Master of Science (Business Analytics)

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Numerical Analytics and Software

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Background to Programming Assignment 2: Linear equations

1. Motivation

In this document I have given some background to motivate the problem in your Programming Assignment. The actual solution does not necessarily involve all of this background: your task is to identify the mathematical content in the problem (*i.e.*, the mathematical model) and use the tools developed in the course (SOR on sparse matrices) to attack this.

It will be of benefit to translate the solution of the model back into the terms of the original problem.

2. Basic concepts from finance

A *(financial) derivative* or *derivative security* is a financial instrument whose value depends on the values of other more basic, underlying variables, such as stock prices, stock indices, foreign currencies, and so on. It is a legal contract. A *portfolio* is a collection or bundle of assets, which may include stocks, derivatives, government bonds, riskless assets, etc. It should be as diverse as possible, to spread risk.

Examples of derivatives include *options*. A *call option* gives the holder the right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is called the exercise price or *strike price*. The date in the contract is called the *expiration date* or *maturity*. There are two classes of options:

- *American* options can be exercised at any time up to the expiration date;
- *European* options can be exercised only on the expiration date itself.

(These terms have nothing to do with the exchange where the option is traded.)

The option gives the holder a *right* but not an *obligation* to trade in the underlying asset on/before maturity. Having this choice is worth something, but the question is: how much? What is a fair price to pay for an option (fair to both the buyer of the option and the seller of the option)? This is the *pricing* problem.

In general, it is not possible to get analytic (or closed form) solutions to this problem: only approximate numerical solutions may be found. However, for certain simple situations (*e.g.*, an option on one underlying risky asset [Black-Scholes, Merton, 1973], an option on the max or min of two underlying risky assets [Stulz, 1982]), closed form solutions do exist.

In this document, we'll consider the option pricing problem in the special case of a single underlying stock.

3. Basic mathematical concepts

A quantity whose value changes over time in an uncertain way is said to follow a *stochastic process*. To allow calculus techniques to be used, the model often adopted is *continuous-time, continuous-variable*: the quantity can take any value in a certain range (not just discrete values), and the value can change at any time (not just at certain fixed times). Technically, a stochastic process z is an ordered set of random variables, one for each time instant considered: $\{z_t : t \in [0, T]\}$ where $[0, T]$ is the range of times under discussion.

A *Markov process* is a particular type of stochastic process with the following property, called the *Markov property*: only the present value of the quantity is relevant for predicting the future, the past history of the quantity is not relevant. Stock prices are often assumed to follow a Markov process: the Markov property implies that the probability distribution of the price at any particular time in the future does not depend on the path followed by the price in the past. [This is consistent with the weak form of the *market efficiency* assumption, which states that the present price of a stock stores all the information contained in a record of past prices.]

Recall that if a random variable X is normal of mean μ and standard deviation σ, we write $X \sim N(\mu, \sigma)$. Suppose that the current value z_0 of a stock is 100 and that the *change* in its value in one day, $\Delta z := z_1 - z_0$, is known to be a random variable $\sim N(0, 1)$. Recall that when two identical independent normal random variables are added, the sum is a normal random variable with mean the sum of the two means, and variance the sum of the variances. Thus over two days the change in the value of z would have mean 0 and variance 2, that is, the change would be $\sim N(0, \sqrt{2})$. In general, the change in value over a time interval Δt is $\sim N(0, \sqrt{\Delta t})$, *i.e.*, normally distributed with standard deviation proportional to the square root of Δt.

A Markov process of this type is called a *Wiener process*[1] with mean change 0 and variance rate of 1 per day. Formally, we say that z follows a Wiener process if it has these two properties:

(a) The *change* Δz_t during a small time interval Δt is $\Delta z_t = \varepsilon\sqrt{\Delta t}$, where $\varepsilon \sim N(0, 1)$. (Thus Δz_t itself is $\sim N(0, \sqrt{\Delta t})$.)

(b) The values of Δz_{t_1} and Δz_{t_2} for any two different short time intervals t_1 and t_2 are independent (this is the Markov property).

As in calculus, one can form limits and talk about stochastic differentials, though this involves a lot of extra machinery. A Wiener process is actually the *limit* as $\Delta t \to 0$ of the process z above. We write $\mathrm{d}t$, $\mathrm{d}z$ etc., though strictly speaking such terms only make sense when they occur in an "Itô integral".

We call a stochastic process x an *Itô process* if

$$\mathrm{d}x = a(x, t)\,\mathrm{d}t + b(x, t)\,\mathrm{d}z,$$

where z is a Wiener process. This should be thought of as a stochastic differential equation, relating the differential of x to those of t and z. Here a and b are functions[2] of the two variables x and t. The component $a(x, t)\,\mathrm{d}t$ is a drift term, a deterministic dependence of x on t; while $b(x, t)\,\mathrm{d}z$ is the stochastic component. The quantity x has a drift rate of a and a variance rate of b^2. We will not discuss in detail the theory of measures, etc., save to state the important *Itô's Lemma*:

Lemma 3.1 (Itô). *Suppose that x follows an Itô process:*

$$\mathrm{d}x = a(x, t)\,\mathrm{d}t + b(x, t)\,\mathrm{d}z$$

[1]The same type of process is used in physics to model the motion of a particle being hit by molecules and so is sometimes called a *Brownian motion* process — Einstein received a Nobel prize for his work on this and the photoelectric effect, not on relativity!

[2]If a and b are both constants, then the Itô process x is called a *generalised Wiener process*.

where z is a Wiener process and a and b are functions of x and t. Let f be a $\mathcal{C}^2$ function[3] *of x and t. Then, to a first-order (Taylor series) approximation, f follows the Itô process*

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2\right)\mathrm{d}t + \frac{\partial f}{\partial x}\,b\,\mathrm{d}z$$

where z is the same Wiener process. Thus f has a drift rate of

$$\frac{\partial f}{\partial x}\,a + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,b^2$$

and a variance rate of

$$\left(\frac{\partial f}{\partial x}\right)^2 b^2.$$

Itô's Lemma allows us to find the stochastic differential of a function f of x, once we know the stochastic differential of x itself. A key point is that the Wiener process underlying the basic variable x is the same as that underlying the function f: each has the same source of uncertainty.

4. Application to option pricing

We denote the stock price by S and the option price by f. In practice we examine not the change in stock price but rather the relative change (ratio of change in price to the price itself):

$$\frac{\mathrm{d}S}{S} = a\,\mathrm{d}t + b\,\mathrm{d}z \quad\Longleftrightarrow\quad \mathrm{d}S = aS\,\mathrm{d}t + bS\,\mathrm{d}z.$$

This is because in reality the percentage rate of return $\mathrm{d}S/S$ required by investors is independent of the stock price S. In this case[4] one may write $a = \mu$, standing for the expected rate of return, and $b = \sigma$, the volatility[5] of the stock price.

Black-Scholes and Merton in 1973 independently modelled stock prices by the Itô process $\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}z$, and derived the Black-Scholes(-Merton) equation[6]: this is a partial differential equation which must be satisfied by any financial derivative f dependent on a non-dividend-paying stock S. The model made assumptions:

- the risk-free interest rate r is constant and independent of maturity date
- μ and σ are constant
- short selling (selling a borrowed holding) of securities is allowed
- there are no transaction costs (including taxes)
- S pays no dividends during the life of the option
- there is no *arbitrage* (*i.e.*, no opportunities to make riskless profits)
- trading is continuous in time

Some of these simplifying assumptions can be relaxed, *e.g.*, σ may be a known function of t, dividends may be allowed for, r may be uncertain, etc.

They used Itô's Lemma to show that f must have the same underlying Wiener process (namely, z) as does S, and so one can make a *riskless* portfolio out of suitable weights of f and S. That is, the uncertainty due to $\mathrm{d}z$ can be *eliminated*[7] — at least for a short time. In a little more detail:

- We know that the stochastic part of the variation $\mathrm{d}S$ of S is $\sigma S\,\mathrm{d}z$

[3] A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called $\mathcal{C}^2$ if all first and second order partials of f both exist and are continuous.

[4] This model of stock price behaviour is sometimes called *geometric Brownian motion.*

[5] Note that if $\sigma = 0$, we would have zero volatility and a purely deterministic process, $\mathrm{d}S/S = \mu\mathrm{d}t$. Then we can show

$$\ln S_T - \ln S_0 = \mu(T-0) \;\Rightarrow\; S_T/S_0 = e^{\mu T} \;\Rightarrow\; S_T = S_0 e^{\mu T}$$

by integrating both sides, the right from 0 to T and the left from $S_0 := S(0)$ to $S_T := S(T)$.

[6] Scholes and Merton received the 1997 Nobel prize in Economics for this (Black died in 1995).

[7] This is called "hedging".

- Itô's Lemma tells us that the stochastic part of the variation $\mathrm{d}f$ of f is $\sigma S \frac{\partial f}{\partial S}$ (since the b in Itô's Lemma is just σS)
- Thus a portfolio $P = -f + \frac{\partial f}{\partial S} S$ of weight -1 in f (a "short" or borrowed holding) and weight $\partial f/\partial S$ in S would have a total random $\mathrm{d}z$ component of

$$\left(-\sigma S \frac{\partial f}{\partial S} + \sigma S \frac{\partial f}{\partial S}\right) \mathrm{d}z = 0,$$

  that is, P is a riskless portfolio, with no uncertainty about its value.

This is only true for a short time however, as $\partial f/\partial S$ is constantly changing.

By the no arbitrage assumption, this portfolio must give the same rate of return as other short-term risk-free securities; that is, over a short time period Δt,

$$\Delta P = rP\Delta t$$

where r is the risk-free interest rate (*e.g.*, a bank rate). This eventually gives

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

This is the Black-Scholes(-Merton) equation: it is a non-stochastic partial differential equation (PDE) which must be satisfied by any financial derivative f dependent on a non-dividend-paying stock S. By changing variables[8] it can be recast as the famous *Heat Equation* $u_{xx} = u_\tau$ from Physics.

5. Solving the Black-Scholes-Merton equation

Solving a partial differential equation means finding the functions f which satisfy it in a certain region. There will be many such functions and we use conditions[9] on the value of f at the boundary of the region to identify the particular f.

The solutions f of the Black-Scholes-Merton PDE correspond to all the different options that can be defined with S as the underlying asset. The particular option that is obtained when the equation is solved depends on the boundary/initial conditions that are used.

The Black-Scholes-Merton equation does not involve any terms (such as μ, the expected return on the stock S) that are affected by the risk preferences of investors. If risk preferences do not enter the equation, they cannot affect its solution. In particular, we can make the simplifying assumption that all investors are risk neutral, and so evaluate f. This leads to the Black-Scholes pricing formulas for the prices of European call and put options, which in turn give approximations for the prices of American call and put options.

As mentioned in the introduction, there are exact closed-form solutions for the original form of the Black-Scholes-Merton equation, using techniques you'll meet/have met in Quantitative Methods, *e.g.*, separation of variables, Fourier series etc. The appropriate technique may depend on the particular boundary conditions. One possible approach is to change variables and transform the Black-Scholes-Merton equation to the Heat equation; alternatively, since the Black-Scholes-Merton equation is fairly simple in its own right, it may be solved directly.

There are also numerical approaches, including tree-based approaches, simulation (especially Monte Carlo simulation) and finite differences, which we now consider.

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[8]Let $\tau = T - t$, $u = Ce^{r\tau}$ and $x = \ln(S/X) + (r - \frac{\sigma^2}{2})\tau$. Here u_τ is shorthand for $\frac{\partial u}{\partial \tau}$, u_{xx} short for $\frac{\partial^2 u}{\partial x^2}$.

[9]There is an enormous area of mathematics concerned with the conditions under which solutions f to a given PDE on a given region with given boundary conditions exist and/or are unique.

6. Finite difference approaches

We will look at the approach known as the *implicit finite difference method.* Finite difference methods use finite approximations to (infinitesimal) differentials, by dividing the region into a grid of closely spaced, equally separated points, to get approximate values for f at those points.

We concentrate on the pricing of a European put option: other situations are similar.

We start by defining the region in which we want to solve the PDE. As there are two independent variables, S and t, this will be a region (domain) in $\mathbb{R}^2$. The value of $f(S,t)$ can be graphed along a third (vertical) axis.

Let the maturity date of the option be T and the strike price (price of option at time T) be X. Let $S_{\max}$ be a stock price sufficiently high so that the option value f is effectively zero when $S = S_{\max}$. (In particular, $S_{\max} > X$.) Then we will get a rectangular region $[0, S_{\max}] \times [0, T]$ in $\mathbb{R}^2$ in which we must solve for f.

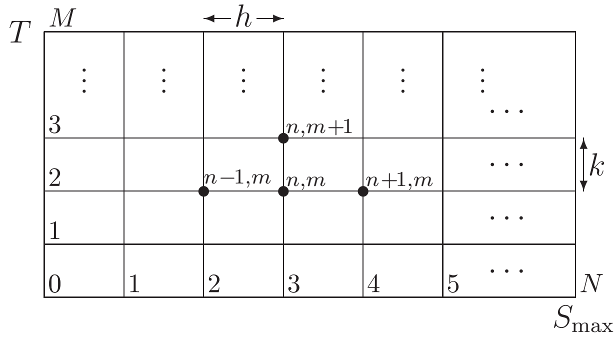
We divide the interval $[0, S_{\max}]$ into N equal subintervals of length h. This defines a "spatial" grid $\{S_n\}_{n=0}^{N}$ in the S-direction by

$$S_0 = 0, \qquad S_n = nh = S_{n-1} + h, \quad 1 \le n \le N-1, \qquad S_N = S_{\max}$$

We divide the interval $[0, T]$ into M equal subintervals (timesteps) of length k, restricting our attention to uniform time intervals

$$t_0 = 0, \qquad t_m = mk = t_{m-1} + k, \quad m = 0, 1, \ldots, M.$$

This will give a two-dimensional mesh or grid of points in our domain, looking something like:



We solve the PDE approximately at these points: that is, we get approximations to the true values of f at the grid points. To get approximate values for f at other points in the domain we can interpolate from the grid point values.

As a shorthand, we denote the approximate solution at the grid point $(S_n, t_m) = (nh, mk)$ by

$$f_{n,m} = f(nh, mk).$$

There are several different solution schemes. An *explicit* scheme allows us to solve for each $f_{n,m}$ as we go along, in terms of the $f_{n,m}$ already worked out, but has stability issues, requiring $k/h^2 < \frac{1}{2}$.

Implicit schemes, as we use here, are unconditionally stable, but we cannot solve at a given timestep until we have relations among all the $f_{n,m}$ for that timestep: then we get a system of algebraic equations in the unknowns $f_{n,m}$, where m is fixed and $0 \le n \le N$. Since the Black-Scholes-Merton equation is a *linear* partial differential equation, this system is a system of linear equations, and so can be solved by the methods we have met already, such as *LU* decomposition, *SOR*, etc. We'll see that the system is very sparse, in fact tridiagonal, so *Sparse-SOR* works well.

The actual process is as follows: we approximate the partial derivatives in the Black-Scholes-Merton equation by finite differences, which are approximations to the derivative (in a similar sense to how the *2-Secant* zero-finding method approximates the tangent in the *Newton* method).

For a function of two variables, we can keep one fixed and take differences in the other. This is analogous to how partial derivatives are taken with respect to one variable, by keeping the other fixed. For a function $f(S,t)$ with spacing h between successive S-values and spacing k between successive t-values we use the first t backward difference to approximate $\partial f/\partial t$

$$\frac{\partial}{\partial t} f(S,t) = \frac{f(S,t) - f(S,t-k)}{k} + O(k),$$

the average of the first S forward and backward differences to approximate $\partial f/\partial S$:

$$\frac{\partial}{\partial S} f(S,t) = \frac{f(S+h,t) - f(S-h,t)}{2h} + O(h),$$

and the second S symmetric central difference[10] to approximate $\partial^2 f/\partial S^2$:

$$\frac{\partial^2}{\partial S^2} f(S,t) = \frac{f(S+h,t) - 2f(S,t) + f(S-h,t)}{h^2} + O(h^2).$$

There is a slight complication: we know the option value at maturity date ($t = T$) but not today ($t = 0$). Thus we will start at $t = T$ and work from T back to 0 in the time dimension, so a backward t difference will actually be a negative forward difference.

If we let $(S_n, t_m) = (nh, mk)$ be our base point (about which we take the differences), we get

$$\begin{aligned}
\frac{\partial}{\partial t} f(nh, mk) &\approx \frac{f(nh, mk) - f(nh, (m+1)k)}{k} \\
&= \frac{f_{n,m} - f_{n,m+1}}{k}, \\
\frac{\partial}{\partial S} f(nh, mk) &\approx \frac{f((n+1)h, mk) - f((n-1)h, mk)}{2h} \\
&= \frac{f_{n+1,m} - f_{n-1,m}}{2h}, \\
\frac{\partial^2}{\partial S^2} f(nh, mk) &\approx \frac{f((n+1)h, mk) - 2f(nh, mk) + f((n-1)h, mk)}{h^2} \\
&= \frac{f_{n+1,m} - 2f_{n,m} + f_{n-1,m}}{h^2}.
\end{aligned}$$

Substituting these in the Black-Scholes-Merton equation (noting that $S = nh$) gives

$$\frac{f_{n,m} - f_{n,m+1}}{k} + rnh\frac{f_{n+1,m} - f_{n-1,m}}{2h} + \frac{1}{2}\sigma^2 n^2 h^2 \frac{f_{n+1,m} - 2f_{n,m} + f_{n-1,m}}{h^2} = rf_{n,m},$$

for $n = 1, \ldots, N-1$, $m = 0, \ldots, M-1$. After rearrangement we get the implicit finite difference equations

$$-\frac{nk}{2}(n\sigma^2 - r)f_{n-1,m} + (1 + kr + k\sigma^2 n^2)f_{n,m} - \frac{nk}{2}(n\sigma^2 + r)f_{n+1,m} = f_{n,m+1} \tag{1}$$

for $n = 1, \ldots, N-1$, $m = 0, \ldots, M-1$.

The four grid points related by this equation are shown in the diagram: notice that the value of f at grid point $(nh, (m+1)k)$ is only related to the values of f at a few neighbouring grid points, the three immediately below it. This means that the coefficients of the system form a *sparse* matrix.

At the time of solving such an equation, $f_{n,m+1}$ will be known, which is why we have put it on the right here (just as in the usual $Ax = b$ we put the known vector b on the right).

To obtain the finite difference solution, we solve the difference equations (1) for $0 < n < N$ and $0 < m \leq M$. The boundary conditions determine the $f_{0,m}$ and the $f_{N,m}$, while the "initial" condition (actually a final condition) determines the $f_{n,M}$. The problem is then to find the $f_{n,m}$ for $0 \leq m < M$ (*i.e.*, $0 \leq t < T$) and $0 < n < N$ (*i.e.*, $0 < S < S_{\max}$).

[10]This may be defined either as the forward difference of backward-difference approximations to the first order partial, or as the backward difference of forward-difference approximations to the first order partial. The symmetric central difference approximation is preferred as its symmetry mirrors the reflectional symmetry of second order partials.

Look at the "final" condition first: the value of the put option at time T is $\max\{X - S_T, 0\}$ where S_T is the stock price at time T. This gives

$$f_{n,M} = \max\{X - nh, 0\}, \quad n = 0, 1, \ldots, N.$$

The value of the put option when $S = 0$ is just the strike price X, giving

$$f_{0,m} = X, \quad m = 0, 1, \ldots, M.$$

As assumed earlier, the value of the put option is zero when $S = S_{\max}$, giving

$$f_{N,m} = 0, \quad m = 0, 1, \ldots, M.$$

These define the value of the option along three edges of our grid (where $S = 0$, $S = S_{\max}$ and $t = T$) and allow us to get started.

We first form a system of linear equations for the timestep $m = M - 1$ and solve the resulting matrix equation $Af = b$. This would give

$$-\frac{nk}{2}(n\sigma^2 - r)f_{n-1,M-1} + (1 + kr + k\sigma^2 n^2)f_{n,M-1} - \frac{nk}{2}(n\sigma^2 + r)f_{n+1,M-1} = f_{n,M}$$

for $n = 1, \ldots, N - 1$ and we know all the numbers on the right hand side from the boundary conditions and "final" condition when $t = T$, so we can solve this. [We also have $f_{n-1,M-1} = f_{0,M-1}$ for $n = 1$ and $f_{n+1,M-1} = f_{N,M-1}$ for $n = N - 1$.]

Now we know all the f values $f_{n,M-1}$ so we can use these to find the values $f_{n,M-2}$ for the $M - 2$ timestep: this means solving another system of linear equations for this timestep.

This gives us all the f values $f_{n,M-2}$; we can use these to find the values $f_{n,M-3}$ for the $M - 3$ timestep… and so on: we solve a system of linear equations for each timestep until we have done timestep 0.

Each system of linear equations as in (1) can be written using a tridiagonal $(N - 1) \times (N - 1)$ matrix $A =$

$$\begin{bmatrix}
1 + kr + k\sigma^2 1^2 & \frac{-k}{2}(\sigma^2 + r) & 0 & \cdots & 0 & 0 \\
\frac{-2k}{2}(2\sigma^2 - r) & 1 + kr + k\sigma^2 2^2 & \frac{-2k}{2}(2\sigma^2 + r) & \cdots & 0 & 0 \\
0 & \frac{-3k}{2}(3\sigma^2 - r) & 1 + kr + k\sigma^2 3^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 + kr + k\sigma^2(N-2)^2 & \frac{-(N-2)k}{2}((N-2)\sigma^2 + r) \\
0 & 0 & 0 & \cdots & \frac{-(N-1)k}{2}((N-1)\sigma^2 - r) & 1 + kr + k\sigma^2(N-1)^2
\end{bmatrix}$$

giving the system of linear equations

$$A \begin{bmatrix} f_{1,m} \\ f_{2,m} \\ f_{3,m} \\ \vdots \\ f_{N-2,m} \\ f_{N-1,m} \end{bmatrix} = \begin{bmatrix} f_{1,m+1} + \frac{k}{2}(\sigma^2 - r)X \\ f_{2,m+1} \\ f_{3,m+1} \\ \vdots \\ f_{N-2,m+1} \\ f_{N-1,m+1} \end{bmatrix}$$

Because the matrix A is tridiagonal (only three non-zero diagonals), it is sparse and so lends itself to efficient algorithms in terms of storage and computation time, as we've seen in class for SOR.[11]

Notation 6.1. Write $f^{(j)}_{n,m}$ for the value of f at mesh point (nh, mk) on iteration j. The matrix equation can be written more compactly as $Af^{(m)} = b^{(m+1)}$, $m = 0, 1, \ldots, M - 1$.

Remark 6.2. The implicit finite difference scheme is unconditionally stable. This implicit method is only accurate to $O(k)$. An improvement which is still unconditionally stable but converges faster (errors are $O(k^2)$), is the Crank-Nicolson scheme, based on central differences; but it involves more work to implement and gives a matrix with 5 non-zeros in each row (bandwidth = 2).

[11] There is also a very fast $O(n)$ version of LU-decomposition called ADI for tridiagonal matrices.