# Classic Flag Algebra

Fix a family of graphs $\mathcal{G}$ and some type $\sigma$. Take the vector space $\mathbb{R}\mathcal{G}^\sigma$. We then quotient this space by the span of vectors of the form

$$F - \sum_{F' \in \mathcal{G}_n^\sigma} p(F; F')F$$

where $n \geq |F|$. Call this space $\mathcal{A}^\sigma$. To turn this space into an algebra we need to define a product: For $\sigma$-flags $F, F'$ define

$$F \cdot F' = \left[\sum_{H \in \mathcal{G}_n^\sigma} p(F, F'; H)H\right]$$

(where $[\cdot]$ means the corresponding coset in the quotient space) for any $n$ large enough s.t. $F, F'$ fit in a $\sigma$-flag of size $n$. This turns $\mathcal{A}^\sigma$ into an algebra after extending bilinearly to the whole space.

We call $\mathcal{A}^\sigma$ **the (classic) flag algebra of type** $\sigma$.

> **Todo**
>
> Is the classic algebra associative?

We extend our classic density function to this space bilinearly (in the first argument).

**Lemma:** *Densities in $\mathcal{A}^\sigma$ are multiplicative in the limit. Meaning for any $f, g \in A^\sigma$ and $\sigma$-flag $G$ we have:*

$$p((f \cdot g); G) = p(f; G)p(g; G) + O\left(\frac{1}{|G|}\right).$$

# Local Flag Algebra

Fix a family of graphs $\mathcal{G}$ and some type $\sigma$. Take the vector space $\mathbb{R}\mathcal{L}^\sigma$, the span of local $\sigma$-flags.

NB: Unlike in the classic flag algebra we do not quotient the space.

> **Todo**
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> Can we quotient by something? e.g. $\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma$

Again we need to define a product. We actually adopt the product from classic algebras with 2 minor adjustments:

1. We don't need a coset as we aren't quotienting the space.
2. This means we need to pick a specific value of $n$. We choose $n$ exactly large enough to fit both flags.

**Definition (Local Product):** Let $F, F' \in \mathcal{L}^\sigma$ be given. Let $n := |F| + |F'| - |\sigma|$. Then define

$$F \cdot F' := \sum_{H \in \mathcal{L}_n^\sigma} p(F, F'; H) \cdot H.$$

> **Note**
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> This is $p$, the classic density, not $\rho$. We've never defined $\rho$ on more than one input flag.

Extend this product bilinearly to the space $\mathbb{R}\mathcal{L}^\sigma$ to make it an algebra.

> **Todo**
>
> Is this associative?

We hope to achieve multiplicative local density in the limit as $\Delta \to \infty$.

**Lemma:** *For fixed $F, F', G \in \mathcal{L}^\sigma$ we have:*

$$\rho(F \cdot F'; G) = \rho(F; G)\rho(F'; G) + O\left(\frac{1}{\Delta}\right).$$

Before proving this we prove a result about embedding pairs of local flags into larger flags.

**Lemma:** *If $F, F'$ are local $\sigma$-flags and $G$ is a $\sigma$-flag of size $n = |F| + |F'| - |\sigma|$ such that $p(F, F'; G) > 0$ then $G$ is also a local $\sigma$-flag.*

*Proof:*

Let $\theta, \theta'$ be the $\sigma$ embeddings for $F, F'$ and $\eta$ the $\sigma$ embedding for $G$.

To prove $G$ is a local flag we first need to show that $\rho(G; \cdot)$ is a bounded function. i.e. $c(G; \cdot) \in O(\Delta^{|G| - |\sigma|})$.

As $p(F, F'; G) > 0$ there is some $U, U' \subseteq V(G)$ such that $U \cap U' = \operatorname{im} \eta$ and $F \cong G[U] \wedge F' \cong G[U']$ as $\sigma$ flags.

Let $(H, \zeta)$ be another $\sigma$-flag. If $c(G; H) = 0$ we're done so assume otherwise and let $\operatorname{im} \zeta \subseteq V \subseteq V(H)$ be such that $G \cong H[V]$ as $\sigma$-flags. In particular then this isomorphism $\phi$ induces an embedding of $U, U'$ into $V(H)$ such that $\operatorname{im} \eta = \zeta(U) \cap \zeta(U')$ and $H[\zeta(U)] \cong F \wedge H[\zeta(U')] \cong F'$ as $\sigma$-flags. Hence any choice of an instance of $G$ in $H$ and choice of instances $U, U'$ of $F, F'$ in $G$ gives rise to a pair of instances of $F$ and $F'$ in $H$. There are at most some constant number $C$ instances of $F, F'$ in $G$ (as the size of $G$ is fixed).

Note then also that any choice of a pair of instances of $F, F'$ in $H$ can be derived from at most 1 instance of $G$, as the size of $G$ was chosen to be the minimum possible such that both $F, F'$ fit. If the two instances overlap (outside of the required intersection at $\operatorname{im} \zeta$) then they don't correspond to an instance of $G$. If they don't overlap then their union corresponds to a single *possible* instance of $G$.

In summary each instance of $G$ gives rise to some non-zero but bounded number of pairs of instances of $F, F'$ in $H$, and each pair of instances is induced by at most 1 instance of $G$. Therefore $c(G; H) \leq \frac{1}{C} \cdot c(F; H) \cdot c(F'; H)$. $c(F; \cdot)$ and $c(F', \cdot)$ are $\in O(\Delta^{|F| - |\sigma|})$

and $\in O(\Delta^{|F'|-|\sigma|})$ respectively as $F, F'$ are local flags hence their product is $\in O(\Delta^{|F|+|F'|-2|\sigma|}) = O(\Delta^{|G|-|\sigma|})$ so $c(G, \cdot) \in O(\Delta^{|G|-|\sigma})$ showing $\rho(G, \cdot)$ is a bounded function as required.

**Todo**

Remains to prove that $G$ still has bounded density after fixing any unlabelled vertices. This might not be needed though if we prove that this property is always implied by the bounded property.

($\blacksquare$)

The value of this lemma is that is tells us that our sum over local flags in the definition of the product does include all flags for which $F, F'$ can be embedded.