Quiz 2

● Graded

Student

SHAURYA JOHARI

Total Points

8 / 30 pts

Question 1

Question 1

4 / 6 pts

+ 6 pts Completely correct answer

✔ **+ 3 pts** Part 1 completely correct

+ 0 pts No submission/incorrect submission/incorrect answer

+ 2 pts Part 1: $\{f(x) \in \mathbb{F}[x] : deg\ f(x) \leq 3, f(1) = 0\}$ has dimension ≤ 3

+ 1 pt Part 1: In the proof by contradiction, argued that $\{A(x), B(x), C(x), D(x)\}$ is a basis of $\{f(x) \in \mathbb{F}[x] : deg\ f(x) \leq 3\}$.

+ 3 pts Part 2 completely correct

+ 0 pts Part 2 completely wrong/no submission

+ 1 pt Part 2: Just mentioned correct example but did not provide any/correct justification

+ 0 pts Click here to replace this description.

+ 1 pt Point adjustment

3 idea is correct but we need some concrete example to say that linearly independent is possible. +1 for the idea.

Question 2

Question 2

3 / 9 pts

+ 9 pts Completely Correct Answer

+ 0 pts No Answer or Completely Wrong Answer

✔ **+ 1 pt** H_n is closed under scalar multiplication

✔ **+ 2 pts** H_n is closed under addition

+ 2 pts Finding a spanning set of H_2

+ 1 pt The spanning set is Linearly Independent

+ 1 pt Answering **no** to part (iii) with partially correct justification

+ 3 pts Correctly justified the answer for part (iii)

+ 2 pts For writing correct basis of H_2 over $\mathbb{R}$ without any justification.

+ 1 pt For writing three element of the basis in part 2 correctly

1 Incorrect

2 Does not belong to H_n

Question 3

Question 3

1 / 10 pts

+ 0 pts Completely Wrong Answer or No Answer

+ 10 pts Completely Correct Answer

+ 1 pt W_σ is closed under scalar multiplication

✔ **+ 1 pt** W_σ is closed under addition

+ 2 pts Finding the dimension of W_σ correctly with some justification

+ 2 pts Finding the dimension of W_η correctly with some justification

+ 2 pts Finding the dimension of $W_\sigma \cap W_\eta$ correctly with some justification

+ 2 pts Finding the dimension of $W_\sigma + W_\eta$ correctly with proper justification

+ 1 pt Writing only $dim(W_\sigma + W_\eta) = dim(W_\sigma) + dim(W_\eta) - dim(W_\sigma \cap W_\eta)$

Question 4

Question 4

0 / 5 pts

- **+ 5 pts** Completely correct
- ✔ **+ 0 pts** Completely wrong/not attempted/no substantial progress
- **+ 2 pts** Invariance of rank/uniqueness of RREF
- **+ 1 pt** Mentioned that RREF has all rational entries without proper justification
- **+ 1 pt** Obtained integral solution from rational solution

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Question 4. Let A be a $m \times n$ matrix with integer entries. Show that if the homogeneous system $A\mathbf{x} = \mathbf{0}$ of linear equations has a non-zero real solution then there exists a non-zero solution $u = \begin{pmatrix} u_1 \\ u_2 \\ .. \\ u_n \end{pmatrix}$ of $A\mathbf{x} = \mathbf{0}$ such that $u_1, u_2, ..., u_n$ are all integers.

[5 Marks]

Answer 4 :

$A = [a_{ij}]_{m \times n} \Rightarrow \quad Au = 0 \Rightarrow \forall i \in \{1, 2 \cdots m\},$

$$\sum_{j=1}^{n} a_{ij} u_j = 0$$

$\Rightarrow a_{ij} \in \mathbb{Z} \Rightarrow$

Quiz 2 : MTH113M/MTH102A

Date : 05/02/2024 | Time : 7:00- 7:40 pm | Total Marks : 30

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$\mathbb{R}, \mathbb{C}$ denote the set of real numbers, complex numbers respectively. Answer ONLY in the specific space provided.

Question 1. Let $A(x), B(x), C(x)$ and $D(x)$ be four polynomials with real coefficients having degree at most 3.

(1) Let $A(1) = B(1) = C(1) = D(1) = 0$. Is the set $\{A(x), B(x), C(x), D(x)\}$ always linearly dependent over $\mathbb{R}$?

(2) Let $A(0) = B(0) = C(0) = D(0) = 1$. Is the set $\{A(x), B(x), C(x), D(x)\}$ always linearly dependent over $\mathbb{R}$?

Justify your answers. [3+3=6 Marks]

Answer 1:

$P_3(X \mid \mathbb{R}) = A_0 + A_1 X + A_2 X^2 + A_3 X^3$ is the vector space of all possible ~~coe~~ polynomials over $\mathbb{R}$ with max degree = 3.

Basis of space : $\{1, X, X^2, X^3\}$

$\{A(x), B(x), C(x), D(x)\}$ is Linearly dependent $\Rightarrow$ $\exists\, p, q, r, s \in \mathbb{R}$ s.t. $(pA + qB + rC + sD)(x) = 0$ & $p = q = r = s = 0$ isn't only sol^n

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3.$

Define C & D in similar fashion.

~~$A(1) = B(1) = C(1) = D(1) = 0 \Rightarrow$~~

$A(0) = B(0) = C(0) = D(0) = 1 \Rightarrow a_0 = b_0 = c_0 = d_0 = (1)$

This doesn't give restriction on other coefs wrt the different polynomial (a_3, b_3, c_3, d_3 etc can be any values $\Rightarrow$ ~~(2)~~ A, B, C & D are Linearly independent (2) is false

(1) $\Rightarrow A(1) \Rightarrow a_0 + a_1 + a_2 + a_3 = 0 \Rightarrow a_3 = -(a_1 + a_2 + a_0)$

~~p(a)~~. $p a_i + q b_i + r c_i + s d_i = 0 \quad i \in \{0, 1, 2, 3\}$

And $\left[\operatorname{Det} \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{bmatrix} = 0 \Rightarrow \exists \right.$ Infinite sol^{n}s for (p, q, r, s)

$\Rightarrow$ A, B, C, D are Linearly Dependant

3

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Question 2. Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices whose entries are complex numbers. With respect to usual matrix addition and usual scalar multiplication $M_n(\mathbb{C})$ is a vector space over both $\mathbb{C}$ and $\mathbb{R}$. For a matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, define $\bar{A} := (\bar{a}_{ij})$, where $\bar{a}_{ij}$ denotes the complex conjugate of of a_{ij}. Consider the set $H_n = \{A \in M_n(\mathbb{C}) : A = \bar{A}^T\}$, where $\bar{A}^T$ is the transpose of $\bar{A}$.
(i) Prove that H_n is a $\mathbb{R}$-subspace of $M_n(\mathbb{C})$, when $M_n(\mathbb{C})$ considered as a vector space over $\mathbb{R}$.
(ii) For $n = 2$, find a basis of the vector space H_2 over $\mathbb{R}$.
(iii) Is H_n a $\mathbb{C}$-subspace of $M_n(\mathbb{C})$, when $M_n(\mathbb{C})$ considered as a vector space over $\mathbb{C}$? Justify your answer. [3+3+3=9 Marks]

Answer 2 : O belongs to H_n $\because \bar{O} = O$

Aans

Also, if we choose $c_1, c_2 \in \mathbb{R}$ & $M_1, M_2 \in M_n(\mathbb{C})$ s.t. $M_3 = c_1M_1 + c_2M_2$ $(\bar{c_1} = c_1$ & $\bar{c_2} = c_2)$

Then $\bar{M_3} = \overline{c_1M_1 + c_2M_2} = c_1\bar{M_1} + c_2\bar{M_2} = c_1M_1 + c_2M_2 = M_3$

$\Rightarrow \bar{M_3} = M_3 \neq$ ~~A~~ $c_1M_1 + c_2M_2 \in M_n(\mathbb{C})$

$\therefore$ Linear combination of any 2 matrices in $M_n(\mathbb{C})$ gives a 3rd matrix in $M_n(\mathbb{C}) \Rightarrow H_n$ is a $\mathbb{R}$-subspace of $\mathbb{C}$

Bans. Note that $\bar{i} \neq i$ & $\bar{z} = z \Rightarrow z \in \mathbb{R}$

2×2 Matrix $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ And the basis of H_2 over $\mathbb{R}$ is

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
$A_1 \quad A_2 \quad A_3 \quad A_4$

$\because \bar{A_i} = A_i$ & $aA_1 + bA_2 + cA_3 + dA_4$ gives a matrix from H_2 $(a,b,c,d \in \mathbb{R})$

Cans. No, given statement is false.

Consider $\begin{bmatrix} 3 & 5i \\ 0 & 0 \end{bmatrix} = M$. $\bar{M} = \begin{bmatrix} 3 & -5i \\ 0 & 0 \end{bmatrix} \neq M$

But $M = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}(1) + (i)\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$

$= M_1 + iM_2$ where $M_1, M_2 \in H$

~~00100~~

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Question 3. Let S_n denote the set of all permutations on the set $\{1, 2, ..., n\}$.
(i) For a fix $\sigma \in S_n$. Consider the set $W_\sigma = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (x_1, x_2, ..., x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})\}$. Prove that W_σ is a subspace of $\mathbb{R}^n$ with respect to usual addition and scalar multiplication.
(ii) Consider $\sigma = (1\,3)(2\,4\,5)$ and $\eta = (2\,3)(4\,5)$ in S_5, where $(1\,3), (2\,3), (4\,5)$ are 2-cycles and $(2\,4\,5)$ is the 3-cycle in S_5. Find the dimensions of the four subspaces W_σ, W_η, $W_\sigma \cap W_\eta$ and $W_\sigma + W_\eta$.

[2+8=10 Marks]

Answer 3 : Let $T_1, T_2 \in W_\sigma$. $(c \in \mathbb{R})$

$T_1 \cancel{+T_2} =$ $T_1 = (x_{\sigma(1)}, x_{\sigma(2)} \cdots x_{\sigma(n)})$
$T_2 = (y_{\sigma(1)}, y_{\sigma(2)} \cdots y_{\sigma(n)})$

$T_1 + T_2 = (x_{\sigma(1)} + y_{\sigma(1)}, x_{\sigma(2)} + y_{\sigma(2)} \cdots x_{\sigma(n)} + y_{\sigma(n)}) =$

$((x+y)_{\sigma(1)}, (x+y)_{\sigma(2)} \cdots (x+y)_{\sigma(n)}) =$

$(z_{\sigma(1)}, z_{\sigma(2)} \cdots z_{\sigma(n)}) = T_3 \in W_\sigma$.

Also $cT_1 = (cx_{\sigma(1)}, cx_{\sigma(2)} \cdots cx_{\sigma(n)}) = (d_{\sigma(1)}, d_{\sigma(2)} \cdots d_{\sigma(n)})$
$(cx_i = d_i)$

~~Highest no. of cycles~~ n-cycles

Dimension of W_σ: ~~(1 3), (2 4 5)~~ 3 $(E_{13} \times E_{24} \times E_{45})$

Dimension of W_η: ~~(1), (2 3), (4 5)~~ 2 $(E_{23} \times E_{45})$

Dimension of $W_\sigma \cap W_\eta$: 1 (

" " $W_\sigma + W_\eta$: 4

Largest value of n-cycle = Dimension

Justification?