Exam: MTH111M (2023-2024 I)

Date: 20 September 2023 | Time: 13:00-15:00 hrs | Maximum marks: 90

Name: | **Roll No.**

Instructions: (Read carefully)

- Please enter your NAME and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 12 pages. Check to see if the print is either faulty or missing on any of the pages. In such a case, ask for a replacement immediately.
- Please answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booket.

Q 1. (a) Discuss the convergence/divergence of the sequence $(\frac{n^4}{e^n})$ as $n \to \infty$. *(Do not use L'Hospital rule for any justification).*

(b) Let $x_1 = 1$ and $x_2 = 2$ and $x_{n+2} = \frac{x_{n+1}+5x_n}{6}$ for all $n \geq 1$. Show that (x_n) converges and find its limit. (3+7)

Solution:

(a) Let $a_n = \dfrac{n^4}{e^n}$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^4}{e^{n+1}} \frac{e^n}{n^4} \to \frac{1}{e}. \qquad [2]$$

Since $\frac{1}{e} < 1$, by Ratio test, the sequence (a_n) converges. [1]

(b) Firstly,

$$|x_{n+2} - x_{n+1}| = |\frac{x_{n+1} + 5x_n}{6} - x_{n+1}| = \frac{5}{6}|x_{n+1} - x_n|. \qquad [3]$$

The sequence satisfies the Cauchy Criterion and hence it converges. [1]

Observe that $x_{n+2} + \frac{5}{6}x_{n+1} = x_{n+1} + \frac{5}{6}x_n$. [2]

Let $\lim\limits_{n\to\infty} x_n = \ell$.

Then

$$\ell + \frac{5}{6}\ell = x_2 + \frac{5}{6}x_1 = 2 + \frac{5}{6} = \frac{17}{6}.$$

Hence

$$\frac{11\ell}{6} = \frac{17}{6},$$

which implies $\ell = \frac{17}{11}$. [1]

Q 2. (a) Let p be an irrational number. Show that there exists a sequence (r_n) of rational numbers such that $r_n \to p$ as $n \to \infty$.

(b) Let $f : [-1, 1] \to \mathbb{R}$ be twice differentiable. Suppose $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Show that $f'(0) = f''(0) = 0$. (3+6)

Solution:

(a) For all $n \in \mathbb{N}$, consider p and $p + \frac{1}{n}$.

For each $n \in \mathbb{N}$, there exists a rational $\gamma_n \in (p, p + \frac{1}{n})$. [2]

By sandwich theorem, $\lim\limits_{n\to\infty} \gamma_n = p$. [1]

(b) By continuity of f, $\lim\limits_{n\to\infty} f(\frac{1}{n}) = f(0) = 0$. [1]

Now,

$$f'(0) = \lim_{n\to\infty} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}} = 0. \quad [2]$$

By Rolle's Theorem, for each n, there exists $c_n \in (0, \frac{1}{n})$, such that $f'(c_n) = 0$. [2]

Now,

$$f''(0) = \lim_{n\to\infty} \frac{f'(c_n) - f'(0)}{c_n} = 0. \quad [1]$$

Q 3 (a) Using Rolle's theorem and intermediate value theorem, find the number of distinct real values of x satisfying the equation $x^2 = x\sin x + \cos x$.

(b) Let $f : [a, b] \to \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that $f^{(n+1)}$ exists on $[a, b]$ and $f^{(n+1)}(x) = 0$ for all $x \in [a, b]$. Using Taylor's theorem, show that f is a polynomial of degree less than or equal to n. (5+3)

Solution:

(a) Let $f(x) = x^2 - x\ sin\ x - cos\ x$. Then $f'(x) = x(2 - cos\ x)$.

Since $f'(x) = 0$ has one real root,
by Rolle's Theorem $f(x) = 0$ has at most two real roots. [2]

Now,

$$f(-\frac{\pi}{2}) > 0, f(0) < 0 \ \ and \ \ f(\frac{\pi}{2}) > 0.$$

By Intermediate value Property, $f(x)$ has at least two real roots. [2]

Hence, $f(x) = 0$ has exactly two real solutions. [1]

(b) Let $x \in [a, b]$. Then, by Taylor's Theorem,

$$f(x) = f(a) + f'(a)(x - a) + \ldots + \frac{f^n(a)}{n!}(x - a)^n + \frac{f^{n+1}(c)}{(n + 1)!}(x - a)^{n+1},$$

for some $c \in (a, x)$. [2]

Hence,

$$f(x) = f(a) + f'(a)(x - a) + \ldots \frac{f^n(a)}{n!}(x - a)^n,$$

which is a polynomial of degree $\leq n$. [1]

Q 4 (a) Let $\sum_{n=1}^{\infty} a_n$ converge. Show that $a_n \to 0$ as $n \to \infty$.

(b) Let $a_n = 1 - n \sin \frac{1}{n}$ and $b_n = \sin \frac{1}{n}$ for all $n \in \mathbb{N}$. Discuss the convergence/divergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} (a_n + b_n)$. (3+6)

Solution:

(a) Let (s_n) be the sequence of partial sums of $\sum a_n$, Since, $\sum a_n$ converges, the sequence (s_n) also converges. Let $\ell = \lim_{n\to\infty} s_n$.

Then $a_{n+1} = s_{n+1} - s_n \to 0$. [3]

(b) Observe that

$$\lim_{n\to\infty} \frac{1 - nsin\ \frac{1}{n}}{\frac{1}{6n^2}} == \lim_{x\to 0^+} \frac{1 - \frac{1}{x} sin\ x}{\frac{x^2}{6}} = \lim_{x\to 0^+} \frac{6(x - sin\ x)}{x^3} = 1.$$

By the limit comparison test, $\sum a_n = \sum 1 - nsin\ \frac{1}{n}$ converges. [4]

Since, $\lim_{n\to\infty} \frac{sin\ \frac{1}{n}}{\frac{1}{n}} = 1$, by the limit comparison test $\sum b_n$ diverges. [1]

If $\sum(a_n + b_n)$ converges, then $\sum(a_n + b_n - a_n) = \sum b_n$ converges, which is a contradiction. Therefore, $\sum(a_n + b_n)$ diverges. [1]

Q 5 (a) Let $a_n = \frac{1}{n \ln n}$ for all $n > 1$. Discuss the convergence/divergence of $\sum\limits_{n=2}^{\infty} a_n$. *(Do not use integral test).* (3+6)

(b) Let $x \geq 0$. Using Taylor's theorem, show that the series $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges to e^x.

Solution:

1. By the Cauchy condensation test,

$$\sum_{n=0}^{\infty} \frac{1}{n\ ln\ n} \text{ converges} \iff \sum_{k=1}^{\infty} 2^k \frac{1}{2^k\ ln\ 2^k} \text{converges} \iff \sum_{k=1}^{\infty} \frac{1}{k\ ln\ 2} \text{ converges.} \qquad [2]$$

Since $\sum \frac{1}{k\ ln\ 2}$ diverges, $\sum\limits_{n=2}^{\infty} a_n$ diverges. [1]

2. Let $x > 0$ and $s_0, s_1, \ldots$ be the sequence of partial sums for $\sum\limits_{0}^{\infty} \frac{x^n}{n!}$.

By Taylor's Theorem, there exist $c_n \in (0, x)$, $n \in \mathbb{N}$, such that

$$|e^x - s_n| = |\frac{e^{c_n} x^{n+1}}{(n+1)!}|. \qquad [2]$$

Further, for each $n \in \mathbb{N}$,

$$|\frac{e^{c_n} x^{n+1}}{(n+1)!}| \leq |\frac{e^x x^{n+1}}{(n+1)!}|. \qquad [2]$$

Since

$$\frac{e^x x^{n+1}}{(n+1)!} \frac{n!}{e^x x^n} = \frac{x}{n+1} \to 0,$$

by the ratio test for sequences, $|\frac{e^x x^{n+1}}{(n+1)!}| \to 0$. [1]

Hence $s_n \to e^x$. [1]

Q 6 (a) Determine all the values of x for which $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}$ converges.

(b) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that $\sum_{n=1}^{\infty} \frac{3^n+|a_n|}{4^n+|a_n|}$ converges. (4+5)

Solution:

(a) Since

$$\left(\frac{|x-2|^n}{n3^n}\right)^{\frac{1}{n}} \to \frac{|x-2|}{3}, \qquad [1]$$

the power series converges on $|x-2| < 3$, *i.e.* for $x \in (-1, 5)$. [1]

At $x = -1$, the series converges. [1]

At $x = 5$, the series diverges. [1]

(b) Since $\sum a_n$ converges, $a_n \to 0$. Thus $|a_n| \to 0$. [1]

Observe that

$$\frac{4^n}{3^n}\left(\frac{3^n + |a_n|}{4^n + |a_n|}\right) \to 1.$$

[2]

By Limit Comparison Test, the series converges. [2]

Q 7 (a) Discuss the convergence/divergence of $\int_1^\infty \frac{x^2 \sin x \cos x}{e^x} dx$.

(b) Let $f : [0,1] \to \mathbb{R}$ be an increasing function. Using the Riemann criterion show that f is integrable. (4+5)

Solution:

(a) Note that

$$|\frac{x^2 sin\ xcos\ x}{e^x}| \leq \frac{x^2}{e^x}. \qquad [1]$$

Now,

$$\frac{\frac{x^2}{e^x}}{\frac{1}{x^2}} = \frac{x^4}{e^x} \to 0, \text{ as } x \to \infty.$$

By Limit Comparison Test,

$$\int_1^\infty \frac{x^2}{e^x} \text{ converges.} \qquad [2]$$

Therefore by comparison test

$$\int_1^\infty |\frac{x^2 sin\ xcos\ x}{e^x}| \text{ converges.}$$

Hence the given integral converges. [1]

(b) Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ and M_i, m_i be the supremum and infimum of f, respectively, on the interval $[\frac{i-1}{n}, \frac{i}{n}]$, $1 \leq i \leq n$. [1]

Now, $M_i = f(\frac{i}{n})$ and $m_i = f(\frac{i-1}{n})$ [2]

$$U(P_n, f) - L(P_n, f) = \sum_{i=1}^{n} (M_i - m_i)\frac{1}{n} = \frac{1}{n}(f(1) - f(0)) \to 0. \qquad [1]$$

Hence, by Riemann Criterion f is integrable. [1]

Q 8 (a) Find $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}}$.

(b) Let $f : [0, 2] \to \mathbb{R}$ be a continuous function such that $\int_0^2 \{xf(x) + \int_0^x f(t)dt\}dx = 4$. Find the value of $\int_0^2 f(x)dx$. (4+5)

Solution:

1. Note that
$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{1+\frac{k}{n}}}. \qquad [2]$$
which is a Riemann sum and hence
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}} = \int_0^1 \frac{1}{\sqrt{1+x}}dx. \qquad [1]$$
Therefore $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}} = 2\sqrt{2} - 2$ [1]

2. Observe that
$$xf(x) + \int_0^x f(t)\ dt = \frac{d}{dx}(x\int_0^x f(t)dt). \qquad [2]$$
So,
$$\int_0^2 [xf(x) + \int_0^x f(t)dt]dx = [x\int_0^x f(t)dt]_0^2 = 2\int_0^2 f(t)dt. \qquad [2]$$
Therefore, $\int\limits_0^2 f(t)dt = 2$. [1]

Q 9 (a) Let $f : [-2, 5] \to \mathbb{R}$ be defined by $f(x) = x^2|x-3|$. Find all the points of local maximum and local minimum of f in $[-2, 5]$. [6+3]

(b) Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x_0 \in [a, b]$. Using the extended mean value theorem show that $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$ for any $x \in [a, b]$.

Solution:

(a) Note that $f(x) = x^2(3 - x)$ on $(-2, 3)$ and $f(x) = x^2(x - 3)$ on $(3, 5)$.

Therefore, $f'(x) = 3x(2 - x)$ on $(-2, 3)$ and $f'(x) = 3x(x - 2)$ on $(3, 5)$. [2]

Thus, the candidates for the points of local maximum and local minimum are $-2, 0, 2, 3$ and 5. [1]

$f'(x) < 0$ on $(-2, 0)$ and $(2, 3)$ and $f'(x) > 0$ on $(0, 2)$ and $(3, 5)$. [2]

Thus, 0 and 3 are points of local minimum and $-2, 2, 5$ are points of local maximum. [1]

(b) Let $x \in [a, b] \setminus \{x_0\}$. By the Extended Mean Value Theorem,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2}(x - x_0)^2,$$

for some c between x and x_0. [2]

Since $\frac{f''(c)}{2}(x - x_0)^2 > 0$, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$. [1]

Answer to Q 9 can be continued here:

Q 10. Let $f(x) = \frac{2x^3+x^2-4}{x^2-4}$. Here $f''(x) = \frac{16x(x^2+12)}{(x^2-4)^3}$.

1. Find the slant asymptote(s) of f (if any).
2. Determine the intervals in which f is increasing or decreasing.
3. Find the points of local maximum or local minimum.
4. Find the intervals of convexity or concavity.
5. Sketch the graph of f. [9]

Solution:

Since $f(x) = 2x + 1 + \frac{8x}{x^2-4}$, $y = 2x + 1$ is a slant asymptote [2]

Note that

$$f'(x) = \frac{2x^2(x^2-12)}{(x^2-4)^2}.$$

So, $f'(x) = 0$ when $x = -2\sqrt{3}, 0, 2\sqrt{3}$. [1]

Since $f'(x) > 0$ on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$, f is increasing on these intervals. [1]

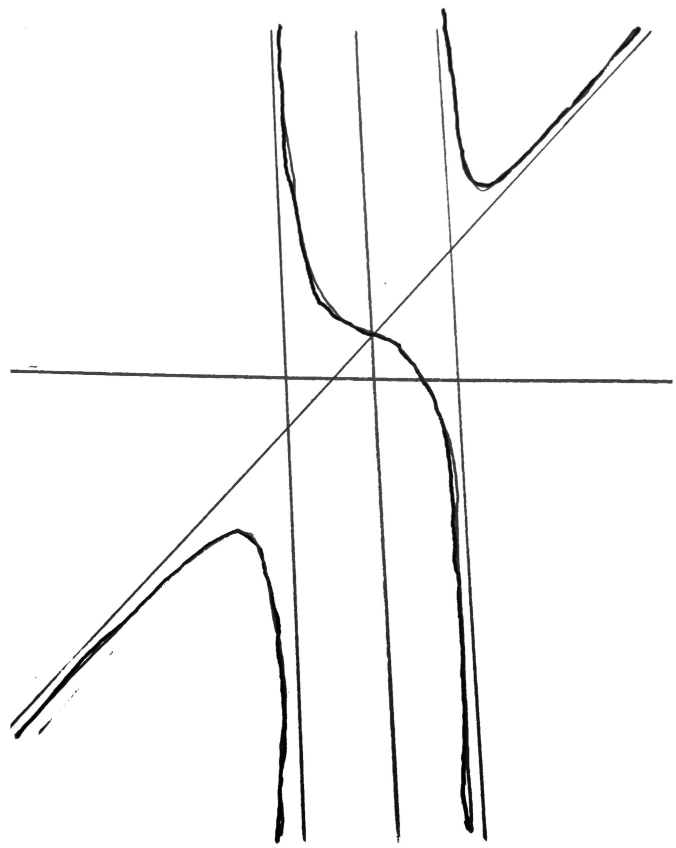
Since $f'(x) < 0$ on $(-2\sqrt{3}, -2)$, $(-2, 2)$ and $(2, 2\sqrt{3})$, f is decreasing on these intervals. [1]

Hence, $-2\sqrt{3}$ is a point of local maximum and $2\sqrt{3}$ is a point of local minimum. [1]

Since, $f''(x) > 0$ on $(-2, 0)$ and $(2, \infty)$, f is convex on these intervals. [1]

Since, $f''(x) < 0$ on $(-\infty, -2)$ and $(0, 2)$, f is concave on these intervals. [1]

Answer to Q 10 can be continued here:



1 mark for the graph.