End Sem SET A

Graded

Student

SHAURYA JOHARI

Total Points

14 / 24 pts

Question 1

Question A.1.

0 / 5 pts

- **+ 5 pts** Completely correct
- ✔ **+ 0 pts** Completely wrong or not attempted
- **+ 3 pts** First part completely correct
- **+ 1 pt** $CS(A) = CS(A^2)$
- **+ 1 pt** By rank-nullity showing $\dim N(A) = \dim N(A^2)$
- **+ 1 pt** $N(A) = N(A^2)$
- **+ 1 pt** Showing $N(A) \cap CS(A) = \{0\}$ with proper justification
- **+ 1 pt** Showing $\dim(N(A) + CS(A)) = n$
- **+ 1 pt** $N(A) + CS(A) = \mathbb{R}^n$

Question 2

Question A.2.

Resolved **5** / 5 pts

+ **5 pts** Completely correct

+ **0 pts** Completely wrong/not attempted/no substantial progress

✔ + **2 pts** Part (i) completely correct

+ **1 pt** In part (i), showed that T preserves vector addition, i.e., $T(A+B) = T(A) + T(B)$.

+ **1 pt** In part (i), showed that T preserves scalar multiplication, i.e., $T(\alpha A) = \alpha T(A)$.

✔ + **3 pts** Part (ii) completely correct

+ **1 pt** In part (ii), showed that $\dim ImT = 1$, i.e., T is surjective.

+ **1 pt** In part (ii), observes that $\dim M_n(\mathbb{R}) = n^2$.

+ **1 pt** Using rank-nullity theorem, deduced that $\dim V = n^2 - 1$.

+ **0 pts** Part (ii) completely incorrect or not using rank-nullity theorem

Regrade Request **Submitted on: Feb 28**

For Part ii, I've assumed T to be as defined in part i of the question and that T(A)= Trace of A.
Thus T(A) maps a n x n matrix to a scalar value, so dimension of image of T is 1

Corrected

Reviewed on: Feb 29

Question 3

Question A.3.

5 / 8 pts

+ **8 pts** Completely correct

+ **0 pts** Completely wrong/not attempted/no substantial progress

✔ + **4 pts** Part (i) completely correct

+ **2 pts** Showed that $M_{ij} = M_{ik}M_{kj} - M_{kj}M_{ik}$, whenever $i \neq j$.

+ **2 pts** Showed that $M_{11} - M_{jj} = M_{1j}M_{j1} - M_{j1}M_{1j}$ for $j > 1$.

+ **4 pts** Part (ii) completely correct

+ **2 pts** that $\{M_{ij} : i, j = 1, \ldots, n, i \neq j\} \cup \{M_{11} - M_{jj} : j = 2, \ldots, n\}$ is linearly independent with justification.

✔ + **1 pt** Observed that W is a proper subspace of $M_n(\mathbb{R})$ or W is contained in the subspace of trace zero matrices.

In part (ii) you have to show, why the given subspace exactly equal to the subspace of all matrices with trace zero. Part marking is given according to the marking scheme.

Question 4

Question A.4.

4 / 6 pts

+ 0 pts Completely wrong/not attempted/no substantial progress

✔ **+ 1 pt** Finding the matrix A correctly

✔ **+ 1 pt** Found correct eigenvalues of A, i.e., 4 and 9, using the characteristic polynomial

+ 6 pts Completely correct

+ 1 pt Finding one eigenvector correctly

✔ **+ 2 pts** Finding two eigenvectors correctly

+ 1 pt The correct choice for P, i.e., $P := Q \begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix}$ has been taken, equivalently made the change of variables $u = z/6, v = w/6$.

+ 1 pt Applying an appropriate ORTHOGONAL change of variables $X = QY$, the equation has been brought to the form $9z^2 + 4w^2 = 1$.

A

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Question A.4. Let $Q(x,y) = 8x^2 - 4xy + 5y^2$. Find a matrix A of order 2×2 such that $Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}$.

Compute the eigenvalues of A and then find a matrix P such that by applying change of variable $\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$ the curve $8x^2 - 4xy + 5y^2 = 1$ changes to the form $\frac{u^2}{4} + \frac{v^2}{9} = 1$. ~~A is rotated~~ [2+4=6 Marks]

Answer A.4.:

$\begin{bmatrix} u \\ v \end{bmatrix} = U$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\therefore$ $[x \ y]\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} =$ $\begin{bmatrix} x \\ y \end{bmatrix} = X$

$U = PX$

$[(ax+cy) \ (bx+dy)]\begin{bmatrix} x \\ y \end{bmatrix} =$

$[(ax^2 + dy^2 + (c+b)xy)] = [8x^2 + 5y^2 - 4xy]$.

$\therefore$ $a = 8, d = 5, c+b = -4 \Rightarrow A = \begin{bmatrix} 8 & c \\ -4-c & 5 \end{bmatrix}$

Eigen value of $A = \lambda \Rightarrow$ Det $(A - \lambda I) = 0$.

$(8-\lambda)(5-\lambda) + c(4+c) = 0$

It is convienent to keep $b = c \Rightarrow b = c = -2$.

$\therefore$ $(\lambda-5)(\lambda-8) = 4 \Rightarrow$ Eigenvalues: 4 & 9.

$\begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow$ $\lambda = 4 \Rightarrow 8a - 2b = 4a \Rightarrow b = 2a$

$\lambda = 9 \Rightarrow -2b = a \Rightarrow$

$\lambda = 4$, Eigenvector: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda = 9$, Eigenvector: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Consider $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1}$ $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$1 = [x \ y] A \begin{bmatrix} x \\ y \end{bmatrix} = [u \ v]\begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = X^T P^T A P X$.

$\therefore$ $P^T A P = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}$ $\boxed{P = A^{-1}Q}$ is our matrix

$AQ = Q\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$ $Q^{-1}AQ = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$ $Q^{-1}A^{-1}Q = \cancel{P^T A P}$. $P = A^{-1}Q \Rightarrow$ $P^T A P = Q^T A^{-1T} Q = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}$

End-Semster Exam MTH113M/MTH102A : SET A

Date : 19/02/2024 | Time : 6:00- 7:30 pm | Will be Collected Back at 7:30 pm

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Question A.1. Let A be a real matrix of order $n \times n$ such that $rank(A) = rank(A^2)$. Let $N(A)$ and $CS(A)$ denote the null space and column space of A respectively. Show that $N(A) \cap CS(A) = \{0\}$ and $N(A) + CS(A) = \mathbb{R}^n$. [3+2=5 Marks]

Answer A.1.:

~~Assume $k \in N(A) \cap CS(A)$. Then $Ak = 0 \Rightarrow$~~

~~If $k \in CS(A) \Rightarrow$ $A(Ak) = 0$~~

~~Colspace of $A^2 \subset$ Colspace of A.~~

Let Rank$(A) = r$. & Let $k \in N(A) \cap CS(A) \neq 0$.

$\therefore$ $Ak = 0 \Rightarrow A^2 k = 0 \Rightarrow k \in N(A^2)$.

~~If Rank(A)~~ Rank(A) = Dim (Colspace(A)) = Dim (Colspace (A^2)) = r.

Colspace$(A) = \{c_1, c_2 \dots c_r\} \Rightarrow k = q_1 c_1 + q_2 c_2 \dots q_r c_r \neq 0$

(Suppose) $\Rightarrow$ $A(Ak) = A(q_1 A c_1 + q_2 A c_2 \dots q_r A c_r) = 0 \Rightarrow$

Rank $(A^2) <$ Rank $(A) \Rightarrow$ Contradiction.

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Question A.2. (i) Prove that the map $T : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, defined by $T(A) = Trace\,(A)$, is a linear map.
(ii) Consider the subspace $V = \{A \in M_n(\mathbb{R}) : Trace\,(A) = 0\}$. Use rank-nullity theorem to prove that the dimension of V is $n^2 - 1$.
[2+3=5 Marks]

Answer A.2.:

Let $A, B \in M_n(\mathbb{R})$, $Tr(A) = c$, $Tr(B) = d$, $\gamma \in \mathbb{R}$

$Trace(A + \gamma B) = Tr([a_{ij} + \gamma b_{ij}]_{n\times n}) = \sum_{i=1}^{n} [a_{ii} + \gamma b_{ii}] = \sum_{i=1}^{n} a_{ii} + \gamma \sum_{i=1}^{n} b_{ii} = Tr(A) + \gamma Tr(B) = T(A) + \gamma T(B)$

$\therefore$ T follows addition property, scalar multiplication etc.
$\Rightarrow$ T is Linear.

ii ans. By ~~Rank~~ ~~$T(A) = Trace(A)$~~ Rank Nullity Thm (RNT):

$Dim(Img(T)) + Dim(Ker(T)) = Dim(Domain(T))$

Transformation: $T: \mathbb{R}_{n\times n} \rightarrow \mathbb{R}$.

$Dim(Domain(T)) = n^2$ $\quad Dim(Img(T)) = 1$

$\therefore Dim(Ker(T)) = n^2 - 1$

But $Ker(T) = \{v \mid T(v) = Trace(v) = 0\}$

$\Rightarrow Ker(T) = V$. $\therefore$ By RNT We've shown that

$Dim(V) = n^2 - 1$

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Question A.3. (i) Let M_{ij} denote the $n \times n$ matrix with 1 only at (i, j)-th position and 0 elsewhere. Show that $M_{ij} = M_{ik}M_{kj} - M_{kj}M_{ik}$ for $i \neq j$ and $M_{11} - M_{jj} = M_{1j}M_{j1} - M_{j1}M_{1j}$ for $2 \leq j \leq n$.
(ii) Consider the set $S = \{AB - BA : A, B \in M_n(\mathbb{R})\}$. Let W be the linear span of S over $\mathbb{R}$ in $M_n(\mathbb{R})$. Prove that the dimension of W is $n^2 - 1$.
[4+4=8 Marks]

Answer A.3.:

ii ans.) Note that $Tr(AB - BA) = 0$ ($\because Tr(AB) = Tr(BA)$)

$\therefore$ Let $AB - BA = [m_{ij}]_{n\times n}$.

Then $AB - BA$ satisfies $\sum_{i=1}^{n} m_{ii} = 0 \Rightarrow \sum_{\substack{i=1 \\ i\neq k}}^{n} m_{ii} = -m_{kk}$

Where $k \in \{1, 2 \dots, n\}$

Other elements of M, i.e m_{ij} $(i \neq j)$ are free, i.e. they don't satisfy any special condition in $AB - BA = M$.

$\therefore$ Dimension of (M) = Number of 'Free variables' in $\{m_{ij} \mid 1 \leq i, j \leq n\} = n \times n - 1$

$\because$ There's only $n-1$ Free variables in $\{m_{ii} \mid 1 \leq i \leq n\}$

$\therefore$ Dimension of $M = n^2 - 1$

i ans.) $M_{ij} = \{\underbrace{0, 0 \dots}_{j-1\ 0s} e_i, \underbrace{0, 0 \dots}_{n-j}\}$. Let O_i = 0 column matrix at ith position.

$M_{ik}M_{kj} = [m_{pq}]_{n\times n}$ Where $m_{pq} = \langle O_p^T, O_q \rangle$ (Usually)

$M_{ik} = \{O_1^T O_2^T \dots \underbrace{O^T}_{k-1} e_k^T O_{k+1} \dots O_n\}$.

$M_{kj} = \{O_1 O_2 \dots O_{j-1} e_k O_{j+1} \dots O_j\}$

$\therefore M_{ik}M_{kj}$ at co-ordinates $(i, j) = \langle e_k^T e_k \rangle = 1 \Rightarrow$

$M_{ik}M_{kj} = M_{ij}$

$M_{kj}M_{ik} = 0$ $\because$ Mismatch at (k, j) wrt. (i, k)

For ~~Part~~ Part B, $M_{ij} = M_{ik}M_{kj} \Rightarrow M_{1j}M_{j1} = M_{11} \Rightarrow M_{11} - M_{jj} = M_{1j}M_{j1} - M_{1j}M_{1j}$

$M_{pp} = M_{pq}M_{qp} \Rightarrow M_{jj} = M_{j1}M_{1j}$ $(q = 1)$