MTH114-End-Sem-PartA

Graded

Student

SHAURYA JOHARI

Total Points

21 / 25 pts

Question 1

Question1

7 / 7 pts

- ✔ **+ 1 pt** Written name in capital + cleanly+ inside box
- ✔ **+ 6 pts** Fully Correct
- **+ 0 pts** Incorrect
- **+ 1 pt** Characteristic equation of the homogeneous part : $m^2 + 2m + 1 = 0$, so $m = -1, -1$
- **+ 1 pt** Two independent solution of the homogeneous part $y_1 = e^{-x}, \quad y_2 = xe^{-x}$.
- **+ 1 pt** Wronskian $W(y_1, y_2) = e^{-2x}$.
- **+ 1 pt** Particular integral $y_p = u(x)y_1 + v(x)y_2$.

  $u(x)' = -\frac{y_2 r}{W} = -1$ so $u(x) = -x$
- **+ 1 pt** $v(x)' = \frac{y_1 r}{W} = 1/x$ so $v(x) = \ln(x)$.
- **+ 1 pt** Found PI and hence the general solution correctly

Question 2

Question2

6 / 6 pts

- ✔ **+ 6 pts** Fully Correct
- **+ 0 pts** Incorrect/Not Answered
- **+ 1 pt** Taken correct comparing equation
- **+ 1 pt** Applied Sturm Comparison properly
- **+ 1 pt** 2nd part: Taken correct interval $[x_1, x_1 + \pi]$
- **+ 1 pt** Applied first part correctly.
- **+ 2 pts** third part: proper correct argument
- **+ 1 pt** argued that $q(x) \to 1$ as $x \to \infty$ and the roots of $v'' + v = 0$ are π apart. Hence $x_2 - x_1 \to \pi$.

Question 3

Question3

4 / 6 pts

+ **3 pts** 3a. fully Correct

+ **0 pts** 3a. Incorrect

+ **2 pts** 3a. Argued properly LC not holds on $\mathbb{R}^2$

✔ + **1 pt** 3a. Argued continuous everywhere

✔ + **3 pts** 3b. Fully correct

+ **2 pts** 3b. Argued LC is satisfied

+ **0 pts** 3b. Incorrect

+ **1 pt** 3b. Argued that discontinuous at integer points

1 asked for R^2. where it does not satisfy LC

Question 4

Question4

4 / 6 pts

+ **3 pts** 4a. fully Correct

+ **0 pts** 4a. Incorrect

✔ + **1 pt** 4a. Calculated wronskian for $x \neq 0$

+ **1 pt** 4a. Calculated wronskian for x =0

+ **1 pt** 4a. Argued properly they are independent

✔ + **3 pts** 4b. Fully correct

+ **0 pts** 4b. Incorrect

+ **1 pt** 4b. Written p(x), q(x), r(x) correctly

+ **1 pt** 4b. Written the discontinuity points correctly

+ **1 pt** 4b. Written the interval correctly

4. (a) Let $f(x)=x^2|x|$ and $g(x)=x^3$, $x \in I=[-1,1]$. Calculate the wronskian $W(f,g)(x)$. Are f,g linearly independent over I ?

(b) Find the largest interval on which a unique solution is guaranteed to exist of the IVP,

$$(x+2)y''+xy'+\cot(x)y=x^2+1,\quad y(2)=11,\quad y'(2)=-2.$$

[3+3]

Answer: $f(x)=x^2|x| = -x^3 \quad x<0$

$= x^3 \quad x>0$

$f'(x) = -3x^2 \quad x<0 \Rightarrow$ Thus $f'(x) = 3x|x|$

$3x^2 \quad x>0$

$g'(x)=3x^2.$

$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} x^2|x| & x^3 \\ 3x|x| & 3x^2 \end{vmatrix} = 3x^4|x| - 3x^4|x| = 0$

$\therefore W(f,g) = 0 \;\forall\; x \in [-1,1]$

If ~~W(f~~ f & g were linearly independent Then $W(f,g) \neq 0$ Over I, which is false.

Thus, f & g are Linearly Dependent.

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ii ans If $y''+p(x)y'+q(x)y = r(x)$ is our ODE if $p(x), q(x)$ & $r(x)$ are continuous & well bounded in (a,b), Then $\exists$ a unique sol^n to a IVP with $y(c)=d$ & $y'(c)=e$ $(c \in (a,b))$ Only within (a,b)

ii ans $y''+\frac{x}{x+2}y'+\frac{\cot x}{x+2}y = \frac{x^2+1}{x+2}$ $\quad p(x)=\frac{x}{x+2}$ $\quad q(x)=\frac{\cot x}{x+2}$ $\quad r(x)=\frac{x^2+1}{x+2}$

$\therefore$ Points of singularity: $x=-2$ And $x=n\pi \;\; n\in\mathbb{Z}$

Unique sol^n to IVP exist over an interval where $p(x)$ & $q(x)$ is continuous & bounded

$(x_0=2)$: Greatest interval containing $x_0=2$: $(0,\pi)$

$\therefore$ Largest interval For sol^n: $(0,\pi)$

ODE : MTH 114M : End-Sem: Part-A

Name: SHAURYA JOHARI   Roll No: 230959

- Write your name cleanly in CAPITAL letters and roll number inside the boxes (1 mark).
- Write answers in the space provided only.   Total marks: 25   Time: 5:00 pm - 6:30 pm.

1. Find the general solution using variation of parameters, $y''+2y'+y=\frac{e^{-x}}{x}$, $x>0$. [6]

Answer: (First Let RHS=0 For Homo sol^n ~~Homogeneous~~ y

Characteristic Polynomial: $m^2+2m+1=0$

$m=-1$ & Double Roots $\Rightarrow$

Solution of homogeneous equation: $Ae^{mx}+Bxe^{mx}$

$= Ae^{-x} + Bxe^{-x} = Ay_1 + By_2 = y_h$

Let particular sol^n to Given ODE be $y_p = uy_1 + vy_2$

$r(x) = e^{-x}/x$ & $W(y_1,y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$

$= (1-x)e^{-2x} + xe^{-2x} = e^{-2x}.$

$u = \int -\frac{y_2 r}{W}\,dx = -\int \frac{xe^{-x}\,xe^{-x}}{e^{-2x}\,x}\,dx = -x$

$v = \int \frac{y_1 r}{W}\,dx = \int \frac{e^{-x}\,xe^{-x}}{e^{-2x}\,x}\,dx = \ln x$

$y_p = (-x)e^{-x} + xe^{-x}(\ln x) = -xe^{-x} + x\ln x\, e^{-x}$

$\therefore$ General solution: $y = y_p + y_h =$

$Ae^{-x} + Bxe^{-x} - xe^{-x} + xe^{-x}\ln x =$

$C_1e^{-x} + C_2xe^{-x} + xe^{-x}\ln x$

C_1 & C_2 are constants $\in \mathbb{R}$

2. Let $0 \le p < 1/2$ and consider the normal form of the Bessel equation $y'' + (1 + \frac{\frac{1}{4} - p^2}{x^2})y = 0,\ x > 0$. Show that any non trivial solution y_p has a zero in $(a, a+\pi)$, $a > 0$. Moreover, if $x_1 < x_2$ be consecutive positive zeros of y_p, then show that $x_2 - x_1$ is less than π and $x_2 - x_1$ approaches π as $x_1 \to \infty$.

[2+2+2]

Answer: Consider ψ to be a sol^n to given ODE &

let $\gamma(x)$ Be sol^n to $y^2 + q(x)y = 0$

Where $q(x) = 1$ & $p(x) = (1 + \frac{\frac{1}{4} - p^2}{x^2})$.

$\because$, $p(x) > q(x)\ \forall\ x > 0 \Rightarrow$ By Strum's Comparision Thm, Between 2 solutions to γ, there's solution to ψ. Choose $a \in \mathbb{R}$. $\gamma = \sin(x-a)$ is a solution to ~~given~~ $y'' + y = 0$ And between a & $a+\pi$ (0s of $\sin(x-a)$), There's a 0 to ψ.

Let $a = x_1$ Then $\because \exists$ sol^n $x_2 \in (a, a+\pi) \equiv (x_1, x_1+\pi)$

$x_2 < x_1 + \pi \Rightarrow$ $x_2 - x_1 = \pi$

Consider $\mathcal{N}$ to be a sol^n to $y'' + (1+\epsilon)y = 0$

Where $\epsilon \in \mathbb{R}$ & $0 < \frac{\frac{1}{4} - p^2}{x^2} < \epsilon$.

$\because$, B^{tw} 2 Roots of ψ, i.e. x_1 & x_2 $\exists$ Root of $\mathcal{N}$ (Strum's Comparision Thm)

Set $\mathcal{N} = \sin((\sqrt{1+\epsilon})(x - x_1))$

Root ~~Roftter~~ after x_1 For $\mathcal{N}$: $x_1 + \frac{\pi}{\sqrt{1+\epsilon}} \in (x_1, x_2)$

$x_1 + \frac{\pi}{\sqrt{1+\epsilon}} < x_2 \Rightarrow$ $x_2 - x_1 > \frac{\pi}{\sqrt{1+\epsilon}}$

As $\epsilon \to 0$, $\frac{\frac{1}{4} - p^2}{x_1^2} \to 0 \Rightarrow x_1 \to \infty$ & $x_2 - x_1 \to \pi$ As $x_1 \to \infty$

Name: SHAURYA JOHARI Roll No: 230959 **PartA**

3. Explain your answers. Here LC = Lipschitz Condition and $[x]$ = the greatest integer $\le x$.

(a) Let $f(x,y) = xy$. Does f satisfies LC on $\mathbb{R}^2$? Yes Is f continuous on $\mathbb{R}^2$? Yes

(b) Let $g(x,y) = y + [x]$. Does g satisfies LC on $\mathbb{R}^2$? Yes Is g continuous on $\mathbb{R}^2$? No

x_0 is any integer & x_p is any x.

[3+3]

Answer: Function $f(x,y)$ satisfy Lipschitz (LC) condition on $[a,b] \times [c,d]$ if $\forall\ y_1, y_2 \in (c,d)$ $\exists$ $M \in \mathbb{R}^+$ s.t.

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} \le M. \quad (y_1 \ne y_2)$$

A ans. $f(x,y) = xy \Rightarrow f(x_0,y_0) = x_0 y_0$ & $\lim_{\substack{h \to 0 \\ k \to 0}} f(x+h, y+k) =$

$\lim_{(h,k)\to(0,0)} (x_0+h)(y_0+k) = x_0 y_0 + \lim_{(h,k)\to(0,0)} (hy_0 + kx_0 + hk) = x_0 y_0 + C$

$f(x,y)$ is continuous over $\mathbb{R}^2$

Also,

$$\frac{|f(x_0,y_1) - f(x_0,y_2)|}{|y_1 - y_2|} = \frac{|x_0(y_1 - y_2)|}{|y_1 - y_2|} = |x_0| \le M$$ For some $M \in \mathbb{R}^+$

$\therefore$ (We can always find $M \in \mathbb{R}^+$ s.t. $M \ge |x_0|$).

f satisfies LC

B ans. $g(x,y) = y + [x]$ $\quad \lim_{\epsilon \to 0} f(x_0 - \epsilon, y) = y + x_0 - 1$

$(x_0 \in \mathbb{Z})$

But $\lim_{\epsilon \to 0} f(x_0 + \epsilon, y) = y + x_0 \ne \lim_{\epsilon \to 0} f(x_0 - \epsilon, y)$

g isn't continuous over $\mathbb{R}^2$ & discont. when $x = x_0$

Also Fixing x at x_p, $\frac{|f(x_p,y_1) - f(x_p,y_2)|}{|y_1 - y_2|} =$

$$\left| \frac{[x_p] + y_1 - [x_p] - y_2}{y_1 - y_2} \right| = 1$$ When $y_1 \ne y_2$ & $1 < 2 \Rightarrow$ Value is Bounded

$g(x,y)$ Satisfies LC But ~~is discontinuous~~ isn't cont over $\mathbb{R}^2$ ~~When~~ At $x \in \mathbb{Z}$