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Since every norm on R^n is convex, ~~and~~ we can derive that

$$f_0(x)=\frac{1}{2}\|Ax-y\|_2^2+\lambda\|x\|_1$$

is convex.

∴ 0 is a minimizer if and only if 0 is a subgradient of f_0 at 0.

Thus, the subgradient optimality can be written as:

$$0\in\partial\left(\frac{1}{2}\|Ax-y\|_2^2+\lambda\|x\|_1\right)\Leftrightarrow 0\in\{A^T(Ax-y)+\lambda\partial\|x\|_1\}$$

$$\Leftrightarrow \lambda\partial\|x\|_1=-A^T(Ax-y)$$

$$\Leftrightarrow \begin{cases}-A_i^T(Ax-y)=\lambda\operatorname{sign}(x_i), & \text{if } x_i\neq 0\\ |-A_i^T(Ax-y)|\leq\lambda, & \text{if } x_i=0\end{cases}$$

~~where~~

$$\therefore (-A_i^T(Ax-y))_i\in\begin{cases}\{+\lambda i\} & x_i>0,\\ \{-\lambda i\} & x_i<0, \quad i=1,\cdots,n\\ [-\lambda i,\lambda i] & x_i=0,\end{cases}$$

∴ The condition that 0 is optimal is that

$$(-A_i^Ty)_i\in[-\lambda,\lambda] \text{ for } i=1,\cdots,n$$

$$\therefore \|A^Ty\|_\infty\leq\lambda$$

$$\lambda_{max}=\|A^Ty\|_\infty$$

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Proof of uniqueness:

Suppose that there is a non-zero minimizer x^*, therefore

$$f_0(x^*) = f_0(0)$$

$$\therefore 0 = f_0(x^*) - f_0(0)$$

$$= \frac{1}{2}\|Ax^* - y\|_2^2 + \lambda\|x^*\|_1 - \frac{1}{2}\|y\|_2^2$$

$$\cancel{= \frac{1}{2}(A^TAx - A^Tb}$$

$$= \frac{1}{2}x^TA^TAx - y^TAx + \frac{1}{2}y^Ty + \lambda\|x\|_1 - \frac{1}{2}y^Ty$$

$$\geq \frac{1}{2}x^TA^TAx - y^TAx + \|A^Ty\|_\infty \|x\|_1$$

$$= \frac{1}{2}\|Ax\|^2 - (A^Ty)^Tx + \|A^Ty\|_\infty\|x\|_1$$

$$= \frac{1}{2}\|Ax\|^2 - \sum_{i=1}^{n}(A^Ty)_i x_i + \sum_{i=1}^{n}\|A^Ty\|_\infty |x_i|$$

$$\cancel{\geq \frac{1}{2}\|Ax\|^2}$$

$$\geq \frac{1}{2}\|Ax\|^2$$

~~1° if $\|Ax\|_2^2 = 0$ then~~

$\therefore \frac{1}{2}\|Ax\|_2^2 \leq 0$

$\therefore Ax = 0 \quad \because x^* \neq 0$

$\therefore f_0(x^*) = \frac{1}{2}\|y\|_2^2 + \lambda\|x^*\|_1 > \frac{1}{2}\|y\|_2^2 = f_0(0)$

this contradicts with ~~the~~ $f_0(x^*) = f_0(0)$

$\therefore 0$ is the unique minimizer when $\lambda_0 \geq \lambda_{max} = \|A^Ty\|_\infty$