A RELATION BETWEEN PERMEABILITY AND SIZE DISTRIBUTION OF PORES

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Summary

An equation has been derived for the relation between permeability and the size distribution of the pores in isotropic material. If the mean radius of the pores in each of n equal fractions of the total pore space is represented in decreasing order of size by $r_1, r_2, \ldots,$ and r_n cm. respectively, then permeability is given by

$$K = \epsilon^2 n^{-2}[r_1^2+3r_2^2+5r_3^2+\ldots+(2n-1)r_n^2]/8,$$

where ϵ is the porosity in cm.^{3}/cm.^{3} of porous material, and K is in units of cm.^{2}

By this means, permeability can be calculated from the curve relating water content to suction. This has been tested on published data for flow of air through porous stones and flow of water through saturated and unsaturated sands. Calculated values have been found to agree satisfactorily with measured values over a wide range of permeability.

The Kozeny equation is discussed as a particular case of this equation.

Introduction

EQUATIONS relating permeability to other measurable properties of porous materials are used in the various fields concerned with flow of water, oil, and gases through soils, porous rocks, and industrial products. The first equations to be developed dealt with an 'ideal' porous system composed of spherical grains of uniform size. In his analysis of the pore geometry of such a system, Slichter (1899) found that a conducting capillary tube was triangular in section and that along its curved length it passed alternately through maximum and minimum areas of cross-section of pore. If, however, the capillary was assumed to be straight instead of sinuous and to have a circular cross-section of the same area as the minimum triangular section, the errors involved in these substitutions were mutually compensating and an equation could be derived which related permeability to porosity and particle diameter. Green and Ampt (1912) found that, although it was derived for an ideal system, Slichter's equation applied better to beds of sand grains of irregular shape (which it fitted very well) than to beds composed of glass spheres. Subsequently Smith (1932) showed that Slichter's analysis was at fault because the ratio of an average to a minimum section depends on porosity, and, using the mean triangular section as the basis, he then obtained an equation which fitted Green and Ampt's data satisfactorily. However, he found that a better fit for the sand grains of irregular shape was obtained when the minimum rather than the mean triangular section was used.

The work of Kozeny, Fair and Hatch, and Carman has led to the wide use of an equation, usually referred to as the Kozeny equation, in which an average size of conducting tube is obtained from the hydraulic radius (as given by porosity, ϵ, in cm.^{3}/cm.^{3} of bed divided by the surface area, S, of the particles in cm.^{2}/cm.^{3} of bed). This equation is fully dealt with by Carman (1937, 1956), Dallavalle (1948), and Childs and Collis-George (1950). It may be written in the form

$$K = \epsilon^3/S^2k, \tag{1}$$

where K is the permeability in cm.^{2} and k is an empirical constant. The constant, k, is usually given a value of 5, but there is evidence of variation especially in consolidated materials of low porosity (Wyllie and Rose, 1950; Wyllie and Spangler, 1952; Carman, 1956). There is serious difficulty in applying the equation to material containing widely different pore sizes especially when numerous small particles which contribute largely to surface area may together act towards fluid flow as a single aggregate of much smaller area.

With the development of suction and associated methods for measuring size distribution of pores (Richards, 1949), there has come the possibility of using pore size instead of particle properties as the basis for a permeability expression. Childs and Collis-George (1950) have developed an equation which takes account of the various sequences of pores of different sizes when pores are randomly arranged as in a porous system. An empirical factor is required in their equation and this has not been widely tested for constancy in different materials.

Derivation of Equation

In the present treatment, the assumption is made that the rate of flow is controlled by the cross-sectional area of the necks connecting the pores. The cross-sectional area of a neck, when the fit of one pore to the next is perfect, is taken to be that of the smaller pore. Since the fit is often imperfect, allowance is made for a reduction in the size of the neck when the two pores, in random alignment, do not fit perfectly on to one another. A mean cross-sectional area of neck is thus obtained and this is taken to represent that of a tube through which flow occurs. The error in neglecting the widening of the path of flow within pores will be in the opposite sense to the errors arising from the assumptions which will also be made that the necks are circular in section, that there is only one neck at the end of the smaller pore of a sequence, and that the path of flow is not lengthened by imperfect alignment of the necks in the apparent direction of flow. It is also assumed that, in isotropic material of porosity ϵ cm.^{3}/cm.^{3}, the fractional pore area in any plane is ϵ cm.^{2}/cm.^{2} This is discussed by Carman (1956, p. 8) as an acceptable assumption when the pore space is randomly distributed.

According to Poiseuille's equation for stream-line flow, the mean velocity, u' cm./sec., of a fluid in a narrow tube of radius r_t cm. is

$$u' = -(r_t^2\, d\phi/dl)(1/8\eta), \tag{1}$$

in which $d\phi/dl$ is the potential gradient causing flow and η is the viscosity. If the cross-section through a porous material shows a portion, ϵ, to be made up of the cross-sections of tubes of radius r_t, the velocity, u', in this portion will be equal to u''/ϵ, where u'' is the apparent linear velocity for the porous medium as a whole. Hence

$$u'' = -(\epsilon r_t^2 d\phi/dl)(1/8\eta). \tag{2}$$

In the present solution a value for r_t is found which is appropriate to isotropic porous materials.

If the two surfaces, A and B, exposed by a section through an isotropic porous material are rejoined randomly, the pores on these surfaces will be connected by necks whose area of cross-section can be considered in the following way. Unit area of each of the exposed surfaces can be regarded as being made up of n portions each of the same area, $1/n$, and porosity, ϵ cm.^{2}/cm.^{2}, and each containing pores of one mean radius $r_1, r_2, \ldots$ and r_n cm. respectively, where $r_1 > r_2 > \ldots > r_n$. That portion of surface A which has pores of radius r_1 will in effect contact n portions of surface B each of area $1/n^2$ and each containing pores of one size $r_1, r_2, \ldots$, and r_n respectively. The area of neck resulting from perfect pore-to-pore contacts will be that of the smaller pore. However, the smaller pore can fit against more or less of the solid matrix instead of wholly against a pore on the opposite surface and, on the average, the area of contact will be ϵ times the area of the smaller pore. Hence the mean neck area for each of the n portions of surface B in contact with this first portion of surface A will be $\epsilon\pi r_1^2, \epsilon\pi r_2^2, \ldots$, and $\epsilon\pi r_n^2$ respectively. Similarly that portion of A containing pores of size r_2 will provide neck areas of $\epsilon\pi r_2^2, \epsilon\pi r_2^2, \epsilon\pi r_3^2, \ldots$, and $\epsilon\pi r_n^2$ respectively for another n portions of surface B (each of area $1/n^2$) with which it makes contact. The series is continued in this way until the nth portion of surface A containing pores of radius r_n has been considered. This last portion will provide a neck area of $\epsilon\pi r_n^2$ for each of the n portions of surface B with which it makes contact. The average area of cross-section of neck for all the n^2 portions of surface B is then

$$\epsilon\pi[(r_1^2+r_2^2+\ldots+r_n^2)+(2r_2^2+r_3^2+\ldots+r_n^2)+\ldots+nr_n^2]/n^2$$

or

$$\epsilon\pi n^{-2}[r_1^2+3r_2^2+5r_3^2+\ldots+(2n-1)r_n^2].$$

This area of neck is taken to represent the cross-sectional area of a tube of radius r_t controlling flow through the porous material. Hence

$$r_t^2 = \epsilon n^{-2}[r_1^2+3r_2^2+5r_3^2+\ldots+(2n-1)r_n^2]. \tag{3}$$

On substituting for r_t^2 in equation (2), the linear velocity, u, in a direction normal to any cross-section of the porous material is obtained and is given by

$$u = -\epsilon^2 n^{-2}(d\phi/dl)[r_1^2+3r_2^2+5r_3^2+\ldots+(2n-1)r_n^2]/8\eta. \tag{4}$$

Since linear velocity represents the volume of fluid passing in unit time across unit cross-sectional area of the porous material, this result of the averaging process will be valid only if the averages are in all cases based on equal areas of the porous material. This condition has been met using equal fractions of the pore space as the basis. It is also met in obtaining the average neck area, $\epsilon\pi r^2$, for each of the n^2 portions of the contacting surfaces because each of the uniform smaller pores occupies with its proportion of solid matrix an equal area of surface and each is assumed to have one neck of radius between o and r.

According to Darcy's law for the flow of fluids through porous materials, the linear velocity is given by

$$u = -K\eta^{-1}\, d\phi/dl,$$

where K cm.^{2} is the permeability. Hence, from equation (4),

$$K = \epsilon^2 n^{-2}[r_1^{\,2}+3r_2^{\,2}+5r_3^{\,2}+\ldots+(2n-1)r_n^{\,2}]/8. \tag{5}$$

Equation (5) makes it possible to calculate permeability of porous material from the size distribution of its pores.

One special application is to the flow of liquids through unsaturated material. Here the liquid-filled pores are the conducting pores and those filled with air are excluded from the calculation so that ϵ in equation (5) is replaced by the volume concentration, c (c.c. liquid/c.c. porous material). Unsaturated permeability is difficult to measure and a relation enabling it to be calculated at various concentrations should be of value. Even saturated permeability poses experimental difficulties due to air blockages, leakage between the sample and the container, and the disturbance of the structure in the process of measuring permeability when water is used as the fluid. Pore-size measurements can be made as a routine matter in non-swelling materials by measuring the water withdrawn when the suction on the water is progressively increased. The radius of the largest pores remaining full of water when a suction of h cm. of water is applied is given by

$$r = 2\gamma/\rho g h, \tag{6}$$

where γ is the surface tension of water, ρ is its density, and g the acceleration due to gravity.

In applying equation (5) it will often be convenient to use the suction in place of the pore radius. From (6), $r^2 = 2{\cdot}25\times 10^{-2}h^{-2}$ at 20° C. Hence substituting in (5),

$$K = 2{\cdot}8\times 10^{-3}\epsilon^2 n^{-2}[h_1^{-2}+3h_2^{-2}+5h_3^{-2}+\ldots+2(n-1)h_n^{-2}], \tag{7}$$

where h_1, h_2, ..., and h_n represent the suction in the equal classes; h_1, corresponding to r_1, belongs to the class with the largest pores and h_n, corresponding to r_n, belongs to the class with the smallest. In equations (5) and (7) K is given as intrinsic permeability in units of cm^2. K may be obtained for flow of water in the units cm./sec. ('hydraulic conductivity') if the numerical constant is multiplied by g/η. This then becomes $2{\cdot}7\times 10^2$ at 20° C. in equation (7).

Relation of Equation (5) to the Kozeny Equation

In the particular case of material of uniform pore radius, r_u, equation (5) reduces to a form similar to the Kozeny equation (1). For pores of uniform size, equation (5) becomes

$$K = \epsilon^2 r_u^{\,2}/8,$$

and, if the pores are cylindrical tubes whose surface area per unit volume of porous material is given by $S = 2\epsilon/r_w$, then

$$K = \epsilon^4/2S^2. \tag{8}$$

Equation (8) resembles equation (1), but if they are to be identical the Kozeny–Carman constant, k, in equation (1) must be equal to $2/\epsilon$. If k is given its usual value of 5 for unconsolidated materials (Carman, 1937; 1956), permeability calculated by the two equations (1) and (8) will be the same when $\epsilon = 0{\cdot}4$.

This raises the question of the constancy of k. It is generally accepted that values of k depart seriously from 5 in consolidated materials which therefore cannot be handled by the Kozeny equation unless k is first determined for the material. Even in unconsolidated materials, Wyllie and Gregory (1955) found that k varied inversely with porosity but was about 5 at $\epsilon = 0{\cdot}4$. The present conclusion that $k = 2/\epsilon$ conforms fairly well to their experimental evidence.

Method of Calculation

Values for r^2 or $1/h^2$ for use in equations (5) and (7) can be obtained for each of n classes from the corresponding areas under the cumulative curve for pore space plotted against r^2 or $1/h^2$. However, if n is sufficiently large, r_1, r_2, ..., and r_n or h_1, h_2..., and h_n are given with little error by the radius or suction corresponding to the mean water content of each class (i.e. the median radius or suction in each class).

The method of calculating permeability may be illustrated using the data of Day and Luthin (1956), for Oso Flaco fine sand. The water content, c, at zero suction in their packed sand column was 0·416 c.c./c.c. In order to calculate the permeability at this water content, the median suction, h, corresponding to the middle of each of n equal classes of porosity is read from the suction curve of Fig. 1. In the example given, n is taken as 14, corresponding to a porosity class interval, c/n, which is approximately equal to 0·03. The calculation then proceeds as in Table 1 and the value obtained for the sum of products is used in equation (7) so that

$$K = 2{\cdot}8\times10^{-3}\times(0{\cdot}416)^2\times(14)^{-2}\times102{\cdot}9\times10^{-3}\text{ cm.}^2$$
$$= 2{\cdot}5\times10^{-7}\text{ cm.}^2$$

The permeability as measured by Day and Luthin was $1{\cdot}6\times10^{-7}$ cm.^{2}

The larger the value taken for n, the more accurate will be the calculation. However, in the case of the Day and Luthin data, the calculated values of K at $c = 0{\cdot}416$ varied very little when a low value of $n = 8$ and a high value of $n = 40$ were used.

The permeability to water at water contents less than 0·416 (i.e. in the unsaturated material) can be calculated from the data of Table 1. For example, if the pores in the first three porosity classes are taken to be air-filled so that the water content, c, is reduced to 0·327 and, if n is taken as 11 instead of 14, the values given for $1/h^2$ in Table 1 can be used directly. The sum of products then becomes

$$(1\times1{\cdot}06+3\times0{\cdot}94+5\times0{\cdot}84+7\times0{\cdot}77+9\times0{\cdot}70+11\times0{\cdot}63+$$
$$+13\times0{\cdot}55+15\times0{\cdot}47+17\times0{\cdot}37)\times10^{-3}\text{ or }47{\cdot}2\times10^{-3}.$$

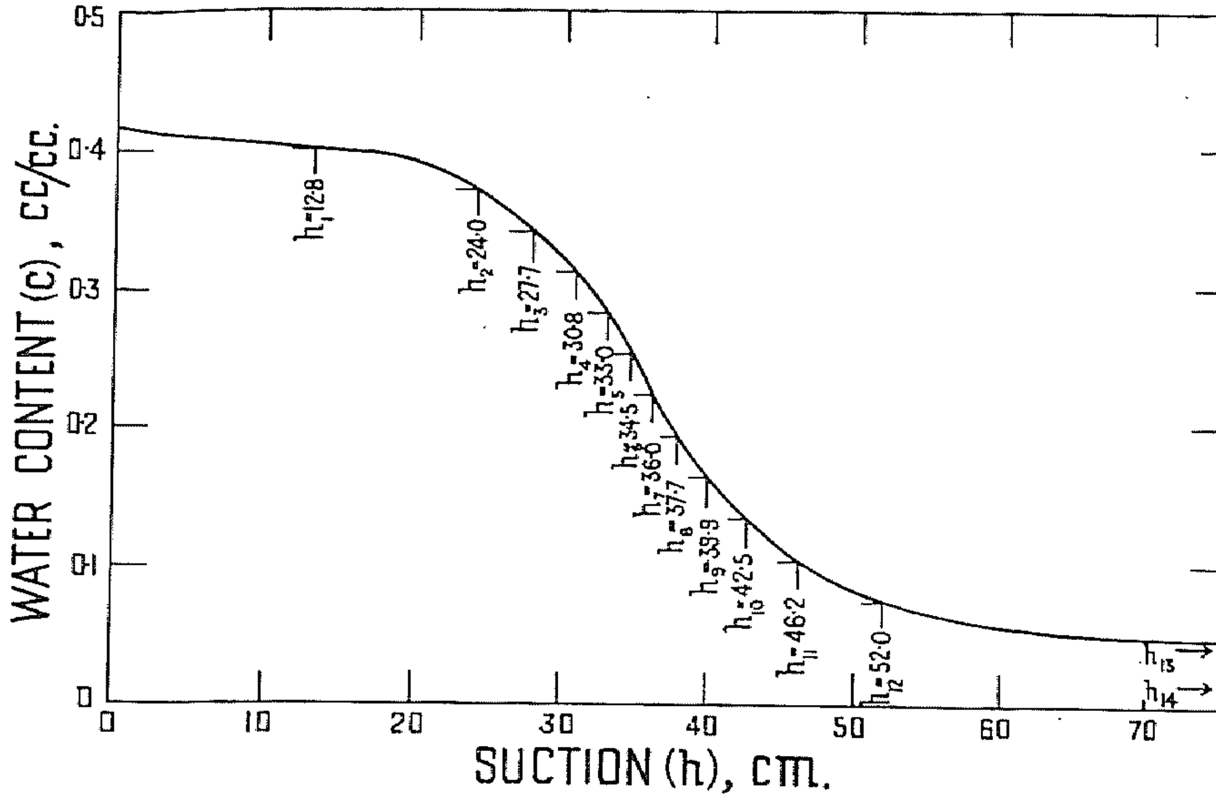

FIG. 1. Water content–suction data for Oso Flaco fine sand. The median suction of each of fourteen equal porosity classes is shown (i.e. n is here taken as 14)

TABLE 1

Calculation of Permeability of a Fine Sand of Porosity 0·416 from the Data of Fig. 1

Porosity class	*h*	$1/h^2$	*Multiplier*	*Product*
1	12·8	$6{\cdot}11 \times 10^{-3}$	1	$6{\cdot}1 \times 10^{-3}$
2	24·0	1·74 ,,	3	5·2 ,,
3	27·7	1·30 ,,	5	6·5 ,,
4	30·8	1·06 ,,	7	7·4 ,,
5	33·0	0·94 ,,	9	8·5 ,,
6	34·5	0·84 ,,	11	9·2 ,,
7	36·0	0·77 ,,	13	10·0 ,,
8	37·7	0·70 ,,	15	10·5 ,,
9	39·9	0·63 ,,	17	10·7 ,,
10	42·5	0·55 ,,	19	10·4 ,,
11	46·2	0·47 ,,	21	9·9 ,,
12	52·0	0·37 ,,	23	8·5 ,,
13	large	. .	25	. .
14	large	. .	27	. .
			Sum of products =	$102{\cdot}9 \times 10^{-3}$

From this, the permeability can be calculated as

$$K = 2{\cdot}8\times10^{-3}\times(0{\cdot}327)^2\times(11)^{-2}\times47{\cdot}2\times10^{-3} = 1{\cdot}15\times10^{-7}\ \text{cm.}^2$$

At $c = 0{\cdot}327$ the measured value of K from Day and Luthin's permeability data is $1{\cdot}0\times10^{-7}$ cm.^{2}

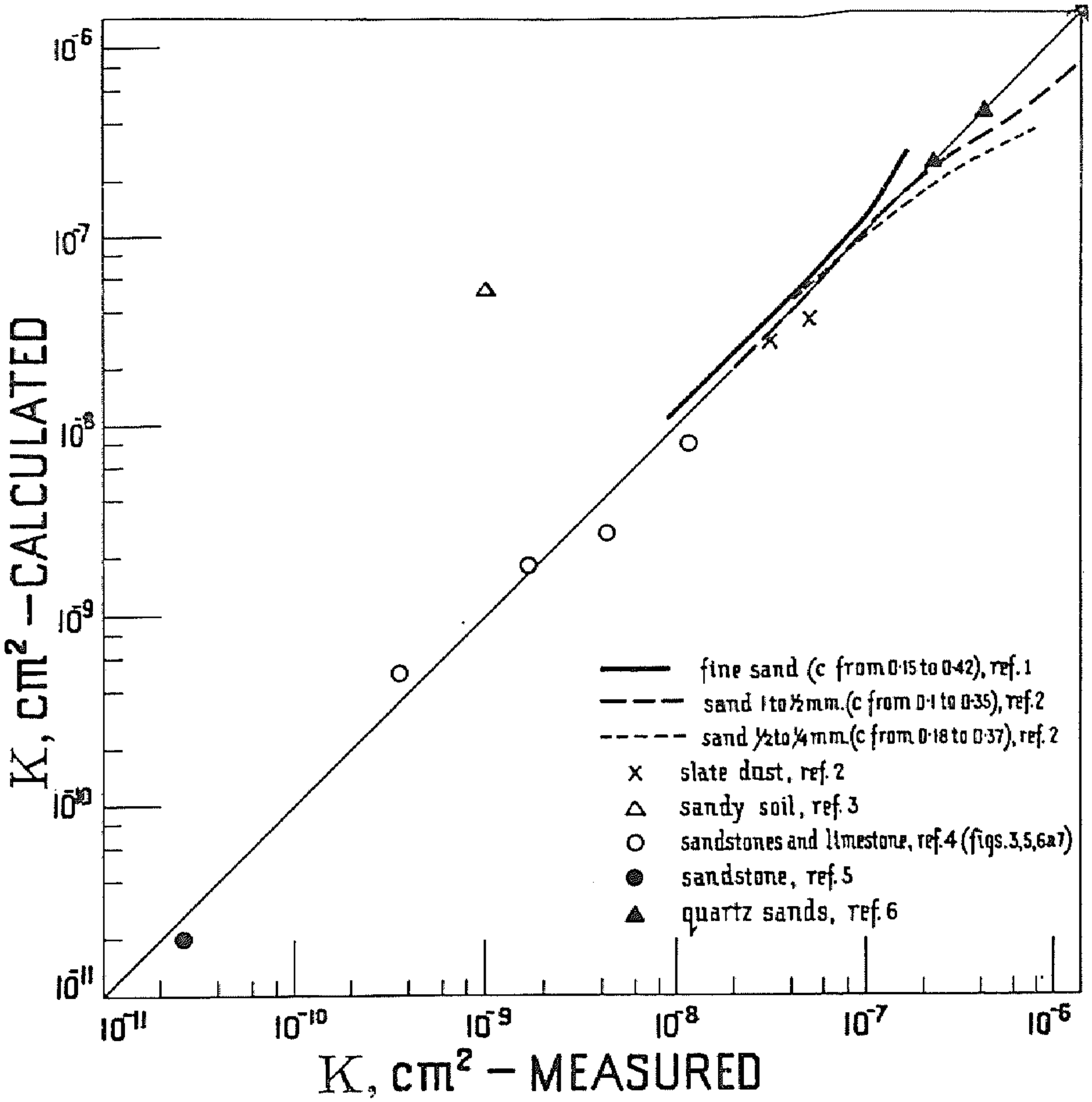

FIG. 2. Comparison of measured permeability with permeability calculated from the size distribution of pores. References: (1) Day and Luthin (1956); (2) Childs and Collis-George (1950); (3) Moore (1939); (4) Purcell (1949); (5) Wyllie and Spangler (1952); (6) Baver (1938), taking 3 in. as the diameter of Baver's permeameter (see Smith, Browning, and Pohlman, 1944, Soil Science, 57, 197–213)

Comparison of Measured and Calculated Permeability

Permeabilities calculated in this way are compared in Fig. 2 with measured values on sands and porous stones. For this purpose, published data have been used in which permeability and a water content–suction curve of sufficient detail are given for the same material. A large range of measured permeability from $2{\cdot}7\times10^{-11}$ to $1{\cdot}3\times10^{-6}$ cm.^{2} is represented. Permeability was variously measured using air (refs. 4 and

5 in Fig. 2) or water (refs. 1, 2, 3, and 6) as the fluid and the data from refs. 1 and 2 cover the permeabilities for a given material to water when at various degrees of unsaturation. Agreement between calculated and measured values is satisfactory as is shown by the fit of the data to the straight line representing equality. In any pair of the values obtained by calculation and by measurement, the larger is not more than 2·3 times the smaller except in one case. The exceptional point is for Oakley sand at $c=0{\cdot}23$, and since Moore's measured permeability up to saturation is so much lower than that of other comparable sands, the discrepancy may be due to error in measuring permeability.

The equation appears to have rather general application within the limitations imposed in its derivation. These are principally that the material should be isotropic and should contain no lengthy conducting channels. In order to apply the equation, it is necessary to have a reliable measurement of size distribution of pores and, since this is usually done by suction methods, the accuracy of these in swelling materials may limit the accuracy of the calculation for soils of moderate or high clay content.

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