An analogous statement holds for (the restrictions of) the function $r_x(\omega)/K(x(\omega),0)$ and then for $Q(x,0)$. Using the functional equation (3), we conclude that $Q(x,y)$ is algebraic in the two variables x,y.

In the section below, we refine the previous statement, by proving that $Q(x,y)$ is algebraic in x,y,z (Problem (B)).

Proof of the algebraicity of the trivariate GF. We start by proving the algebraicity of $Q(0,y)$ as a function of y,z. We consider the representation of $r_y(\omega)$ given in Theorem 3 and apply eight times the addition theorem (P4) for ζ-functions, namely (for suitable values of $k\in\mathbf{Z}$ that can be deduced from (21))

$$\zeta_{1,3}(\omega-k\omega_2/8)=\zeta_{1,3}(\omega)-\zeta_{1,3}(k\omega_2/8)+\frac{1}{2}\frac{\wp_{1,3}'(\omega)+\wp_{1,3}'(k\omega_2/8)}{\wp_{1,3}(\omega)-\wp_{1,3}(k\omega_2/8)}.$$

We then make the weighted sum of the eight identities above (corresponding to the good values of k in (21)); this way, we obtain

$$r_y(\omega)=U_1(\omega)+U_2+U_3(\omega),$$

where $U_1(\omega)$ is the weighted sum of the eight functions $\zeta_{1,3}(\omega)$, U_2 is the sum of c and of the weighted sum of the eight quantities $\zeta_{1,3}(k\omega_2/8)$, and $U_3(\omega)$ is the weighted sum of the eight quantities

$$\frac{\wp_{1,3}'(\omega)+\wp_{1,3}'(k\omega_2/8)}{\wp_{1,3}(\omega)-\wp_{1,3}(k\omega_2/8)}. \tag{41}$$

Since the sum of the residues in the formula (21) equals 0, the coefficients in front of $\zeta_{1,3}(\omega)$ is 0, so that $U_1(\omega)$ is identically zero. To prove that U_2 is algebraic in z, it suffices to use similar arguments as we did to prove that $Q(0,0)$ is algebraic (see Section 4: we group together different ζ-functions and we use standard identities as the Frobenius-Stickelberger equality (P9) or the addition formula for the ζ-function (P4)); we do not repeat the arguments here. Finally, we show that $U_3(\omega)$ is algebraic in $y(\omega)$ over the field of algebraic functions in z. In other words, we show that there exists a non-zero bivariate polynomial P such that

$$P(U_3(\omega),y(\omega))=0,$$

where the coefficients of P are algebraic functions in z. This is enough to conclude to the algebraicity of $Q(0,y)$ as a function of y,z.

To prove the latter fact, we shall prove that each term (41) satisfies the property above (with different polynomials P, of course). First, Lemma 13 below implies that $\wp_{1,3}(k\omega_2/8)$ and $\wp_{1,3}'(k\omega_2/8)$ are both algebraic in z. Further, it follows from (P7) that the function $\wp_{1,3}(\omega)$ is algebraic in $y(\omega)$ over the field of algebraic functions in z. The same property holds for $\wp_{1,3}'(\omega)$; this comes from the fact above together with the differential equation (32) satisfied by the Weierstrass elliptic functions.

The proof of the algebraicity of $Q(x,0)$ as a function of x,z is analogous. With equation (3) the algebraicity of $Q(x,y)$ as a function of x,y,z is proved.

Lemma 13. *For any $k\in\mathbf{Z}$ and any $\ell\in\mathbf{Z}_+$, $\wp^{(\ell)}(k\omega_2/8)$ and $\wp_{1,3}^{(\ell)}(k\omega_2/8)$ are (infinite or) algebraic functions of z.*