Neural-Fly Enables Rapid Learning for Agile Flight in Strong Winds

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1 Euler–Lagrange dynamics with aerodynamic force + error-based control equation

1.1 Full dynamics (including aerodynamic force)

We consider an Euler–Lagrange (EL) mechanical system with generalized coordinates and velocities:

$$q(t) \in \mathbb{R}^n, \qquad \dot{q}(t) \in \mathbb{R}^n.$$

The dynamics (with aerodynamic / disturbance force) are:

$$M(q)\,\ddot{q} + C(q, \dot{q})\,\dot{q} + g(q) = u + f(q, \dot{q}, w),$$

where:

$M(q) \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $\quad C(q, \dot{q}) \in \mathbb{R}^{n\times n}, \quad g(q) \in \mathbb{R}^n.$

The unknown aerodynamic force is denoted:

$$f(q, \dot{q}, w) \in \mathbb{R}^n,$$

Key EL structural identity

A classical property of EL systems is that: $\dot{M}(q) - 2C(q, \dot{q})$ is skew-symmetric

which implies for any vector $x \in \mathbb{R}^n$

$x^\top \left(\dot{M} - 2C\right) x = 0 \quad \iff \quad x^\top \dot{M} x = 2x^\top C x.$

We will use this to simplify the Lyapunov derivative later.

1.2 Tracking errors and the "composite" error variable s

Let $q_d(t)$ be a desired trajectory (at least twice differentiable). Define tracking errors:

$$\tilde{q} := q - q_d, \qquad \dot{\tilde{q}} := \dot{q} - \dot{q}_d.$$

Choose a constant gain matrix:

$$\Lambda \in \mathbb{R}^{n\times n}, \quad \Lambda \succ 0.$$

Define the "composite" tracking error:

$$s := \dot{\tilde{q}} + \Lambda \tilde{q}.$$

Define the reference velocity/acceleration (standard in EL control):

$$\dot{q}_r := \dot{q}_d - \Lambda\tilde{q}, \qquad \ddot{q}_r := \ddot{q}_d - \Lambda\dot{\tilde{q}}.$$

Then you can verify:

$$s = \dot{q} - \dot{q}_r, \qquad \dot{s} = \ddot{q} - \ddot{q}_r.$$

1.3 Aerodynamic force model, estimation, and why parameter-error divergence is dangerous

We assume the unknown aerodynamic force admits a **low-dimensional linear-in-parameters representation** plus a residual:

$$f(q, \dot{q}, w) = \phi(q, \dot{q})\, a(t) + d(t).$$

- $\phi(q, \dot{q}) \in \mathbb{R}^{n \times h}$: known "basis/features" (e.g., a neural network output evaluated online)

  $a(t) \in \mathbb{R}^h$: unknown time-varying coefficient (captures wind-dependent effects)

  $d(t) \in \mathbb{R}^n$: unmodeled remainder (representation error, unmodeled physics, etc.)

We maintain an online estimate $\hat{a}(t)$, Define parameter estimation error

$$\tilde{a} = \hat{a} - a.$$

Then the estimated aerodynamic force is:

$$\hat{f} = \phi(q, \dot{q})\, \hat{a}.$$

The **force estimation error** becomes:

$$f - \hat{f} = \phi a + d - \phi\hat{a} = -\phi\tilde{a} + d.$$

Why $\tilde{a}$ can destabilize the system

If $\tilde{a}(t)$ is not guaranteed bounded, then the term $-\phi\tilde{a}$ can grow without bound and acts like an **unbounded input** injected into the closed-loop error dynamics. Even if we add stabilizing feedback $-Ks$, an unbounded forcing term can dominate and drive s large, breaking tracking (and potentially saturating actuators in practice).

So: **stability requires controlling both s and $\tilde{a}$**

1.4 Control law and the closed-loop error equation in s

Choose a feedback gain:

$$K \in \mathbb{R}^{n \times n}, \quad K \succ 0.$$

Define the control input:

$$u = M(q)\,\ddot{q}_r + C(q, \dot{q})\,\dot{q}_r + g(q) \ - \ Ks \ - \ \phi(q, \dot{q})\,\hat{a}.$$

Now derive the s dynamics.

Start from the plant:

$$M\ddot{q} + C\dot{q} + g = u + f.$$

Substitute u and cancel terms:

$$M\ddot{q} + C\dot{q} + g = (M\ddot{q}_r + C\dot{q}_r + g - Ks - \phi\hat{a}) + f.$$

Bring $M\ddot{q}_r + C\dot{q}_r + g$ to the left:

$$M(\ddot{q}-\ddot{q}_r)+C(\dot{q}-\dot{q}_r)=-Ks-\phi\hat{a}+f.$$

Use:

$$\dot{s}=\ddot{q}-\ddot{q}_r, \qquad s=\dot{q}-\dot{q}_r,$$

to obtain:

$$M\dot{s}+Cs=-Ks-\phi\hat{a}+f.$$

Replace $f=\phi a+d$, and group $\tilde{a}=\hat{a}-a$

$$M\dot{s}+Cs=-Ks-\phi(\hat{a}-a)+d$$

$$M\dot{s}+(C+K)s=-\phi\tilde{a}+d$$

This is the key "tracking-error dynamics with parameter error injection.

2.Why we introduce a joint Lyapunov function + exponential contraction to an error ball

2.1 If we only use tracking energy, we need $\tilde{a}$ bounded (not automatic)

A natural tracking-only Lyapunov candidate is:

$$V_s := \frac{1}{2}s^\top M(q)\,s.$$

Its derivative is:

$$\dot{V}_s = s^\top M\dot{s}+\frac{1}{2}s^\top\dot{M}s.$$

Using the closed-loop equation:

$$M\dot{s}=-(C+K)s-\phi\tilde{a}+d,$$

we get:

$$\dot{V}_s=s^\top\left(-(C+K)s-\phi\tilde{a}+d\right)+\frac{1}{2}s^\top\dot{M}s=-s^\top Ks\;+\;s^\top(d-\phi\tilde{a})\;+\;\left(-s^\top Cs+\frac{1}{2}s^\top\dot{M}s\right).$$

Using EL identity:

$$-s^\top Cs+\frac{1}{2}s^\top\dot{M}s=0,$$

$$\dot{V}_s=-s^\top Ks+s^\top(d-\phi\tilde{a}).$$

Bounding:

$$s^\top Ks\ge\lambda_{\min}(K)\|s\|^2,$$

$$s^\top(d-\phi\tilde{a})\le\|s\|\,\|d-\phi\tilde{a}\|.$$

$$\dot{V}_s\le-\lambda_{\min}(K)\|s\|^2+\|s\|\,\|d-\phi\tilde{a}\|.$$

This implies "ultimate boundedness" **only if** $\|d-\phi\tilde{a}\|$ is bounded—i.e., you need $\tilde{a}$ bounded. But **boundedness of** $\tilde{a}$ is *not automatic* unless we design an update law for $\hat{a}$ that enforces it.

Hence we introduce a **joint** Lyapunov function including $\tilde{a}$

2.2 Joint Lyapunov function (tracking + parameter-error energy)

We choose:

$$V := \frac{1}{2}s^\top Ms \;+\; \frac{1}{2}\tilde{a}^\top P^{-1}\tilde{a},$$

where:

$$P(t) \in \mathbb{R}^{h\times h}, \quad P(t) \succ 0$$

is a time-varying positive definite matrix (later we will define its dynamics).

This is "equivalent energy" because:

- $s^\top Ms$ is kinetic-like energy in the tracking error
- $\tilde{a}^\top P^{-1}\tilde{a}$ measures parameter error magnitude under the metric P^{-1}

2.3 Derivative of the joint Lyapunov function and the key coupling term

Compute:

$$\dot{V} = \dot{V}_s + \dot{V}_a,$$

where $\dot{V}_s$ is already known:

$$\dot{V}_s = -s^\top Ks + s^\top(d - \phi\tilde{a}).$$

Now handle:

$$V_a := \frac{1}{2}\tilde{a}^\top P^{-1}\tilde{a}.$$

Differentiate (product rule; both $\tilde{a}$ and P^{-1} vary):

$$\dot{V}_a = \tilde{a}^\top P^{-1}\dot{\tilde{a}} + \frac{1}{2}\tilde{a}^\top(\dot{P^{-1}})\tilde{a}.$$

$$\dot{V} = -s^\top Ks + s^\top d - s^\top\phi\tilde{a} + \tilde{a}^\top P^{-1}\dot{\tilde{a}} + \frac{1}{2}\tilde{a}^\top(\dot{P^{-1}})\tilde{a}.$$

The problematic coupling term is:

$$-s^\top\phi\tilde{a}.$$

To make $\dot{V}$ negative (up to bounded disturbance terms), we want **an opposite term** $+\,s^\top\phi\tilde{a}$ appear from $\tilde{a}^\top P^{-1}\dot{\tilde{a}}$

This motivates including a **tracking-driven** component in $\dot{\hat{a}}$ such that:

$$\dot{\hat{a}} \supset P\phi^\top s.$$

Because then:

$$\tilde{a}^\top P^{-1}(P\phi^\top s) = \tilde{a}^\top\phi^\top s = s^\top\phi\tilde{a},$$

which cancels the dangerous coupling exactly.

This is the core "why tracking term is necessary" from the Lyapunov viewpoint.

2.4 Resulting contraction to an error ball (high-level inequality form)

After designing $\dot{\hat{a}}$ and $\dot{P}$ appropriately (next section), $\dot{V}$ can be arranged into the form:

$$\dot{V} \le -\alpha V + \beta,$$

with constants $\alpha > 0$ and $\beta \geq 0$ depending on:

- K (damping)
- boundedness of $d(t)$, measurement noise, and time-variation of $a(t)$

Solving the scalar differential inequality yields:

$$V(t) \leq e^{-\alpha t} V(0) + \frac{\beta}{\alpha}\left(1 - e^{-\alpha t}\right),$$

hence:

$$\limsup_{t \to \infty} V(t) \leq \frac{\beta}{\alpha},$$

which implies$[s; \tilde{a}]$ converges exponentially into a ball whose radius is proportional to $\sqrt{\beta/\alpha}$,This is the precise meaning of "equivalent energy shrinks to an error ball."

3. Optimal parameter estimation: obtain $\hat{a}$, $\dot{P}$ and the gain (Kalman–Bucy structure)

3.1 Measurement and process models for the unknown coefficient $a(t)$

We assume we can form a (noisy) measurement $y(t) \in \mathbb{R}^n$ of the aerodynamic force component represented by ϕa

$$y(t) = \phi(q, \dot{q})\, a(t) + \epsilon(t),$$

where $\epsilon(t)$ is measurement noise with covariance:

$$\mathbb{E}[\epsilon(t)\epsilon(\tau)^\top] = R\,\delta(t - \tau), \qquad R \succ 0.$$

We also model the time-variation of $a(t)$ as a stable drift + process noise:

$$\dot{a}(t) = -\lambda a(t) + \nu(t),$$

with:

$$\mathbb{E}[\nu(t)\nu(\tau)^\top] = Q\,\delta(t - \tau), \qquad Q \succeq 0.$$

Here:

- $\lambda \geq 0$ enforces "forgetting / leakage"
- Q encodes how fast we allow$a(t)$ to vary (wind variability)

3.2 Kalman–Bucy (continuous-time) estimator for $a(t)$

i. Stochastic formulation of the parameter–observation system

We consider an unknown scalar parameter $a(t)$ evolving in time and observed indirectly through noisy measurements.The deterministic rate equation

$$\dot{a}(t) = -\lambda a(t) + \nu(t)$$

is interpreted as a stochastic differential equation by modeling the uncertainty term $\nu(t)$ as white noise.Specifically, we write the parameter dynamics in Itô form as

$$\mathrm{d}a(t) = -\lambda a(t)\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t$$

where W_t is a standard Brownian motion satisfying

$$\mathbb{E}[\mathrm{d}W_t] = 0, \qquad \mathbb{E}[(\mathrm{d}W_t)^2] = \mathrm{d}t$$

and $Q \geq 0$ represents the process noise intensity.

The observation model is given in algebraic form as

$$y(t) = \phi(t)a(t) + \epsilon(t)$$

where $\epsilon(t)$ is zero-mean measurement noise with variance $R > 0$.In continuous time, this is expressed in differential form as

$$\mathrm{d}y(t) = \phi(t)a(t)\mathrm{d}t + \sqrt{R}\mathrm{d}V_t$$

where V_t is an independent standard Brownian motion with

$$\mathbb{E}[\mathrm{d}V_t] = 0, \qquad \mathbb{E}[(\mathrm{d}V_t)^2] = \mathrm{d}t$$

We assume W_t and V_t are independent.

ii Gaussianity of the stochastic processes

We assume a Gaussian prior for the initial parameter value,

$$a(0) \sim \mathcal{N}(\bar{a}_0, P_0).$$

Since the SDE for $a(t)$ is linear and driven by Gaussian noise, its explicit solution can be written as

$$a(t) = \mathrm{e}^{-\lambda t}a(0) + \int_0^t \mathrm{e}^{-\lambda(t-s)}\sqrt{Q}\mathrm{d}W_s$$

Both terms on the right-hand side are linear functionals of Gaussian random variables; therefore $a(t)$ is Gaussian for all t

Similarly, the observation increment $\mathrm{d}y(t)$ is a linear function of $a(t)$ plus Gaussian noise.

Hence, for any finite collection of times, the random variables

$$\big(a(t),, y(t_1), \ldots, y(t_n)\big)$$

are jointly Gaussian.

Let $\mathcal{Y}_t := \sigma y(s) \mid 0 \leq s \leq t$ denote the observation history.

Then $a(t)$ and $\mathcal{Y}_t$ are jointly Gaussian in the sense that all finite-dimensional distributions are jointly Gaussian.

iii Estimation objective and MMSE criterion

At time t , we seek an estimate of $a(t)$ using only the available observation history $\mathcal{Y}_t$.
Any admissible estimator must be of the form $\eta(\mathcal{Y}_t)$.

The estimation objective is to minimize the mean-square error

$$\mathbb{E}\left[(a(t) - \eta(\mathcal{Y}_t))^2\right]$$

This defines a projection problem in the Hilbert space $L^2(\Omega)$of square-integrable random variables, equipped with the inner product

$$\langle X, Z\rangle := \mathbb{E}[XZ].$$

The set of all $\mathcal{Y}_t$-measurable random variables forms a closed linear subspace of $L^2(\Omega)$.

iv Orthogonal decomposition and Pythagorean identity

For any admissible estimator $\eta(\mathcal{Y}_t)$, we decompose the estimation error as

$$a(t)-\eta=\big(a(t)-\mathbb{E}[a(t)\mid\mathcal{Y}_t]\big)+\big(\mathbb{E}[a(t)\mid\mathcal{Y}_t]-\eta\big)$$

Squaring and taking expectations yields:

$$\begin{aligned}\mathbb{E}[(a-\eta)^2]=\mathbb{E}[(a-\mathbb{E}[a\mid\mathcal{Y}_t])^2]+\mathbb{E}[(\mathbb{E}[a\mid\mathcal{Y}_t]-\eta)^2]\\ +2\mathbb{E}\big[(a-\mathbb{E}[a\mid\mathcal{Y}_t])(\mathbb{E}[a\mid\mathcal{Y}_t]-\eta)\big]\end{aligned}$$

Using the tower property,

$$\mathbb{E}[Z]=\mathbb{E}[\mathbb{E}[Z\mid\mathcal{Y}_t]],$$

The cross term:

$$\mathbb{E}\big[(a-\mathbb{E}[a\mid\mathcal{Y}_t])(\mathbb{E}[a\mid\mathcal{Y}_t]-\eta)\big]=\mathbb{E}\big[\mathbb{E}\big[(a-\mathbb{E}[a\mid\mathcal{Y}_t])(\mathbb{E}[a\mid\mathcal{Y}_t]-\eta)\mid\mathcal{Y}_t\big]\big]$$

Given Y， $\mathbb{E}[a\mid\mathcal{Y}_t]-\eta$ can be confirmed，and

$$\mathbb{E}\big[a-\mathbb{E}[a\mid\mathcal{Y}_t]\mid\mathcal{Y}_t\big]=0$$

we obtain

$$\mathbb{E}[(a-\eta)^2]=\mathbb{E}[(a-\mathbb{E}[a\mid\mathcal{Y}_t])^2]+\mathbb{E}[(\mathbb{E}[a\mid\mathcal{Y}_t]-\eta)^2]$$

This is the Pythagorean identity in $L^2(\Omega)$.

v. Optimal estimator as conditional expectation

The first term in the above decomposition is independent of η, while the second term is nonnegative and vanishes if and only if

$$\eta(\mathcal{Y}_t)=\mathbb{E}[a(t)\mid\mathcal{Y}_t].$$

Therefore, the unique minimum–mean–square error (MMSE) estimator is

$$\hat{a}(t)=\mathbb{E}[a(t)\mid\mathcal{Y}_t]$$

This result does not rely on Gaussianity; it follows solely from the quadratic loss criterion and the geometry of $L^2(\Omega)$

To compute $\hat{a}(t)$ in continuous time, we exploit the fact that new information arrives through the observation increment $\mathrm{d}y(t)$.

The observation history satisfies

$$\mathcal{Y}_{t+\mathrm{d}t}=\sigma\big(\mathcal{Y}_t,\mathrm{d}y(t)\big)$$

By definition,

$$\hat{a}(t+\mathrm{d}t)=\mathbb{E}[a(t+\mathrm{d}t)\mid\mathcal{Y}_{t+\mathrm{d}t}]=\mathbb{E}[a(t+\mathrm{d}t)\mid\mathcal{Y}_t,\mathrm{d}y(t)]$$

Writing $a(t+\mathrm{d}t)=a(t)+\mathrm{d}a(t)$ and subtracting $\hat{a}(t)$, we obtain

$$\mathrm{d}\hat{a}=\hat{a}(t+\mathrm{d}t)-\hat{a}(t)=\mathbb{E}[a(t+\mathrm{d}t)\mid\mathcal{Y}_t,\mathrm{d}y(t)]-\mathbb{E}[a(t)\mid\mathcal{Y}_t]$$

$$\boxed{\mathrm{d}\hat{a}=\mathbb{E}[\mathrm{d}a\mid\mathcal{Y}_t,\mathrm{d}y]+\big(\mathbb{E}[a(t)\mid\mathcal{Y}_t,\mathrm{d}y]-\mathbb{E}[a(t)\mid\mathcal{Y}_t]\big)}$$

The first term represents the predicted drift of the estimate, while the second term represents the correction induced by the newly acquired observation increment.

Regardless of whether the condition is $\mathcal{Y}_t$ or $\mathrm{d}y$,the Brownian increment mean is 0.

$$\boxed{\mathbb{E}[\mathrm{d}a \mid \mathcal{Y}_t, \mathrm{d}y] = \mathbb{E}[-\lambda a(t)\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t \mid \mathcal{Y}_t, \mathrm{d}y] = -\lambda\mathbb{E}[a \mid \mathcal{Y}_t, \mathrm{d}y]\mathrm{d}t = -\lambda\hat{a}dt}$$

By the joint Gaussian conditional mean formula，

$$\mathbb{E}[\,x \mid y\,] = \mu_x + \Sigma_{xy}\,\Sigma_{yy}^{-1}\,(y - \mu_y)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}\right)$$

$$\mathbb{E}[\,x \mid y\,] = \mu_x + \frac{\mathrm{Cov}(x,y)}{\mathrm{Var}(y)}\,(y - \mu_y)$$

Conditioned on $\mathcal{Y}_t$, the pair (a(t) ,$\mathrm{d}y(t)$) is jointly Gaussian, since $\mathrm{d}y(t)$ is a linear function of $a(t)$ plus Gaussian noise

$$a(t) \mid \mathcal{Y}_t \sim \mathrm{Gaussian}$$

$$dy \mid \mathcal{Y}_t = \phi(t)\,a(t)\,dt + R^{1/2}\,dV_t, dV_t \sim \mathcal{N}(0,\ dt)$$

Hence,$\big(a(t),\ dy\big) \mid \mathcal{Y}_t$ is jointly Gaussian

$$\mu_a = \mathbb{E}[a(t) \mid \mathcal{Y}_t] = \hat{a}(t)$$

$$\mu_y = \mathbb{E}[dy \mid \mathcal{Y}_t]$$

$$P = \mathrm{Var}(a(t) \mid \mathcal{Y}_t)$$

$$R = \mathrm{Var}(dy \mid \mathcal{Y}_t)$$

$$C = \mathrm{Cov}(a(t), dy \mid \mathcal{Y}_t)$$

$$\begin{pmatrix} a(t) \\ dy \end{pmatrix} \Bigg| \mathcal{Y}_t \sim \mathcal{N}\left(\begin{pmatrix} \mu_a \\ \mu_y \end{pmatrix}, \begin{pmatrix} P & C \\ C^\top & R \end{pmatrix}\right)$$

$$\mathbb{E}[a(t) \mid \mathcal{Y}_t, dy] = \hat{a}(t) + \mathrm{Cov}(a(t), dy \mid \mathcal{Y}_t)\,\mathrm{Var}(dy \mid \mathcal{Y}_t)^{-1}\,(dy - \mathbb{E}[dy \mid \mathcal{Y}_t])$$

so,

$$\mathbb{E}[a(t) \mid \mathcal{Y}_t, \mathrm{d}y] - \mathbb{E}[a(t) \mid \mathcal{Y}_t] = \mathrm{Cov}(a(t), dy \mid \mathcal{Y}_t)\,\mathrm{Var}(dy \mid \mathcal{Y}_t)^{-1}\,(dy - \mathbb{E}[dy \mid \mathcal{Y}_t])$$

Hence, the conditional expectation $\mathbb{E}[a(t) \mid \mathcal{Y}_t, \mathrm{d}y]$ is linear in $\mathrm{d}y$ and can be expressed as a local regression on the innovation

$$\mathrm{d}\nu(t) = \mathrm{d}y(t) - \mathbb{E}[\mathrm{d}y(t) \mid \mathcal{Y}_t] = \mathrm{d}y(t) - \phi(t)\hat{a}(t)\mathrm{d}t$$

This yields the continuous-time update structure for the optimal estimator.

Conditioned on $\mathcal{Y}_t$,Noise increment $\mathrm{d}V_t$ and $a(t)$ are independent, Calculate the two conditional covariances:

$$\mathrm{d}y(t) = \phi(t)a(t)\mathrm{d}t + \sqrt{R}\mathrm{d}V_t$$

$$\boxed{\mathrm{Cov}(a(t), dy \mid \mathcal{Y}_t) = \mathrm{Cov}(a(t), \phi(t)a(t)\mathrm{d}t \mid \mathcal{Y}_t) = \mathrm{Cov}(a(t), a(t)\mathrm{d}t \mid \mathcal{Y}_t)\phi(t)^\top = P(t)\phi(t)^\top\mathrm{d}t}$$

where $P(t)\mathrm{d}t = \mathrm{Cov}(a(t), a(t)\mathrm{d}t \mid \mathcal{Y}_t)$

$$\mathrm{Cov}(a(t), a(t)\mathrm{d}t \mid \mathcal{Y}_t) = \mathrm{d}t\ \ \mathrm{Cov}(a(t), a(t) \mid \mathcal{Y}_t) = \mathrm{d}t\ \ \mathrm{Var}(a(t) \mid \mathcal{Y}_t).$$

calculate $R = \mathrm{Var}(dy \mid \mathcal{Y}_t)$

$$\boxed{R = \mathrm{Var}(dy \mid \mathcal{Y}_t) = \phi(t)a(t)\mathrm{d}t + \sqrt{R}\mathrm{d}V_t = Rdt + o(dt^2)}$$

Finally, we obtain the differential of the $\mathrm{d}\hat{a}$

$$\boxed{\mathrm{d}\hat{a} = -\lambda\hat{a}dt + P(t)\phi(t)^\top \mathrm{d}t(Rdt)^{-1}(\mathrm{d}y(t) - \phi(t)\hat{a}(t)\mathrm{d}t)}$$

vi Define the mean-square estimation error and derive $\dot{P}(t)$

Define the estimation error

$$e(t) = a(t) - \hat{a}(t)$$

We define the (scalar) error mean-square as

$$P(t) = \mathbb{E}\big[e(t)^2 \mid \mathcal{Y}_t\big]$$

$$P(t) = \mathbb{E}\big[e(t)^2 \mid \mathcal{Y}_t\big] = \mathrm{Var}(a(t) \mid \mathcal{Y}_t)\,.$$

(Equivalently, one may use the conditional version $P(t) := \mathbb{E}[e(t)^2 \mid \mathcal{Y}_t]$; in the linear–Gaussian setting this $P(t)$ turns out to be deterministic and satisfies the same Riccati ODE.

$$\mathrm{d}e(t) = \mathrm{d}a(t) - \mathrm{d}\hat{a}(t)$$

$$e_y(t) = y(t) - \phi(t)\hat{a}(t).$$

$$\mathrm{d}y(t) = \phi(t)a(t)\mathrm{d}t + \sqrt{R}\mathrm{d}V_t$$

Substitute the expressions above:

$$\mathrm{d}e(t) = \Big(-\lambda a(t)\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t\Big) - \Big(-\lambda\hat{a}(t)\mathrm{d}t + K(t)\big(\mathrm{d}y(t) - \phi(t)\hat{a}(t)\mathrm{d}t\big)\Big)$$

Group terms. First the drift terms:

$$-\lambda a(t)\mathrm{d}t + \lambda\hat{a}(t)\mathrm{d}t = -\lambda(a(t) - \hat{a}(t))\mathrm{d}t = -\lambda e(t)\mathrm{d}t$$

Then the correction drift:

$$-K(t)\big(\mathrm{d}y(t) - \phi(t)\hat{a}(t)\mathrm{d}t\big) = -K(t)\big(\phi(t)a(t)\mathrm{d}t + \sqrt{R}\mathrm{d}V_t - \phi(t)\hat{a}(t)\mathrm{d}t\big) = -K(t)\phi(t)e(t)\mathrm{d}t - K(t)\sqrt{R}\mathrm{d}V_t$$

Therefore the error SDE is

$$\mathrm{d}e(t) = -\big(\lambda + K(t)\phi(t)\big)e(t)\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t - K(t)\sqrt{R}\mathrm{d}V_t$$

$$\mathrm{d}(e^2) = 2e\mathrm{d}e + (\mathrm{d}e)^2$$

The term $2e\mathrm{d}e$. Using the error SDE,

$$2e\mathrm{d}e = 2e\Big(-(\lambda + K\phi)e\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t - K\sqrt{R}\mathrm{d}V_t\Big) = -2(\lambda + K\phi)e^2\mathrm{d}t + 2e\sqrt{Q}\mathrm{d}W_t - 2eK\sqrt{R}\mathrm{d}V_t$$

The quadratic variation term $(\mathrm{d}e)^2$. Since $\mathrm{d}t$is higher order compared to Brownian increments, we keep only the $\mathrm{d}W_t$ and $\mathrm{d}V_t$ parts:

$$\mathrm{d}e = \cdots + \sqrt{Q}\mathrm{d}W_t - K\sqrt{R}\mathrm{d}V_t$$

Thus,

$$(\mathrm{d}e)^2 = (\sqrt{Q}\mathrm{d}W_t - K\sqrt{R}\mathrm{d}V_t)^2 == Q(\mathrm{d}W_t)^2 - 2K\sqrt{QR}\mathrm{d}W_t\mathrm{d}V_t + K^2R(\mathrm{d}V_t)^2$$

Using independence of W_t andV_t,

$$\mathrm{d}W_t \mathrm{d}V_t = 0$$

and the Itô identities

$$(\mathrm{d}W_t)^2 = \mathrm{d}t, \qquad (\mathrm{d}V_t)^2 = \mathrm{d}t$$

we obtain

$$(\mathrm{d}e)^2 = (Q + K^2 R)\mathrm{d}t$$

$$\boxed{\mathrm{d}(e^2) = -2(\lambda + K\phi)e^2\mathrm{d}t + (Q + K^2 R)\mathrm{d}t + 2e\sqrt{Q}\mathrm{d}W_t - 2eK\sqrt{R}\mathrm{d}V_t}$$

Recall $P(t) = \mathbb{E}[e(t)^2]$. Taking expectation on both sides:

$$\mathrm{d}\mathbb{E}[e^2] = \mathbb{E}\big[\mathrm{d}(e^2)\big]$$

Now use the fact that Itô integrals have zero mean (under standard integrability conditions):

$$\mathbb{E}\big[e(t)\mathrm{d}W_t\big] = 0, \qquad \mathbb{E}\big[e(t)\mathrm{d}V_t\big] = 0.$$

Hence the stochastic terms vanish in expectation, yielding

$$\mathrm{d}P(t) = \Big(-2(\lambda + K(t)\phi(t))\mathbb{E}[e(t)^2] + Q + K(t)^2 R\Big)\mathrm{d}t$$

$$\dot{P}(t) = -2(\lambda + K(t)\phi(t))P(t) + Q + K(t)^2 R.$$

For the linear–Gaussian MMSE filter, the gain is

$$K(t) = \frac{P(t)\phi(t)}{R}.$$

Substitute into the ODE:

$$\dot{P}(t) = -2\lambda P(t) - 2\Big(\frac{P(t)\phi(t)}{R}\Big)\phi(t)P(t) + Q + \Big(\frac{P(t)\phi(t)}{R}\Big)^2 R.$$

Compute each term carefully:

The middle drift correction term:

$$-2\Big(\frac{P\phi}{R}\Big)\phi P = -\frac{2\phi(t)^2}{R}P(t)^2$$

The last term:

$$\Big(\frac{P\phi}{R}\Big)^2 R = \frac{\phi(t)^2}{R}P(t)^2$$

Combine them:

$$-\frac{2\phi(t)^2}{R}P(t)^2 + \frac{\phi(t)^2}{R}P(t)^2 = -\frac{\phi(t)^2}{R}P(t)^2$$

Therefore the final Riccati equation for the mean-square estimation error is

$$\boxed{\dot{P}(t) = -2\lambda P(t) + Q - \frac{\phi(t)^2}{R}P(t)^2.}$$

This equation shows explicitly how the uncertainty evolves:

- $-2\lambda P$ comes from the stable drift $-\lambda a$ (forgetting/decay of uncertainty),
- $+Q$ comes from process noise injection
- $-(\phi^2/R)P^2$ is the information gain from measurements (stronger sensing ϕ or smaller noise R reduces P faster).

vii Matrix generalization

Now let:

$$a(t) \in \mathbb{R}^h, \qquad \hat{a}(t) \in \mathbb{R}^h, \qquad \tilde{a} = \hat{a} - a \in \mathbb{R}^h,$$

and measurement: $y(t) \in \mathbb{R}^n, \qquad y = \phi\, a + \epsilon, \qquad \phi \in \mathbb{R}^{n \times h}.$

Noise covariances:

$$\mathbb{E}[\epsilon\epsilon^\top] = R, \quad R \in \mathbb{R}^{n\times n},\ R \succ 0, \qquad \mathbb{E}[\nu\nu^\top] = Q, \quad Q \in \mathbb{R}^{h\times h},\ Q \succeq 0.$$

The continuous-time Kalman–Bucy estimator has the form:

$$\boxed{\dot{\hat{a}} = -\lambda\hat{a} + P\phi^\top R^{-1}\big(y - \phi\hat{a}\big)}$$

and the covariance evolution becomes the matrix Riccati equation:

$$\boxed{\dot{P} = -2\lambda P + Q - P\phi^\top R^{-1}\phi P.}$$

The matrix Kalman gain is:

$$\boxed{K_a(t) = P(t)\phi(t)^\top R^{-1}.}$$

In our composite adaptation law, we will then add the tracking-driven term:

$$\dot{\hat{a}} = -\lambda\hat{a} + P\phi^\top R^{-1}\big(y - \phi\hat{a}\big) + P\phi^\top s.$$