Math 2D

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Prerequisite Knowledge

Derivatives

- $(fg)' = f'g + fg'$
- $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$
- $\tan x' = \sec^2 x$
- $\sec x' = \sec x \tan x$
- $\csc x' = -\csc x \cos x$

Trigonometry

- $\sin^2 x + \cos^2 x = 1$
- $1 + \cot^2 x = \csc^2 x$

- $\tan^2 x + 1 = \sec^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - \sin^2 x$

Integration by parts

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u$$

Chapter 10.1

Concavity Test

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

Chapter 10.2

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Cycloid

$$\begin{cases} x_p = r(\theta - \sin\theta) \\ y_p = r(1 - \cos\theta) \end{cases}$$

Arc length

$$\int_a^b \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2} \mathrm{d}t$$

Chapter 10.3

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} x^2+y^2=r^2 \\ \tan\theta=\frac{y}{x} \end{cases}$$

Symetric rules

- Unchanges when θ replaced by $-\theta$:
  - symetric about x axis
- Unchanged when r replaced by -r or θ replaced by $\theta+\pi$:
  - symetric about the pole(origin)
- Unchanged when θ replaced by $(\pi-\theta)$:
  - symetric about y axis

Tangents

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta+r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta-r\sin\theta}$$

Chapter 12.1-12.4

Right Hand Rule

Distance

$$P_1(x_1,y_1,z_1),P_2(x_2,y_2,z_2)$$

$$P_1P_2=l=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Scalar

$$\vec{v}=<a,b,c>|\vec{v}|=\sqrt{a^2+b^2+c^2}$$

Unit Vector ($|\vec{u}|=1$)

$$\vec{u}=\frac{\vec{v}}{|\vec{v}|}$$

Dot product

$\vec{a}=<a_1,a_2,a_3>,\vec{b}=<b_1,b_2,b_3>$

1. $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3=|\vec{a}||\vec{b}|\cos\theta$

2. $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

Orthogonal=> perpendicular=> $\theta = \frac{\pi}{2}$

Projection

Scalar projection of $\vec{b}$ onto $\vec{a}$

$$Comp_{\vec{a}}^{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Vector projection of $\vec{b}$ onto $\vec{a}$

$$Proj_{\vec{a}}^{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}$$

Cross Product

$\vec{a} =< a_1, a_2, a_3 >, \vec{b} =< b_1, b_2, b_3 >$

$$\vec{a}\cdot\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =< a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

The vector $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

Chapter 12.5

Line:

$$\vec{r} = \vec{r}_0 + t\cdot\vec{v}$$

$$< x, y, z >=< x_0, y_0, z_0 > +t\cdot < a, b, c >$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Skew Lines:

Neither intersect nor parallel

Plane

$\vec{n}$=>Normal Vector A plane through $P(x_0, y_0, z_0)$ with $\vec{n} =< a, b, c >$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Find Angles between two planes:

Find the angle between normal vectors

Find plane pass through P_1 P_2 and P_3

$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3}$ and with P_1

Distance from $P(x_1, y_1, z_1)$ to plane $ax + by + cz + d = 0$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Find the distance between two lines:

1. Find normal vector (By cross product the two velocity vectors of two lines)
2. Establish a plane
3. Use formula

Chapter 13.3

Arc Length in 3D

$$l = \int_a^b \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2 + (\frac{\mathrm{d}z}{\mathrm{d}t})}\mathrm{d}t$$

if we assume $\overrightarrow{r(t)} =< f(t), g(t), h(t) >$=>$\overrightarrow{r'(t)} =< f'(t), g'(t), h'(t) >$

$$l = \int_a^b |\overrightarrow{r'(t)}|\mathrm{d}t$$

Tangent Vector

$$\overrightarrow{T(t)} = \frac{\overrightarrow{r'(t)}}{|\overrightarrow{r'(t)}|} \quad |\overrightarrow{T(t)}| = 1$$

Normal Vector

$$\overrightarrow{N(t)} = \frac{\overrightarrow{T'(t)}}{|\overrightarrow{T'(t)}|}$$

Binomal Vector

$$\overrightarrow{B(t)} = \overrightarrow{T(t)} \times \overrightarrow{N(t)}$$

Normal Plane

Normal plane: The normal plane consists of all lines that are orthogonal to the tangent vector.

The normal vector of the normal plane is the tangent vector

$$\vec{n} = \vec{T}$$

Osculating plane

The plane that comes the closest to containing the part of the curve near point P.

The normal vector of the osculating plane is the binomal vecotr

$$\vec{n} = \vec{B}$$

Chapter 13.4

Velocity

$$\overrightarrow{v(t)} = \overrightarrow{r'(t)}$$

Speed

$$Speed = |\overrightarrow{v(t)}| = |\overrightarrow{r'(t)}|$$

Chapter 14.1

Level Curves / Contour Map

The level curves of a function f of two variables are the curve with equations $f(x, y) = k$, where k is a constant (in the range of f).

Functions of Three or more variables

Chapter 14.2

Limit of multi-var functions

Prove the limit does not exist

Pick two different paths and show that the height or the value if the function is different if we transverse along these different paths.

Tips

- Make sure the path you choose contains the point (a,b)
- It is highly recommended that one of the paths you choose is either by forcing x=0 and moving along y axis or forcing y=0 and moving along x axis.
- It is helpful to choose a path that makes the degree of the numerator and the denominator equal.

Continuity

Sometimes we can use the concept of continuity for proving that the limit exist.

- If we have a region (two dimensional region in xy-plane) that the function is defined in that region, the function is continuous in that region. In other words, a function is continuous at any point inside its domain.
- Polynomials are continuous everywhere.
- Rational functions are continuous everywhere except where the denominator is zero.

Chapter 14.3

Partial Derivative

To Find f_x

Regard y as a constant and differentiate f(x,y) with respect to x.

To Find f_y

Regard x as a constant and differentiate f(x,y) with respect to y.

Higher Derivatives

The notation f_{xy} means, first we take the derivative with respect to x, and then we take the derivative with respect to y.

$f_{xy} = f_{yx}$

Chapter 14.4

Tangent Plane

The tangent plane to the surface S at $P(x_0, y_0, z_0)$ is

$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Linear Approximation

$f(x, y) = z \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Theorem: If partial derivatives of f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b)

Chapter 14.5

Case1: Theorem:

Suppose that $z = f(x, y)$ is a differentiable function of x and y. Where $x = g(t)$ and $y = g(t)$ are both differentiable functions of t.

Then:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$$

Case2: Theorem:

$z = f(x, y)$ such that $x = g(s, t)$ and $y = h(s, t)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Implicit Differentiation

Suppose that an equation of form F(x,y)=0 defines y implicitly as a differentiable function of x, that is, y=f(x) where F(x,y)=0 or F(x, f(x))=0

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

Chapter 14.6

Directional derivative

Theorem:

if f is a differentiable function of x and y, then f has a directional derivative in the direction of any vector $\vec{u} =< a, b >$ and:

$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$

Gradient Vector:$< f_x, f_y >$

$D_u f(x, y) =< f_x(x, y), f_y(x, y) > \cdot < a, b >=< f_x, f_y > \cdot \vec{u}$

Maximizing the directional derivative

Theorem:

If f is a differentiable function of two or three variables, the Maximum value of directional derivative is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector

Tangent planes to surfaces

An arbitrary point on the surface, the gradient vector at that point is perpendicular to any tangent vector to any curve C on S that passes through P.

Tangent Plane:

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

Normal Line:

$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$

Chapter 14.7

First Derivative Test

if f has a local Max or Min at (a,b), and the first partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Second Derivative Test

$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2$

(a) If D>0 and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local Minimum

(b) If D>0 and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local Maximum

(c) If D<0, then $f(a, b)$ is a saddle Point.

(d) If D=0, then $f(a, b)$ could be a local Maximum or a local Minimum or a saddle point

Chapter 14.8

The method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to constraint $g(x, y, z) = k$

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$$

Two Constraints

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

Chapter 15.1

Double Integral & Fubini's Theorem

$$\iint_R f(x, y) \mathrm{d}A = \int_a^b \int_c^d f(x, y) \mathrm{d}y \mathrm{d}x$$

$$\iint_R f(x, y) \mathrm{d}A = \int_c^d \int_a^b f(x, y) \mathrm{d}x \mathrm{d}y$$

Special Case

$$\iint_R f(x, y) \mathrm{d}A = \iint_R g(x) h(y) \mathrm{d}A = \int_a^b g(x) \mathrm{d}x \int_c^d h(y) \mathrm{d}y$$

Average value

$$f_{average} = \frac{1}{A(R)} \iint f(x, y) \mathrm{d}A$$

Chapter 15.2

Type I

$$\iint_D f(x, y) \mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \mathrm{d}y \mathrm{d}x$$

Type II

$$\iint_D f(x,y)\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\mathrm{d}x\mathrm{d}y$$

Chapter 15.3

Change to polar coordinates in a double integral

$$\iint f(x,y)\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$

More Complicated Regions in polar Coordinates

if f is continuous on polar region of the form $D = \{(r,\theta)|\alpha < \theta < \beta, h_1(\theta) < r < h_2(\theta)\}$ Then

$$\iint_D f(x,y)\mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$$