

Logistic Regression

Data Science: Jordan Boyd-Graber

University of Maryland

SLIDES ADAPTED FROM WILLIAM COHEN

To ease notation, let's define

$$\pi_i = \frac{\exp \boldsymbol{\beta}^T x_i}{1 + \exp \boldsymbol{\beta}^T x_i} \tag{1}$$

Our objective function is

$$\ell = \sum_{i} \log p(y_i | x_i) = \sum_{i} \ell_i = \sum_{i} \begin{cases} \log \pi_i & \text{if } y_i = 1 \\ \log(1 - \pi_i) & \text{if } y_i = 0 \end{cases}$$
 (2)

Taking the Derivative

Apply chain rule:

$$\frac{\partial \ell}{\partial \beta_j} = \sum_{i} \frac{\partial \ell_i(\vec{\beta})}{\partial \beta_j} = \sum_{i} \begin{cases} \frac{1}{\pi_i} \frac{\partial \pi_i}{\partial \beta_j} & \text{if } y_i = 1\\ \frac{1}{1 - \pi_i} \left(-\frac{\partial \pi_i}{\partial \beta_j} \right) & \text{if } y_i = 0 \end{cases}$$
(3)

If we plug in the derivative,

$$\frac{\partial \pi_i}{\partial \beta_j} = \pi_i (1 - \pi_i) x_j, \tag{4}$$

we can merge these two cases

$$\frac{\partial \ell_i}{\partial \beta_i} = (y_i - \pi_i) x_j. \tag{5}$$

Gradient

$$\nabla_{\beta} \ell(\vec{\beta}) = \left[\frac{\partial \ell(\vec{\beta})}{\partial \beta_0}, \dots, \frac{\partial \ell(\vec{\beta})}{\partial \beta_n} \right]$$
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Update

$$\Delta \beta \equiv \eta \nabla_{\beta} \ell(\vec{\beta}) \tag{7}$$

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Why are we adding? What would well do if we wanted to do **descent**?

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 η : step size, must be greater than zero

Gradient

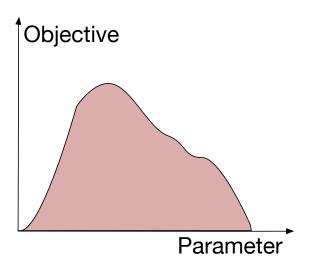
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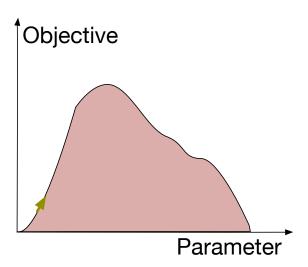
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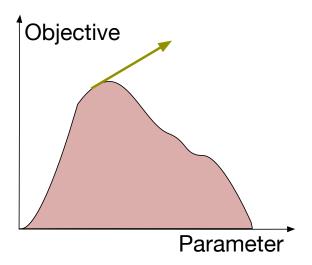
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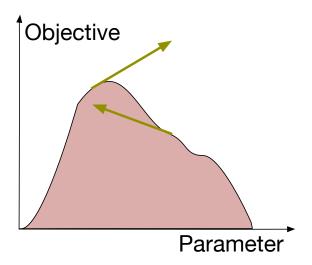
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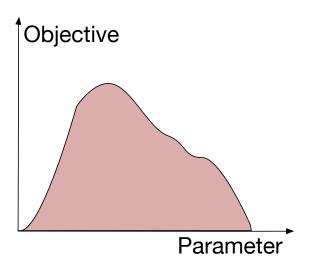
NB: Conjugate gradient is usually better, but harder to implement











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- Average over all observations
- What if we compute an update just from one observation?

Getting to Union Station

Pretend it's a pre-smartphone world and you want to get to Union Station





Stochastic Gradient for Logistic Regression

Given a **single observation** x_i chosen at random from the dataset,

$$\beta_{j} \leftarrow \beta_{j}' + \eta \left[y_{i} - \pi_{i} \right] x_{i,j} \tag{10}$$

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Examples in class.

Algorithm

- 1. Initialize a vector B to be all zeros
- 2. For t = 1, ..., T
 - □ For each example \vec{x}_i , y_i and feature j:
 - Compute $\pi_i \equiv \Pr(y_i = 1 | \vec{x}_i)$
 - Set $\beta[j] = \beta[j]' + \lambda(y_i \pi_i)x_i$
- 3. Output the parameters β_1, \ldots, β_d .

Wrapup

- Logistic Regression: Regression for outputting Probabilities
- Intuitions similar to linear regression
- We'll talk about feature engineering for both next time