Classic Flag Algebras

Fix a family of graphs $\mathcal{G}$ and some type σ. Take the vector space $\mathbb{R}\mathcal{G}^\sigma$. Define a subspace $\mathcal{K}^\sigma \subseteq \mathbb{R}\mathcal{G}^\sigma$ as the span of vectors of the form

$$F - \sum_{F' \in \mathcal{G}_n^\sigma} p(F; F')F$$

for F a σ-flag and $n \geq |F|$. Define the quotient space $\mathcal{A}^\sigma := \mathbb{R}\mathcal{G}^\sigma / \mathcal{K}^\sigma$. To turn this space into an algebra we need to define a product: For σ-flags F, F' define

$$F \cdot F' = \left[\sum_{H \in \mathcal{G}_n^\sigma} p(F, F'; H)H \right]$$

(where $[\cdot]$ means the corresponding coset in the quotient space) for any n large enough s.t. F, F' fit in a σ-flag of size n. This turns $\mathcal{A}^\sigma$ into an algebra after extending bilinearly to the whole space.

We call $\mathcal{A}^\sigma$ **the (classic) flag algebra of type** σ.

This algebra is commutative, associative and unital if σ is **non-degenerate** meaning $\mathcal{G}^\sigma$ is infinite. This implies $\sigma \notin \mathcal{K}^\sigma$: Lemma 2.4 (Razborov - 2007).

We extend our classic density function to this space bilinearly (in the first argument).

Lemma: *Densities in A^σ are multiplicative in the limit. Meaning for any $f, g \in A^\sigma$ and σ-flag G we have:*

$$p((f \cdot g); G) = p(f; G)p(g; G) + O\left(\frac{1}{|G|}\right).$$

Local Flag Algebras

Fix a family of graphs $\mathcal{G}$ and some type σ. Take the vector space $\mathbb{R}\mathcal{L}^\sigma$, the span of local σ-flags.

NB: Unlike in the classic flag algebra we do not quotient the space.

> **Todo**
>
> Can we quotient by something? e.g. $\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma$

Again we need to define a product. We actually adopt the product from classic algebras with 2 minor adjustments:

1. We don't need a coset as we aren't quotienting the space.
2. This means we need to pick a specific value of n. We choose n exactly large enough to fit both flags.

Definition (Local Product): Let $F, F' \in \mathcal{L}^\sigma$ be given. Let $n := |F| + |F'| - |\sigma|$. Then define

$$F \cdot F' := \sum_{H \in \mathcal{L}_n^\sigma} p(F, F'; H) \cdot H.$$

Note

This is p, the classic density, not ρ. We've never defined ρ on more than one input flag.

Extend this product bilinearly to the space $\mathbb{R}\mathcal{L}^\sigma$ to make it an algebra. This algebra is associative and unital which we prove later.

We hope to achieve multiplicative local density in the limit as $\Delta \to \infty$.

Theorem 1: *For fixed $F, F', G \in \mathcal{L}^\sigma$ we have:*

$$\rho(F \cdot F'; G) = \rho(F; G)\rho(F'; G) + O\left(\frac{1}{\Delta}\right).$$

Before proving this we prove a result about embedding pairs of local flags into larger flags.

Lemma: *If F, F' are local σ-flags and G is a σ-flag of size $n = |F| + |F'| - |\sigma|$ such that $p(F, F'; G) > 0$ then G is also a local σ-flag.*

Proof

Let θ, θ' be the σ embeddings for F, F' and η the σ embedding for G.

To prove G is a local flag we first need to show that $\rho(G; \cdot)$ is a bounded function. i.e. $c(G; \cdot) \in O(\Delta^{|G|-|\sigma|})$.

As $p(F, F'; G) > 0$ there is some $U, U' \subseteq V(G)$ such that $U \cap U' = \operatorname{im}\eta$ and $F \cong G[U] \wedge F' \cong G[U']$ as σ flags.

Let (H, ζ) be another σ-flag. If $c(G; H) = 0$ we're done so assume otherwise and let $\operatorname{im}\zeta \subseteq V \subseteq V(H)$ be such that $G \cong H[V]$ as σ-flags. In particular then this isomorphism ϕ induces an embedding of U, U' into $V(H)$ such that $\operatorname{im}\eta = \zeta(U) \cap \zeta(U')$ and $H[\zeta(U)] \cong F \wedge H[\zeta(U')] \cong F'$ as σ-flags. Hence any choice of an instance of G in H and choice of instances U, U' of F, F' in G gives rise to a pair of instances of F and F' in H. There are at most some constant number C instances of F, F' in G (as the size of G is fixed).

Note then also that any choice of a pair of instances of F, F' in H can be derived from at most 1 instance of G, as the size of G was chosen to be the minimum possible such that both F, F' fit. If the two instances overlap (outside of the required intersection at $\operatorname{im}\zeta$) then they don't correspond to an instance of G. If they don't overlap then their union corresponds to a single *possible* instance of G.

In summary each instance of G gives rise to some non-zero but bounded number of pairs of instances of F, F' in H, and each pair of instances is induced by at most 1 instance of G. Therefore $c(G; H) \leq \frac{1}{C} \cdot c(F; H) \cdot c(F'; H)$. $c(F; \cdot)$ and $c(F', \cdot)$ are $\in O(\Delta^{|F|-|\sigma|})$ and $\in O(\Delta^{|F'|-|\sigma|})$ respectively as F, F' are local flags hence their product is $\in O(\Delta^{|F|+|F'|-2|\sigma|}) = O(\Delta^{|G|-|\sigma|})$ so $c(G, \cdot) \in O(\Delta^{|G|-|\sigma})$ showing $\rho(G, \cdot)$ is a bounded function as required.

It remains to prove that G still has bounded density after fixing any unlabelled vertices. This argument is extremely similar to what we saw already. Label some new vertex v in G, extending η to η'. We know $V(G)$ can be split into U, U' disjoint outside $\operatorname{im}\eta$ such that $G[U] \cong F, G[U'] \cong F'$ in some constant number of ways. Then for each such split $v \in U \setminus \operatorname{im}\eta$ or $v \in U' \setminus \operatorname{im}\eta$. WLOG assume $v \in U \setminus \operatorname{im}\eta$. Then for every embedding of (G, η') into H we can again use the U, U' split to derive copies of F, F' in H. But we know $v \in U$ so as F is a local flag we have $\in O(\Delta^{|F|-|\sigma|-1})$ possible copies. There are $O(\Delta^{|F'|-|\sigma|})$ copies of F' leading to a bound of order $O(\Delta^{|G|-|\sigma|-1})$ as required.

(■)

The value of this lemma is that is tells us that our sum over local flags in the definition of the product does include all flags for which F, F' can be embedded.

Proof of Theorem 1

Todo

Address assuming $F, F' \neq \sigma$

Let F, F' be local σ-flags, and G a σ-flag. We can compute

$$\begin{aligned} &\rho(F;G)\cdot\rho(F';G) - \rho(F\cdot F';G) \\ &= \rho(F;G)\cdot\rho(F';G) - \left(\sum_{H\in\mathcal{L}_n^\sigma} p(F,F';H)\rho(H;G)\right) \\ &= \frac{c(F;G)\cdot c(F';G)}{\binom{\Delta}{|F|-k}\binom{\Delta}{|F'|-k}} - \frac{\sum_{H\in\mathcal{L}_n^\sigma} c(F,F';H)c(H;G)}{\binom{n-k}{|F|-k}\binom{\Delta}{n-k}} \end{aligned} \tag{†}$$

where $k = |\sigma|$ and $n = |F| + |F'| - k$. First we note both denominators are asymptotically equivalent:

$$\binom{\Delta}{|F|-k}\binom{\Delta}{|F'|-k} = \frac{\Delta!}{(|F|-k)!(\Delta-(|F|-k))!}\frac{\Delta!}{(|F'|-k)!(\Delta-(|F'|-k))!}$$

and

$$\begin{aligned} \binom{n-k}{|F|-k}\binom{\Delta}{n-k} &= \frac{(n-k)!}{(|F|-k)!(n-k-(|F|-k))!}\frac{\Delta!}{(n-k)!(\Delta-(n-k))!} \\ &= \frac{\Delta!}{(|F|-k)!(n-|F|)!(\Delta-(n-k))!} \\ &= \frac{\Delta!}{(|F|-k)!(|F'|-k)!(\Delta-(n-k))!}. \end{aligned}$$

Hence

$$\frac{\binom{\Delta}{|F|-k}\binom{\Delta}{|F'|-k}}{\binom{n-k}{|F|-k}\binom{\Delta}{n-k}} = \frac{\Delta!(\Delta-(n-k))!}{(\Delta-(|F|-k))!(\Delta-(|F'|-k))!}$$

Remembering that $(n-k) = (|F|-k) + (|F'|-k)$ we see that this is of the form

$$\frac{\Delta(\Delta-1)\dots(\Delta-(a-1))}{(\Delta-b)(\Delta-b-1)\dots(\Delta-b-(a-1))}$$

where $a = |F|-k, b = |F'|-k$. Both numerator and denominator are composed of a terms. This expression has limit 1 as $\Delta \to \infty$ showing

$$\binom{\Delta}{|F|-k}\binom{\Delta}{|F'|-k} \sim \binom{n-k}{|F|-k}\binom{\Delta}{n-k}.$$

Todo

Do I need this whole argument at all? Is it too verbose?

As the denominators in (†) are asymptotically equivalent we can write

$$\frac{c(F;G)\cdot c(F';G)}{\binom{\Delta}{|F|-k}\binom{\Delta}{|F'|-k}} - \frac{\sum_{H\in\mathcal{L}_n^\sigma} c(F,F';H)c(H;G)}{\binom{n-k}{|F|-k}\binom{\Delta}{n-k}}$$
$$\sim \frac{c(F;G)\cdot c(F';G) - \sum_{H\in\mathcal{L}_n^\sigma} c(F,F';H)c(H;G)}{\binom{n-k}{|F|-k}\binom{\Delta}{n-k}}$$

Now we look the asymptotics of the numerator:

$$c(F;G)\cdot c(F';G) - \left(\sum_{H\in\mathcal{L}_n^\sigma} c(F,F';H)c(H;G)\right) \qquad (\dagger\dagger)$$

Let θ, θ', η be the embedding of σ into F, F', G resp. The product $c(F;G)\cdot c(F';G)$ counts the number of pairs $\operatorname{im}\eta \subseteq U, U' \subseteq V(G)$ such that $G[U] \cong F \wedge G[U'] \cong F'$ as σ-flags.

Comparatively the sum $\sum_{H\in\mathcal{L}_n^\sigma} c(F,F';H)c(H;G)$ counts the number of pairs $U, U' \subseteq V(G)$ such that $G[U] \cong F \wedge G[U'] \cong F'$ as σ-flags *and* $U \cap U' = \operatorname{im}\eta$. This fact depends on our previous lemma which guarantees that any σ-flag H of size n inducing instances of F and F' is a local flag so is included in $\mathcal{L}_n^\sigma$.

Therefore $(\dagger\dagger)$ counts those pairs $\operatorname{im}\eta \subseteq U, U' \subseteq V(G)$ such that $G[U] \cong F \wedge G[U'] \cong F'$ as σ-flags but $U \cap U' \neq \operatorname{im}\eta$ meaning U, U' have a nonempty intersection outside of $\operatorname{im}\eta$.

Todo

This part needs diagrams.

Let $\operatorname{im}\eta \subseteq U \subseteq V(G)$ inducing F be fixed. Pick any $v \in U \setminus \operatorname{im}\eta$ and ask how many $\operatorname{im}\eta \subseteq U' \subseteq V(G)$ are there inducing F' such that $v \in U'$?

Let σ' be the type obtained by labelling v: $\sigma \cong G[\operatorname{im}\eta \cup \{v\}]$.

If there are any $\operatorname{im}\eta \subseteq U' \subseteq V(G)$ inducing F' with $v \in U'$ then $\operatorname{im}\eta \cup \{v\} \subseteq U'$ so there is some induced embedding of σ' in F' meaning F' is a local σ'-flag. Otherwise there are no such U' and our answer to the question is 0 (for that choice of v).

Let θ' be any σ'-embedding into F'. Now we can express the question of how many embeddings of F' into G there are including v by extending the σ-embedding η to a σ'-embedding η' and asking for the value of $c((F',\theta'),(G,\eta'))$. We know (F',θ') is a local flag so we have $c((F',\theta'),\cdot) \in O(\Delta^{|F'|-(k+1)})$. Let C be a corresponding constant so we have $c((F',\theta'),(G,\eta')) \leq C\Delta^{|F|-(k+1)}$.

There was a constant number of choices of θ' so we get some larger constant C' bounding the number of induced copies of F' including v by $C'\Delta^{|F|-(k+1)}$.

Then as $v \in U$ there was some constant number of choices of v meaning the total number of induced copies of F' that can overlap with a fixed induced copy of F is $\in O(\Delta^{|F'|-(k+1)})$. There are $O(\Delta^{|F|-k})$ induced copies of F in G leaving a total of $O(\Delta^{|F|-k}\Delta^{|F'|-(k+1)}) = O(\Delta^{n-k-1})$ possible pairs U, U' inducing copies of F, F' which overlap outside of $\operatorname{im}\eta$.

$$\therefore (\dagger\dagger) \in O(\Delta^{n-k-1}).$$

Now we finally can bound $(\dagger)$ by realising $\binom{n-k}{|F|-k}\binom{\Delta}{n-k} \in \Omega(\Delta^{n-k})$ so we get

$$\begin{aligned} \rho(F;G)\cdot\rho(F';G) - \rho(F\cdot F';G) &\sim \frac{c(F;G)\cdot c(F';G) - \sum_{H\in\mathcal{L}_n^\sigma} c(F,F';H)c(H;G)}{\binom{n-k}{|F|-k}\binom{\Delta}{n-k}} \\ &= \frac{O(\Delta^{n-k-1})}{\Omega(\Delta^{n-k})} \\ &= O\left(\frac{1}{\Delta}\right). \end{aligned}$$

(■)

Lemma: *For local type σ the local flag algebra $\mathcal{L}^\sigma$ is associative.*

Proof:

Let F_1, F_2, F_3 be local σ-flags. Then let $k := |\sigma|$, $\ell := |F_1| + |F_2| - k$, and $n := |F_1| + |F_2| + |F_3| - 2k$. We calculate

$$\begin{aligned} (F_1\cdot F_2)\cdot F_3 &= \left(\sum_{H\in\mathcal{L}_\ell^\sigma} p(F_1,F_2;H)H\right)\cdot F_3 \\ &= \sum_{H\in\mathcal{L}_\ell^\sigma} p(F_1,F_2;H)(H\cdot F_3) \\ &= \sum_{H\in\mathcal{L}_\ell^\sigma} p(F_1,F_2;H)\left(\sum_{G\in\mathcal{L}_n^\sigma} p(H,F_3;G)G\right) \\ &= \sum_{G\in\mathcal{L}_n^\sigma}\left(\sum_{H\in\mathcal{L}_\ell^\sigma} p(F_1,F_2;H)p(H,F_3;G)\right)G \\ &= \sum_{G\in\mathcal{L}_n^\sigma} p(F_1,F_2,F_3;G)G \end{aligned}$$

where we use the chain rule for classic densities. This term is symmetric in all 3 of F_1, F_2, F_3 and the product is commutative so $(F_1\cdot F_2)\cdot F_3 = F_1\cdot(F_2\cdot F_3)$

(■)

Lemma: *For local type σ the local flag algebra $\mathcal{L}^\sigma$ has unit σ.*

Proof

For any $F, G \in \mathcal{L}^\sigma$ we have $p(\sigma, F; G) = p(F;G)$. Also if $F, F' \in \mathcal{L}^\sigma$ have the same size then $p(F;F') = p(F';F) = 0$ hence

$$\sigma\cdot F = \sum_{H\in\mathcal{L}_{|F|}^\sigma} p(\sigma,F;H)H = \sum_{H\in\mathcal{L}_{|F|}^\sigma} p(F;H)H = F.$$

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Averaging Operator

Todo

This theory can be generalised to partial unlabellings (See definition 10 in (Razborov - 2007: Flag Algebras).

Averaging Classic Flags

Definition ($\downarrow F$): If F is a σ-flag then $\downarrow F$ is the graph underlying the flag (equivalently a $\emptyset$-flag).

Definition ($q_\sigma(F)$): For a σ-flag F define $q_\sigma(F)$ to be the probability that a random injective $\theta\colon[|\sigma|] \to V(F)$ is such that $(\downarrow F, \theta) \cong F$.

Todo

Probably easily related to the size of the automorphism group.

Definition: ($[\![\cdot]\!]$): We define the **downward operator** or **averaging operator** as follows: For σ-flag F define

$$[\![F]\!] := q_\sigma(F) \cdot \downarrow F.$$

Extend this operator linearly to get a linear map $\mathbb{R}\mathcal{G}^\sigma \to \mathbb{R}\mathcal{G}$.

Lemma: *The averaging operator maps $\mathcal{K}^\sigma$ to $\mathcal{K}^\emptyset$ so forms a valid linear map $\mathcal{A}^\sigma \to \mathcal{A}^\emptyset$.*

Lemma: *Let F be a σ-flag and G a graph with $|G| \geq |F|$ such that $p(\sigma; G) > 0$. Let $\theta\colon[|\sigma|] \to V(G)$ be a uniformly random σ-embedding*[1] *then:*

$$\mathbb{E}_\theta[p(F;(G,\theta)] = \frac{p([\![F]\!];G)}{p([\![\sigma]\!];G)} = \frac{q_\sigma(F) \cdot p(\downarrow F;G)}{q_\sigma(\sigma) \cdot p(\sigma;G)}$$

Averaging Local Flags

We adopt the same definitions for $\downarrow F$, $q_\sigma(F)$ and $[\![F]\!]$ from the classic case.

Note: unlabelling is not always well behaved in the local case.

Lemma: *Let F be a local σ-flag. Then $\downarrow F$ is not necessarily a local $\emptyset$-flag.*

Proof

Consider the family $\mathcal{G}$ of 2 coloured (red/black) graphs where there are $\leq \Delta$ black vertices and the rest are red. Let $\sigma = \bullet$. σ is a local type as for any $G \in \mathcal{G}$ and $\bullet$-embedding η into G there is exactly 1 label preserving map $\bullet \to (G;\eta)$. Hence $c(\bullet;(G,\eta) = 1$ and $\rho(\bullet;(G,\eta) = \frac{1}{\binom{\Delta}{0}} = 1$ is bounded so σ is a local type, (and implicitly a local flag).

However, $[\![\sigma]\!] = \downarrow\bullet$ and so for any 2-coloured graph $G \in \mathcal{G}$ $c([\![\sigma]\!];G)$ counts the number of red vertices which by our choice of $\mathcal{G}$ is $\geq |G| - \Delta$. Hence $\rho([\![\sigma]\!];G) \geq \frac{|G|-\Delta}{\Delta} = \frac{|G|}{\Delta} - 1$ which is not bounded as $|G|$ can be arbitrarily large relative to Δ. Hence $[\![\sigma]\!]$ is not a local $\emptyset$-flag.

(■)

This doesn't mean however that we cannot use the averaging operator, it still forms a valid linear map into $\emptyset$-flags (maybe non-local) $\mathcal{L}^\sigma \to \mathcal{G}^\emptyset$.

We have the averaging behaviour as in the classic case, but this time only in the limit.

Theorem 2: *Let F be a (possibly non local) σ-flag and G a graph with $|G| \geq |F|$ such that $\rho(\sigma; G) > 0$. Let $\theta\colon [|\sigma|] \to V(G)$ be a uniformly random σ-embedding*[1-1] *then:*

$$\mathbb{E}_\theta[\rho(F;(G,\theta)] \sim \frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} = \frac{q_\sigma(F)\cdot\rho(\downarrow F;G)}{q_\sigma(\sigma)\cdot\rho(\sigma;G)}$$

Proof:

We use the following identity relating the classic and local densities: For any σ-flag H we have

$$\rho([\![H]\!];G) = q_\sigma(H)\rho(\downarrow H;G) = \frac{q_\sigma(H)c(\downarrow H;G)}{\binom{\Delta}{|H|}} = \frac{\binom{|G|}{|H|}q_\sigma(H)c(\downarrow H;G)}{\binom{|G|}{|H|}\binom{\Delta}{|H|}}$$

$$= \frac{\binom{|G|}{|H|}}{\binom{\Delta}{|H|}}\frac{q_\sigma(H)c(\downarrow H;G)}{\binom{|G|}{|H|}} = \frac{\binom{|G|}{|H|}}{\binom{\Delta}{|H|}}q_\sigma(H)p(\downarrow H;G) = \frac{\binom{|G|}{|H|}}{\binom{\Delta}{|H|}}p([\![H]\!];G).$$

Then we compute

$$\frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} = \frac{p([\![F]\!];G)}{p([\![\sigma]\!];G)}\frac{\binom{|G|}{|F|}}{\binom{\Delta}{|F|}}\frac{\binom{\Delta}{|\sigma|}}{\binom{|G|}{|\sigma|}}.$$

Using the averaging result for classic algebras we get

$$\frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} = \frac{p([\![F]\!];G)}{p([\![\sigma]\!];G)}\frac{\binom{|G|}{|F|}}{\binom{\Delta}{|F|}}\frac{\binom{\Delta}{|\sigma|}}{\binom{|G|}{|\sigma|}} = \mathbb{E}_\theta[p(F;(G,\theta))]\frac{\binom{|G|}{|F|}}{\binom{\Delta}{|F|}}\frac{\binom{\Delta}{|\sigma|}}{\binom{|G|}{|\sigma|}}$$

where θ is a uniformly random σ-embedding. Then we use a similar identity which says

$$p(F;(G,\theta)) = \rho(F;(G,\theta))\frac{\binom{\Delta}{|F|-|\sigma|}}{\binom{|G|-|\sigma|}{|F|-|\sigma|}}$$

to get

$$\frac{\rho([\![F]\!])}{\rho([\![\sigma]\!])} = \mathbb{E}_\theta[p(F;(G,\theta))]\frac{\binom{|G|}{|F|}}{\binom{\Delta}{|F|}}\frac{\binom{\Delta}{|\sigma|}}{\binom{|G|}{|\sigma|}} = \mathbb{E}_\theta[\rho(F;(G,\theta))]\frac{\binom{\Delta}{|F|-|\sigma|}}{\binom{|G|-|\sigma|}{|F|-|\sigma|}}\frac{\binom{|G|}{|F|}}{\binom{\Delta}{|F|}}\frac{\binom{\Delta}{|\sigma|}}{\binom{|G|}{|\sigma|}}.$$

We now show this multiplicative factor is of the form $(1+o(1))$:

$$\frac{\binom{\Delta}{|F|-|\sigma|}}{\binom{|G|-|\sigma|}{|F|-|\sigma|}}\frac{\binom{|G|}{|F|}}{\binom{\Delta}{|F|}}\frac{\binom{\Delta}{|\sigma|}}{\binom{|G|}{|\sigma|}} = \frac{\binom{\Delta}{|F|-|\sigma|}\binom{\Delta}{|\sigma|}}{\binom{\Delta}{|F|}}\frac{\binom{|G|}{|F|}}{\binom{|G|-|\sigma|}{|F|-|\sigma|}\binom{|G|}{|\sigma|}}.$$

We use the following binomial relation $\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}$ to reduce this to

$$\frac{\binom{\Delta}{|F|-|\sigma|}\binom{\Delta}{|\sigma|}}{\binom{\Delta}{|F|}}\frac{1}{\binom{|F|}{|\sigma|}}.$$

We then have the following relation in the limit $\binom{n}{a}\binom{n}{b} \sim \binom{n}{a+b}\binom{a+b}{a}$ so we get

$$\frac{\binom{\Delta}{|F|-|\sigma|}\binom{\Delta}{|\sigma|}}{\binom{\Delta}{|F|}}\frac{1}{\binom{|F|}{|\sigma|}} \sim \frac{\binom{|F|}{|\sigma|}}{\binom{|F|}{|\sigma|}} = 1.$$

Therefore

$$\mathbb{E}_\theta[\rho(F;(G,\theta))] = (1+o(1))\frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)}. \qquad (\blacksquare)$$

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1. Note, different to a uniforly random injective map. This is conditioned on the property that the map is an actual embedding.↩↩