MTH114-End-Sem-PartB

● Graded

Student

SHAURYA JOHARI

Total Points

24 / 25 pts

Question 1

Question1

6 / 7 pts

✔	**+ 1 pt** Name written in capital + cleanly + inside box

+ 1 pt 1a. Correct formula for convolution

✔	**+ 1 pt** 1a. Correct calculations

+ 2 pts 1a. Fully correct

+ 0 pts 1a. Incorrect

+ 1 pt 1b. correct formula of LT for y'' and cos

+ 1 pt 1b. Correct calculation of Y = LT(y)

+ 2 pts 1b. Correct calculation of inverse Laplace transform to find y

✔	**+ 4 pts** 1b. Fully correct

+ 0 pts 1b. Incorrect

Sign error.

Question 2

Question2

6 / 6 pts

+ 0 pts 2a. Incorrect

+ 1 pt 2a. -1 irregular singular

+ 1 pt 2a. Found ALL singular points 0, 1, -1

+ 1 pt 2a. 0,1 regular singular

✔	**+ 3 pts** 2a. Fully Correct

+ 0 pts 2b INcorrect

+ 1 pt 2b. correctly found p(0), q(0) (or b(0, c(0))

+ 1 pt 2b. Only one Frobenious solution due to repeated roots

✔	**+ 3 pts** 2b. Fully correct

+ 1 pt 2b. Indicial equation general formula correct

Question 3

Question3

6 / 6 pts

✔ **+ 3 pts** 3a. Fully Correct

+ 0 pts 3a. Incorrect

+ 1 pt 3a. argued not all a_n zero.

+ 1 pt 3a. written $f(x) = (x - x_0)^m g(x)$

+ 1 pt 3a. argued g(x) non-zero in a nbd

✔ **+ 3 pts** 3b. Fully correct

+ 0 pts 3b. Incirrect

+ 1 pt 3b. Given that $fg' - f'g = 0$ on an interval I. Since zeros of f are isolated points we can choose an interval $I' \subset I$ such that $f \neq 0$ on I'.

+ 1 pt 3bThen on I', we have $(fg' - f'g)/f^2 = 0$, implies $(g/f)' = 0$, imples $g = cf$ on I'.

+ 1 pt 3b. Now h= g-cf is analytic on I and h is zero on an interval I' i.e. h has non isolated zero. Hence by (i), we must have h =0 on I.

Question 4

Question4

6 / 6 pts

+ 6 pts Fully Correct

+ 0 pts Incorrect

✔ **+ 1 pt** $\lambda > 0$ case: general soution correct

✔ **+ 1 pt** $\lambda > 0$ case: correct caclution using boundary conditions

✔ **+ 1 pt** $\lambda = 0$ case: General solution correct

✔ **+ 1 pt** $\lambda = 0$ case: correct calculation using boundary conditions

✔ **+ 1 pt** $\lambda < 0$ case: General solution correct

✔ **+ 1 pt** $\lambda < 0$ case: correct calculation using boundary conditions

4. Find the eigenvalues and the eigenfunctions of the following boundary value problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(2\pi) = 0.$$

[6]

Answer: Assume $\lambda = -p^2$ Where $p > 0$

$y'' - p^2 y = 0 \Rightarrow$ Possible Eigen Function;

$Ae^{px} + Be^{-px}$. But we have $A+B=0$ Boundary cond.

$Ae^{2\pi p} + Be^{-2\pi p} = 0 \Rightarrow A = B = 0$

$\therefore$ No eigenfunction if $\lambda < 0$

If $\lambda = 0$, Then $y'' = 0 \Rightarrow y = A + Bx$

But boundary condition $\Rightarrow A = 0$ & $B = 0 \Rightarrow y = 0$

$y \equiv 0$ Can't be Taken as an eigen function

$\lambda > 0 \Rightarrow \lambda = q^2 \quad q \in \mathbb{R}$

$y'' + q^2 y = 0 \Rightarrow$ Possible Functions: $A\cos qx + B\sin qx$

Boundary condition: $y(0) = 0 \Rightarrow A = 0$.

$y(2\pi) = 0 \Rightarrow B \sin 2q\pi = 0$.

But $\sin n\pi = 0 \quad \forall\, n \in \mathbb{Z}^+$

$n^2 = 4q^2 = 4\lambda \Rightarrow$

$\lambda = \frac{n^2}{4}$ Where $n \in \mathbb{N}$ Gives us all possible eigen values for BVP

And Eigen Functions; ~~A B~~

$P \sin \frac{n}{2}x + Q\sin(-\frac{n}{2}x) = (P-Q)\sin\frac{nx}{2} \equiv$

$(C \in \mathbb{R}$ is const.$)$ $\quad C\sin\frac{nx}{2}$ Where $n \in \mathbb{N}$

Eigen values $\lambda = n^2/4$, Eigen Function: $C\sin\frac{nx}{2} = y(x)$

H ... $\mathcal{L}(h(x)) = \mathcal{L}(f(x))\,\mathcal{L}(g(x))$ then $h = f * g$ convolution (For Part B)

ODE : MTH 114M : End-Sem-B

Name: SHAURYA JOHARI Roll No: 230959

- Write your name cleanly in CAPITAL letters and roll number inside the boxes (1 mark).
- Write answers in the space provided only. Total marks: 25 Time: 5:00 pm - 7:00 pm.

1. (a) Find the convolution $e^{at} * e^{bt}$ for $a \neq b$.
   (b) Use Laplace transform to solve the ODE: $y'' + 4y = \cos 2t$, $y(0) = 0$, $y'(0) = 1$. [2+4]

$\mathcal{L}(f(x)) = \int_0^\infty e^{-st} f(t)\,dt$

Ans

Answer: $f(t) = e^{at} \quad g(t) = e^{bt}$.

$f * g = \int_0^x f(t)\, g(x-t)\,dt \quad (x \to t,\ t \to \tau)$

Convolution $\Rightarrow \int_0^t e^{a\tau} e^{b(t-\tau)}\,d\tau = e^{bt}\int_0^t e^{(a-b)\tau}\,d\tau = \frac{e^{bt} - e^{at}}{a-b}$

B ans $\mathcal{L}(\cos pt) = \frac{s}{s^2+p^2} \quad p \in \mathbb{R} \quad p = 2$

$\mathcal{L}(y) = Y$, Then $\mathcal{L}(y'') = s^2 Y - sy(0) - y'(0) \Rightarrow$

$\mathcal{L}(y'' + 4y) = s^2 Y - s(0) - (1) + 4Y = \frac{s}{s^2+4}$

$(s^2+4)Y = 1 + \frac{s}{s^2+4} \Rightarrow Y = \frac{1}{s^2+4} + \frac{s}{(s^2+4)^2}$

$\mathcal{L}(\sin px) = \frac{p}{s^2+p^2} \quad \forall p \in \mathbb{R}$

$\mathcal{L}^{-1}(Y) = y(x) = \mathcal{L}^{-1}\left(\frac{1 \times 2}{2\, s^2+4} + \frac{s}{s^2+4} \times \frac{1}{2} \times \frac{2}{s^2+4}\right) =$

$\frac{\sin 2t}{2} + \frac{1}{2}\int_0^t \sin 2\tau \cos(2t - 2\tau)\,d\tau$ (Convolution)

$= \frac{\sin 2t}{2} + \frac{1}{4}\int_0^t (\sin 2t)\,d\tau + (\sin(4\tau - 2t))\,d\tau =$

$\frac{\sin 2t}{2} + \frac{t\sin 2t}{4} = y(t)$

$\int_0^t \sin(4\tau - 2t)\,d\tau = \int_0^t \sin(4(t-\tau) - 2t)\,d\tau = \int_0^t \sin(2t - 4\tau)\,d\tau = 0.$

2. (a) Locate and classify the singular points of the ODE: $x^2(x^2-1)^2y'' - x(1-x)y' + y = 0$.

(b) What is the indicial equation for singular point $x_0 = 0$? How many Frobenius series solution does it have about $x_0 = 0$?

[3+3]

Aans.

Answer: $y'' + \frac{x(x-1)}{x^2(x-1)^2(x+1)^2}y' + \frac{y}{x^2(x-1)^2(x+1)^2} = 0$

$p(x) = \frac{x(x-1)}{x^2(x-1)^2(x+1)^2}$ $q(x) = \frac{1}{x^2(x-1)^2(x+1)^2}$

$p(x)$ & $q(x)$ are undef when $x = 0$ OR ± 1. These are our sing. pts. (singular points)

$\lim_{x \to x_0} (x-x_0)p(x)$ Exists when $x_0 = 0$ OR 1 BUT NOT -1

$\lim_{x \to x_0} (x-x_0)^2 q(x)$ " ~~whe~~ for all: $x_0 = -1, 0$ OR 1

$\therefore$, $x_0 = 0$ & $x_0 = 1$ Are Regular Singular Points

Meanwhile $x_0 = -1$ Is irregular singular point.

Bans $x^2 y'' + x\frac{\cancel{(x-1)}}{(x^2-1)(x+1)^2}y' + \frac{y}{(x-1)^2(x+1)^2} = 0$

Power series of ?

$b(x) = -(1+x+x^2\cdots)(1-x+x^2-x^3\cdots)^2$

$c(x) = (1+x+x^2\cdots)^2(1-x+x^2\cdots)^2$

$b_0 = b(0) = -1$ $c_0 = c(0) = 1$

Indical eq^n: $r(r-1) + b_0 r + c_0 = 0$

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$

$\therefore$ We have double root $r=1$ For indical eq^n

There's only 1 Frobenius series sol^n about $x_0 = 0$

(Frobenius Form: $x^2y'' + xb(x)y' + c(x)y = 0$)

($\frac{1}{1-z} = 1+z+z^2+z^3\cdots$ & $\frac{1}{1+z} = 1-z+z^2-z^3\cdots$) For Part (A)

* - Basically x_1 & x_2 are consecutive 0s of $h(x)$ & $\exists$ ∞ many $x \in (x_1, x_2)$ s.t. $h(x) \neq 0$

(For Part A)

Name: SHAURYA JOHARI Roll No: ~~230959~~ 230959 PartB

3. (a) Let $f(x): I = (a,b) \to \mathbb{R}$ be a (real) analytic function which is not identically zero. Show that if $f(x_0) = 0$ then $\exists\ \epsilon > 0$ such that $f(x) \neq 0$ for all $0 < |x - x_0| < \epsilon$ (i.e., zeros of f are isolated).

(x_1, x_2)

(b) Deduce that if f, g are analytic functions on an interval $I = (a,b)$ and the Wronskian $W(f,g) = 0$ on I then f, g are linearly dependent on I.

[3+3]

Bans

Answer: $W(f,g) = fg' - gf' = 0$ $\forall$ $x \in I$.

We can choose $I' \subseteq I$ s.t. $f(x) \neq 0$ When $x \in I'$

$\therefore$, Over I', $\frac{-gf' + fg'}{f^2} = 0 \Rightarrow \frac{d}{dx}\left(\frac{g}{f}\right) = 0 \Rightarrow$

$\frac{g}{f} = c \in \mathbb{R}$.

Now consider $h: g(x) - cf(x)$

If f & g were Linearly independent over I (& hence I')

Then isolated 0s would've occured over (a,b) for $h(x)$

But now $h(x)$ is analytic over I' & non isolated 0s exist (over I'),

contradicting our assumption

Thus f & g are linearly dependent.

A ans $f(x)$ is analytic about $x = x_0 \Rightarrow$

$f(x) = \cancel{a_0 +} \sum_{n=0}^{\infty} a_n (x-x_0)^n$ Where $a_0 = f(x_0) = 0$

$\because$ $f(x) \neq 0$, $\exists$ $k \in \mathbb{N}$ s.t. $a_k \neq 0$.

Now $f(x) = (x-x_0)^k \sum_{n=0}^{\infty} a_{k+n}(x-x_0)^n = (x-x_0)^k b(x)$

$b(x)$ is analytic about x_0 $\rightarrow$ Where $b(x)$ is another power series & $b(x) \neq 0$

And $b(x_0) = a_k \neq 0$. $\Rightarrow$ If $x_1 < x_0 < x_2$ are consecutive 0s of $a(x)$ Then $b(x_1) = b(x_2) = 0$*.

Choose $\epsilon = \min\{x_2 - x_0, x_0 - x_1\}$. Then $\forall$ $x \in (x_0 - \epsilon, x_0 + \epsilon)$ $(x \neq x_0)$, $\rightarrow$ $(x-x_0)^k \neq 0$ & $b(x) \neq 0 \Rightarrow$ $f(x) \neq 0$ $\forall$ $x \in (x_0 - \epsilon, x_0 + \epsilon)$ $x \neq x_0$

$\therefore$ $\exists$ infinite x b/w 2 consecutive 0s of $f(x)$ s.t. $f(x) \neq 0$

$f(x)$ has isolated 0s, HP.