Proof For the correctness it suffices to observe that the matrices M are set up in such a way that finally $M[n, j(r+1)+i]$ is precisely $n^j a_{n+i}$ in Algorithm 1.33 and the coefficient of x^n in $x^j a^{(i)}(x)$ in Algorithm 1.34.

For the cost estimate, note that in both algorithms the construction of M takes at most $(d+1)(r+1)(N-r+1) = \mathrm{O}(d^2r^2)$ operations, and that a nullspace basis of an $(N-r+1)\times(r+1)(d+1)$ matrix can be computed using $\mathrm{O}(d^\omega r^\omega)$ operations. ∎

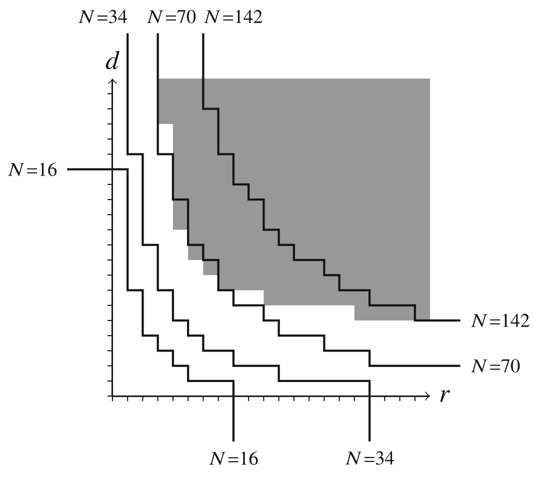
Algorithms 1.33 and 1.34 work for any choice of r, d, N, but the results are interesting only if $N \geq (r+1)(d+2)-2$. Typical implementations therefore contain an additional instruction at the beginning which aborts with an error message "not enough data" if $N < (r+1)(d+2)-2$.

If the number of known terms is fixed, then only a certain finite set of points (r, d) can be reasonably tested for possible recurrences or differential equations. It is instructive to compare the location of these points to the location of the points (r, d) for which there actually exist equations.

Example 1.36 The formal power series

$$\frac{1+x^5}{\sqrt{x+1}} + \frac{2x+3}{\sqrt{1-x}} + (3x^4-4x^3+8)\exp\left(\frac{x}{1-x}\right) = 12 + 11x + \tfrac{29}{2}x^2 + \cdots$$

satisfies a differential equation of order r with polynomial coefficients of degree d for every point (r, d) in the gray region in the figure below.



From the $N+1$ coefficients $a_0, \ldots, a_N$ of the series, such an equation can be reconstructed if $N \geq (r+1)(d+2)-2$. For some choices of N, the regions which can be investigated are those underneath the black curves in the figure. All of these regions are finite, but only the region for $N = 16$ completely fits into the picture. For $N = 16$ and $N = 34$ no equations can be