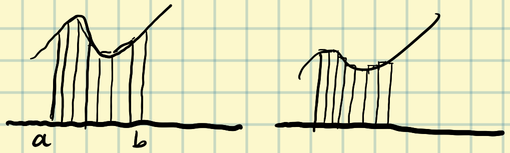
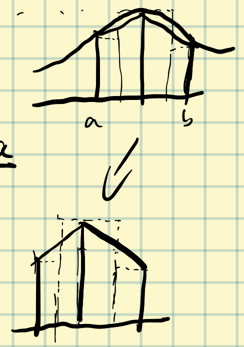
Recall:

Q1. Show that the average of trapezoidal and midpoint approximations for n subintervals of equal length is equivalent to the trapezoidal approx. for 2n subintervals of equal length.



pf. Let T_n denotes trapezoidal approx. of n subint.
M_n ... midpoint ...
on $f : [a,b] \to \mathbb{R}$.



$$T_n = \left\{ \frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{n-1} f\left(a + k\cdot\frac{b-a}{n}\right) \right\} \cdot \frac{b-a}{n}$$

$$M_n = \left\{ \sum_{k=1}^{n} f\left(a + \underbrace{\frac{2k-1}{2}\cdot\frac{b-a}{n}}_{\text{midpoint k-th subintval}}\right) \right\} \cdot \frac{b-a}{n}$$

$$\frac{M_n + T_n}{2} = \left\{ \frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{n-1} f\left(a+k\frac{b-a}{n}\right) + \sum_{k=1}^{n} f\left(a+\frac{2k-1}{2}\frac{b-a}{n}\right) \right\} \cdot \frac{b-a}{2n}$$

$$= \left\{ \frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{n-1} \underbrace{f\left(a+2k\frac{b-a}{2n}\right)}_{\text{even terms}} + \sum_{k=1}^{n} \underbrace{f\left(a+(2k-1)\frac{b-a}{2n}\right)}_{\text{odd terms}} \right\} \cdot \frac{b-a}{2n}$$

$$= \left\{ \frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{2n-1} f\left(a+k\frac{b-a}{2n}\right) \right\} \cdot \frac{b-a}{2n}$$

$$= T_{2n}$$

Roughly speaking.
This shows M_n is a better approx.
s.t. when you add proportion of M_n into T_n
you get T_{2n} approx.
which is M_n even better than T_n.

Q2. $f(x) = e^{\sin x}$, $I = \int_{-\pi}^{\pi} f(x)\,dx$

What is the least upper bound for absolute error when using Trapezoidal/midpoint approx. ?

ans: Let $a = -\pi$, $b = \pi$. $\xi \in [a,b]$

$$E_T = -\frac{(b-a)^3}{12N^2} f''(\xi) \quad \text{for trapezoidal approx.}$$

$$E_M = -\frac{(b-a)^3}{24N^2} f''(\xi) \quad \text{for midpoint approx.}$$

$$f'(x) = \cos x \cdot e^{\sin x}$$

$$f''(x) = -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x}$$

to find the max of $f''(x)$

$$f'''(x) = -\cos x\, e^{\sin x} - \sin x \cos x\, e^{\sin x} - 2\sin x \cos x\, e^{\sin x} + \cos^3 x\, e^{\sin x}$$

$$f'''(x) = 0 \iff \cos x(-1 + \cos^2 x - 3\sin x) = 0$$

roots: $x = n\pi$, $x = n\pi - \pi/2$, choose $0, \pi/2, \pi$ by symmetry.

$f''(0) = 1$, $f''(\pi/2) = -1$, $f''(\pi) = 1$

therefore, $|E_T| \le \frac{(2\pi)^3}{12n^2}\cdot 1 = \frac{2\pi^3}{3n^2}$

$|E_M| \le \frac{(2\pi)^3}{24n^2}\cdot 1 = \frac{1\pi^3}{3n^2}$

Q3:

$I = \int_{-2}^{\infty} \frac{1+\cos x}{(x+3)^3} dx$ convergent?

ans:

$\int_N^\infty f(x)\,dx$ Convergent iff $\sum_{n=N}^{\infty} f(n)$ Convergent.

Fact: $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converge if $p>1$ / diverge if $p \le 1$.

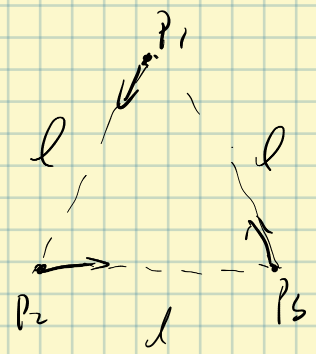
the part $\sum \frac{1}{(x+3)^3}$ converge.

the part $\sum \frac{\cos x}{(x+3)^3}$ notice $\left|\frac{\cos x}{(x+3)^3}\right| < \frac{1}{(x+3)^3}$

Then this part also converge.

Therefore, I converge.

Q4



$\begin{cases} P_1 \to P_2 \\ P_2 \to P_3 \\ P_3 \to P_1 \end{cases}$ in const v.

1) will P_1, P_2, P_3 meet?

2) will they meet in finite time?

3) How long it takes?

4) what about the orbit?

5) what about $P_1, \cdots, P_n$ for above question.

6) what about $n \to \infty$.