CME 100 Tutorials 8 and 9

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1 Tutorial 8

1.1 Del operator

$\vec{\nabla}$ is a vector of partial derivatives. In 3D, it looks like

$$\vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

If you think about it like this, taken with the vector field

$$\vec{F} = \begin{bmatrix} M & N & P \end{bmatrix}$$

the expressions we've seen make a lot of sense. Divergence for example, is a dot product, which produces *a scalar*. Gradient, for example, is a scalar times a vector, so we multiply each component $\vec{\nabla} f$. Today, we're going to learn about something called curl, which is a cross product. Let's set it up!

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Oh look, we've seen this before, when I showed you the trick for conservative vector fields. But don't worry about it, the point is that I want to make the del operator more intuitive. You won't be tested on this theory, but hopefully this builds confidence that you've chosen the correct form.

1.2 Example with finding scalar potential

16.3, #19: Compute the following integral:

$$\int_{(1,1,1)}^{(1,2,3)} 3x^2 \, \mathrm{d}x + \frac{z^2}{y} \, \mathrm{d}y + 2z \ln(y) \, \mathrm{d}z$$

There are many ways to do this integral. Note that we cannot readily integrate it because it's with respect to multiple variables.

Hard way: Parameterize in a single shot: $\vec{r}(t) = \begin{bmatrix} 1 & t+1 & 2t+1 \end{bmatrix}$, $0 \le t \le 1$.

Medium way: Parameterize in two pieces: $\vec{r}_1(t) = \begin{bmatrix} 1 & t & 1 \end{bmatrix}$, $1 \le t \le 2$ and $\vec{r}_2(t) = \begin{bmatrix} 1 & 2 & t \end{bmatrix}$, $1 \le t \le 3$. If you have to parameterize, do it this way. Piecewise is usually clean, but still, two integrals.

Easy way(?): Find scalar potential and plug in end points. We do this in multiple steps.

(1) Check to see if it's conservative. Note that $M = 3x^2$, $N = \frac{z^2}{y}$, and $P = 2z\ln(y)$. Use my trick with the curl.

(2) Integrate one part at a time to find $\vec{\nabla} f$.

(3) Finally, plug in the end points. You should get $\boxed{9\ln(2)}$.

1.3 Example with Green's Theorem

Consider the vector field $\vec{F} = \begin{bmatrix} x - y & x \end{bmatrix}$ and the unit circle $\vec{r} = \begin{bmatrix} \cos(t) & \sin(t) \end{bmatrix}$. Compute the flux of the field and the work done over the path (we've already solved this using parametrization).

First, we write down everything we know:

$$\begin{aligned}
M &= x - y = \cos(t) - \sin(t) \\
N &= x = \cos(t) \\
\mathrm{d}x &= \frac{\mathrm{d}\cos(t)}{\mathrm{d}t} = -\sin(t)\,\mathrm{d}t \\
\mathrm{d}y &= \frac{\mathrm{d}\sin(t)}{\mathrm{d}t} = \cos(t)\,\mathrm{d}t \\
\frac{\partial M}{\partial x} &= 1, \qquad \frac{\partial M}{\partial y} = -1 \\
\frac{\partial N}{\partial x} &= 1, \qquad \frac{\partial N}{\partial y} = 0
\end{aligned}$$

Then we compute the flux:

$$\begin{aligned}
\iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,\mathrm{d}A &= \iint_R 1 + 0\,\mathrm{d}A \\
&= \pi \\
\oint_C M\,\mathrm{d}y - N\,\mathrm{d}x &= \oint_C (\cos(t) - \sin(t))\cos(t)\,\mathrm{d}t - \cos(t)(-\sin(t))\,\mathrm{d}t \\
&= \int_0^{2\pi} \cos^2(t)\,\mathrm{d}t \\
&= \pi
\end{aligned}$$

Now we compute the work:

$$\begin{aligned}
\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,\mathrm{d}A &= \iint_R 1 - (-1)\,\mathrm{d}A \\
&= 2\pi \\
\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y &= \oint_C (\cos(t) - \sin(t))(-\sin(t))\,\mathrm{d}t + \cos(t)\cos(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (-\sin(t)\cos(t) + 1)\,\mathrm{d}t \\
&= 2\pi
\end{aligned}$$

Note: Green's theorem does not say that flux and work are the same thing(!!), we got different answers after all. It's simply giving us two ways to calculate each quantity, and we get a choice depending on the situation. Again, it relates line integrals over a closed path to double integrals over a region; that's all it says.

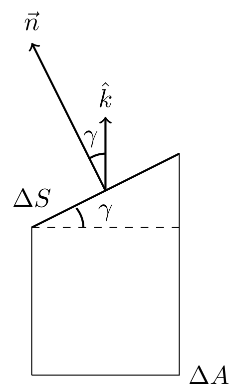

Figure 1: Here's an illustration of simple geometry. Hold up a sheet of paper, looking at it from the side.

1.4 Surface integrals motivation

Remember what integrals are: Infinite sum of really small pieces. We now have an integral of the form

$$\boxed{\iint_S g(x,y,z)\,\mathrm{d}S}$$

where the "small piece" is a piece of the surface and $g(x,y,z)$ is a function of 3D space (it is *not* the surface itself!).

Since our surface has the implicit parametrization $f(x,y,z) = 0$, we have that $\vec{n} = \vec{\nabla} f$ (see Lecture 8, page 63). This means our surface integral is now

$$\boxed{\iint_S g(x,y,z)\,\mathrm{d}S = \iint_R g(x,y,z)\frac{\left|\vec{\nabla} f\right|}{\left|\vec{\nabla} f \cdot \hat{k}\right|}\,\mathrm{d}A}$$

1.5 Divergence theorem motivation

Talk about exploding volcanoes.

$$\boxed{\oiint_S \vec{F} \cdot \vec{n}\,\mathrm{d}S = \iiint_V \vec{\nabla} \cdot \vec{F}\,\mathrm{d}V}$$

Surface integrals are usually very messy (there are gradients and norms), so this gives us another way to compute a surface integral over a *closed* surface. It is useful in cases where the surface is hard to specify, or perhaps easy to specify but has multiple parts (like a box).

2 Tutorial 9

2.1 Surface integral/Flux example

$$\boxed{\iint_S g(x,y,z)\,\mathrm{d}S = \iint_R g(x,y,z)\frac{\left|\vec{\nabla} f\right|}{\left|\vec{\nabla} f\cdot\hat{k}\right|}\,\mathrm{d}A}$$

Exercise: Find the surface area of the paraboloid cup defined by $z = 4 - x^2 - y^2,\ z \geq 0$.

Solution: Find f, $\vec{\nabla} f$, $\left\|\vec{\nabla} f\right\|$, and $\left|\vec{\nabla} f\cdot\hat{k}\right|$. Then integrate. The answer is $\frac{\pi}{6}(17^{3/2}-1)$.

2.2 Stokes' theorem and Curl review

$$\boxed{\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = \oint_C M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,\mathrm{d}S}$$

$$\text{curl} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

2.3 Divergence theorem example

$$\boxed{\oiint_S \vec{F}\cdot\vec{n}\,\mathrm{d}S = \iiint_V \vec{\nabla}\cdot\vec{F}\,\mathrm{d}V}$$

Exercise: Find the flux of the field

$$\vec{F} = \begin{bmatrix} x^2yz & xy^2z & xyz^2 \end{bmatrix}$$

going out of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, and $(1,1,1)$.

Solution: Instead of integrating four separate faces, we can find the divergence $\vec{\nabla}\cdot\vec{F} = 6xyz$ and then integrate over the volume

$$\int_0^1\int_0^x\int_0^y \vec{\nabla}\cdot\vec{F}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{8}$$

2.4 Stokes' Theorem example

Exercise: Find the work done by the field

$$\vec{F} = \begin{bmatrix} x-y+z & x+2y+3z & 1-xy \end{bmatrix}$$

moving ccw around the curve formed by cutting the paraboloid $z = x^2 + y^2$ by the planes $x = 1$, $x = -1$, $y = 1$, $y = -1$.

Solution: Find f, $\vec{\nabla} f$, $\left\|\vec{\nabla} f\right\|$, $\left|\vec{\nabla} f\cdot\hat{k}\right|$, and $\vec{\nabla}\times\vec{F}$. Compute the work integral to get $\boxed{-8}$.

3 Finals review 2018/12/04

3.1 Jacobian and coordinate transformations

What is integration? Adding up a bunch of little parts. When we have a double integral, this little part is an infinitesimal area. When we have a volume integral, this little part is an infinitesimal volume. Now, this infinitesimal area, as we've seen, can be expressed in many different ways, depending on the coordinate system.

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta = J(u,v)\,\mathrm{d}u\,\mathrm{d}v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u\,\mathrm{d}v$$

In 3D the result is analogous.

Example

Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,\mathrm{d}x\,\mathrm{d}y$$

by applying the transformation

$$u = \frac{2x-y}{2},\ v = \frac{y}{2}$$

Solution: We need to do several things. First we solve for x and y in terms of u, v to get $y = 2v$ and $x = u + v$. Then we use the equations to determine the bounds on our new variables. Lastly we transform

$$\begin{array}{c|c|c} x = y/2 & u+v = 2v/2 & u = 0 \\ x = y/2+1 & u+v = 2v/2+1 & u = 1 \\ y = 0 & 2v = 0 & v = 0 \\ y = 4 & 2v = 4 & v = 2 \end{array}$$

the area element.

$$J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Thus the integral becomes

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_0^1 2u\,\mathrm{d}u\,\mathrm{d}v = \int_0^2 1\,\mathrm{d}v = 2$$

3.2 Line integrals and Vector fields

The reason we do line integrals (also called path integrals) is because we want to add together the value of a function applied along a path where the function value is changing. Hence, the formula is

$$\int_C f(x,y,z)\,\mathrm{d}s$$

Let's handle the special cases, when $f = 1$, the integral gives us arc length. When $f = \delta$, the integral gives us mass. When $f = x\delta$ then it gives us $M_{yz} = M\bar{x}$. When $f = (y^2+z^2)\delta$ then it gives us I_x.

A **vector field** $\vec{F}$ is a collection of arrows (vectors) in space. Each component of the vector field could depend on $\{x, y, z\}$. Therefore,

$$\vec{F} = \begin{bmatrix} M(x,y,z) & N(x,y,z) & P(x,y,z) \end{bmatrix}$$

3.3 Work / Circulation

Why do vector fields and line integrals go so well together? It's because the vector field specifies a new vector at each point along your path. It allows us to easily calculate things like work, which can be written as

$$W = \int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\,\mathrm{d}s = \int_C \vec{F} \cdot \vec{v}/\|\vec{v}\|\,\mathrm{d}s = \int_C \vec{F} \cdot \vec{T}\,\mathrm{d}s = \int_C \vec{F} \cdot \vec{v}\,\mathrm{d}t$$

It is perhaps more conveniently written as

$$W = \int_C M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$$

Example

Evaluate the integral

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}$$

where $\vec{F} = \begin{bmatrix} xz & 0 & -yz \end{bmatrix}$ and C is the line segment from $(-1, 2, 0)$ to $(3, 0, 1)$.

Solution: First we need to parameterize the path. Since this is fairly tricky and important, I'll illustrate two ways. The first way to think of it is you're starting at the point $(-1, 2, 0)$, and as you scale up your parameter t (chosen to range from 0 to 1), you will eventually arrive at $(3, 0, 1)$. Therefore, the scaling should be t times the difference in each coordinate.

$$\vec{r}(t) = \begin{bmatrix} -1 + 4t & 2 - 2t & t \end{bmatrix}$$

Another way to think about it is you're given two points and you have to connect them. That's like taking a weighted average, where you're splitting $(1-t)$ of the weight on the starting point (we need to be there when $t = 0$) and t of the weight on the ending point. Therefore, we get $\vec{r}(t) = (1-t)\begin{bmatrix} -1 & 2 & 0 \end{bmatrix} + t\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. You can verify this simplifies to the above.

Now that we have the parametrization, we can plug into $\vec{F}$ and solve the integral.

$$\begin{aligned}
\int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_C \vec{F} \cdot \vec{v}\,\mathrm{d}t \\
&= \int_C 4(-1 + 4t)(t) - (2 - 2t)(t)\,\mathrm{d}t \\
&= \int_0^1 18t^2 - 6t\,\mathrm{d}t \\
&= 6t^3 - 3t^2 \Big|_0^1 = 3
\end{aligned}$$

3.4 Conservative fields

Conservative fields have many nice properties. They're also pretty naturally occurring.

- Examples: Gravitational field, static electric field, heat flow
- True if vector field is the gradient of a scalar potential. $\vec{F} = \vec{\nabla} f$.
- Line integral over a path depends only on the endpoints of the path. Possibly easier than integrating a parametrization. $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\,\mathrm{d}t = f(\vec{r}(b)) - f(\vec{r}(a))$.
- Integral over a closed path must equal 0
- Condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \quad \text{or} \quad \boxed{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{0}} \ (\textbf{hack})$$

Example

(a) Is the vector field $\vec{F} = \begin{bmatrix} 2xy^2 + x & 2x^2y + y \end{bmatrix}$ conservative? (point is, know the formula)

Solution: We compute

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4xy - 4xy = 0$$

and see that the field is conservative. The above value is called the k-component of the curl, but only FYI.

(b) Find the potential function (point is, can you integrate)

We integrate first along the x-coordinate then y to get

$$\begin{aligned}
\frac{\partial f}{\partial x} &= M \\
f(x,y) &= \int M \,\mathrm{d}x \\
f(x,y) &= x^2y^2 + \frac{1}{2}x^2 + g(y) && \text{(candidate } f\text{)} \\
\frac{\partial f}{\partial y} &= N && \text{(equate } \hat{\jmath} \text{ components, use candidate } f\text{)} \\
2x^2y + \frac{\partial g}{\partial y} &= 2x^2y + y \\
g(y) &= \int y \,\mathrm{d}y && \text{(integrate wrt } y\text{)} \\
g(y) &= \frac{1}{2}y^2 + C \\
f(x,y) &= x^2y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + C && \text{(put it all together)}
\end{aligned}$$

(c) Find circulation over the unit circle in the xy-plane (point is, know your properties)

Solution: Since the vector field is conservative, the circulation over a closed path is $\boxed{0}$.

(d) Obtain the same answer as above using Green's Theorem (point is, know the formula)

Solution: Green's Theorem says

$$\begin{aligned}
\oint_C \vec{F} \cdot \mathrm{d}\vec{r} &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}A \\
&= \iint_R 4xy - 4xy \,\mathrm{d}A = 0
\end{aligned}$$

We get the same answer. yay.

(e) Obtain the same answer as above by actually doing a line integral (point is, know how to parameterize)

Solution: The parameterization of the circle is $\vec{r}(t) = \begin{bmatrix} \cos(t) & \sin(t) \end{bmatrix}$, $t \in [0, 2\pi]$. The vector field is

$$\vec{F} = \begin{bmatrix} 2xy^2 + x & 2x^2y + y \end{bmatrix} = \begin{bmatrix} 2\cos(t)\sin^2(t) + \cos(t) & 2\cos^2(t)\sin(t) + \sin(t) \end{bmatrix}$$

Now we do the integral, wheeee

$$
\begin{aligned}
\int_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_0^{2\pi} \left[2\cos(t)\sin^2(t) + \cos(t) \quad 2\cos^2(t)\sin(t) + \sin(t)\right] \cdot \left[-\sin(t) \quad \cos(t)\right] \mathrm{d}t \\
&= \int_0^{2\pi} -2\cos(t)\sin^3(t) - \cos(t)\sin(t) + 2\cos^3(t)\sin(t) + \sin(t)\cos(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} -2\cos(t)\sin^3(t) + 2\cos^3(t)\sin(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} 2\cos(t)\sin(t)\left(\cos^2(t) - \sin^2(t)\right)\mathrm{d}t \\
&= \int_0^{2\pi} \sin(2t)\cos(2t)\,\mathrm{d}t && \text{(use double angle formulas)} \\
&= \frac{1}{2}\int_0^{2\pi} \sin(4t)\,\mathrm{d}t && \text{(double angle again)} \\
&= 0
\end{aligned}
$$

Well, this was trickier than I imagined. But we get the same answer, yay.

3.5 Flux

Flux is the amount of stuff passing through a boundary. This can be a 2D curve, or a 3D surface. Doesn't matter. In 2D this is

$$\Phi = \int_C \vec{F} \cdot \hat{n}\,\mathrm{d}s = \int_C \left(M\frac{\mathrm{d}y}{\mathrm{d}s} - N\frac{\mathrm{d}x}{\mathrm{d}s}\right)\mathrm{d}s = \int_C M\,\mathrm{d}y - N\,\mathrm{d}x$$

Conveniently there's a Green's Theorem for this too:

$$\oint_C \vec{F} \cdot \hat{n}\,\mathrm{d}s = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,\mathrm{d}A = \oint_C M\,\mathrm{d}y - N\,\mathrm{d}x$$