A Comparative Study of Wiener Process and Geometric Brownian Motion

710882199 - Enzo HAEGEL

Abstract:

This report presents a comparison between two stochastic processes, the Wiener process (also known as Brownian motion) and the Geometric Brownian Motion (GBM). The study utilizes numerical simulations to generate paths for both processes. Additionally, it investigates their statistical properties at the end of the time horizon, T = 1.

Introduction:

The Wiener process, named after American mathematician Norbert Wiener, is a continuous-time stochastic process fundamental to the mathematical theory of Brownian motion. This process has wide-ranging applications in physics, mathematics, and engineering, particularly in areas that involve modeling of random behavior over time, such as the motion of particles suspended in a liquid medium.

The Geometric Brownian Motion (GBM), a variant of the Wiener process, plays a pivotal role in financial mathematics. It is frequently used in modeling asset or stock prices in financial markets due to its key properties: continuity and independence of increments. Specifically, models based on GBM are foundational in the Black-Scholes option pricing model, a landmark in modern financial theory.

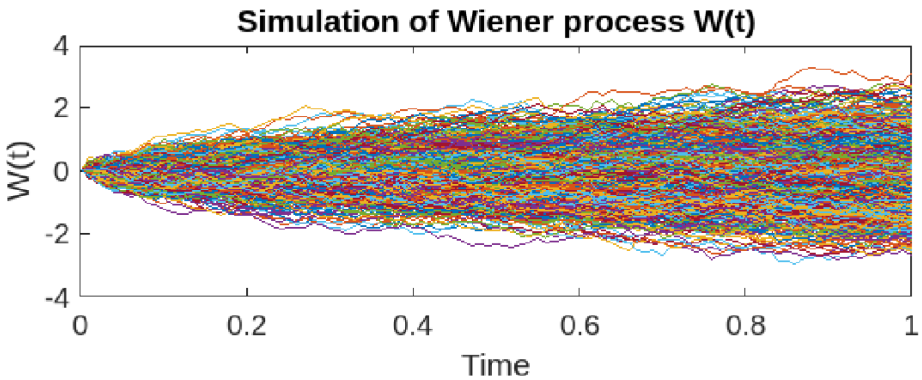
Method:

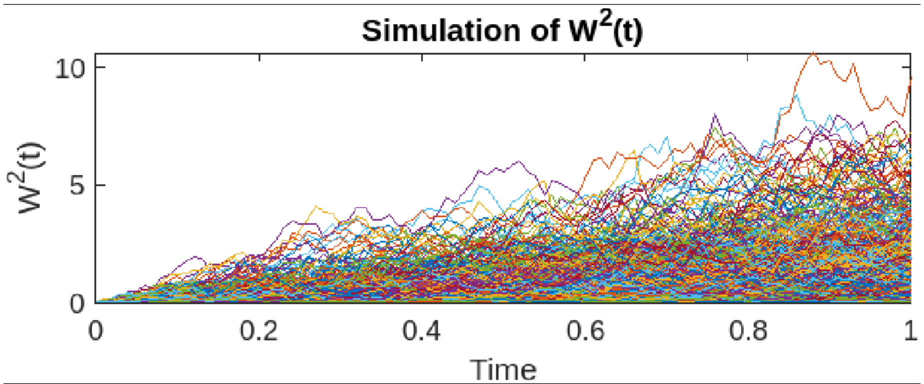
To perform the comparison, 1000 paths for each process were generated with a time step, dt = 0.01, for a time horizon, T = 1.

For the Wiener process, the incremental change dW was calculated for each time step using the formula `sqrt(dt)*randn`, where `randn` generates a random number from the standard normal distribution. The process W(t) and its square W^2(t) were stored for each path at each time step. The expected value of W^2(T) and several statistical properties of W(T) were calculated and reported.

For the Geometric Brownian Motion, the asset price S(t) for each path at each time step was calculated using the formula `S0 * exp((mu - 0.5 * sigma^2) * (iStep*dt) + sigma * sqrt(iStep*dt) * epsilon)`, where `epsilon` is a standard normal random number, mu = 0.05 is the expected return, and sigma = 0.2 is the volatility. The statistics of the asset price S(T) at time T were computed and reported.

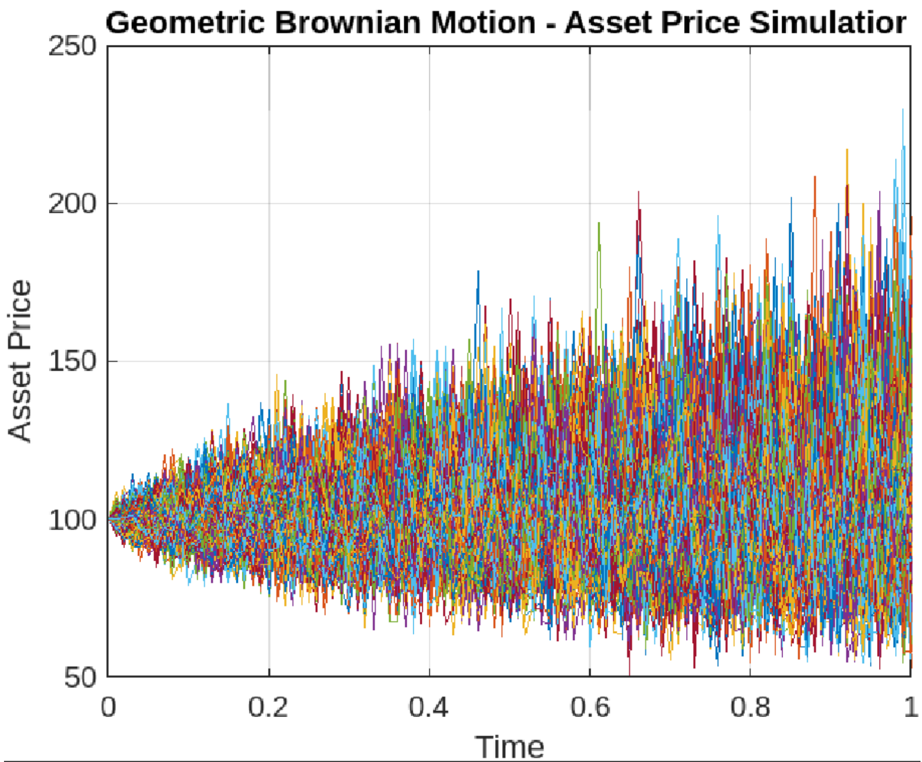
Results:





For the Wiener process, the following statistics were reported at time T:

- Expected value of W^2(T): 1.02
- Mean of W(T): -0.02
- Standard deviation of W(T): 1.01
- Median of W(T): -0.00
- Minimum of W(T): -2.65
- Maximum of W(T): 3.10



For the GBM, the following statistics for the asset price S(T) at time T were computed:

- Mean asset price at T: 105.43
- Standard deviation of asset price at T: 21.58
- Median asset price at T: 102.98
- Minimum asset price at T: 52.59
- Maximum asset price at T: 196.30

Discussion:

The simulation of the Wiener process exhibited properties consistent with theoretical expectations, i.e., the expected value of W^2(T) is close to T and the process W(T) is normally distributed around 0 with standard deviation sqrt(T).

The simulation of the Geometric Brownian Motion, a model often used for asset price evolution in financial markets, generated paths that show a distribution of prices at time T with a mean around 105.43. The variability of prices, reflected in the standard deviation, minimum, and maximum values, underscores the inherent uncertainty and risk in asset prices.

Conclusion:

The study showcased the use of numerical simulations for understanding the properties of two essential stochastic processes, the Wiener process, and the Geometric Brownian Motion. The numerical results corroborate the theoretical properties of these processes, confirming the utility of these mathematical models in various fields, especially in finance for the latter.

Future work could extend these simulations to higher dimensions, include jumps or other forms of price discontinuities, or consider different stochastic processes such as the Poisson process or Levy processes. Another valuable direction would be applying these models to real-world financial data and examining their effectiveness in predicting asset price dynamics.