CME 100 Lecture 17

Green's Theorem, Surface Integrals, Flux

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***** Write out what we're going to do on the board! Lots of fun things in store.

1 Logistics

- Homework 6 is due today.
- Homework 7 is posted on Canvas, due the *Tuesday* after Thanksgiving break.
- Homework 8 is posted on Canvas, due the *Thursday* after Thanksgiving break.
- I don't like using microphones, and I'm pretty loud, but let me know in the back if you can't hear me. Also, there's 242 seats and 150 of you, so, *come forward*.
- Teaching is tough—and there's a lot I need to work on—so *please* let me know if I can explain something better!
- Ask questions! Please feel comfortable asking any and all questions, this is an open and inclusive learning environment. Also, I would like to see as many of you participate as possible, so please be respectful of your classmates and raise your hand.

2 Recap

- Flux: Stuff passing across a curve per unit time. **DRAW A PICTURE**.

$$\boxed{\text{flux} = \int_C \vec{F}\cdot\hat{n}\,\mathrm{d}s = \int_C \left(M\frac{\mathrm{d}y}{\mathrm{d}s} - N\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \int_C M\,\mathrm{d}y - N\,\mathrm{d}x}$$

  We choose $\hat{n}$ to be the outward normal, $\hat{n} = \vec{T}\times\hat{k}$.

- Circulation: The amount of pushing force along a path. Another version of the line integral for work. **DRAW A PICTURE**.

$$\Gamma = \oint_C \vec{v}\cdot\vec{T}\,\mathrm{d}s$$

- Conservative fields:
  - True if vector field is the gradient of a scalar potential. $\vec{F} = \nabla f$.
  - Line integral over a path depends only on the endpoints of the path. Usually easier than integrating a parametrization. $\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{v}(t)\,\mathrm{d}t = f(\vec{r}(b)) - f(\vec{r}(a))$.
  - Integral over a closed path must equal 0
  - Examples: Gravitational field, static electric field, heat flow
  - Condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \quad \text{or} \quad \boxed{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{0}} \ (\textbf{hack})$$

2.1 Example 1

(a) Determine if the field given by $\vec{F}(x,y,z) = \begin{bmatrix} y & x+z^2 & 2yz+2z \end{bmatrix}$ is conservative.

We set up the curl and see that

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \begin{bmatrix} 2z-2z & 0-0 & 1-1 \end{bmatrix} = \vec{0}$$

(b) Find the potential function $f(x,y,z)$.

We integrate M to get

$$f(x,y,z) = \int y \, \mathrm{d}x = xy + g(y,z)$$

Now we differentiate this expression with respect to y and equate to N to get

$$\frac{\mathrm{d}f}{\mathrm{d}y} = x + \frac{\mathrm{d}g(y,z)}{\mathrm{d}y} \implies g(y,z) = yz^2 + h(z)$$

Finally we substitute this into f and differentiate with respect to z to get

$$\frac{\mathrm{d}f}{\mathrm{d}z} = 2yz + \frac{\mathrm{d}h(z)}{\mathrm{d}z} = 2yz + 2z \implies h(z) = z^2$$

Thus

$$f(x,y,z) = xy + yz^2 + z^2 + C$$

(c) Find the work done on a particle along the path $\vec{r}(t) = \begin{bmatrix} t\cos(\pi\sqrt{t}) & t^2\sin(\pi t) & t^{3/2} \end{bmatrix}$ for $t \in [0,4]$ by this force field.

We've seen that $\vec{F}$ is conservative and $f(x,y,z) = xy + yz^2 + z^2 + C$ is its potential field. Therefore, we just evaluate the two endpoints to get

$$\text{work} = f(\vec{r}(4)) - f(\vec{r}(0)) = f(4,0,8) - f(0,0,0) = 64$$

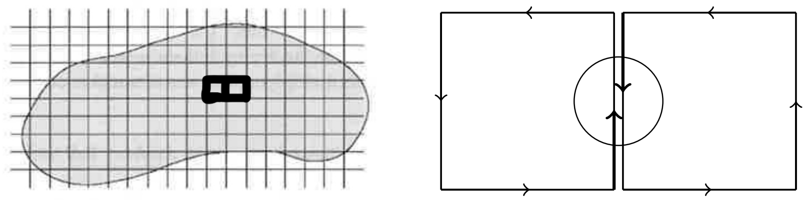
(d) If the field was conservative but we did not know its potential, what can we do?

Answer: We find a simpler path that goes from $(0,0,0)$ and $(4,0,8)$ and integrate along that path.

3 Green's Theorem

3.1 Introduction

Now, let's look at line integrals performed over a closed path, where the field may not be conservative (**DRAW A PICTURE**). Since it's a closed path, we'd like to see if we can relate the line integral on the boundary to a double integral over the region. *Emphasize* this region is in 2D—it's flat.

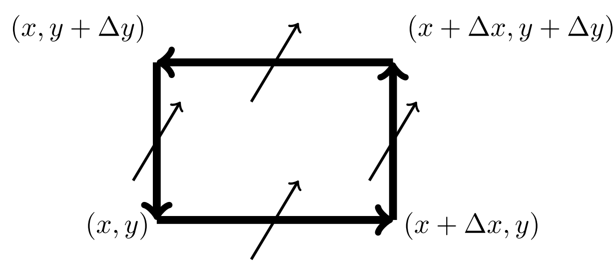


(a) (b)

So, first, because calculus, what will we do? Chop up the area into small regions. Now, we're going to zoom in on these two squares right here, and compute a line integral over their boundaries. Note that we will adopt a **counterclockwise orientation**. Vadim drew it incorrectly in the coursereader. We see that the two shared sides cancel each other out! This will happen with all interior boundaries, and so the original integral over the whole region is equal to the sum of integrals over each smaller square.

3.2 Flux and Divergence

So, what is an individual line integral equal to?



If we label a small square like so and assume the vector field flows as drawn $\vec{F} = \begin{bmatrix} M & N \end{bmatrix}$, with *outward* normal vector, we get

$$\begin{aligned}
\Delta \text{flux} &= \text{right} + \text{left} + \text{top} + \text{bottom} \\
&= M(x + \Delta x, y)\Delta y - M(x, y)\Delta y + N(x, y + \Delta y)\Delta x - N(x, y)\Delta x \\
&= \frac{M(x + \Delta x, y)\Delta y \Delta x - M(x, y)\Delta y \Delta x}{\Delta x} + \frac{N(x, y + \Delta y)\Delta x \Delta y - N(x, y)\Delta x \Delta y}{\Delta y} \\
&= \frac{M(x + \Delta x, y) - M(x, y)}{\Delta x}\Delta x \Delta y + \frac{N(x, y + \Delta y) - N(x, y)}{\Delta y}\Delta x \Delta y \\
\mathrm{d}flux &= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y && \text{(limit as } \Delta x, \Delta y \to 0) \\
\frac{\partial^2 flux}{\partial x \partial y} &= \boxed{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}} = \text{flux density} = \text{divergence} = \boxed{\nabla \cdot \vec{F}} = \text{div } \vec{F}
\end{aligned}$$

If we integrate the above equation over the region, we get **Green's Theorem for flux**:

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,\mathrm{d}s = \int\int_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,\mathrm{d}A = \oint_C M\,\mathrm{d}y - N\,\mathrm{d}x}$$

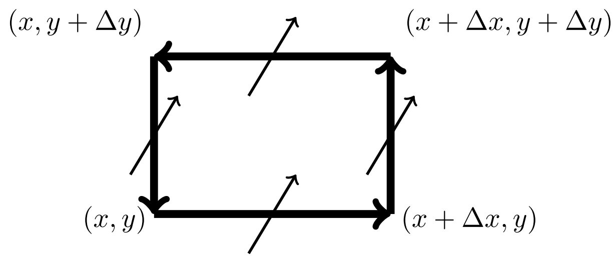
This works for *closed paths* only. If it's just a section, you still have to parameterize and integrate.

What the hell is divergence? Divergence can be thought of as the amount of expansion or compression of a fluid at a point, depending on the sign (**DRAW A PICTURE**). A vector field with zero divergence is called *divergence-free* and a fluid velocity field with zero divergence is said to be *incompressible*. Note that divergence is an operator that transforms a *vector* into a *scalar*.

Big picture: Note that what Green's Theorem is saying should be intuitive, we have some phenomenon happening across a curve (e.g. flux), and the amount of outward flux is equal to the integral of its rate of change over the region R enclosed by C. Whatever is leaving the enclosed area must be passing through the boundary.

3.3 Circulation and Work

Last time we looked at the normal components of the vector field, so now we can look at the tangential components. What would such a line integral give us? Work! We use the same infinitesimal element (reproduced below) and approximate the line integral along the boundary as:



$$\begin{aligned}
\Delta\text{work} &= \text{right} + \text{left} + \text{top} + \text{bottom}\\
&= N(x+\Delta x, y)\Delta y - N(x,y)\Delta y - M(x, y+\Delta y)\Delta x + M(x,y)\Delta x\\
&= \frac{N(x+\Delta x, y)\Delta y\Delta x - N(x,y)\Delta y\Delta x}{\Delta x} - \frac{M(x, y+\Delta y)\Delta x\Delta y - M(x,y)\Delta x\Delta y}{\Delta y}\\
&= \frac{N(x+\Delta x, y) - N(x,y)}{\Delta x}\Delta x\Delta y - \frac{M(x,y+\Delta y) - M(x,y)}{\Delta y}\Delta x\Delta y\\
\Delta\text{work} &= \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}}\mathrm{d}x\,\mathrm{d}y && (\text{limit as } \Delta x, \Delta y \to 0)
\end{aligned}$$

If we integrate the above equation over the region, we get **Green's Theorem for work**:

$$\boxed{\oint_C \vec{F}\cdot\vec{T}\,\mathrm{d}s = \int\int_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,\mathrm{d}A = \oint_C M\,\mathrm{d}x + N\,\mathrm{d}y}$$

These two "Green's Theorems" are equivalent! We can replace N with M and M with $-N$ and get the same relationships. The only thing Green's theorem is saying is that given a line integral over a closed path, we can instead compute the double integral over the enclosed region. That's it, and it doesn't matter what the integral is actually calculating.

3.4 Example 1

Consider the vector field $\vec{F} = \begin{bmatrix} x-y & x \end{bmatrix}$ and the unit circle $\vec{r} = \begin{bmatrix} \cos(t) & \sin(t) \end{bmatrix}$. Compute the flux of the field and the work done over the path (we've already solved this using parametrization).

First, we write down everything we know:

$$\begin{aligned}
M &= x - y = \cos(t) - \sin(t) \\
N &= x = \cos(t) \\
\mathrm{d}x &= \frac{\mathrm{d}\cos(t)}{\mathrm{d}t} = -\sin(t)\,\mathrm{d}t \\
\mathrm{d}y &= \frac{\mathrm{d}\sin(t)}{\mathrm{d}t} = \cos(t)\,\mathrm{d}t \\
\frac{\partial M}{\partial x} &= 1, \qquad \frac{\partial M}{\partial y} = -1 \\
\frac{\partial N}{\partial x} &= 1, \qquad \frac{\partial N}{\partial y} = 0
\end{aligned}$$

Then we compute the flux:

$$\begin{aligned}
\int\int_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,\mathrm{d}A &= \int\int_R 1 + 0\,\mathrm{d}A \\
&= \pi \\
\oint_C M\,\mathrm{d}y - N\,\mathrm{d}x &= \oint_C (\cos(t) - \sin(t))\cos(t)\,\mathrm{d}t - \cos(t)(-\sin(t))\,\mathrm{d}t \\
&= \int_0^{2\pi} \cos^2(t)\,\mathrm{d}t \\
&= \pi
\end{aligned}$$

Now we compute the work:

$$\begin{aligned}
\int\int_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,\mathrm{d}A &= \int\int_R 1 - (-1)\,\mathrm{d}A \\
&= 2\pi \\
\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y &= \oint_C (\cos(t) - \sin(t))(-\sin(t))\,\mathrm{d}t + \cos(t)\cos(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (-\sin(t)\cos(t) + 1)\,\mathrm{d}t \\
&= 2\pi
\end{aligned}$$

Note: Green's theorem does not say that flux and work are the same thing(!!), we got different answers after all. It's simply giving us two ways to calculate each quantity, and we get a choice depending on the situation. Again, it relates line integrals over a closed path to double integrals over a region; that's all it says.

3.5 Example 2

Evaluate the line integral

$$\oint_C xy\,\mathrm{d}y - y^2\,\mathrm{d}x$$

over the unit square in the first quadrant.

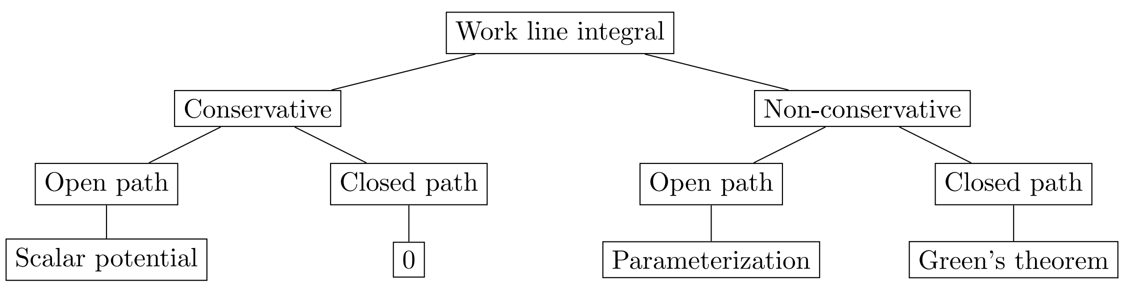
We can use either form of Green's theorem to solve this problem, as it's just asking for us to evaluate an integral. Using the flux equation, we can set $M = xy$ and $N = y^2$ to get

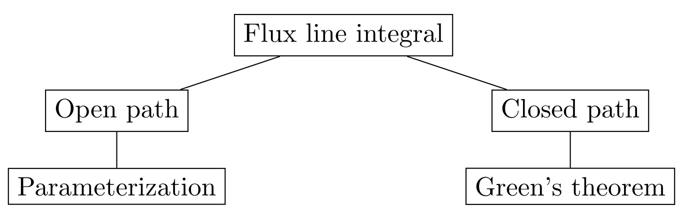
$$\oint_C xy\,\mathrm{d}y - y^2\,\mathrm{d}x = \int_0^1 \int_0^1 y + 2y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 3y\,\mathrm{d}y = \frac{3}{2}$$

Similarly, we can use the work equation and set $M = -y^2$ and $N = xy$ to get

$$\oint_C xy\,\mathrm{d}y - y^2\,\mathrm{d}x = \int_0^1 \int_0^1 y - (-2y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 3y\,\mathrm{d}y = \frac{3}{2}$$

We get the same answer. Whew. And we didn't have to split up any paths! See the following flow chart for a summary of the different techniques used to evaluate line integrals.





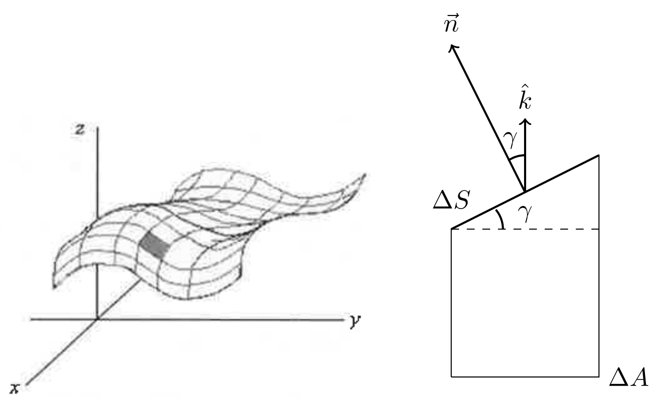
The best part of Green's theorem is that we can apply it to regions that are normally tricky to deal with. For an example, see http://tutorial.math.lamar.edu/Classes/CalcIII/GreensTheorem.aspx or just search "Paul's Online Math Notes Green's Theorem."

*** 1-MINUTE STRETCH BREAK ***

4 Surface Integrals

Recall that for double integrals like the ones in Green's Theorem, we evaluated the integral of a function $f(x, y)$ over a flat area in the xy-plane. Now, there might be cases where we want to evaluate a function of three variables, $g(x, y, z)$, over a *surface* $f(x, y, z) = 0$ (**DRAW A PICTURE** and show using $z = f(x, y)$ parametrization). For example, maybe we have this curved surface and we want to find the total mass, given its density. Similar to before, we now have an integral of the form

$$\boxed{\int\int_S g(x, y, z)\, \mathrm{d}S}$$



(a) (b)

For example, letting $g(x, y, z) = 1$ gives the surface area and $g(x, y, z) = \delta(x, y, z)$ gives the mass. However, $\mathrm{d}S$ is pretty tricky to evaluate, so we'll project the surface onto the xy-plane. In 2D, we would have a diagram that looks like the above. We see that the relationship between ΔS and ΔA is

$$\Delta S \cos(\gamma) = \Delta A$$

$$\mathrm{d}S \frac{\left|\vec{n} \cdot \hat{k}\right|}{|\vec{n}|\left|\hat{k}\right|} = \mathrm{d}A$$

$$\boxed{\mathrm{d}S = \frac{|\vec{n}|}{\left|\vec{n} \cdot \hat{k}\right|}\, \mathrm{d}A}$$

Since our surface has the implicit parametrization $f(x, y, z) = 0$, we have that $\vec{n} = \nabla f$ (see Lecture 8, page 63). This means our surface integral is now

$$\boxed{\int\int_S g(x, y, z)\, \mathrm{d}S = \int\int_R g(x, y, z) \frac{|\nabla f|}{\left|\nabla f \cdot \hat{k}\right|}\, \mathrm{d}A}$$

4.1 Example 1

Find the surface area of a hemisphere with radius R (not counting the bottom surface).

The equation for a hemisphere is $f(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$, and we want to integrate the function $g = 1$ over S. The integral is

$$\int\int_S \mathrm{d}S = \int\int_R \frac{|\nabla f|}{\left|\nabla f \cdot \hat{k}\right|}\, \mathrm{d}A$$

Now we just find what we need.

$$\begin{aligned}
\nabla f &= \begin{bmatrix} 2x & 2y & 2z \end{bmatrix} \\
|\nabla f| &= \sqrt{4x^2 + 4y^2 + 4z^2} = 2R \\
\left|\nabla f \cdot \hat{k}\right| &= 2z
\end{aligned}$$

And so performing the integral in polar coordinates (not cylindrical/spherical because we're in 2D!), we get

$$\begin{aligned}
\int\int_R \frac{|\nabla f|}{\left|\nabla f \cdot \hat{k}\right|} \,\mathrm{d}A &= \int_0^{2\pi} \int_0^R \frac{2R}{2z} r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= 2\pi \int_0^R \frac{Rr}{\sqrt{R^2 - r^2}} \,\mathrm{d}r \\
&= 2\pi \left(-R\sqrt{R^2 - r^2}\right) \Big|_0^R \\
&= 2\pi R^2
\end{aligned}$$

□

5 Flux

You will see surface integrals a lot more after the break. One reason we care about evaluating surface integrals is so we can find the flux of a vector field through a surface. Like before, this measures how much stuff is passing a boundary. We previously looked at normal components of a vector field across a curve, and now we can consider the normal components of a vector field across a surface. If we integrate this normal component using a surface integral, we can get (where $\hat{n}$ is the unit normal):

$$\boxed{\text{flux}_{\text{surface}} = \int\int_S \vec{F} \cdot \hat{n} \,\mathrm{d}S}$$

5.1 Example 1

Find the flux of the vector field $\vec{F} = z\hat{k}$ over the top surface of the hemisphere with radius R.

We again find that $f(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$ describes the surface and compute all the parts as

$$\begin{aligned}
\nabla f &= \begin{bmatrix} 2x & 2y & 2z \end{bmatrix} \\
|\nabla f| &= \sqrt{4x^2 + 4y^2 + 4z^2} = 2R \\
\left|\nabla f \cdot \hat{k}\right| &= 2z \\
\hat{n} &= \frac{\nabla f}{|\nabla f|} = \begin{bmatrix} \frac{x}{R} & \frac{y}{R} & \frac{z}{R} \end{bmatrix} \\
\vec{F} \cdot \hat{n} &= \frac{z^2}{R} \\
\int\int_R \frac{z^2}{R} \frac{2R}{2z} \,\mathrm{d}A &= \int_0^{2\pi} \int_0^R \sqrt{R^2 - r^2} r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= 2\pi \cdot \left(-\frac{1}{3}(R^2 - r^2)^{3/2}\right) \Big|_0^R \\
&= \frac{2}{3}\pi R^3
\end{aligned}$$

□