Graphs

Definition ((c_e, c_v)-graph): We define a **(c_e, c_v)-graph** to be a simple undirected graph where each edge and vertex is assigned a colour from $[c_e], [c_v]$ respectively.

Definition (Isomorphism): A function $f: V(G) \to V(H)$ for G, H (c_e, c_v)-graphs is an isomorphism if it is a classic graph isomorphism and preserves colours. We write $G \cong H$.

Definition (Graph Class): If $\mathbb{G}^{c_e, c_v}$ is the class of all (c_e, c_v)-graphs then a **graph class** $\mathcal{G} \subseteq \mathbb{G}^{c_e, c_v}$ is just some subclass.

Definition (Hereditary Class): A graph class $\mathcal{G}$ is **hereditary** if the class is closed under taking *induced* subgraphs: $G \in \mathcal{G} \wedge U \subseteq V(G) \implies G[U] \in \mathcal{G}$.

Definition (Hereditary Closure): Given a graph class $\mathcal{G}$ define the **hereditary closure** as the minimal hereditary class $\overline{\mathcal{G}}$ containing $\mathcal{G}$. Intuitively this is just $\mathcal{G}$ and all possible induced subgraphs of $G \in \mathcal{G}$.

Flags

The choice of $\mathcal{G}$ informs all the following definitions.

> **Important**
>
> In Razborov's original formulation[1] he defines flags in a very general way in terms of finite model theory. In particular his definitions place requirements on $\mathcal{G}$ that mean it must be hereditary. For this reason we don't reference hereditary closures in the classic flag case.

Modified from (Silva, Filho, Sato - 2016: [Flag Algebras: A First Glance](#)).

Original definitions: Section 2.1 of (Razborov - 2007: [Flag Algebras](#)).

Definition (Type): A **type of size** $k \in \mathbb{N}_0$ is some $\sigma \in \mathcal{G}$ with $V(\sigma) = [k]$. We write $|\sigma|$ for the size of the type.

Definition (Embedding): An **embedding** of a type σ of size k into $F \in \mathcal{G}$ is an injective function $\theta: [k] \to V(F)$ such that θ is an isomorphism between σ and $F[\operatorname{im} \theta]$.

Definition (Flag): A pair (F, θ) where $F \in \mathcal{G}$ and θ is an embedding of σ into F is a **σ-flag.**

Notation: We denote the collection of all σ-flags up to isomorphism with $\mathcal{G}^\sigma$. Use a subscript $\mathcal{G}_n^\sigma$ to denote only flags of size n. If $\sigma = \emptyset$ we can drop the σ superscript.

Notation: We will often implicitly treat a type σ as a σ-flag. Specifically one with the $\mathrm{id}: [|\sigma|] \to [|\sigma|]$ embedding.

Definition (Labelled): We refer to a vertex in (F, θ) as labelled if it is in the image of θ, and often call $\operatorname{im} \theta$ the *labelled part of F.*

Definition (Flag Isomorphism): $f: V(F) \to V(F')$ is a σ-flag isomorphism from $(F,\theta) \to (F',\theta')$ if it is an isomorphism $F \to F'$ which preserves labels, meaning $f(\theta(i)) = \theta'(i) \,\forall\, i \in [|\sigma|]$. We can write $(F,\theta) \cong (F',\theta')$ if such an f exists.

Empty-flags: If $\sigma = \emptyset$, the empty graph, then flags are just graphs with trivial embeddings. We usually just write F rather than (F,θ) and treat graphs as implicit $\emptyset$-flags.

Induced Count

Definition (Induced Count): Fix two σ-flags $(F,\theta),(G,\eta)$. We define the induced count of (F,θ) in (G,η) written $c((F,\theta);(G,\eta))$ (abbreviated as $c(F;G)$ if θ,η are implicit) as the number of induced copies of F in G s.t. the labelled part of F maps to the labelled part of G.

Specifically this is the number of subsets $\mathrm{im}(\eta) \subseteq U \subseteq V(G)$ such that $(F,\theta) \cong (G[U],\eta)$. Clearly $c(F,G) = 0$ if $|G| < |F|$.

Extension to multiple flags: Take a finite number of σ-flags $(F_i,\theta_i), i \in [t]$ and another σ-flag (G,η). We now define $c(F_1,\dots,F_t;G)$ as the number of t-tuples of induced copies of $F_1,\dots,F_t$ in G which are disjoint except precisely at the labelled part of G.

Precisely this is the number of subsets $U_1,\dots,U_t \subseteq V(G)$ such that $\mathrm{im}\,\eta = U_i \cap U_j \,\forall\, i,j \in [t]$ where $i \neq j$ and $(F_i,\theta_i) \cong (G[U_i],\eta)$ for all $i \in [t]$.

Definition (Fit): Note that if $c(F_1,\dots,F_t;G) > 0$ then $|G| - |\sigma| \geq \sum_{i=1}^{t} |F_i| - |\sigma|$. If this holds we say $F_1,\dots,F_t$ **fit in** G.

Classic Density

This is the classic density used in Razborov's flag algebras: [(Razborov - 2007)](#).

Definition (Induced Density): For σ-flags (F,θ) and (G,η) define $p((F,\theta),(G,\eta))$ (abbreviated as $p(F,G)$) as the proportion of subsets $\mathrm{im}\,\eta \subseteq U \subseteq V(G)$ such that $(F,\theta) \cong (G[U],\eta)$:

$$p(F;G) = \frac{c(F;G)}{\binom{|G|-|\sigma|}{|F|-|\sigma|}}.$$

Extension to finite set of flags: As with $c(F,G)$ we can extend to a finite collection of flags (F_i,θ_i), $i \in [t]$ where we normalise $c(F_1,\dots,F_t;G)$ by the total number of possible subsets $U_1,\dots,U_t \subseteq V(G)$ such that $U_i \cap U_j = \mathrm{im}\,\eta \,\forall i \neq j$.

$$p(F_1,\dots,F_t;G) = \frac{c(F_1,\dots,F_t;G)}{\binom{|G|-|\sigma|}{|F_1|-|\sigma|,\dots,|F_t|-|\sigma|,R}}$$

using multinomial coefficient notation where $R = (|G| - |\sigma|) - \sum_{i=1}^{t} |F_i| - |\sigma|$.

Probabilistic Interpretation: You can interpret $p(F_1,\dots,F_t;G)$ as the probability that a uniformly random tuple $U_1,\dots,U_t \subseteq V(G)$ such that $U_i \cap U_j = \mathrm{im}\,\eta \,\forall\, i \neq j$ has the property that $(F_i,\theta_i) \cong (G[U_i],\eta) \,\forall\, i \in [t]$.

Lemma (Chain Rule): *If $F_1,\dots,F_t$ are σ-flags which fit in G then for all $1 \leq s \leq t$ and every n such that $F_1,\dots,F_s$ fit into a σ flag of size n and a σ-flag and $F_{s+1},\dots,F_t$ fit in G we have:*

$$p(F_1, \dots, F_t; G) = \sum_{F \in \mathcal{G}_n^\sigma} p(F_1, \dots, F_s; F) p(F, F_{s+1}, \dots, F_t; G).$$

Labelling

Definition (Label Extension): Given a σ-flag (F, θ) and some unlabelled vertex $v \in V(F) \setminus \operatorname{im}\theta$ we can construct a **label extension** $\theta' : [|\sigma| + 1] \to V(F)$ as

$$\theta'(i) = \begin{cases} \theta(i) & \text{if } i \in [|\sigma|] \\ v & \text{if } i = |\sigma| + 1. \end{cases}$$

This is an embedding of the type σ' with vertices $[|\sigma| + 1]$ obtained by adding a new vertex $|\sigma| + 1$ to σ such that $\sigma' \cong F[\operatorname{im}\theta']$.

(F, θ') is then a σ'-flag.

Local Flags

Todo

Needs an example of a density hard to describe with classic flags. Maybe something like paths of length 2 in a Δ-regular graph.

Let $\mathcal{G}$ be some graph class and consider its hereditary closure $\overline{\mathcal{G}}$. We inherit the definition of flags and induced count from the classic case, where we take underlying graphs from $\overline{\mathcal{G}}$.

Example

If the class $\mathcal{G}$ is the class of regular graphs then this class is not hereditary. It's hereditary closure $\overline{\mathcal{G}}$ is just the class of all graphs.

Let Δ be some graph parameter $\Delta : \mathcal{G} \to \mathbb{N}_0$ such that $\Delta(G) \to \infty$ implies $|G| \to \infty$. (More formally there is some monotone increasing unbounded function f such that for all $G \in \mathcal{G}$ we have $|G| \geq f(\Delta(G))$).

Important

We almost exclusively use the maximum degree function Δ so you can effectively always interpret $\Delta(G)$ in this way.

Definition (Local Density): Rather than normalising the induced count of σ-flags $c((F, \theta); (G, \eta))$ by $\binom{|G| - |\sigma|}{|F| - |\sigma|}$ we instead normalise by $\binom{\Delta(G)}{|F| - |\sigma|}$.

$$\rho(F; G) := \frac{c(F; G)}{\binom{\Delta(G)}{|F| - |\sigma|}}.$$

Note the $\rho \neq p$ notation. We consider this as a function $\overline{\mathcal{G}}^\sigma \times \mathcal{G}^\sigma \to \mathbb{R}_{\geq 0}$.

Because of our choice of "normalisation" we are no longer guaranteed a $[0, 1]$ codomain. The full range for ρ is $\mathbb{R}_{\geq 0}$.

Todo

Example of why ρ can be unbounded.

This unboundedness is undesirable. We will now define a restricted subset of the classic flags to the case where we do not have this divergent behaviour.

Todo

Needs an elucidatory example of a family of graphs $\mathcal{G}$ where we describe which structures have divergent density and which do not.

Definition (Local σ-flag): Fix some graph class $\mathcal{G}$ and takes its hereditary closure $\overline{\mathcal{G}}$. This class gives us a collection of flags and types. Let σ be a type. Then a σ-flag $(F,\theta) \in \overline{\mathcal{G}}^\sigma$ is a **local σ-flag** if we have the following properties:

1. $(G,\eta) \to \rho((F,\theta);(G,\eta))$ is a bounded function as a function $\mathcal{G}^\sigma \to \mathbb{R}_{\geq 0}$. (*We are very intentionally using $\mathcal{G}$ and not its closure here*).
2. If we label any of F's unlabelled vertices we get another local flag.

To state this 2nd property precisely: We require that for any label extension θ' of θ, the induced extended flag (F,θ') is also a local flag.

This is not a circular definition as any label extension of (F,θ) reduces the number of unlabelled vertices by 1; We could define inductively starting with those flags with no unlabelled vertices.

What we're trying to capture here is that any "subflag" of F is also a local flag, meaning we can pin down F's vertices and continue to get bounded behaviour.

Note: This second property is not necessarily implied by the first, so it is a required part of the definition of a local flag.

Lemma: *There exists an infinite class of graphs $\mathcal{G}$ and a flag (F,θ) such that $\rho((F,\theta);\cdot)$ is a bounded function but a label extension (F,θ') has unbounded density.*

Proof

Consider 3-vertex-coloured graphs (red, blue, black) and take $\mathcal{G}$ to be the class of graphs with a single red vertex, $\Delta(G)^2$ blue vertices and the rest black vertices such that there are no red-blue edges.

Relative to this $\mathcal{G}$ then the following $\emptyset$-flag is local: $F = (\bullet \quad \bullet)$. We can calculate that for $G \in \mathcal{G}$ we have $c(F;G) = \Delta(G)^2$ as we have 1 choice for the red vertex and $\Delta(G)^2$ for the blue. Hence $\rho(F;G) = c(F;G)/\binom{\Delta(G)}{2} \leq 2.$

Consider labelling the red-vertex in F, giving $F' = (\circ \quad \bullet)$. with type $\sigma = \bullet$. For any $G \in \mathcal{G}$ there is a unique embedding η_0 of σ, mapping this red vertex to the single red vertex. However we still have $c(F';(G,\eta_0)) = \Delta(G)^2$ so $\rho(F';(G,\eta_0)) = c(F';(G,\eta_0))/\Delta \sim \Delta(G)$ which is unbounded.

(■)

Notation: Write $\mathcal{G}^{\sigma}_{\mathrm{loc},n}$ to be the set of all local σ-flags of size n up to isomorphism. Write $\mathcal{G}^{\sigma}_{\mathrm{loc}}$ for all local σ-flags. If $\sigma = \emptyset$ then we can drop the σ superscript.

Warning

The chain rule does not hold in general for local flags.

Warning

Just because valid local σ-flags exist does not necessarily mean σ itself is implicitly a local σ-flag.

Note

By our above definitions if (F, θ) is a local σ-flag then the function $\Delta(G)$ somehow "bounds" the degrees of freedom for embeddings of unlabelled vertices of F into G.

e.g. If $\Delta(G)$ is the maximum degree function, then $\Delta(G)$ bounds the degree of freedom of picking a vertex which is connected to some fixed vertex. This gives intuition to why the local σ-flags in the class of graphs with this choice of $\Delta(G)$ is precisely those flags F where each vertex is connected to some labelled vertex.

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1. Razborov - 2007: Flag Algebras↩