Quiz: MTH111M (2023-2024 I)

Date: 30 August 2023 Time: 07:30 pm-08:10 pm Maximum marks: 30

Name: ______________________ **Roll No.** ____________

Instructions: (Read carefully)

- Please enter your NAME and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 6 pages. Check to see if the print is either faulty or missing on any of the pages. In such a case, ask for a replacement immediately.
- Please answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booket.

Q 1. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n^2}$ for all $n \in \mathbb{N}$. Show that (x_n) converges. (5)

Soultion:

$$|x_{n+1} - x_n| = |\frac{1}{2+x_n^2} - \frac{1}{2+x_{n-1}^2}| \qquad [1]$$

$$= \frac{|x_{n-1}^2 - x_n^2|}{(2+x_n^2)(2+x_{n-1}^2)} \leq \frac{|x_n - x_{n-1}||x_{n+1} + x_n|}{(2+x_n^2)(2+x_{n-1}^2)} \qquad [1]$$

$$\leq \frac{2}{4}|x_n - x_{n-1}| \qquad [3]$$

Hence, (x_n) satisfies the Cauchy criterion and is therefore convergent.

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Q 2. Find the number of distinct real solutions of the equation $4x^2 - \cos x - \sin 2x = 0$.
(*Do not use graphs to justify your answer*). (6)

Solution:

Let $f(x) = 4x^2 - cos\ x - sin\ 2x$. Then

$$f'(x) = 8x + sin\ x - 2cos\ 2x$$

$$f''(x) = 8 + cos\ x + 4sin\ 2x > 0.$$ [2]

By Rolle's Theorem, we conclude that $f'(x) = 0$ has at most one real solution and hence $f(x) = 0$ has at most two real solutions. [2]

We observe that $f\big(-\frac{\pi}{2}\big) > 0$, $f(0) < 0$ and $f\big(\frac{\pi}{2}\big) > 0$.

Therefore, by the IVT, $f(x) = 0$ has at least two real solutions. [2]

Hence $f(x) = 0$ has exactly two solutions.

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Q 3 (a) Let $f(x) = \cos\frac{1}{x} + \sin\frac{1}{x}$ for all $x \neq 0$ and $f(0) = 0$. Show that f is not contiunous at 0.

(b) Let $f : [a, b] \to \mathbb{R}$ be differentiable and $f'(x) = 0$ for all $x \in [a, b]$. Show that f is a constant function. (4+3)

Solution:

(a)

Consider the sequence $(x_n) = (\dfrac{1}{2n\pi})$. [2]

Observe that $f(x_n) = 1$, for all n. As, $x_n \to 0$, we get $f(x_n) \to 1 \neq f(0)$. [2]

Therefore, f is not continuous at 0.

(b)

Let $x_0 \in [a, b]$ and $x \in [a, b] \setminus \{x_0\}$.

By the mean value theorem, there exists $c \in (x, x_0)$ (or (x_0, x)), such that
$f(x) - f(x_0) = f'(c)(x - x_0) = 0$.

Hence $f(x) = f(x_0)$. [3]

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Q 4 (a) Let $x, y \in \mathbb{R}$. Show that $(|x|^n + |y|^n)^{\frac{1}{n}} \to M$ as $n \to \infty$, where M is the maximum of $|x|$ and $|y|$.

(b) Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. Suppose that f is differentiable on $[a, b]$ and $\lim_{x \to x_0} f'(x)$ exists. Show that $f'(x_0) = \lim_{x \to x_0} f'(x)$. (4+3)

(a) Observe that,

$$(M^n)^{\frac{1}{n}} \leq (|x|^n + |y|^n)^{\frac{1}{n}} \leq (2M^n)^{\frac{1}{n}}$$

[2]

By sandwich theorem and the fact that $2^{\frac{1}{n}} \to 1$,

$$(|x|^n + |y|^n)^{\frac{1}{n}} \to M$$

[2]

(b) Since f is contiuous at x_0, $\lim_{x \to x_0} [f(x) - f(x_0)] = 0$ and hence by L'Hospital's Rule,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} f'(x).$$

[3]

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Q 5. Let $f : [0,4] \to \mathbb{R}$ be a continuous function such that $f(0) = f(4)$. Show that there exists $x_0 \in [0,2]$ and y_0 such that $x_0 + 2 \leq y_0 \leq 4$ and $f(y_0) = \frac{f(x_0)+f(0)}{2}$. (5)

Solution:

Define a function $g : [0,2] \to \mathbb{R}$ by $g(x) = f(x+2) - f(x)$. [1]

Here, $g(0) = f(2) - f(0)$ and $g(2) = f(4) - f(2) = f(0) - f(2)$. Hence $g(0) = -g(2)$.

Therefore, by the Intermediate Value Theorem, there exists $x_0 \in [0,2]$ such that $g(x_0) = 0$.

Thus, $f(x_0) = f(x_0 + 2)$. [2]

Now

$$\frac{f(x_0) + f(0)}{2} = \frac{f(x_0 + 2) + f(4)}{2}.$$

Using the IVT, there exists $y_0 \in [x_0 + 2, 4]$ such that $f(y_0) = \dfrac{f(x_0 + 2) + f(4)}{2}$. [2]