CME 104 midterm review

Enze Chen (enze)

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1 Purpose

In this review session I hope to accomplish the following:

- Review the content that we've covered so far
- Give some test taking strategies or tips.
- Answer any remaining questions you might have.

2 Midterm logistics

- Plan for (unless something happens that I can't foresee) Thursday, May 10, in class from 4:30pm - 6:20pm
- One double-sided page of notes. There are some formulas on the back of the test (same as last year). *No* calculator.
- Take a look at last year's midterm. The questions and test will likely be very similar in format. Content is everything through lecture 9.
- Mon/Tue OH are the same, but we will hold different OH on Wednesday. Homework 5 is due *next* Thursday.
  - Enze: Wednesday, 3:00-4:20pm in 200-124 (instead of 7:00-9:00pm)
  - Elise: Wednesday, 5:30-7:00pm in 200-219

3 Linear Algebra

Problem 1 - LU decomposition

We are given a matrix

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 6 & 8 \\ 3 & 2 & 18 \end{bmatrix}$$

a) Find a LU decomposition of A.

Solution:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 4 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & -8 \end{bmatrix}$$

How do we know if this is right? Multiply and check!

b) Find a basis for the row space and a basis for the column space of A.

Solution: **basis** $\stackrel{\text{def}}{:=}$ A linearly independent set of vectors that span a vector space.

The row space is given by the rows of A corresponding to independent rows of U, and the basis for column space is the columns of A corresponding to independent columns of U.

c) What is the rank of A?

Solution: 3.

d) What is the dimension of the null space of A? What is the null space of A?

Solution: $\dim(\text{null}(A)) = 0$. $\text{rank}(A) + \dim(\text{null}(A)) = n$, **always**. The null space is $\{\vec{0}\}$.

e) Solve the system $A\vec{x} = \vec{b}$ for $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

Solution: $\vec{x} = \begin{bmatrix} -\frac{5}{2} \\ 1 \\ \frac{1}{4} \end{bmatrix}$.

f) What is the determinant of $A^T A$?

Solution: $\det\left(A^T\right) = \det\left(A^T\right)\det(A) = [\det(A)]^2 = [\det(L)\det(U)]^2 = [\det(U)]^2 = 256$.

Problem 2 - Quickies

Vector space $\stackrel{\text{def}}{:=}$ Collection of vectors in $\mathbb{R}^n$ that satisfy three properties:

1. $\vec{0}$ is in the space.
2. Closed under scalar multiplication.
3. Closed under vector addition.

a) Is the space defined by $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; x_1 \leq x_2 \right\}$ a subspace of $\mathbb{R}^2$?

Solution: **No**, negative scalar multiplication.

b) Is the space defined by $V = \left\{ \begin{bmatrix} a + 2b \\ b \\ 0 \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$?

Solution: **Yes**, we can always redefine a and b to express new vectors in this form.

c) Which of the following must be true about square matrix $A \in \mathbb{R}^{n\times n}$ if A is invertible?

- It is full rank. TRUE.
- Its null space is more than just $\{\vec{0}\}$. FALSE.
- $\det(A) = 0$. FALSE.
- A has a zero eigenvalue. FALSE.
- A is non-singular. TRUE.
- The columns of A form a linearly independent set of vectors. TRUE.
- The dimension of the row space of A is n. TRUE.
- The equation $A\vec{x} = \vec{b}$ always has a (unique) solution for $\vec{x}$ given any $\vec{b}$. TRUE.

Problem 3 - Fibonacci sequence

The Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$.

a) Formulate this as a matrix-vector equation.

Solution: We currently have $F_{n+2} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$, which isn't quite enough, but we note that the vector $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$ is like incrementing the vector $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ forward one step. Therefore, we can add the trivial relationship $F_{n+1} = F_{n+1} + (0 \cdot F_n)$ to get the system

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

b) Diagonalize the matrix (i.e. find D **and** Q).

Solution: We subtract λ from the diagonal elements and find the determinant to obtain the *characteristic polynomial* of the matrix.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \longrightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 \implies \lambda = \frac{1 \pm \sqrt{5}}{2}$$

The eigenvectors are found as follows:

- $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$: $A - \lambda_1 I = \begin{bmatrix} 1 - \dfrac{1+\sqrt{5}}{2} & 1 \\ 1 & -\dfrac{1+\sqrt{5}}{2} \end{bmatrix} \longrightarrow \vec{v}_1 = \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$
- $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$: $A - \lambda_2 I = \begin{bmatrix} 1 - \dfrac{1-\sqrt{5}}{2} & 1 \\ 1 & -\dfrac{1-\sqrt{5}}{2} \end{bmatrix} \longrightarrow \vec{v}_1 = \begin{bmatrix} \dfrac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Consequently,

$$D = \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} & 0 \\ 0 & \dfrac{1-\sqrt{5}}{2} \end{bmatrix}, \qquad Q = \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} & \dfrac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

Problem 4 - ODEs and complex eigenvalues

Solve the following system: $\dfrac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix} \vec{x}$.

Solution: Assume $\vec{x} = C\vec{v}e^{\lambda t}$ to get eigenvalue problem. The eigenvalues are $\lambda = 2 \pm 2i$, the eigenvectors are $v_1 = \begin{bmatrix} 2 \\ i \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -i \end{bmatrix}$.

We only need to take one eigenvalue-eigenvector pair, so we'll choose the first one, giving us $\vec{x} = \begin{bmatrix} 2 \\ i \end{bmatrix} e^{(2+2i)t} = e^{2t} \begin{bmatrix} 2 \\ i \end{bmatrix} (\cos(2t) + i\sin(2t)) = C_1 e^{2t} \begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix}$

4 Physics

We've covered a lot of physics applications so far, so let's go over them. I don't know what's on the test, but I believe it will just be the basics (e.g. Ohm's law instead of RLC circuit, simple springs instead of angular momentum). More advanced equations should be supplied if necessary.

Mechanics

Forces are given by:

- Newton's second law: $F = m\dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2}$.
- Hooke's law: $F = -kx$
- Drag: $F = -c\dfrac{\mathrm{d}x}{\mathrm{d}t}$ (for slow movement)
- Torque: $\tau = F \times d$.

Electromagnetism

In circuits, we have

- **Ohm's law** $:\stackrel{\text{def}}{=} \Delta V = IR$.
- **current** $:\stackrel{\text{def}}{=} I = \dfrac{\mathrm{d}Q}{\mathrm{d}t}$.
- **Kirchhoff's current law** $:\stackrel{\text{def}}{=}$ At a junction, $\sum_k I_k = 0$. Sum of currents in = sum of currents out.
- **Kirchhoff's voltage law** $:\stackrel{\text{def}}{=}$ In a loop, $\sum_k V_k = 0$. By convention, voltage increases going up a voltage source, then decreases crossing a resistor when going with the current, and increases when going against the current.

Heat

- **heat flow** $:\stackrel{\text{def}}{=} q = -\dfrac{\partial Q}{\partial t} = -mc\dfrac{\partial T}{\partial t}$
- **Fourier's law** $:\stackrel{\text{def}}{=} q_x = -KA\left.\dfrac{\partial T}{\partial x}\right|_x$
- We assume if $q > 0$ heat is flowing *out*, and if $q < 0$, heat is flowing *into* the region.

Some things to keep in mind:

- In pretty much all cases, the absolute direction does not matter, but relative direction does matter.
- Recipe:
  1. Clearly define your system boundaries.
  2. Label all the parts (forces, current paths, heat sources/sinks, etc).
  3. Governing equation (Newton's second law, heat, etc.)
  4. Plug in expressions.

5 Fourier Series and PDEs

Problem 5 - Absolute value Fourier series

Find the Fourier series representation of $f(x) = L - |x|$ defined on the interval $-L < x < L$.

Solution: We're not actually going to do this. We'll simply make the following observations:

- Parity: This function is even. Therefore, there will not be a sine component to the Fourier series.
- Constant term: a_0 will be $\dfrac{L}{2}$ because the average value of this function is $\dfrac{L}{2}$.
- Symmetry: In fact, we can do something like $a_n = \dfrac{2}{L}\displaystyle\int_0^L (L-x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x$.

Some general tips for Fourier series:

- Leverage symmetry, especially to save time.
- Integration by parts, do tabular integration if you know it.
- Remember that $\left\{\cos\left(\dfrac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ and $\left\{\sin\left(\dfrac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ are mutually orthogonal.
- If the function you're trying to approximate is already sinusoidal, then there will only be a single term (i.e. frequencies always match). It's not just an arbitrary cancellation of sums and trig functions.
- Be aware to split up the domain if different functions are given.

Problem 5 - Simple 1D heat equation

Solve the following PDE:

$$\frac{\partial u}{\partial t} = \lambda^2 \frac{\partial^2 u}{\partial x^2}$$

with BCs and IC

$$u(x=0,t) = 0, \quad \frac{\partial u(x=L,t)}{\partial x} = 0, \quad u(x,t=0) = 1$$

Let's break this down into some principled steps:

(1) **Use separation of variables and solve each variable independently (as an ODE)**. We use separation of variables ($u(x,t) = F(x)G(t)$) and assume each part is equal to $-k^2$. This gives us

$$F''(x) + k^2 F(x) = 0$$
$$G'(t) + \lambda^2 k^2 G(t) = 0$$

(2) **Construct a "candidate" full solution**. The full solution is

$$u(x,t) = (A\cos(kx) + B\sin(kx))e^{-\lambda^2 k^2 t} + Cx + D$$

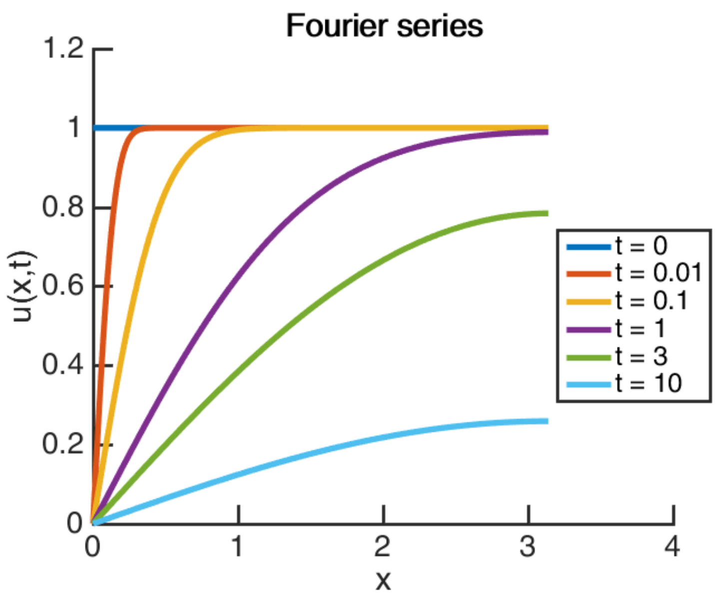
(3) **Apply BCs to eliminate / simplify constants** We use the BCs to get that $A = D = 0$, then $C = 0$ and $k = \left(n - \frac{1}{2}\right)\frac{\pi}{L}$.

(4) **Write down the actual, simplified full solution with the unknown coefficient.** The overall solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n^* \sin\left(\frac{(n-1/2)\pi x}{L}\right) \exp\left(-\frac{\lambda^2 (n-1/2)^2 \pi^2}{L^2} t\right)$$

(5) **Use integration of IC to figure out said unknown coefficient.** ...where

$$B_n^* = \frac{2}{L}\int_0^L \sin\left(\frac{(n-1/2)\pi x}{L}\right) \mathrm{d}x$$



General tips for PDEs:

- If a figure isn't provided, draw the domain and label all BCs.
- Figure out the correct governing PDE, if it isn't provided. Is there time dependence? Is there a non-homogeneous term?
- Use BCs to eliminate / simplify constants. Start with the BCs that equal zero. Probably Dirichlet before Neumann. Then constants before functions.
- If you solve for a constant, such as k, you use that value for k everywhere it appears.
- Then use IC (for time-dependent problems) to solve for Fourier coefficients.
- There are many moving parts, so write slowly and clearly. **Keep the end goal in mind** (solution for $u(x,t)$) even as you're deriving the sub-parts (i.e. solving for $F(x)$ / $G(t)$, B_n^*, etc).

6 Summary

What you need to know:

- Row reduction / Gaussian elimination / LU decomposition. Whether you're given a matrix, a set of vectors, or system of equations, it's all the same problem. You need to know how to do this, quickly and error-free.
- Matrix and vector properties and their disguises.
- A little bit of physics. Go back to basics and isolate your system.
- How to obtain Fourier series coefficients. Look for symmetry in the problem. Integration by parts.
- How to solve "simple" PDEs using separation of variables. Apply BCs first to eliminate constants, then ICs.
- Need more problems? Search online. Paul's Online Math Notes are also pretty good.

What we did not cover:

- Matrix inverse: Gauss-Jordan elimination—lame.

  But let's go over $(AB)^{-1} = B^{-1}A^{-1}$. Similar with transposes. Also $(A^T)^{-1} = (A^{-1})^T$ and $(A^n)^{-1} = (A^{-1})^n$ which you showed on the homework.
- Pseudoinverse—see notes. Discussed Midterm Problem 1g during review.
- Complex Fourier coefficients—a worked example is on the last page.

Bonus Material

Tabular integration

Given an integral $\int u \, \mathrm{d}v$, integration by parts tells us that $\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$. When we keep expanding this, it can be tricky to keep track of signs, constants of integration, and integration limits. **Tabular integration** is a simplified way to do integration by parts when certain conditions are met. Given an integral that can be expressed as

$$\int_a^b f(x)g(x) \, \mathrm{d}x$$

we should consider tabular integration if the following hold:

- $f(x)$ has derivatives that eventually go to 0.
- $g(x)$ is integrable (and hopefully "nice").

The steps to tabular integration are as follows:

(1) Put $f(x)$ and $g(x)$ into two columns, $f(x)$ on the left.

(2) Take derivatives of $f(x)$ until you get to 0.

(3) Integrate of $g(x)$ the same number of times, without including integration constants.

(4) Starting from the top, alternately assign $+$ and $-$ signs for $f(x)$ and its derivatives, flipping signs on the terms as necessary.

(5) Starting from the top, multiply the ith term in the left column (now with correct sign) with the $(i+1)$th term in the right column.

(6) Sum all the products.

(7) Evaluate the expression at the limits $[a, b]$.

Example: $\displaystyle\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2L^3}{n^2\pi^2} \cos(n\pi)$

Complex Fourier series

Find the complex Fourier series of the sign function

$$f(x) = \textbf{sign}\,(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 0 & x = 0 \\ 1 & 0 < x \leq \pi \end{cases}$$

The general equation for complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{L}\right)$$

We calculate the coefficients c_0 and c_n. Since the average value of the sign function is 0, we have $c_0 = 0$. For c_n, we have

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(-\frac{in\pi x}{L}\right)\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-inx}\,\mathrm{d}x + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\left[-\frac{e^{-inx}}{in}\bigg|_{-\pi}^{0} + \frac{e^{-inx}}{-in}\bigg|_{0}^{\pi}\right] \\ &= \frac{i}{2n\pi}\left[-(1-e^{in\pi}) + e^{-in\pi} - 1\right] \\ &= \frac{i}{2n\pi}\left[2\cos(n\pi) - 2\right] \\ &= \boxed{-\frac{2i}{n\pi}, \quad n \text{ odd}} \end{aligned}$$

Therefore,

$$f(x) = \textbf{sign}\,(x) = \boxed{\sum_{n=-\infty;\ \text{odd}}^{\infty} -\frac{2i}{n\pi}\exp(inx)}$$