left(\sqrt{\cdots}\right)\right) > 0$

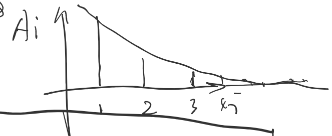
$\Rightarrow ab \not< 0$

Combination of normal modes.

$i\uparrow$, $a_i \downarrow$

amplitude normal modes

FT $\Rightarrow$ A_i



$\longrightarrow$ Fourier transformation

normal

$$C \cdot \int_0^L f_i(x)\, f_j(x)\, dx = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

norm

e_2

e_1

$$e_i \cdot e_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

inner product