# Notes on the Null Space

Hi all,

Thanks for all of your great participation and questions in class today, and sorry we didn't have time to cover a concrete example of column/row/null space before the end of class. I did want to follow up and address a question regarding the null space and equation

$$A\vec{x} = \vec{0} \tag{1}$$

First, to reiterate, the **null space** is defined as the set of all possible vectors $\{\vec{x}\}$ that satisfy the above equation. We also saw how the trivial solution, $\vec{x} = \vec{0}$, is *always* in the null space.

Now, if $\vec{0}$ is the *only* vector in the null space, then the system $A\vec{x} = \vec{0}$ has a unique solution (that of $\vec{x} = \vec{0}$). We saw with linear independence how this means that the columns of $A$ form a **linearly independent** set of vectors, and we have that $\text{rank}(A) = n$.

*The potential confusion I caused*: We also learned last week that if $r = n$ and *if the matrix $A$ is square*, then the general equation $A\vec{x} = \vec{b}$ will also have a unique solution when the null space only contains the zero vector. We can see this by applying what we learned today about the **basis**. Since the $n$ columns of $A$ form a linearly independent set of vectors that **span** $\mathbb{R}^n$ when $A$ is square, these vectors are thus a basis for $\mathbb{R}^n$ and any vector $\vec{b}$ can be formed (uniquely) by setting the entries of $\vec{x}$ to form the appropriate linear combination of the columns of $A$. If $A$ is not square, then all bets are off, but hopefully this clarifies my confusion during lecture.

We didn't have a whole lot of time to address this part during lecture, but the null space is also a subspace of $\mathbb{R}^n$, and for a matrix of rank $r$, the dimension of its null space (i.e. the number of basis vectors that span $\text{null}(A)$) is equal to $n - r$. I also mentioned that if the null space has more than just the zero vector, then we have infinitely many solutions to the system $A\vec{x} = \vec{0}$. I hope this helps summarize a bit of what was covered today. Today's lecture was definitely theory heavy, and if you have any questions, please do not hesitate to ask in OH or post to Piazza.

Cheers,

Enze

P.S. If you were curious as to why having a non-zero vector in the null space leads to infinitely many solutions, note that we now have some vector $\vec{w} \in \text{null}(A),\ \vec{w} \neq \vec{0}$ such that $A(\alpha \cdot \vec{w}) = \vec{0}$. I'm being general with the $\alpha$ here because of subspace property #3. So, if we found a particular solution to $A\vec{x}_p = \vec{b}$, we can manipulate it to get:

$$\begin{aligned} A\vec{x}_p &= \vec{b} \\ A\vec{x}_p + \vec{0} &= \vec{b} \\ A\vec{x}_p + A(\alpha \cdot \vec{w}) &= \vec{b} \\ A(\vec{x}_p + \alpha \cdot \vec{w}) &= \vec{b} \end{aligned}$$

This shows that if we have one solution $\vec{x}_p$, we can add *any scalar multiple* of $\vec{w}$ to our solution to obtain another solution. Since there are infinite choices for $\alpha \in \mathbb{R}$, we have infinite solutions to $A\vec{x} = \vec{b}$. This flexibility can be leveraged in system design, for example, by optimizing $\alpha$ and choosing the best solution $\vec{u} = \vec{x} + \alpha \cdot \vec{w}$ to optimize some objective, such as energy consumption, cost, weight, etc. $\square$