Product Distributions

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1 The Product Distribution

Suppose two independent, continuous random variables, X and Y described by densities $f_X(x)$ and $f_Y(y)$. Consider the random variable, Z, formed by

$$Z = XY$$

Then we may find its pdf in the following way[1]

$$F_Z(z) = Pr\{Z \leq z\} = Pr\{XY \leq z\} = Pr\{XY \leq z, X \geq 0\} + Pr\{XY \leq z, X \leq 0\}$$

$$F_Z(z) = Pr\{Y \leq \frac{z}{X}, X \geq 0\} + Pr\{Y \geq \frac{z}{X}, X \leq 0\}$$

$$F_Z(z) = \int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y) dy dx + \int_{-\infty}^0 f_X(x) \int_{z/x}^\infty f_Y(y) dy dx$$

All that remains is to take the derivative and use the Fundamental Theorem of Calculus.

$$f_Z(z) = \int_0^\infty f_X(x) f_Y(z/x) \frac{1}{x} dx - \int_{-\infty}^0 f_X(x) f_Y(z/x) \frac{1}{x} dx$$

This may be written

$$f_Z(z) = \int_{-\infty}^\infty f_X(x) f_Y(z/x) \frac{1}{|x|} dx \tag{1}$$

[1] This proof can be found on Wikipedia, but it is a standard method of proof in probability theory.

2 The Product of 1-d Gaussian Random Variables

To start with, let us consider the product of two zero-mean normally distributed random variables, X and Y, with respective variances σ_x and σ_y. We will need the following identity[2] which holds for $\mathrm{Re}(x) > 0$.

$$K_\alpha(x) = \int_0^\infty e^{-x\cosh t}\cosh(\alpha t)dt \tag{2}$$

Here, $K_\alpha(x)$ refers to the αth modified Bessel function of the second kind. Going back to our random variables, we form their product distribution according to equation 1.

$$f_Z(z) = \frac{1}{2\pi\sigma_x\sigma_y}\int_{-\infty}^{\infty}\exp\Big(-\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2x^2\sigma_y^2}\Big)\frac{1}{|x|}dx$$

This can be purposefully obfuscated in the following way

$$f_Z(z) = \frac{1}{\pi\sigma_x\sigma_y}\int_0^\infty \exp\Big\{-\frac{|z|}{2\sigma_x\sigma_y}\big[e^{\ln(\frac{x^2\sigma_y}{|z|\sigma_x})} + e^{-\ln(\frac{x^2\sigma_y}{|z|\sigma_x})}\big]\Big\}\frac{1}{x}dx \tag{3}$$

From which we may make the substitution $t = \ln(\frac{x^2\sigma_y}{|z|\sigma_x})$, which yields

$$f_Z(z) = \frac{1}{2\pi\sigma_x\sigma_y}\int_{-\infty}^{\infty}\exp\big(-\frac{|z|}{\sigma_x\sigma_y}\cosh t\big)dt \tag{4}$$

Using the fact that the integrand is even, and applying the identity in equation 2 for $\alpha = 0$, we obtain

$$f_Z(z) = \frac{1}{\pi\sigma_x\sigma_y}K_0\bigg(\frac{|z|}{\sigma_x\sigma_y}\bigg) \tag{5}$$

3 The Product of Circularly Symmetric Complex Gaussian Random Variables

Now let us consider the random variables Z_1 and Z_2, which are circularly symmetric gaussian random variables. Then letting

$$Z_k = X_k + iY_k$$

[2] This identity can be found on p. 181 of Watson, G. N., A Treatise on the Theory of Bessel Functions, Second Edition, (1995) Cambridge University Press. ISBN 0-521-48391-3.

X and Y real, then

$$Z_1^* Z_2 = X_1 X_2 + Y_1 Y_2 + i(X_1 Y_2 - X_2 Y_1)$$

Since Z_1 and Z_2 are circularly symmetric and equation 5 shows an even distribution from the product of two one-dimensional gaussians, the following calculations are identical for both the real and imaginary components. The distribution for the real part would be given by a convolution. For ease of notation let us denote

$$X = \operatorname{Re} Z_1^* Z_2$$

Then,

$$f_X(x) = \frac{1}{(\pi\sigma_1\sigma_2)^2} \int_{-\infty}^{+\infty} K_0\left(\frac{|y|}{\sigma_1\sigma_2}\right) K_0\left(\frac{|x-y|}{\sigma_1\sigma_2}\right) dy \tag{6}$$

Let us not attempt to directly compute this convolution. Instead, we will extract its moments with the following identity[3], which holds for $\operatorname{Re}\alpha > |\operatorname{Re}\nu|$

$$\int_0^\infty t^{\alpha-1} K_\nu(t) dt = 2^{\alpha-2} \Gamma\left(\frac{\alpha-\nu}{2}\right) \Gamma\left(\frac{\alpha+\nu}{2}\right) \tag{7}$$

From this we may obtain

$$E[X^n] = \frac{1}{(\pi\sigma_1\sigma_2)^2} \int_{-\infty}^{+\infty} dy K_0\left(\frac{|y|}{\sigma_1\sigma_2}\right) \int_{-\infty}^{+\infty} dx x^n K_0\left(\frac{|x-y|}{\sigma_1\sigma_2}\right) \tag{8}$$

We can perform a substitution to obtain

$$E[X^n] = \frac{1}{(\pi\sigma_1\sigma_2)^2} \int_{-\infty}^{+\infty} dy K_0\left(\frac{|y|}{\sigma_1\sigma_2}\right) \int_{-\infty}^{+\infty} du (u+y)^n K_0\left(\frac{|u|}{\sigma_1\sigma_2}\right) \tag{9}$$

Then, using the binomial expansion, exchanging the sum and the integral, and collecting terms in y outside the integral over u, we have

$$E[X^n] = \frac{1}{(\pi\sigma_1\sigma_2)^2} \sum_{j=0}^{n} \binom{n}{j} \int_{-\infty}^{+\infty} dy y^j K_0\left(\frac{|y|}{\sigma_1\sigma_2}\right) \int_{-\infty}^{+\infty} du u^{n-j} K_0\left(\frac{|u|}{\sigma_1\sigma_2}\right) \tag{10}$$

The integral over y shows that terms with odd j vanish, since the Bessel Function in question is even. Also, if n is odd, then terms with even j also vanish due to the integral over u. Thus, we only have left to consider even n and j. Rewriting the sum with this knowledge, performing another substitution and adjusting the bounds so that the integration may be carried out according to the identity in equation 7, we have

$$E[X^n] = 4\frac{(\sigma_1\sigma_2)^n}{\pi^2} \sum_{j=0}^{n/2} \binom{n}{2j} \int_0^\infty ds s^{2j} K_0(s) \int_0^\infty dt t^{n-2j} K_0(t) \tag{11}$$

[3] This one is available at http://functions.wolfram.com/BesselTypeFunctions/BesselK/21/02/01/

Applying our identity, we have for even n,

$$E[X^n] = \frac{(\sigma_1\sigma_2)^n}{\pi^2} \sum_{j=0}^{n/2} 2^n \left[\Gamma\left(\frac{2j+1}{2}\right) \Gamma\left(\frac{n-2j+1}{2}\right) \right]^2 \tag{12}$$

In particular, the mean is 0, and the variance is

$$E[X^2] = 4\frac{(\sigma_1\sigma_2)^2}{\pi^2} \left\{ \left[\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right) \right]^2 + \left[\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right) \right]^2 \right\} \tag{13}$$

or, using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $z\Gamma(z) = \Gamma(z+1)$,

$$E[X^2] = 2(\sigma_1\sigma_2)^2 \tag{14}$$