End Sem SET B

Graded

Student

SHAURYA JOHARI

Total Points

15 / 26 pts

Question 1

Question B.1.

Resolved **1** / 7 pts

+ 7 pts Completely correct

+ 0 pts Completely wrong/not attempted/no substantial progress

+ 4 pts Part (i) completely correct

+ 2 pts Showing inner product of odd and even function is zero

✔ **+ 1 pt** Showing a function is sum of odd and even function

+ 1 pt Showing $W^\perp$ is equal to the subspace of even functions

+ 3 pts Part (ii) completely correct

+ 2 pts Part (ii) almost correct

+ 0 pts Click here to replace this description.

+ 0 pts Click here to replace this description.

Regrade Request **Submitted on: Feb 28**

I've mentioned that a function can be represented as a linear combination of odd and even function and showed that inner product space of g(x) with an odd function need not be zero, taking an example, if coefficient of odd function in linear combination is non zero . I also showed that inner product of f(x) with g(x) as 0 implies that g(x)=g(-x), thus implying g(x) is an even function

1 mark given.

Reviewed on: Feb 29

Question 2

Question B.2.

7 / 7 pts

- ✔ **+ 7 pts** Completely correct
- **+ 5 pts** The whole solution is correct provided there are few computation errors
- **+ 0 pts** Completely wrong/not attempted/no substantial progress
- **+ 4 pts** Part (i) completely correct
- **+ 1 pt** First two vectors computed correctly using Gram-Schmidt
- **+ 1 pt** Third or fourth vector computed correctly using Gram-Schmidt
- **+ 1 pt** Some steps correct while finding the orthogonal matrix using the vectors obtained by Gram-Schmidt
- **+ 3 pts** Part (ii) completely correct
- **+ 1 pt** Providing some correct steps to find the matrix R
- **+ 2 pts** R is computed correctly even if there is a possible error in computing Q.

Question 3

Question B.3.

4 / 6 pts

+ **6 pts** Completely correct

+ **0 pts** Completely wrong/not attempted/no substantial progress

✔ + **2 pts** Finding two vectors of orthogonal basis of W correctly using Gram-Schmidt

3

✔ + **1 pt** Finding the last vector of an orthogonal basis of W correctly using Gram-Schmidt

4

✔ + **1 pt** Writing the projection formula correctly

2

+ **2 pts** The projection computed correctly except one coordinate w.r.t. standard basis, i.e., $\frac{1}{7}(3, 18, 25, 20)$, /orthogonal/orthonormal basis obtained from Gram-Schmidt process.

+ **2 pts** Observing the projection can be obtained by solving the equation $A^T AX = A^T b$ where the column vectors of A are $(1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 1, 1)$.

+ **2 pts** Providing some correct steps to find a solution of the above system of linear equations

+ **1 pt** The solution of the above system of linear equations is correct

+ **1 pt** Observing $(1, -1, -1, 2)$ is orthogonal to W

+ **2 pts** Finding the equation of the hyperplane corresponding to the subspace W

+ **2 pts** Writing $P_W(b)$ correctly in terms of the normal to W

+ **1 pt** Finding the projection point correctly

1 Incomplete calculations.

Question 4

Question B.4.

3 / 6 pts

+ **6 pts** Completely correct

+ **0 pts** Completely wrong/not attempted/no substantial progress

+ **3 pts** Part (i) completely correct

✔ + **0 pts** Part (i) completely wrong

✔ + **3 pts** Part (ii) completely correct

+ **0 pts** Part (ii) completely wrong

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Question B.4. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Show the following:
(i) All eigenvalues of A are real.
(ii) Eigenvectors of A corresponding to distinct eigenvalues are orthogonal to each other. [3+3=6 Marks]

Answer B.4. :

ii ans Let (λ_1, v_1) & (λ_2, v_2) Be pairs of eigenvalues- eigenvectors wrt A s.t. $\lambda_1 \neq \lambda_2$

$$A v_1 = \lambda_1 v_1 \quad \& \quad A v_2 = \lambda_2 v_2 \in$$

(λ_i, v_i) are all possible n pairs of vectors with values

Let $C = [v_1, v_2 \dots v_n]$ be $n \times n$ Matrix ~~wrt~~ with colums v_i. You

$\cancel{A C = C \operatorname{Diag}(\lambda_1, \lambda_2 \dots \lambda_n)}$ $\quad AC = C \operatorname{Diag}(\lambda_1 \dots \lambda_n)$

$$v_2^T A v_1 = \lambda_1 v_2^T v_1 . \quad (v_2^T A v_1)^T = v_1^T A^T v_2 = v_1^T A v_2 = \lambda_2 v_1^T v_2 .$$

$\therefore$, ~~We have~~ $\because$ Dimension of $v_2^T A v_1 = 1 \times 1$ We have $\lambda_1 v_2^T v_1 = \lambda_2 v_1^T v_2$ i.e. possible iff $v_2^T v_1 = v_1^T v_2 = 0$ OR $\langle v_1, v_2 \rangle = 0$.

i ans. $(Av)^H = (\lambda v)^H \Rightarrow$

$$v^T A^H = v \bar{\lambda} \Rightarrow$$

$$\lambda = \bar{\lambda} \quad \because v^T, A^H = A$$

Are all Real

$\Rightarrow$ E.val & vectors are real

End-Semster Exam MTH113M/MTH102A : SET B

Date : 19/02/2024 | Time : 6:00- 8:00 pm

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Question B.1. Consider the vector space $C[-1, 1]$ of all real valued continuous functions defined over $[-1, 1]$. Define the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$, where $f, g \in C[-1, 1]$. Let $W = \{f \in C[-1, 1] : f \text{ is an odd function}\}$.
(i) Show that the orthogonal complement $W^\perp$ of W is the subspace of all even functions in $C[-1, 1]$.
(ii) Let $f(x) = e^x$, where $x \in [-1, 1]$. Find $\inf\{\|f - g\| : g \in W\}$. [4+3=7 Marks]

Answer B.1.:

Every function is a sum of odd & even function on $\mathbb{R}^2$.

i ans. Let $g(x) \in W^\perp$ Then.

$$\langle f, g \rangle = 0 \Rightarrow \int_{-1}^{1} f(x) g(x) dx$$

$$\Rightarrow \int_a^b f(x) dx = \int_a^b f(a+b-x) dx .$$

$$\therefore 0 = \int_{-1}^{1} f(x) g(x) dx = \int_{-1}^{1} f(-x) g(-x) dx = -\int_{-1}^{1} f(x) g(-x) dx .$$

$$\int_{-1}^{1} f(x) (g(x) - g(-x)) dx = 0 .$$

If $g(x) = \cancel{a(x) + a(-x)} + g(-x)$, Then above integral is always 0.

Else, if $g(x) = A p(x) + B q(x)$ p & q: Odd & Even function $A \neq 0$.

$\int_{-1}^{1} A f(x) p(x) dx = \int_{-1}^{1} A f(-x) p(-x) dx$ & Thus need not be 0. ($f(x) = x^3$, $p(x) = x$ give, No 0)

$\Rightarrow \therefore A = 0$. & $g(x)$ is even function $\in W^\perp$.

ii ~~ans~~

$$\|f - g\|^2 = \int_{-1}^{1} |f(x) - g(x)|^2 dx$$

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} \dots \frac{x^n}{n!} \Rightarrow$$

For min values $g(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} \dots$

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Question B.2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$. [4+3=7 Marks]

$u_1\ u_2\ u_3\ u_4$

Gram ~~Sch~~ r

Answer B.2.:

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{21} & a_{24} \\ 0 & 0 & a_{33} & a_{24} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

~~Otho~~ Orthogonal basis of columns of A;

1st 2 Basis:

$$\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}_{w_1} \ \& \ \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}}_{w_2} \qquad (\langle u_1 u_2 \rangle = 0)$$

Gram schmidt Orthogonalization Proces

$$v_3 = u_3 - \langle u_3 w_1 \rangle w_1 - \langle u_3 w_2 \rangle w_2 = \begin{bmatrix}1/2\\-1\\1/2\\1\end{bmatrix} \Rightarrow \cancel{w_3 = \sqrt{\tfrac{2}{5}}\begin{bmatrix}1/2\\-1\\1/2\\1\end{bmatrix}} \quad w_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$$

Likewise. $w_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} \Rightarrow$ Orthogonal matrix Q: $\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1\\0&1&-1&0\\1&0&0&-1\\0&1&1&0\end{bmatrix}$

$$A = QR \Rightarrow Q^{-1}A = R = Q^{T}A \Rightarrow$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&-1&0&1\\1&0&-1&0\end{bmatrix}\begin{bmatrix}1&0&1&2\\0&1&-1&1\\1&0&1&1\\0&1&1&1\end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}2&0&2&3\\0&2&0&2\\0&0&2&0\\0&0&0&1\end{bmatrix}}_{\text{Diagonal matrix } R}$$

Orthogonal matrix:

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1\\0&1&-1&0\\1&0&0&-1\\0&1&1&0\end{bmatrix}$$

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Question B.3. Consider the point $b = (1, 2, 3, 4) \in \mathbb{R}^4$. Let W be the linear span of the set $\{(1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 1, 1)\}$ in $\mathbb{R}^4$. Find the orthogonal projection $P_W(b)$ of b on the subspace W. [6 Marks]

$u_1 \quad u_2 \quad u_3$

Answer B.3.:

Consider w_1, w_2 & w_3 to be orthogonal basis of W. $\|w_i\| = 1$ & $\langle w_i, w_j \rangle = 0$ For $1 \le i, j \le 3$, $i \ne j$

$\|v_1\| = \left\|\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\| = \sqrt{2} \Rightarrow$ Orthogonal Base $w_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$

$\left\|\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\| = \sqrt{2} \Rightarrow$ $v_2 = u_2 - \langle u_2 w_1 \rangle w_1$ (Gram Schmidt Orthogonalization Proces)

$$v_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \left\langle \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\rangle \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix}$$

$$\|v_2\| = \sqrt{3/2} \Rightarrow w_2 = \begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix} \cdot \sqrt{\frac{2}{3}}$$

$$v_3 = u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2 = \begin{bmatrix}-7/12\\13/12\\4/12\\12/12\end{bmatrix} \begin{bmatrix}-2/3\\2/3\\2/3\\1\end{bmatrix}$$

$$\cancel{w_3 = \frac{v_3}{\|v_3\|} = \frac{12}{\sqrt{378}}\begin{bmatrix}-7/12\\13/12\\4/12\\12/12\end{bmatrix}} \quad w_3 = \sqrt{\tfrac{3}{7}}\begin{bmatrix}-2/3\\2/3\\2/3\\1\end{bmatrix}$$

Projection of b on W $= \langle b\, w_1 \rangle w_1 + \langle b\, w_2 \rangle w_2 + \langle b\, w_3 \rangle w_3$

$$= \frac{3}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \frac{5}{3}\begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix} + \frac{20}{7}\begin{bmatrix}-2/3\\2/3\\2/3\\1\end{bmatrix}$$