End Semester Examination MTH112M

Graded

Student

SHAURYA JOHARI

Total Points

29 / 40 pts

Question 1

Question 1

3 / 5 pts

+ **0 pts** Wrong answer/ Not attempted

✔ **+ 1 pt** Correctly showing that limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist.

✔ **+ 1 pt** $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

+ **1 pt** Writing correct steps like providing suitable inequalities towards showing $\lim_{(x,y)\to(0,0)} f(x,y)$ exist.

✔ **+ 1 pt** Computation of $f_y(0,0)$ is correct

+ **1 pt** Correct computation of $f_y(x,y)$ for some $(x,y) \neq (0,0)$

+ **2 pts** Correctly showing $\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist

+ **5 pts** Full correct

+ **0 pts** Click here to replace this description.

Question 2

Question 2

2 / 5 pts

+ **0 pts** Wrong answer/ Not attempted

+ **1 pt** Some steps provided to find the equation of normal

✔ **+ 2 pts** Correct equation is provided for the normal to the given curve

+ **1 pt** Finding a condition to show the curve satisfies an equation of circle

+ **2 pts** Correctly showing that an arbitrary point on the given curve satisfies an equation of a circle

+ **1 pt** For the correct observation that $R(t)$ lies on a circle of constant curvature.

+ **5 pts** Full correct answer

Question 3

Question 3

5 / 5 pts

+ 0 pts Wrong answer/ Not attempted

+ 1 pt The given surface is a level surface $\left\{(x,y,z)\in\mathbb{R}^3 : f(x,y,z)=0\right\}$ where $f(x,y,z)=x^2+2xy-2y-z.$

+ 1 pt The normal direction of the tangent plane at an arbitrary point on the given level surface is given by $\nabla f(x,y,z)$

+ 1 pt $\nabla f(x,y,z)=(2x+2y,\ 2x-2,\ -1)$

+ 1 pt Finding condition for the tangent plane parallel to xy-plane

+ 1 pt Computation of the required point $(1,-1,1)$ on the given surface is correct.

✔ **+ 5 pts** All correct

Question 4

Question 4

4 / 7 pts

+ 0 pts Wrong answer/ Not attempted

✔ **+ 1 pt** Observing that $(\frac{2}{3},\frac{1}{3})$ is a critical point of the given function which is in the interior of D.

+ 1 pt Observing $(\frac{1}{\sqrt{3}},0)$ is an extremum point of f

✔ **+ 1 pt** Observing that $(0,0)$ is an is an extremum point of f

✔ **+ 1 pt** Observing $(2,0)$ is an extremum point of f

✔ **+ 1 pt** Observing that $(0,2)$ is an is an extremum point of f

+ 1 pt Observing that $(1,1)$ is an is an extremum point of f

+ 1 pt $(2,0)$ is a point of absolute maximum of the given function.

+ 7 pts All correct

You have not found all the extremum points.

1 Wrong calculation.

Question 5

Question 5

5 / 5 pts

- **+ 0 pts** Wrong answer/ Not attempted
- **+ 1 pt** Using Green's theorem the given line integral $\frac{1}{2}\int_C -ydx + xdy = \int\int_D (N_x - M_y)\ dxdy$
- **+ 1 pt** Functions $M(x,y)$ and $N(x,y)$ are given correctly to apply Green's theorem
- **+ 1 pt** Shown that $N_x - M_y = 1$
- **+ 1 pt** $\int\int_D (N_x - M_y)\ dxdy = \int\int_D\ dxdy = Area(D)$
- **+ 1 pt** Area(D) = $\int_a^b f(x)\ dx$
- ✔ **+ 5 pts** All correct

Question 6

Question 6

6 / 6 pts

- **+ 0 pts** Wrong answer/ Not attempted
- **+ 1 pt** Writing $div(F)$ formula correctly
- **+ 1 pt** div$(F) = 15(x^2 + y^2)$
- **+ 1 pt** By divergence theorem we have the required surface integral is a triple integral $\int\int\int_D div(F)dV$
- **+ 1 pt** Change of variables in the computation is correct
- **+ 2 pts** Correct computation of the triple integral is 160π
- ✔ **+ 6 pts** All correct

Question 7

Question 7

4 / 7 pts

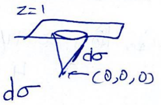
- **+ 0 pts** Wrong answer/ Not attempted
- ✔ **+ 1 pt** By Stokes' theorem the line integral is equal to the surface integral $\int\int_S Curl(F).\overrightarrow{n}d\sigma$
- ✔ **+ 2 pts** Curl$(F) = (xz, -yz, x^2 + y^2)$
- ✔ **+ 1 pt** Computation of $\overrightarrow{n}$ is correct
- **+ 1 pt** Change of variables in the computation is correct
- **+ 2 pts** Correct computation of the surface integral is $\int\int_S Curl(F).\overrightarrow{n}d\sigma = \frac{\pi}{2}$
- **+ 7 pts** All correct

Name: SHAURYA JOHARI Roll: 230959

Question 7. Consider the surface $S = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2,\ 0 \le z \le 1\}$ and let $F(x, y, z) = (\sin x - \frac{y^3}{3},\ \cos y + \frac{x^3}{3},\ xyz)$ be the function defined on $\mathbb{R}^3$. Use Stokes' Theorem to evaluate the line integral $\int_C F.dR$ where C is the boundary curve of S. [7 Marks]

Answer :

A/c to Stokes Thm,



$$\oint_C \bar{F}\cdot dR = \iint_{S} (\vec{\nabla}\times\vec{F})\cdot\bar{n}\, d\sigma$$

S: Region containing all pts of surface

$d\sigma = r_1 dr_2 d\theta$

$r_2 = r_1\cos\theta$

$\bar{F} = M\hat{i} + N\hat{j} + P\hat{k}$

$$\nabla\times\bar{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$\Rightarrow \bar{\nabla}\times\bar{F} = xz\hat{i} - yz\hat{j} + (x^2+y^2)\hat{k}$$

Unit normal vector $\bar{n}$ (Outward) For $S : x^2+y^2-z^2=0$:

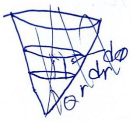
$$\frac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{\sqrt{(2x)^2+(2y)^2+(-2z)^2}} = \frac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$ on curved surface

& $\hat{k}$ for Top Base

Parameterize S ⇒ $x: r\cos\theta$, $y: r\sin\theta$, $z: r$ $\theta\in[0,2\pi]$, $r\in[0,1]$

$$\nabla\times F\cdot\hat{n} = -\frac{2y^2z}{\sqrt{x^2+y^2+z^2}}$$

$d\sigma =$ ~~$r\cos\phi\, dr\, d\theta$~~



$d\sigma = r\,dr$ ~~$\cos\theta\, d\sigma$~~

$$\iint (\bar{\nabla}\times\bar{F})\cdot\bar{n}\, d\sigma = \iint_S \frac{2y^2z}{\sqrt{x^2+y^2+z^2}}\, d\sigma =$$

$$\iint \frac{}{\sqrt{2}r}$$

End Semester Examination
MTH112M/MTH101A-Part II

Date : 19/11/2023 | Time : 5:30 - 7:15 pm | Total Marks : 40

NAME : SHAURYA JOHARI	ROLL : 230959

INSTRUCTIONS: (Read carefully)

- Please enter your NAME in CAPITAL LETTERS and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 8 pages with 7 questions. In case you have received a wrongly printed or missing pages question paper, ask for the replacement immediately.
- Answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booklet.

Name: SHAURYA JOHARI Roll: 23095, 9

Question 1. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y-y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Check whether the functions $f(x,y)$ and $f_y(x,y)$ are continuous at $(0,0)$? [2+3=5 Marks]

Answer :

$f(x,y)$ is continuous at $(0,0)$ if $\forall\, h,k \in \mathbb{R}$, $\lim_{(h,k)\to(0,0)} |f(h,k) - f(0,0)| = 0$

$$f(h,k) = \frac{h^2k - k^3}{h^2+k^2} = \frac{k(h^2-k^2)}{h^2+k^2}$$

$\therefore$ $\lim_{(h,k)\to(0,0)} |f(h,k) - f(0,0)| = \lim_{(h,k)\to(0,0)} \left| \frac{k(h^2-k^2)}{(h^2+k^2)} \right|$

$\leq \lim_{(h,k)\to(0,0)} |k|$

$\therefore$ $0 \leq |f(h,k)| \leq |k|$ And as $k \to 0$, $|k| \to 0$

$\therefore$ By Sandwich thm, $\lim_{(h,k)\to(0,0)} |f(h,k)| = 0 = f(0,0)$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$

$f_y(x,y)$ ~~is conti~~ $= \frac{d}{dy} f(x,y)$

$f_y(x,y)$ is cont. at $(0,0)$ if $\lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$ exists

$$f(0,k) = \frac{0^2k - k^3}{0^2+k^2} = -k \Rightarrow$$

$\lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = -\frac{k}{k} = -1$, It exists & is real

$\therefore$ $f_y(x,y)$ is continuous at $(0,0)$

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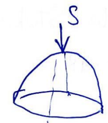
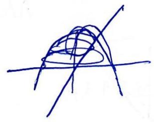
Question 6. Let $D = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq 4 - x^2 - y^2\}$ and S be the boundary surface of D. Consider $F(x,y,z) = (5x^3 + z^{19},\ 5y^3 + xe^{11},\ x + \cos xy^{23})$ defined on $\mathbb{R}^3$. Evaluate the surface integral $\iint_S F.\vec{n}\, d\sigma$ where $\vec{n}$ is the outward unit normal to the surface S. [6 Marks]

Consider S_2: $x^2+y^2+z=4$, $4 \geq z \geq 0$





Consider surface S_1: $x^2+y^2=4$, $z=0$

$S = S_1 \cup S_2$ is the surface that bounds D

Applying divergence thm,

$$\iint_{S_2 \cup S_1} \bar{F}\cdot\bar{n}\, d\sigma = \iiint_D \nabla\cdot\bar{F}\, dV$$

$$\operatorname{div}(F) = \nabla\cdot\bar{F} = 15x^2 + 15y^2$$

$dV = \pi r^2 dz$. Parameterization:

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Jacobian $= r$

$$\iiint_D 15(x^2+y^2)\, dV = \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} 15r^2 \cdot r\, dz\, dr\, d\theta$$

$$= 2\pi\left(\int_0^2 15r^3(4-r^2)dr\right) = 2\pi\left(\frac{60r^4}{4} - \frac{15r^6}{6}\right)\Big|_{r=0}^{r=2}$$

$2\pi(240 - 160) = 160\pi$.

~~Outward normal to S_1: $-\hat{k}$~~

~~Jacobian~~

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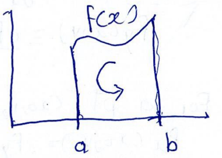
Question 5. Let $f : [a, b] \to \mathbb{R}$ be a real valued smooth function with $f(x) > 0$ for all $x \in [a, b]$. Consider the closed region $D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ 0 \leq y \leq f(x)\}$ in xy-plane. Let C be the boundary curve of D oriented counterclockwise. Prove that the line integral $\frac{1}{2}\int_C (-ydx + xdy) = \int_a^b f(x)dx$. [5 Marks]

Answer :

$M = -y \quad N = x$



M & N are 2 continuous & Differentiable Functions over $\mathbb{R}^2$ & both functions are defined at all pts.

∴ Green's Thm can be used ⇒

$$\frac{1}{2}\oint_C Mdx + Ndy = \frac{1}{2}\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \frac{1}{2}\iint_D (1+1)\, dx\,dy = \iint_D dx\,dy$$

$\iint_D dx\,dy$ Represents ~~closed integ~~ area of Region D (In integral form)

5

Area of D can be thought of as sum of area of infinite smaller strips of width dx s.t. a strip at $x = x_0$ will have height $f(x_0)$

$$\therefore \iint_D dx\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} \left(\int_0^{f(x_i)} dy\right) \Delta x_i$$

$$= \lim_{N\to\infty} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

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Question 2. Let $R : (a, b) \to \mathbb{R}^2$ be a smooth map with $R'(t) \neq (0, 0)$ for all $t \in (a, b)$ and let C be the smooth curve in xy-plane defined by $C = \{R(t) : t \in (a, b)\}$. Find the equation of the normal line to the curve C at $R(t)$. Prove that if all the normal lines to the curve C pass through a fixed point $(p, q) \in \mathbb{R}^2$ then the curve C is contained in a curve of constant curvature. [2+3=5 Marks]

Answer : Tangent to the curve at C : R(t):

$$\frac{dR(t)}{dt} = T(t) \qquad R(t) = p(t)\hat{i} + q(t)\hat{j}$$

$$T(t) = p'(t)\hat{i} + q'(t)\hat{j}$$

And Normal ~~line~~ vector to the curve C at R(t) $= \hat{k} \times T(t)$ ⇒

Eq^n of Normal line:

$R(t) + \lambda\left(\hat{k} \times \frac{dR(t)}{dt}\right)$ For $\lambda \in \mathbb{R}$

⇒ Normal line: $(p(t) - \lambda q'(t))\hat{i} + (q(t) + \lambda p'(t))\hat{j}$

Let it pass thro (p, q) always.

$\therefore \lambda\left(\hat{k} \times \frac{dR(t)}{dt}\right) = p\hat{i} + q\hat{j} - R(t)$

∴ ~~$(p\hat{i}+q\hat{j} - R(t))\cdot(p\hat{i}+q\hat{j} - R(t))$~~ =

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Question 3. Let $S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + 2xy - 2y\}$ be a surface in $\mathbb{R}^3$. Find a point on S at which the tangent plane is parallel to the xy plane. [5 Marks]

Answer : Let $f(x,y) = x^2 + 2xy - 2y$ $\therefore z = f(x,y) = f$

If $z_0 = f(x_0, y_0)$, then equation of

Tgt plane: $z - z_0 = (x - x_0)\frac{\partial f(x_0,y_0)}{\partial x} + (y-y_0)\frac{\partial f(x_0,y_0)}{\partial y}$

Tgt plane is $\perp$ to

Vector $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} - \hat{k}$.

If plane is $\parallel$ to xy plane, then its $\perp$ to $\pm\hat{k}$ $\Rightarrow$

$\therefore$ $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ At some x_0, y_0 for tgt plane

$\frac{\partial f}{\partial x}(x_0,y_0) = 2x_0 + 2y_0 = 0 = 2x_0 - 2 = \frac{\partial f}{\partial y}(x_0, y_0)$

$\therefore$ $x_0 = 1$, $y_0 = -1$

$f(1,-1) = 1 - 2 + 2 = 1 = z_0$

$\therefore$ Pt where tgt plane is parallel to X-axis: $(1,-1,1)$

Tgt plane: $z = 1$

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Question 4. Consider $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \text{ and } x + y \leq 2\}$. Find the absolute maximum of $f : D \longrightarrow \mathbb{R}$ defined by $f(x, y) = x^3 + y^2 - xy - x$. [7 Marks]

Answer : $f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = 3x^2 - y - 1$

$f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = 2y - x$

For a pt (x_0, y_0) to be absolute max OR min,

$f_x(x_0,y_0) = f_y(x_0,y_0) = 0$

$\therefore$ $2y_0 - x_0 = 0 \Rightarrow y_0 = \frac{x_0}{2}$ & $3x_0^2 - \frac{x_0}{2} - 1 = 0$

$6x_0^2 - x_0 - 2 = 0$

$(2x_0+1)(3x_0-2) = 0 \Rightarrow x_0 = -\frac{1}{2}$ OR $\frac{2}{3}$

$\therefore$ $x_0 > 0$, $x_0 = \frac{2}{3}$ & $y_0 = \frac{1}{3}$ could be our pt.

$f_{xx}(x_0,y_0) = 6x_0 = 4 > 0$; $f_{yy}(x_0,y_0) = 2 > 0 \Rightarrow$

$f_{xy} = -1$, $f_{yx} = -1$ And $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 7 > 0$

At $\frac{2}{3}, \frac{1}{3} \Rightarrow (x_0, y_0)$ is a pt of local min.

REJECTED

Boundary analysis: $f(0,0) = 0$

$f(2,0) = 8$ $f(0,2) = 4$

$f_x(x,0) = 3x^2 - 1 \Rightarrow f(x,0)$ increases from $[\frac{1}{\sqrt{3}}$ to $2]$

$f_y(0,y) = 2y \Rightarrow f(0,y)$ increases from $(0,0)$ to $(0,2)$

Along $x+y=2$: $f(x, 2-x) = x^3 + (2-x)^2 - x(2-x) - x = x^3 + 2x^2 - 7x + 4$

$f_x(x,y) = 3x^2 + 4x - 7 > 0$ When $x > 1$ $\Rightarrow$

$f(x,y)$ Decreases from $(0,2)$ to $(1,1)$ & increases from $(1,1)$ to $(2,0)$

$\therefore$ Absolute Maximum $= 8$ At $(2,0)$