Quiz_MTH114

Graded

Student

SHAURYA JOHARI

Total Points

20 / 20 pts

Question 1

question1

4 / 4 pts

- **+ 1 pt** has written the general fomula of integrating factor for 1st order liner ode or the correct p, r
- **+ 1 pt** Found out the integrating factor correctly
- **+ 1 pt** Written the step after multiplying the integrating factor
- **+ 1 pt** Final answer correct
- ✔ **+ 4 pts** fully correct
- **+ 0 pts** incorrect

Question 2

Question2

5 / 5 pts

- **+ 1 pt** 1st Picard iterate Correct
- **+ 1 pt** 2nd Picard iterate Correct
- **+ 1 pt** 3rd Picard iterate Correct
- **+ 1 pt** Written general formaula for n-th integrate
- **+ 1 pt** Taken limit to find the solution
- ✔ **+ 5 pts** fully correct
- **+ 0 pts** Completely wrong

Question 3

Question3

5 / 5 pts

- **+ 1 pt** Differentiated the given equationCorrect
- **+ 2 pts** Eliminated c to find the ODE of the given family
- **+ 1 pt** Written the ODE of the orthogonal family
- **+ 1 pt** Found the orthogonal family
- ✔ **+ 5 pts** fully correct
- **+ 0 pts** incorrect

Question 4

Question4

6 / 6 pts

+ 1 pt calculated f_y correctly

+ 1 pt Applied Picard theorem correctly to find unique solutions

+ 2 pts Solved the ODE correctly

+ 1 pt Correct analysis of no solution

+ 1 pt correct analysis of multiple solution

✔ **+ 6 pts** fully correct

+ 0 pts incorrect

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4. Consider the IVP $(x^2 - 2x)y' = 2(x-1)y, \qquad y(x_0) = y_0$.
   (a) For which values of (x_0, y_0), Picard's theorem implies a unique solution of the IVP?
   (b) Determine all values of (x_0, y_0) such that the IVP has no solution.
   (c) Determine all values of (x_0, y_0) such that the IVP has more than one solution. [2+2+2]

Answer: $y' = f(x,y) = f = \dfrac{2(x-1)y}{x(x-2)}$

$$\frac{\partial f}{\partial y} = \frac{2(x-1)}{x(x-2)}.$$

Note that for $x \neq 0$ OR 2, f & $\dfrac{\partial f}{\partial y}$ is continuous & bounded

Also, We have $\dfrac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \dfrac{|2(x-1)(y_1-y_2)|}{|x(x-2)||(y_1-y_2)|} = \dfrac{|2(x-1)|}{|x(x-2)|} \leq L$ for some $L \in \mathbb{R}^+$.

$\because$, Picard's Theorem guarantees ~~sol~~ Unique sol^n for $f(x,y)$ in the vicinity of (x_0, y_0) when $x_0 \neq 0$ OR 2

If denominator $= 0$, Picard's Thm CAN'T BE applied directly.

$$\frac{dy}{dx} = \frac{2(x-1)y}{x(x-2)} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} + \frac{dx}{x-2} \Rightarrow y = Cx(x-2)$$

~~Can~~ $y = Cx(x-2)$ can be obtained ~~us~~ by directly solving for y.

We see that if $x_0 = 0$ OR 2 & $y_0 \neq 0$, $\nexists$ No sol^n to given diff. eq.

Else, if $y_0 = 0$, then ANY value of C can satisfy & so $y = \alpha x(x-2)$ $\forall$ $\alpha, x \in \mathbb{R}$

To conclude:

If $x_0 \neq 0$ OR 2, By Picard's Thm, $\exists$ unique sol^n for IVP
If $x_0 = 0$ OR 2 & $y_0 \neq 0$, NO SOLUTIONS
If $x_0 = 0$ OR 2 & $y_0 = 0$, infinite sol^{n}s exist & $y = \alpha x(x-2)$

ODE : MTH 114M : QUIZ

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- **Write your name (in capital letters) and roll number in the boxes.**
- Write answers in the space provided only. Total marks 20 Time: 6:45 pm - 7:30 pm.

1. Find a suitable integrating factor and solve the equation $xy' + (2-4x)y = \frac{e^{4x}}{x}$, $x > 0$. [2+2]

Answer: $xy' + (2-4x)y = \dfrac{e^{4x}}{x} \Rightarrow$

$\dfrac{dy}{dx} + \dfrac{2-4x}{x} y = \dfrac{e^{4x}}{x^2}$ Comparing with Bernoulli Eq^n $y' + p(x)y = q(x)$

We have $p(x) = \dfrac{2-4x}{x} = \dfrac{2}{x} - 4$ & $q(x) = \dfrac{e^{4x}}{x^2}$

Multiplying both sides by $\mu = e^{\int p(x)dx} = e^{\int(\frac{2}{x}dx - 4dx)} = x^2 e^{-4x}$, we get

$$x^2 e^{-4x} y' + y(2xe^{-4x} - 4x^2 e^{-4x}) = x^2 e^{-4x} \cdot e^{4x}/x^2$$

$$\Rightarrow x^2 e^{-4x} dy + y(2xe^{-4x} - 4x^2 e^{-4x})dx = (1)dx$$

But $\dfrac{d\mu}{dx} = 2xe^{-4x} - 4x^2 e^{-4x} \Rightarrow \mu dy + y d\mu = dx$

$$\int d(\mu y) = \int dx \Rightarrow x + C = y(x^2 e^{-4x})$$

$\because$, $\boxed{\text{Integrating Factor is } \mu = x^2 e^{-4x} \quad \text{And solution is } y = \dfrac{e^{4x}}{x} + \dfrac{Ce^{4x}}{x^2}}$

Where ~~C is constant (C ∈ R)~~

Where $C \in \mathbb{R}$ & C is constant factor.

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2. Consider the IVP $\frac{dy}{dx} = -y, y(0) = 1$. Calculate the first three Picard iterates $y_1(x), y_2(x), y_3(x)$. Find the solution by Picard iteration method. [3+2]

$y_0 = 1 \ \forall x \in \mathbb{R}$

Answer:

Picard's Iteration Method is a way to solve IVP (Given it satisfies Picard's Thm for existance & uniqueness of a ~~sol~~ sol^n), by ~~suc~~ integrating functions $y_1(x), y_2(x) \cdots y_n(x)$ s.t. $\lim_{n\to\infty} y_n(x) = y(x)$, $y(x)$ being the proper unique sol^n.

$\frac{dy}{dx} = f(x,y) = -y$ & $\left|\frac{\partial f}{\partial y}\right| = 1 \leq L \Rightarrow$ Let $L \in \mathbb{R}, L \geq 1$.

$\because f(x,y)$ & $\frac{\partial f}{\partial y}$ is continuous & bounded in

Let $K \in \mathbb{R}$ ~~$[-L,L]\times[-L,L]$~~ $[-K,K]^2$, $\exists$ unique sol^n for IVP in $[-M,M]$ where $M = \min\left\{\frac{K}{|1|}, \frac{K}{|1|}\right\}$

$T = \sup\left(\frac{\partial f}{\partial y}\right) = 1 \Rightarrow M = K$.

$\therefore$ Unique sol^n exist over $\mathbb{R}^2$, $\because K \in \mathbb{R}$

Picard's Iteration: $y_{n+1}(x) = \int_0^x f(x, y_n(x))\,dx + y_0$

Let $y_0 = 1$, $y_1(x) = 1 + \int_0^x (-1)\,dx = \boxed{1 - x = y_1(x)}$

$y_2(x) = 1 + \int_0^x -(1-x)\,dx = \boxed{1 - x + \frac{x^2}{2!} = y_2(x)}$

$y_3(x) = 1 + \int_0^x -\left(1 - x + \frac{x^2}{2}\right)dx = \boxed{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} = y_3(x)}$

$\boxed{e^x = \sum_{i=0}^{\infty} \frac{x^i}{(i)!} \text{ Power Series}}$

Assume $y_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^i}{(i)!}$. Then $y_{n+1}(x) = 1 - \int_0^x \sum_{i=0}^{n} \frac{(-1)^i x^i}{(i)!}\,dx$

$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} = \sum_{i=0}^{n+1} \frac{(-x)^i}{(i)!} = y_{n+1}(x)$

$\therefore y_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^i}{(i)!}$ & $\lim_{n\to\infty} y_n(x) = \boxed{e^{-x} = y \text{ is our sol}^n}$

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3. Consider the family of curves $y^2 = cx^3$ where c is an arbitrary constant. Find a family of curves orthogonal to it (orthogonal trajectory). [5]

Answer:

$y^2 = Cx^3 \Rightarrow \ln|y^2| = 2\ln|y| = \ln|Cx^3| = \ln|C| + 3\ln|x|$

$\Rightarrow \frac{2dy}{y} = 0 + \frac{3dx}{x}$ ($\because C$ is constant)

$\Rightarrow \frac{dy}{dx} = \frac{3y}{2x} = f(x,y)$

For family of curves satisfying $\frac{dy}{dx} = f(x,y)$, the family of orthogonal trajectories wrt. 1st family of curves is $\frac{dy}{dx} = g(x,y)$ s.t. $f(x,y) \times g(x,y) = -1$ $\forall (x,y) \in \mathbb{R}^2$

$\Rightarrow$ Orthogonal Trajectories satisfy $\frac{dy}{dx} = -\frac{1}{f(x,y)} = \frac{-2x}{3y}$

$\frac{dy}{dx} = \frac{-2x}{3y} \Rightarrow 3y\,dy + 2x\,dx = 0 \Rightarrow$

$d(2x^2 + 3y^2) = 0 \Rightarrow 2x^2 + 3y^2 = C$ (Constant.)

$\therefore$ Orthogonal family of curves wrt $y^2 = C_1 x^3$ is $2x^2 + 3y^2 = C_2$

C_1 & C_2 are const.

But if from $y^2 = Cx^3$, $C = 0$, Then we get the line $y = 0$ & the orthogonal family of lines wrt $y=0$ is $x = C$. $C \in \mathbb{R}$ is const.