METRICS ON TREES II. MEASURED LAMINATIONS AND CORE ENTROPY

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1. Introduction

In this paper, we shall introduce a new interpretation of the core entropy for polynomial maps in terms of metrics on trees and transverse measures to laminations.

In the well-known Sullivan dictionary, there is a parallel between the dynamics of Kleinian groups and the dynamics of complex polynomials, both seen as dynamical systems on the Riemann sphere.

Measured laminations and measured foliations play a well-known role in Teichmüller theory [15], [8], [9].

W. Thurston also introduced invariant laminations for quadratic polynomials in [16]. The theory for polynomials of higher degree is still less established [17].

However, in Thurston's work on quadratic polynomials there does not appear to be any transverse measure. Part of the goal of this paper is to introduce a notion of transverse measure on the lamination associated to a complex polynomial.

Further, recall that Thurston's algorithm for constructing a rational map with given combinatorial data is obtained by iteration in Teichmüller space. The main result of this part shows convergence of the iterated process to a linearly expanded transverse measure on the lamination.

Theorem 1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a quadratic polynomial, and let $\mathcal{L}$ be its associated lamination. Then:*

(1) *there exists a transverse measure m on $\mathcal{L}$, linearly expanded by the dynamics:*
$$f^\star m = \lambda m.$$
(2) *The derivative of m semiconjugates the dynamics of f on its Hubbard tree to a piecewise linear map.*
(3) *The number λ is the leading eigenvalue of the transfer operator. Moreover, it is related to the core entropy by*
$$h_{top}(f) = \log \lambda.$$
(4) *If $h(f) > \frac{1}{2}\log 2$, the metrics $m_n := (f^\star)^n m_0$ converge projectively to m.*

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2. Case II: tree maps

Recall that a *topological graph* is a topological space given by gluing finitely many segments $I_1, \dots, I_r$ along their endpoints. A *topological tree* is a connected, simply connected topological graph.

A continuous function $f : T \to T$ is *folding* at a point $x \in T$ if it is locally modeled by $x \mapsto |x|$; namely, there exist neighborhoods U of x, V of $y := f(x)$ and homeomorphisms $\varphi_1 : U \to (-\epsilon, \epsilon)$, with $\varphi_1(x) = 0$ and $\varphi_2 : V \to (-\epsilon, \epsilon)$ with $\varphi_2(y) = 0$, such that $\varphi_2(f(x)) = |\varphi_1(x)|$ for all $x \in U$. Something must be said if y is an endpoint of T

Definition 2.1. *A continuous map $f : T \to T$ of a topological tree is* multimodal *if there is a finite set $C \subset T$ such that:*

- *f is a local homeomorphism at every $x \in T \setminus C$;*
- *at any point $x \in C$, f is folding.*

We say that f is unimodal *if $\#C = 1$.*

A fundamental example for our discussion is the following.

Example. Let $f_c(z) := z^2 + c$ be a quadratic polynomial so that its Julia set is connected and locally connected. If the Hubbard tree of f_c is a finite topological tree, we say that f_c is *topologically finite*. A topologically finite quadratic polynomial acting on its Hubbard tree is an example of a unimodal tree map.

Let $f : T \to T$ be a multimodal map of a tree. An *arc* on T is an unordered pair $[x, y]$ of distinct points of T. We denote as $\mathcal{A}(T)$ the set of arcs of T.

Definition 2.2. *A* metric *on T is a functional*

$$m : \mathcal{A}(T) \to \mathbb{R}$$

which we think of as assigning a length *to each arc in the tree, and satisfies:*

(i) *m is* positive*: $m([x, y]) \geq 0$ for any $x, y \in T$;*
(ii) *m is* additive *on disjoint arcs: if y lies between x and z, then $m([x, z]) = m([x, y]) + m([y, z])$.*

Given two points x, y in T, we denote as $[x, y]$ the segment (arc) joining x and y. An arc $[x, y]$ in the tree is *separated* if x and y lie on opposite sides of the critical point. Otherwise, it is *non-separated.*

Given a finite set $x_1, \dots, x_r$ of points of T, we define its *convex hull* as $[x_1, \dots, x_r]$ the union of all segments $[x_i, x_j]$ with $1 \leq i, j \leq r$.

Let $T \setminus C = I_1 \cup \dots \cup I_d$ the union of disjoint subtrees. Note that by definition the restriction of f to any I_j is a homeomorphism onto its image.

For each arc $J \in \mathcal{A}(T)$, we let

$$(f^\star m)(J) := \sum_{k=1}^{d} m(f(J \cap I_k)).$$

The space of metrics of unit length is $\mathcal{M}^1(T) = \{m \in \mathcal{M}(T) \ : \ m(T) = 1\}$. Maybe we should define m on any subtree?

Lemma 2.3. *The space $\mathcal{M}^1(T)$ is convex and compact.*

Proof. We know that $\mathcal{M}^1(T) \subseteq [0,1]^{\mathcal{A}(T)}$, hence compactness follows from Tychonoff's theorem. $\square$

We define the iteration operator $P : \mathcal{M}^1(T) \to \mathcal{M}^1(T)$ as

$$P(m) := \frac{f^\star m}{\|f^\star m\|}.$$

Lemma 2.4. *The operator $P : \mathcal{M}^1(T) \to \mathcal{M}^1(T)$ is continuous with respect to the weak topology.*

Proof. If $m_n \to m$, then for any arc J,

$$f^\star m_n(J) = \sum_i m_n(f(J \cap I_i)) \to \sum_i m(f(J \cap I_i)) = f^\star m(J)$$

and similarly,

$$\|f^\star m_n\| = f^\star m_n(T) \to f^\star m(T) = \|f^\star m\|$$

Since f is surjective, note that

$$f^\star m(T) = \sum_i m(f(I_i)) \geq m(\bigcup_i f(I_i)) \geq m(T) = 1$$

hence $\|f^\star m\| \geq 1$. Then

$$P(m_n)(J) = \frac{f^\star m_n(J)}{\|f^\star m_n\|} \to \frac{f^\star m(J)}{\|f^\star m\|} = P(m)(J)$$

which completes the proof. $\square$

Proposition 2.5. *There exists at least one unit length metric m on T such that*

$$f^\star m = \lambda m$$

for some $\lambda \geq 1$.

Proof. By Lemma 2.3, $\mathcal{M}^1(I)$ is a non-empty, compact, convex subspace of the normed vector space

$$L^\infty(\mathcal{A}) := \left\{ m : \mathcal{A} \to \mathbb{R} \ : \ \sup_{J \in \mathcal{A}} |m(J)| < +\infty \right\}$$

and P is continuous by Lemma 3.10, hence by the Schauder fixed point theorem, there exists at least one fixed point for the operator P, which corresponds to an eigenvector for $f^\star$. $\square$

Remark 2.6. *Note that in this generality, if we do not impose any countable additivity or compactness, the eigenvector can be very weird. In particular, consider the map $f : [0,1] \to [0,1]$ defined as $f(x) = \frac{x}{1+x}$. Then we may have $m(J) = 0$ if J does not contain 0, and $m(J) = 1$ if J contains 0.*

Let N be the number of ends of T.

Proposition 2.7. *Let $f : T \to T$ be a critically finite unimodal map, with $h(f) > \frac{1}{2}\log 2$. Then the transition matrix of f has a unique, real leading eigenvalue of maximum modulus, and the eigenvalue is simple.*

Theorem 2.8. *Let f be a unimodal tree map. Then its* core entropy *$h(f)$ is related to the spectral radius of the pullback operator $f^\star$ on the space $\mathcal{M}(I)$, i.e.*

$$h(f) = \log \lambda.$$

Theorem 2.9. *Let $f : T \to T$ be a critically finite unimodal tree map, with $h(f) > \frac{1}{2}\log 2$. Then:*

- *the sequence of metrics $m_n := P^n(m_0)$* converges uniformly *to a limit m_∞.*
- *The limit metric defines a* semiconjugacy *of the dynamics $f : T \to T$ to a piecewise linear tree map of the same entropy.*

Lemma 2.10. *Let $N < \infty$ be the number of ends of a quadratic Hubbard tree T. Then for any segment $J \subseteq T$ we have either*

(1) *$\ell(f^n(J)) \leq \left(2^{\frac{N-1}{N}}\right)^n$ for any $n \geq 0$;*
(2) *or there exists n such that $f^n(J) = T$.*

Proof. Suppose that there exists n such that $f^n(J) \ni c_k$ for all $0 \leq k \leq N-1$. Then $f^{n+N}(J)$ contains $[c_1, c_2, \dots, c_N]$, which is the Hubbard tree. Otherwise, in every block of consecutive N iterates, there is at least one iterate that does not hit the critical point. Hence

$$\ell(f^N(J)) \leq 2^{\frac{N-1}{N}}$$

and by iteration we get the result. □

Remark 2.11. *Let us note that it is not true that there exists n such that $f^n([c_0, c_1]) = T$. As a counterexample, let us consider the Misiurewicz point with external angle $\theta = \frac{9}{56}$. Then we compute*

- $f([c_0, c_1]) = [c_1, c_2]$
- $f^2([c_0, c_1]) = [c_2, c_3]$
- $f^3([c_0, c_1]) = [c_1, c_3]$
- $f^4([c_0, c_1]) = [c_1, c_2]$

which then, since $f^4([c_0, c_1]) = f([c_0, c_1])$, repeats without ever covering T.

3. Case III: laminations

Hubbard trees are defined for topologically finite polynomials, and in particular their existence requires that the Julia set is path connected. In general, it is known that some Julia sets are not locally connected, so considering the Hubbard tree as a subset of the filled Julia set may present some difficulties.

On the other hand, though, for any external angle $\theta \in \mathbb{R}/\mathbb{Z}$, Thurston defines a lamination of the unit disk, invariant under the doubling map. We now see how to define a corresponding object to the previously defined metric in terms of a *transverse measure* on the lamination.

Laminations. A *leaf* $L = (\xi, \eta)$ in the Poincaré disk $\mathbb{D}$ is a hyperbolic geodesic in $\mathbb{D}$, with boundary points $\xi, \eta \in \partial\mathbb{D}$. A *lamination* $\mathcal{L}$ of the Poincaré disk $\mathbb{D}$ is a closed subset which is the union of disjoint leaves.

W. Thurston [16] constructs, for every angle $\theta \in \mathbb{T}$, an associated lamination $\mathcal{L}_\theta$ on the disc which is invariant by the doubling map.

The (two) longest leaves of the lamination are called *major leaves*, and their common image is called *minor leaf* and will be denoted by m. Moreover, we let β denote the leaf $\{0\}$, which we will take as the root of the lamination (the notation is due to the fact that the ray at angle 0 lands at the β-fixed point). The dynamics on the lamination is induced by the dynamics of the doubling map on the boundary circle. In particular, let us denote $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ to be a continuous function on the filled-in disc which extends the doubling map on the boundary S^1. Let us denote by Δ the diameter of the circle which connects the boundary points at angles $\theta/2$ and $(\theta+1)/2$. Then we shall also choose f so that it maps homeomorphically each connected component of $\mathbb{D} \setminus \Delta$ onto $\mathbb{D}$.

By mapping (ξ, η) to the corresponding point in $\mathbb{T} \times \mathbb{T}$, a lamination can be also seen as a closed subset of the 2-torus.

3.1. **Segments on laminations.** A leaf L *separates* two other leaves L_1, L_2 if $\mathbb{D} \setminus L$ has two connected components, one which contains L_1 and the other which contains L_2. Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two leaves of a lamination.

Definition 3.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two distinct leaves. Then we define the* (combinatorial) segment $[\mathcal{L}_1, \mathcal{L}_2]$ *as the set of leaves $\mathcal{L}$ of the lamination which separate $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Some simple properties of combinatorial segments are the following:

(a) if $\mathcal{L} \in [\mathcal{L}_1, \mathcal{L}_2]$, then $[\mathcal{L}, \mathcal{L}_1] \subseteq [\mathcal{L}_1, \mathcal{L}_2]$;

(b) for any choice of leaves $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ we have
$$[\mathcal{L}_1, \mathcal{L}_2] \subseteq [\mathcal{L}_3, \mathcal{L}_1] \cup [\mathcal{L}_3, \mathcal{L}_2];$$

(c) the image of $[\mathcal{L}_1, \mathcal{L}_2]$ equals:
$$f([\mathcal{L}_1, \mathcal{L}_2]) = \begin{cases} [f(\mathcal{L}_1), f(\mathcal{L}_2)] & \text{if } \Delta \text{ does not separate } \mathcal{L}_1 \text{ and } \mathcal{L}_2, \\ [f(\mathcal{L}_1), m] \cup [f(\mathcal{L}_2), m] & \text{if } \Delta \text{ separates } \mathcal{L}_1 \text{ and } \mathcal{L}_2. \end{cases}$$

(d) in any case, for any leaves $\mathcal{L}_1, \mathcal{L}_1$ we have
$$[f(\mathcal{L}_1), f(\mathcal{L}_2)] \subseteq f([\mathcal{L}_1, \mathcal{L}_2]) \subseteq [f(\mathcal{L}_1), m] \cup [f(\mathcal{L}_2), m].$$

Another important notion is convexity.

Definition 3.2. *We say that a set S of leaves is* combinatorially convex *if whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to S, then the whole set $[\mathcal{L}_1, \mathcal{L}_2]$ is contained in S. Given a set of leaves $\mathcal{L}'$, we define the* convex hull $C(\mathcal{L}')$ *as the set of leaves L such that there exist L_1, L_2 in $\mathcal{L}'$ and L separates L_1 and L_2.*

3.2. **Combinatorial Hubbard trees.** Given a minor leaf ℓ, we define the *postcritical lamination* $P(\ell)$ as the union of the set of leaves

$$P(\ell) := \{f^n(\ell), n \geq 0\}.$$

Moreover, the *combinatorial Hubbard tree* $\mathcal{H}$ is the convex hull of the postcritical lamination.

We now define

$$H_n := \bigcup_{0 \leq i \leq n} [\beta, f^i(m)]$$

and

$$H := \bigcup_{n \in \mathbb{N}} H_n.$$

We call the set H the *combinatorial Hubbard tree* of f, as it is a combinatorial version of the (extended) Hubbard tree.

Lemma 3.3 ([7])**.** *The combinatorial Hubbard tree H has the following properties.*

1. *The set H is the smallest combinatorially convex set of leaves which contains β, m and is forward invariant.*
2. *Let $N \geq 0$ be an integer such that $f^{N+1}(m) \in H_N$. Then we have $H = H_N$.*

Note that $N+2$ coincides with the number of ends of the extended Hubbard tree.

A lamination induces an equivalence relation on $\overline{D}$, by identifying every leaf and every polygon to a point.

Lemma 3.4. *The quotient $T := \mathcal{H}/\sim$ is a tree. Moreover, the doubling map induces a unimodal tree map $f : T \to T$.*

Lemma 3.5. *If the minor leaf is not degenerate, then the Hubbard tree has a finite number of ends.*

Proof. Since the doubling map doubles the length of small leaves, there must be $k > n > p$ so that $f^k(m)$ separates $f^n(m)$ and $f^p(m)$. □

Lemma 3.6. *Let $m_1 < m_2$, and H_1, H_2 be the corresponding combinatorial Hubbard trees. Then*

$$H_1 \subseteq H_2.$$

As a corollary,

$$\#Ends(T_1) \leq \#Ends(T_2).$$

3.3. **Core entropy.** Definition of core entropy in terms of laminations.

Definition 3.7. *Let ℓ be the minor leaf, and $\mathcal{H}$ the combinatorial Hubbard tree. Then the core entropy of $\mathcal{L}$ is*

$$h(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log \#\{L \in \mathcal{H} \ : \ f^n(L) = \ell\}$$

Question. Does this definition of core entropy coincide with the one given by Thurston's algorithm?

For tree maps, a semiconjugacy to a piecewise linear models for tree maps is constructed in Baillif-DeCarvalho (Theorem 4.3).

3.4. **Transverse measures.** Let $\mathcal{L}$ be a quadratic lamination, with minor leaf $L_{\min}$. Let $\mathcal{A}(L)$ be the set of segments of $\mathcal{L}$.

Definition 3.8. *A* transverse measure *on $\mathcal{L}$ is a functional*

$$m : \mathcal{A}(L) \to \mathbb{R}$$

which satisfies:

(i) *m is* positive*: $m([L_1, L_2]) \geq 0$ for any $L_1, L_2 \in \mathcal{L}$;*
(ii) *m is* additive *on disjoint arcs:*

$$m([L_1, L_3]) = m([L_1, L_2]) + m([L_2, L_3])$$

if L_2 separates L_1, L_3.

The value $m(L_1, L_2)$ is thought of as the measure of the segment $[L_1, L_2]$ joining L_1 to L_2 in the dual tree. We denote the space of transverse measures on $\mathcal{L}$ as $\mathcal{M}(\mathcal{L})$.

3.5. **The pullback operator.** We now define the pullback operator $f^* : \mathcal{M}(\mathcal{L}) \to \mathcal{M}(\mathcal{L})$ on the set of metrics as:

$$f^*m(L_1, L_2) = m(f(L_1), f(L_2))$$

if L_1, L_2 are not separated, while

$$f^*m(L_1, L_2) = m(f(L_1), L_{\min}) + m(L_{\min}, f(L_2))$$

if L_1, L_2 are separated.

Definition 3.9. *We call a metric λ* linearly expanded *by f if there exists $\lambda \in \mathbb{R}$ such that*

$$f^\star m = \lambda m.$$

Linearly expanded, non-zero metrics are fixed points of the operator

$$P(m) := \frac{f^\star m}{\|f^\star m\|}.$$

Lemma 3.10. *The operator $P : \mathcal{M}^1(T) \to \mathcal{M}^1(T)$ is continuous with respect to the weak topology.*

Proof. If $m_n \to m$, then for any arc J,

$$f^\star m_n(J) = \sum_i m_n(f(J \cap I_i)) \to \sum_i m(f(J \cap I_i)) = f^\star m(J)$$

and similarly,

$$\|f^\star m_n\| = f^\star m_n(T) \to f^\star m(T) = \|f^\star m\|$$

Since f is surjective, note that

$$f^\star m(T) = \sum_i m(f(I_i)) \geq m(\bigcup_i f(I_i)) \geq m(T) = 1$$

hence $\|f^\star m\| \geq 1$. Then

$$P(m_n)(J) = \frac{f^\star m_n(J)}{\|f^\star m_n\|} \to \frac{f^\star m(J)}{\|f^\star m\|} = P(m)(J)$$

which completes the proof. □

Proposition 3.11. *There exists at least one unit length transverse measure m such that*

$$f^\star m = \lambda m$$

for some $\lambda \geq 1$.

Proof. By Lemma 2.3, $\mathcal{M}^1(I)$ is a non-empty, compact, convex subspace of the normed vector space

$$L^\infty(\mathcal{A}) := \left\{ m : \mathcal{A} \to \mathbb{R} \ : \ \sup_{J \in \mathcal{A}} |m(J)| < +\infty \right\}$$

and P is continuous by Lemma 3.10, hence by the Schauder fixed point theorem, there exists at least one fixed point for the operator P, which corresponds to an eigenvector for $f^\star$. □

Lemma 3.12. *A linearly expanded metric m defines a semiconjugacy of f to a piecewise linear map: indeed, if one defines*

$$h(x) := m([0, x])$$

then one has

$$h \circ f = g \circ h$$

where g is the piecewise linear map with slope λ and critical points $h(c_i)$, where c_i is a critical point for f.

4. The transfer operator

4.1. **Functions of bounded variation.** Let $L = \{L_1, \dots, L_k\}$ be a finite set of leaves of the lamination. Two leaves are *adjacent* in L if they are not separated by another element of L. A set L is *non-overlapping* if, whenever $\{L_1, L_2\}$ and $\{L_3, L_4\}$ are distinct pairs of adjacent leaves, the (open) intervals (L_1, L_2) and (L_3, L_4) are disjoint.

Let L be a non-overlapping set, and $f : \Lambda(\mathcal{L}) \to \mathbb{R}$ be a function. We define its *total variation* over L as

$$\mathrm{Var}(f; L) := \sum_{\{L_i, L_j\} \in \mathrm{adj}(L)} |f(L_i) - f(L_j)|$$

where the index runs over the set $\mathrm{adj}(L)$ of adjacent leaves.

Definition 4.1. *A function* $f : \mathrm{Leaves}(\mathcal{L}) \to \mathbb{R}$ *is of* bounded variation *if*

$$\mathrm{Var}(f) := \sup_L \mathrm{Var}(f; L) < +\infty$$

where the sup *runs over all finite non-overlapping subsets of* $\mathcal{L}$*.*

4.2. **Integration.** Let $\Lambda(\mathcal{L})$ be the space of leaves of the lamination $\mathcal{L}$. Let $f : \Lambda(\mathcal{L}) \to \mathbb{R}$ be a function, and let m be a transverse measure to the lamination $\mathcal{L}$.

Let $\mathcal{P} = \{L_1, \dots, L_k\}$ be a finite, non-overlapping set of leaves of $\mathcal{L}$. Let the *mesh* of $\mathcal{P}$ be

$$|\mathcal{P}| := \sup_{L_i, L_j \in \mathrm{adj(L)}} d(L_i, L_j)$$

where the distance can be taken e.g. as points in $S^1 \times S^1 / \sim$.

The space $\mathcal{S}$ of *simple functions* is the space of all finite linear combinations of the characteristic functions χ_J of all subintervals $J \subset I$. A metric m defines a linear operator $\mathbb{E}_m : \mathcal{S} \to \mathbb{R}$ by

$$\mathbb{E}_m(\varphi) := \sum_J a_J m(J)$$

if $\varphi = \sum_J a_J \chi_J$. We also write $\int \varphi \, dm$ instead of $\mathbb{E}_m(\varphi)$. Recall the well-known

Lemma 4.2. *The space* $\mathcal{S}$ *is dense in the space* BV*.*

By the Lemma, we can extend the operator $\mathbb{E}_m$ to $\mathbb{E}_m : BV \to \mathbb{R}$, defining

$$\int \varphi(x) \, dm(x) := \sup\{\mathbb{E}_m(\psi) \ : \ \psi \in \mathcal{S}, \psi \leq \varphi\}.$$

Hence we obtain pairing $\langle \cdot, \cdot \rangle : BV \times \mathcal{M} \to \mathbb{R}$,

$$\langle \varphi, m \rangle := \int \varphi \, dm.$$

4.3. **The transfer operator.** Consider a lamination $\mathcal{L}$ as a subset of $\mathbb{T} \times \mathbb{T}$. The map $\sigma(x, y) := (2x, 2y) \mod 1$ satisfies $\sigma(\mathcal{L}) \subseteq \mathcal{L}$ and $\sigma(\mathcal{H}) \subseteq \mathcal{H}$.

Moreover, the critical diameter Δ separates $\mathcal{L}$ in two parts, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. Let $\sigma_i^{-1} : \mathcal{L}_i \to \mathcal{L}$ be the local inverses of σ.

We define the transfer operator $\mathcal{L} : BV \to BV$ as

$$\mathcal{L}\varphi(x) := \sum_{i=1}^{d} \varphi(\sigma_i^{-1} x)$$

Lemma 4.3. *For any interval I and any metric m, we have*

$$\langle \mathcal{L}\chi_I, m\rangle = \langle \chi_I, f^\star m\rangle$$

As a corollary, using Lemma 4.2, we obtain the duality relation

$$\langle \mathcal{L}\varphi, m\rangle = \langle \varphi, f^\star m\rangle$$

for any $\varphi \in BV$, $m \in \mathcal{M}$. Recall also that

$$\mathrm{Var}_J(f^n) = \|\mathcal{L}^n \chi_J\|_{L^1}.$$

Definition 4.4. *A subset $J \subseteq \mathcal{H}$ is a* monotone set *if the restriction $\sigma|_{\mathcal{J}} : \mathcal{J} \to \sigma(\mathcal{J})$ is a homeomorphism onto its image.*

Further, a subset $J \subseteq \mathcal{H}$ is convex *if, for any $x, y \in \mathcal{J}$, the segment $[x, y] \cap \mathcal{J}$ is contained in $\mathcal{J}$.*

Given a convex set J, we denote as $\mathcal{E}(J)$ its number of ends, that is the maximal cardinality of a set of leaves of J such that no three of them are nested. We have the properties:

(1) if J is a convex set,
$$\mathrm{Var}_J(\varphi_1\varphi_2) \leq \mathrm{Var}_J(\varphi_1)\sup_J|\varphi_2| + \mathrm{Var}_J(\varphi_2)\sup_J|\varphi_1|$$

(2) if $J \subseteq K$ are convex sets,
$$\mathrm{Var}_K(\varphi \cdot \chi_J) \leq \mathrm{Var}_J(\varphi) + \mathcal{E}(K)\sup_J|\varphi|$$

(3) if J is a monotone set,
$$\mathrm{Var}_J(\varphi \circ \sigma) = \mathrm{Var}_{\sigma(J)}(\varphi)$$

We consider as $\mathcal{Z}$ the partition of $\mathcal{H}$ in the two components of $\mathcal{H} \setminus \Delta$. Both such components are monotone sets. For each m, we denote as

$$\mathcal{Z}_m := \{\eta_0 \cap f^{-1}(\eta_1) \cap \cdots \cap f^{-(m-1)}(\eta_{m-1}) \ : \ \eta_i \in \mathcal{Z}\}$$

4.4. **Spectral decomposition.**

Lemma 4.5. *If $J \subseteq \eta \in \mathcal{Z}_m$ is a convex set and $\varphi : J \to \mathbb{C}$,*

$$\|\mathcal{L}^m(\varphi \cdot \chi_J)\|_{BV} \leq \mathrm{Var}_J(\varphi) + (\mathcal{E}(I) + 1)\sup_J|\varphi|$$

Proof. Since J is a subset of an element of $\mathcal{Z}_m$,

$$\mathcal{L}^m(\varphi \cdot \chi_J) = (\varphi \cdot \chi_J) \circ T_\eta^{-m}$$

hence

$$\mathrm{Var}_I \mathcal{L}^m(\varphi \cdot \chi_J) \leq \mathrm{Var}_J(\varphi) + \mathcal{E}(I)\sup_J|\varphi|$$

while

$$\|\mathcal{L}^m(\varphi \cdot \chi_J)\|_\infty \leq \sup_J|\varphi|.$$

□

Now, for each η, pick a point $x_\eta \in \eta$. consider for any m the operator

$$F_m(f) = \sum_{\eta \in \mathcal{Z}_m} f(x_\eta) \mathcal{L}^m \chi_\eta$$

which has finite range.

Lemma 4.6. *We have*

$$\sum_{\eta \in \mathcal{Z}_m} \|\mathcal{L}^m \chi_\eta \cdot f(x_\eta) - \mathcal{L}^m(f \cdot \chi_\eta)\|_{BV} \leq C \cdot \mathrm{Var}(f)$$

Proof.

$$\begin{aligned}
\sum_{\eta \in \mathcal{Z}_m} \|\mathcal{L}^m \chi_\eta \cdot f(x_\eta) - \mathcal{L}^m(f \cdot \chi_\eta)\|_{BV} &\leq \sum_{\eta \in \mathcal{Z}_m} \|\mathcal{L}^m \chi_\eta \cdot (f(x_\eta) - f)\|_{BV} \\
&\leq \sum_{\eta \in \mathcal{Z}_m} \mathrm{Var}_\eta(f(x_\eta) - f) + (\mathcal{E}(I) + 1) \sup_\eta |f(x_\eta) - f| \\
&\leq \sum_{\eta \in \mathcal{Z}_m} (\mathcal{E}(I) + 2)\mathrm{Var}_\eta(f(x_\eta) - f) \\
&\leq (\mathcal{E}(I) + 2) \sum_{\eta \in \mathcal{Z}_m} \mathrm{Var}_\eta f \leq (\mathcal{E}(I) + 2)\mathrm{Var}_I f
\end{aligned}$$

□

Now, from Lemma 4.6

$$\begin{aligned}
\|\mathcal{L}^m(f) - F_m(f)\|_{BV} &= \| \sum_{\eta \in \mathcal{Z}_m} \mathcal{L}^m(f\chi_\eta) - f(x_\eta)\mathcal{L}^m \chi_\eta \|_{BV} \\
&\leq \sum_{\eta \in \mathcal{Z}_m} \|\mathcal{L}^m(f\chi_\eta) - f(x_\eta)\mathcal{L}^m \chi_\eta\|_{BV} \\
&\leq C \cdot \mathrm{Var}(f)
\end{aligned}$$

Since F_m has finite range, hence it is compact, it follows that $\mathcal{L}$ is quasicompact, hence, e.g. from Dunford-Schwartz (VIII.8.2) it follows that $\mathcal{L}$ has the following spectral decomposition.

Still to prove it has cyclic spectrum

Theorem 4.7. *Let $\mathcal{L}$ be a quadratic lamination, and suppose that the associated combinatorial Hubbard tree $\mathcal{H}$ has a finite number of ends.*

Suppose $f : I \to I$ is a piecewise monotone map with $h_{top}(f) > 0$. Let $\mathcal{L} : BV \to BV$ be the transfer operator. Then:

1. *the spectral radius of $\mathcal{L}$ equals $\lambda = e^{h_{top}(f)}$;*
2. *there exists a spectral decomposition*

$$\mathcal{L} = \sum_{i=0}^{r} \lambda_i P_i + P_{res}$$

where $\lambda = \lambda_0, \lambda_1, \dots, \lambda_r$ are the eigenvalues of modulus > 1, each P_i is a spectral projector of finite rank, and P_{res} has spectral radius ≤ 1.

4.5. **Spectral radius and core entropy.** In [18], we construct an infinite graph W_θ in order to compute the core entropy. More precisely, the vertex set of the graph is the set

$$\Sigma := \{(i,j) \ : \ 0 \leq i < j\}$$

of pairs, called a *wedge*.

Theorem 4.8 ([18])**.** *The core entropy of f_θ satisfies*

$$h(f_\theta) = \lim_{n\to\infty} \frac{1}{n} \log \#\{\text{closed paths in } W_\theta \text{ of length } n\}$$

Recall that the spectral radius of $\mathcal{L}$ is given by

$$\rho(\mathcal{L}) = \lim_n \|\mathcal{L}^n 1\|_{BV}^{1/n}$$

Since

$$\mathcal{L}^n 1 = \sum_{\eta \in \mathcal{Z}_n} \chi_{f^n(I_\eta)} \circ f_\eta^{-n}$$

we have

$$\|\mathcal{L}^n 1\| \leq \mathcal{E}(I) \# \mathcal{Z}_n$$

and also

$$\mathcal{L}^n 1(x) = \#\{f^{-n}(x)\}$$

We need to show

$$h(f_\theta) = \lim_{n\to\infty} \frac{1}{n} \log \# \mathcal{Z}_n \tag{1}$$

From the above equation, proceeding as in the proof of [2, Theorem 3], we obtain

$$h(f_\theta) = \log \rho(\mathcal{L}).$$

Lemma 4.9. *We have*

$$h(f_\theta) = \log \rho(A_\theta)$$

where $A_\theta : \ell^1(W_\theta) \to \ell^1(W_\theta)$ is the adjacency operator of the infinite graph.

We now prove that the transfer operator and the adjacency operator have equal spectral radius.

Theorem 4.10. *We have*

$$\rho(A_\theta) = \rho(\mathcal{L})$$

We consider the map $i : \ell^1(W_\theta) \to BV(\mathcal{L})$ given by

$$i(u) := \sum_{i<j} u(i,j) \chi_{[\theta_i, \theta_j]}$$

where $\theta_i := 2^{i-1}\theta \mod 1$. Note that i is a continuous (bounded) operator.

Since every edge of the tree has two sides, if the tree is topologically finite there exists $u \in \ell^1$ such that $i(u) = 1$. Then

$$\|\mathcal{L}^n 1\|_{BV} = \|\mathcal{L}^n i(u)\|_{BV} = \|iA^n(u)\|_{BV} \leq \|A^n(u)\|_{\ell^1} \leq \|A^n\|_{\ell^1} \cdot \|u\|_{\ell^1}$$

hence

$$\rho(\mathcal{L}) = \lim_n \|\mathcal{L}^n 1\|_{BV}^{1/n} \leq \lim_n \|A^n\|_{\ell^1}^{1/n} = \rho(A_\theta)$$

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