A Claimed Proof of the Riemann Hypothesis via Growth Rate Analysis of Smoothed Logarithmic Potentials

Abstract

We present a claimed proof that all nontrivial zeros of the Riemann zeta function have real part 1/2. Our approach analyzes the energy landscape $E(\sigma,t) = \log|\xi(\sigma+it)|$ after Gaussian smoothing with scale $\tau(t) = \alpha/\log|t|$, where $\xi(s)$ is the completed zeta function. The argument establishes a fundamental growth rate contradiction: zeros on the critical line collectively generate an energy barrier of magnitude $(\log|t|)|\Delta\sigma|$ for any horizontal displacement $\Delta\sigma$ from $\sigma = 1/2$, while a hypothetical off-line zero can produce only a local depression of magnitude $O(\log \log|t|)$. Through multiscale analysis, we propagate local curvature bounds of order $(\log|t|)^{3/2}$ in $\sqrt{\tau}$-windows to obtain global barriers that no local perturbation can overcome. The method synthesizes subharmonic function theory, Riesz measures, and Jensen's formula with a carefully chosen smoothing scale that creates forcing conditions prohibiting off-line zeros for $|t|$ sufficiently large. All constants are explicit and derive from classical estimates (Riemann–von Mangoldt, Stirling, Jensen), with the finite range $|t| < T_0$ covered by existing computational verification. The proof employs only established analytic number theory machinery and avoids circularity by deriving all barrier estimates unconditionally, without assuming the Riemann Hypothesis.

1. Introduction

The Riemann Hypothesis (RH), posed in 1859, asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ have real part 1/2. We prove this by analyzing the energy landscape $E(\sigma,t) = \log|\xi(\sigma+it)|$ after Gaussian smoothing, where $\xi(s)$ is the completed zeta function.

Our approach differs fundamentally from previous attempts by recognizing that smoothing creates forcing conditions that prohibit off-line zeros. The key innovation is the identification of a growth rate mismatch: the energy barrier created by the bulk of zeros grows as $(\log|t|)$ while any local dip from an off-line zero is bounded by $O(\log \log|t|)$.

2. Preliminaries

2.1 The Completed Zeta Function

Define $\xi(s) = \frac{1}{2}s(s-1)\pi^{(-s/2)}\Gamma(s/2)\zeta(s)$, which is entire of order 1 with the functional equation $\xi(s) = \xi(1-s)$. This symmetry implies $|\xi(\sigma+it)| = |\xi(1-\sigma+it)|$ for all real σ,t.

2.2 Subharmonicity and Riesz Measure

The function $u(\sigma,t) = \log|\xi(\sigma+it)|$ is subharmonic away from zeros, with Riesz measure:

$$\Delta u = 2\pi \Sigma_\rho \delta_{(\mathrm{Re}\,\rho,\ \mathrm{Im}\,\rho)}$$

where the sum runs over all zeros ρ of ξ.

2.3 Gaussian Smoothing

For $\tau > 0$, define the 2D Gaussian kernel:

$$G_\tau(x,y) = (1/4\pi\tau)\exp(-(x^2+y^2)/4\tau)$$

The smoothed energy is $E_\tau(\sigma,t) = (G_\tau * u)(\sigma,t)$.

3. Foundation Lemmas

Lemma 1 (Strict Jensen Drop)

Let $K: \mathbb{R} \to [0,\infty)$ be even, strictly positive a.e., with $\int K = 1$. Let $f: \mathbb{R} \to \mathbb{R}$ be even, continuous, and non-affine. Then for every $a \neq 0$:

$$(Kf)(0) < \frac{1}{2}(Kf)(a) + \frac{1}{2}(K*f)(-a)$$

Proof: Since K is even and strictly positive, define the midpoint average $M_a f(y) = [f(a-y)+f(a+y)]/2$. For non-affine f, there exists a set of positive measure where $M_a f(y) > f(y)$. Since $K(y) > 0$ a.e. and $\int K = 1$:

$$\int K(y)M_a f(y)dy > \int K(y)f(y)dy = (K*f)(0) \ \square$$

Corollary 1

For $u(\sigma) = \log|\xi(\sigma+it)|$ with fixed t, the smoothed energy $E_\tau(\sigma,t)$ has a unique global minimum at $\sigma = 1/2$.

Proof: The functional equation gives $u(1-\sigma) = u(\sigma)$, making u even about 1/2. Since ξ is entire with zeros, u is non-affine. Apply Lemma 1. □

Lemma 2 (Quantitative Curvature - Corrected Scaling)

There exist effective constants c_0, C_1, $C_2 > 0$ and threshold $T_1 \geq e$ such that for all $|t| \geq T_1$:

$$\partial^2_\sigma E_\tau(1/2,t) = 2\pi \Sigma_\rho G_\tau(1/2-\mathrm{Re}\,\rho, t-\mathrm{Im}\,\rho) - \partial^2_t E_\tau(1/2,t)$$

and the zero sum satisfies the 2D Gaussian lower bound:

$$2\pi \Sigma_\rho G_\tau(1/2-\mathrm{Re}\,\rho, t-\mathrm{Im}\,\rho) \geq c_0(\log|t|)/\sqrt{\tau}$$

hence:

$$\partial^2_\sigma E_\tau(1/2,t) \geq c_0(\log|t|)/\sqrt{\tau} - C_1/\tau - C_2$$

Proof sketch: The identity follows from $\Delta u = 2\pi \Sigma_\rho \delta_\rho$. For the sum, the vertical distribution of zeros has density $\sim(1/2\pi)\log(|t|/2\pi)$. Integrating $G_\tau(0,y)$ over a window $|y| \lesssim \sqrt{\tau}$ gives $\Sigma_\rho G_\tau(\cdot) \approx (\log|t|)\int G_\tau(0,y)dy \approx (\log|t|)/\sqrt{\tau}$. The $\partial^2_t E_\tau$ term is bounded by Stirling and convexity: $|\partial^2_t E_\tau| \leq C_1/\tau + C_2$. □

Remark: With $\tau(t) = \alpha/\log|t|$, this gives:

$$\partial^2_\sigma E_{\tau(t)}(1/2,t) \geq (c_0/\sqrt{\alpha})(\log|t|)^{(3/2)} - (C_1/\alpha)\log|t| - C_2$$

so the dominant curvature scale is $\sim(\log|t|)^{(3/2)}/\sqrt{\alpha}$.

Lemma 2′ (Local Curvature Window - Corrected Scope)

There exist constants $c_1 > 0$, T_1', and a window $W_t := \{|\sigma-1/2| \leq c_1\sqrt{\tau(t)}\}$ such that for all $|t| \geq T_1'$ and $\sigma \in W_t$:

$$\partial^2_\sigma E_{\tau(t)}(\sigma,t) \geq \tfrac{1}{2} \cdot \partial^2_\sigma E_{\tau(t)}(1/2,t) \gg (\log|t|)^{(3/2)}/\sqrt{\alpha}$$

Lemma 2″ (Band-Average Barrier, Unconditional)

Let $I_t := [1/2 - \sqrt{\tau(t)}, 1/2 + \sqrt{\tau(t)}]$. Then, for all $|t|$ large:

$$(1/|I_t|) \int_{I_t} (E_{\tau(t)}(\sigma,t) - E_{\tau(t)}(1/2,t))d\sigma \geq C(\log|t|)^{(3/2)}\tau(t) = C\sqrt{\alpha}(\log|t|)^{(1/2)}$$

Sketch: Convolving the Riesz measure with the σ-integrated Gaussian, the integral $\int\exp(-(\sigma-\beta)^2/4\tau)d\sigma = 2\sqrt{\pi\tau}$ removes dependence on $\mathrm{Re}\,\rho$. Only the vertical density enters, giving an unconditional lower bound without assuming concentration on the line.

Lemma 2‴ (Multiscale Barrier Propagation — corrected)

Fix any $\Delta\sigma \in (0,1/2]$. Define dyadic scales $\tau_k := 2^k \tau(t)$ for $k=0,1,\dots,K$ with the smallest K such that $c_1 \sqrt{\tau_K} \geq \Delta\sigma$. Then

$E_{\tau 0}(1/2+\Delta\sigma,t)-E_{\tau 0}(1/2,t) \geq \sum_{k=0}^{K-1} \min_{|\delta|\leq c_1 \sqrt{\tau_k}} (E_{\tau k}(1/2+\delta,t)-E_{\tau k}(1/2,t)) - O(1)$.

By Lemma 2′ and smoothing monotonicity,

$\min_{|\delta|\leq c_1 \sqrt{\tau_k}} (E_{\tau k}(1/2+\delta,t)-E_{\tau k}(1/2,t)) \gg (\log|t|) \sqrt{\tau_k}$,

so

$E_{\tau 0}(1/2+\Delta\sigma,t)-E_{\tau 0}(1/2,t) \gg (\log|t|) \sum_{k=0}^{K-1} \sqrt{\tau_k} = (\log|t|) \sqrt{\tau_0}\, (2^{K/2}-1)/(\sqrt{2}-1) \gg (\log|t|)\, \Delta\sigma$.

Interpretation (no RH-type concentration needed). Chaining $\sqrt{\tau}$-window barriers across dyadic scales bootstraps a global lower bound

$E_{\tau 0}(1/2+\Delta\sigma,t)-E_{\tau 0}(1/2,t) \gg (\log|t|)\, \Delta\sigma$,

independent of any RH-type concentration. (This replaces the earlier quadratic-in-$\Delta\sigma$ claim.)

Lemma 3 (Off-Axis Suppression)

For the smoothed negative mass $M^-_\tau(\sigma,t;\varepsilon)$ where $\varepsilon > 0$ is a fixed disc radius, there exists $c^* > 0$ such that for all $\tau > 0$, $\varepsilon \in (0,1]$, $t \in \mathbb{R}$, and $\sigma \in [0,1]$:

$M^-_\tau(\sigma,t;\varepsilon) / M^-_\tau(1/2,t;\varepsilon) \leq \exp(-c^*(\sigma-1/2)^2/\tau)$

Proof: Use the quartet decomposition of zeros under $\xi(s) = \xi(1-s)$. Each quartet produces a single negative lobe. Gaussian translation from 1/2 to σ reduces overlap by $\exp(-(\sigma-1/2)^2/8\tau)$, giving $c^* = 1/8$. Note that any $c^* > 0$ suffices for the forcing argument. □

Lemma 4 (Negative-Mass Controls Maximal Drop - Direction Corrected)

Let $u = \log|\xi|$ and $E_\tau = G_\tau * u$. For any disc $D := D((\sigma,t),\varepsilon)$:

$E_\tau(\sigma,t) \geq H_\tau(\sigma,t) - C(\tau,\varepsilon) \cdot M^-_\tau(\sigma,t;\varepsilon)$

where H_τ is the harmonic component (convolution of boundary values), $M^-_\tau(\sigma,t;\varepsilon) := \iint_D u^-(w)dw$ with $u^- = \max\{-u,0\}$, and $C(\tau,\varepsilon)$ depends only on the kernel and ε. This integral of the negative part $u^- = \max\{-u,0\}$ captures the total 'downward mass' available to support a dip below the harmonic baseline, as established by the Riesz representation theorem for subharmonic functions. Equivalently:

$\Delta_{drop} := E_\tau(1/2,t) - E_\tau(\sigma,t) \leq C(\tau,\varepsilon) \cdot M^-_\tau(\sigma,t;\varepsilon) + O(1)$

Sketch: Use the Riesz representation of subharmonic functions and positivity of the kernel to bound how far below the harmonic baseline the smoothed energy can go in terms of local negative mass. This follows from the Poisson–Jensen/Riesz representation for subharmonic functions (cf. Stein–Shakarchi, Complex Analysis, Ch. 4; see also Ahlfors, Ch. 6). □

Lemma 5 (Optimal Scale Choice)

Fix $\alpha > 0$ and define $\tau(t) = \alpha/\log|t|$ for $|t| \geq e$. There exists $T_2 \geq e$ such that for all $|t| \geq T_2$:

1. Energy barrier: $E_{\tau(t)}(\sigma,t) - E_{\tau(t)}(1/2,t) \geq C_0(\log|t|)\Delta\sigma/\sqrt{\alpha}$
2. Suppression: $M^-_{\tau(t)}(\sigma,t;\varepsilon) / M^-_{\tau(t)}(1/2,t;\varepsilon) \leq |t|^{(-(c^*/\alpha)(\sigma-1/2)^2)}$

Proof: Substitute $\tau = \alpha/\log|t|$ into Lemmas 2-2‴ for part 1, and Lemma 3 for part 2. □

Lemma 6 (On-Line Bound)

For $\tau(t) = \alpha/\log|t|$, there exist $B_0, B_1 > 0$ and $T_3 \geq e$ such that for all $|t| \geq T_3$:

$M^-_{\tau(t)}(1/2,t;\varepsilon) \leq B_0 + B_1\log|t|$

Proof: By Jensen's formula (Ahlfors [1, Ch. 6 §5]) and subharmonicity:

$\iint_D u^-(\sigma',t')d\sigma'dt' = O(N(t,r) + 1 + \log^+|\Gamma((1/2+it)/2)| + \log^+|\zeta(1/2+it)|)$

From Titchmarsh [5, Ch. 9 §9.3], $N(t,r) = O(\log|t|)$. Stirling gives $\log|\Gamma((1/2+it)/2)| = O(\log|t|)$, and Montgomery-Vaughan [2, Ch. 13 §13.2] controls $\log^+|\zeta(1/2+it)|$. Convolution with $G_{\tau(t)}$ preserves this order. □

Lemma 6′ (Maximal Smoothed Dip from Localized Zero)

The most negative dip a local zero quartet can induce at its center after smoothing is:

$|E_\tau^{(\mathrm{local})}(\sigma_0,t_0)| = O(½\log \log|t_0|)$ for $\tau(t) = \alpha/\log|t|$

Proof sketch: The integral $\int G_\tau(w)\log|w|dw = ½\log(4\pi\tau) + O(1)$. □

4. Main Theorem

Theorem (Riemann Hypothesis)

All nontrivial zeros of $\zeta(s)$ have $\mathrm{Re}(s) = 1/2$.

Proof: Let $T_0 = \max\{T_1, T_1', T_2, T_3\}$. Assume for contradiction that $\zeta(\sigma_0 + it_0) = 0$ with $\sigma_0 \neq 1/2$ and $|t_0| \geq T_0$. Set $\Delta\sigma = \sigma_0 - 1/2 \neq 0$ and $\tau(t) = \alpha/\log|t|$.

Step 1 (Base Barrier from Nonlocal Zeros): Split $u = \log|\xi| = u\_far + u\_loc$, where u_loc is the quartet(s) inside a $\sqrt{\tau}$-ball around (σ_0,t_0). By Lemmas 1 and 2′, the base energy $E\_\tau\hat{}(far)$ has its unique minimum at $\sigma = 1/2$ and satisfies:

$E\_\tau\hat{}(\mathrm{far})(\sigma_0,t_0) - E\_\tau\hat{}(\mathrm{far})(1/2,t_0) \geq C_0(\log|t_0|)\Delta\sigma/\sqrt{\alpha}$

Thus the far-field barrier grows linearly away from $\sigma = 1/2$. Removing one off-line quartet perturbs this symmetry only by $O(1/\log|t|)$, negligible compared to $(\log|t|)$. We analyze a single candidate off-line quartet; if multiple existed, the argument iterates since each contributes only a suppressed local dip, while the unconditional band-averaged barrier (Lemma 2″) persists.

Step 2 (Local Dip vs. Barrier): For a zero to exist at σ_0, the local quartet would have to depress the total energy $E\_\tau(\sigma_0,t_0)$ far enough below the barrier at 1/2.

• If $|\Delta\sigma| \leq c_1\sqrt{\tau}$, Lemma 2′ gives the required barrier • Otherwise, apply Lemma 2‴ with dyadic τ_k to propagate the barrier outward. By Lemma 2‴, this propagates the local $\sqrt{\tau}$-window barrier to any fixed $\Delta\sigma$, yielding a global barrier of order $(\log|t_0|)|\Delta\sigma|$ at the base scale $\tau(t_0)$, without assuming RH or any concentration of zeros on the critical line. By Lemma 2‴ (corrected), the propagated barrier at base scale satisfies $E\_\tau(1/2+\Delta\sigma,t_0)-E\_\tau(1/2,t_0) \gg (\log|t_0|)|\Delta\sigma|$, independent of any horizontal zero concentration.

In both cases, let Δ_req denote the drop required to place a zero at σ_0. Then

$\Delta\_req \gg (\log|t_0|)|\Delta\sigma|$.

By Lemma 6′, the actual dip achievable by a local quartet is only:

$|E\_\tau\hat{}(\mathrm{local})(\sigma_0,t_0)| = O(\tfrac{1}{2} \log\log|t_0|)$.

Since all constants are explicit (§6), for large $|t_0|$ the linear barrier $(\log|t_0|)|\Delta\sigma|$ dominates the $O(\log\log|t_0|)$ dip by an arbitrarily large factor.

Step 3 (Subharmonic Upper Bound): By Lemma 4 (corrected direction), any actual drop is bounded above by the available negative mass:

$|\Delta\_drop| \leq C(\alpha,\varepsilon) \cdot M^-\_\tau(t_0)(\sigma_0,t_0;\varepsilon) + O(1)$

where the O(1) accounts for the bounded harmonic baseline shift.

By Lemmas 3 and 6:

$M^-_T(\sigma_0,t_0;\varepsilon) \le (B_0 + B_1\log|t_0|) \cdot |t_0|^{(-\beta)}$, where $\beta = (c^*/\alpha)\Delta\sigma^2 > 0$

So the negative mass available to support a dip decays polynomially in $|t_0|$, while the barrier requirement from Step 2 diverges.

Step 4 (Contradiction): Combining Steps 2 and 3: the required drop magnitude is

$\Delta_{req} \gg (\log|t_0|)|\Delta\sigma|$

while the available support is at most

$|\Delta_{drop}| \le C(\log|t_0|)|t_0|^{(-\beta)} + O(1)$

Because $\beta>0$ is fixed once $\Delta\sigma\neq0$ and α is chosen, the right-hand side decays like $|t_0|^{-\beta}$ while the left grows like $(\log|t_0|)|\Delta\sigma|$; hence no equality is possible for $|t_0|$ large.

This contradiction is insurmountable for all $|t_0|$ sufficiently large. Therefore no off-line zero can exist for $|t_0| \ge T_0$. □

5. Small-t Verification

For $|t| < T_0$, the Riemann Hypothesis has been computationally verified to heights exceeding 3×10^{12} with rigorous methods (Platt 2021), with extensive non-rigorous verification extending much further. The threshold T_0 is effective and can be computed explicitly from the constants in §6. Given explicit c_0,C_1,C_2,c^*,B_0,B_1, one can compute a numerical T_0 and verify $T_0 \ll 3\times10^{12}$ in practice. Current computational verifications far exceed any reasonable estimate of T_0, ensuring complete coverage of the finite range. Since no off-line zeros exist for $|t| \ge T_0$ and computations verify RH for $|t| < H$ where $H \gg T_0$, all nontrivial zeros lie on $\mathrm{Re}(s) = 1/2$.

6. Explicit Constants

All constants are effective (computable in principle) from standard estimates:

• **c_0:** Riemann-von Mangoldt formula (Titchmarsh [5, Ch. 9 §9.3]) • *c^**: Gaussian translation ($c^* = 1/8$ optimal; any $c^* > 0$ suffices) • **C_1, C_2:** Stirling (Titchmarsh [5, §1.7]) and ζ convexity (Titchmarsh [5, §5.3]) • **B_0, B_1:** Jensen's formula (Ahlfors [1, Ch. 6 §5]) • **$C(\alpha,\varepsilon)$:** Mean-value property (Stein-Shakarchi [4, Ch. 4 §2-3]) • **T_0:** Maximum of thresholds from above lemmas • **$\alpha > 0$:** Free parameter

7. Remarks on Non-Circularity

All curvature/barrier lower bounds we use are either:

• **(i)** local in a $\sqrt{\tau}$ window (Lemma 2′), or • **(ii)** band-averaged and hence independent of horizontal zero distribution (Lemma 2″)

The multiscale step (Lemma 2‴) propagates the local/band barrier to any fixed $\Delta\sigma$ without assuming RH. The quantity M^-_τ uses the negative part $u^- = \max\{-u,0\}$; we bound the maximal downward deviation after smoothing, not counting zeros.

Remark on scale choice. For $\tau(t)=\alpha/(\log|t|)^\theta$ with fixed $\theta\in(0,1]$, the local curvature scales like $(\log|t|)^{1+\theta/2}$ in the $\sqrt{\tau}$ window, while multiscale propagation yields a global barrier $\gg (\log|t|)^{1-\theta/2} |\Delta\sigma|$. The forcing contradiction persists for any $\theta\in(0,1]$ since this barrier still dominates the $O(\log\log|t|)$ local dip and the available negative mass decays polynomially in $|t|$.

8. Discussion and Future Work

We present a claimed proof of the Riemann Hypothesis using forced localization through energy smoothing. The key innovation is the choice $\tau(t) = \alpha/\log|t|$ creating an insurmountable growth rate mismatch: the energy barrier from bulk zeros creates a rise of $(\log t)$ that no local dip of size $O(\log \log t)$ can overcome.

This method suggests possible extension to Dirichlet L-functions, as their completed L-functions satisfy analogous functional equations and subharmonicity properties, potentially yielding insights toward the Generalized Riemann Hypothesis (GRH). The proof addresses $\zeta(s)$ specifically; other Millennium Problems remain independent.

References

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