I will describe everything here in terms of classic flag algebras, then discuss how it transfers to local flags at the end.

Bounding Densities

Many combinatorial results about graphs can be stated in terms of an upper/lower bound on some density function.

Example (Mantel's Theorem)

Theorem (Mantel's): A triangle free graph has $\leq \frac{n^2}{4}$ edges.

Asymptotically this means for any sequence of triangle free graphs $(G_k)_{k\in\mathbb{N}}$ we have $\lim_{k\to\infty} p(\circ\!-\!\circ; G_k) \leq \frac{1}{2}$ or equivalently for any limit functional ϕ we have $\phi(\circ\!-\!\circ) \leq \frac{1}{2}$.

Assume that for some fixed linear combination of flags f we want to find $\lambda \in \mathbb{R}$ such that for all limit functionals ϕ we have $\phi(f) \leq \lambda$ for some lambda.

Note that for the empty graph $\emptyset$ we always have $p(\emptyset; G) = 1$ and so $\phi(\emptyset) = 1$ for all limit functionals ϕ. Then:

$$\phi(f) \leq \lambda \iff 0 \leq \lambda - \phi(f) \iff 0 \leq \phi(\lambda\emptyset) - \phi(f) \iff 0 \leq \phi(\lambda\emptyset - f).$$

so we have $\phi(f) \leq \lambda$ for all limit functionals ϕ iff $\phi(\lambda\emptyset - f) \geq 0$ for all ϕ. This is exactly the condition that $\lambda\emptyset - f \in \mathcal{C}^{\emptyset}_{\text{sem}}$. Hence we can prove bounds on densities by finding specific elements of the semantic cone.

Relaxation

It suffices to find any $\lambda\emptyset - (f + r) \geq 0$ where $r \geq 0$, this allows us to relax the equality constraint slightly.

Example (Mantel's Proof)

Assume all graphs are triangle free ($\underset{\circ\!-\!\circ}{\overset{\circ}{/\,\backslash}} = 0$). Then we want to capture the density of edges. Note that $\circ\!-\!\circ = \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} + \frac{2}{3}\underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}$ (choosing representatives of the coset).

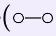
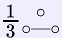
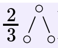
We know the following are all elements of the semantic cone $\mathcal{C}^{\emptyset}_{\text{sem}}$.

- $\emptyset$, $\overset{\circ}{\circ\ \circ}$, $\overset{\circ}{\circ\!-\!\circ}$, $\underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}$, as $\phi(F) \geq 0$ for all flags F, limit functionals ϕ.
- $\emptyset - \overset{\circ}{\circ\ \circ} - \overset{\circ}{\circ\!-\!\circ} - \underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}$. This is true as $\phi(\overset{\circ}{\circ\ \circ} + \overset{\circ}{\circ\!-\!\circ} + \underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}) = 1 = \phi(\emptyset)$ so $\phi(\emptyset - \overset{\circ}{\circ\ \circ} - \overset{\circ}{\circ\!-\!\circ} - \underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}) = 0.$

Consider then the type $\sigma = \bullet$. We must have $(\bullet\!-\!\circ - \bullet\ \circ)^2 \in \mathcal{C}^{\sigma}_{\text{sem}}$ so we have $[\![(\bullet\!-\!\circ - \bullet\ \circ)^2]\!] \in \mathcal{C}^{\emptyset}_{\text{sem}}$. We can compute that $[\![(\bullet\!-\!\circ - \bullet\ \circ)^2]\!] = \overset{\circ}{\circ\ \circ} - \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} - \frac{1}{3}\underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}$.

Therefore

$$\begin{aligned}
&\frac{1}{2}(\emptyset - \overset{\circ}{\circ\ \circ} - \overset{\circ}{\circ\!-\!\circ} - \underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}) + \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} + \frac{1}{2}(\overset{\circ}{\circ\ \circ} - \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} - \frac{1}{3}\underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}) \\
&= \frac{1}{2}\emptyset - \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} - \frac{2}{3}\underset{\circ\ \ \circ}{\overset{\circ}{/\,\backslash}}
\end{aligned}$$

is an element of $\mathcal{C}^{\emptyset}_{\text{sem}}$ (as its a convex cone). Therefore $\phi($$) = \phi(\frac{1}{3}$$+\frac{2}{3}$$) \leq \frac{1}{2}$ which is Mantel's theorem.

(■)

(Technically $\emptyset =$  $+$  $+$  as they are in the same coset in $\mathcal{A}^{\emptyset}$ so this is not quite right, but easily fixed).

Linear Programming

As discussed above we are interested in taking some linear combination of flags f and solving $\min\{\lambda\colon\, \lambda\emptyset - f \in \mathcal{C}^{\emptyset}_{\text{sem}}\}$.

The method we used above was taking linear combinations of some known fixed elements of the cone and trying to optimise over possible non-negative combinations. This corresponds to a method called linear programming.

Let $v_1, \ldots, v_k \in \mathcal{A}^{\emptyset}$ be a known set of vectors in the semantic cone.

There is some finite ordered basis $\mathcal{B}$ consisting of individual flags[1] such that $v_i \in \operatorname{span} \mathcal{B} \,\forall\, i$ and $f \in \operatorname{span} \mathcal{B}$. WLOG assume the last basis element is $\emptyset$.

$$\mathcal{B} = (F_1, \ldots, F_\ell, \emptyset).$$

In particular each basis element here is in the semantic cone.

Write each vector v_i in this basis:

$$v_i = \sum_{j=1}^{\ell} \alpha_{i,j} F_j + \beta_i \emptyset$$

Standard vector notation allows writing $v_i = [\alpha_{i,1}, \ldots, \alpha_{i,\ell}, \beta_i]^T$. We also expand our objective vector f as

$$f = \sum_{i=1}^{\ell} f_i F_i = [f_1, \ldots, f_\ell]^T$$

(WLOG f has no $\emptyset$ component).

Let $c_1, \ldots, c_k \in \mathbb{R}_{\geq 0}$ be non-negative real coefficients. Then $\sum_{i=1}^{k} c_i v_i$ is an element of $\mathcal{C}^{\emptyset}_{\text{sem}}$. We can expand

$$\sum_{i=1}^{k} c_i v_i = \left[\sum_{i=1}^{k} c_i \alpha_{i,1},\ \ldots,\ \sum_{i=1}^{k} c_i \alpha_{i,2},\ \sum_{i=1}^{k} c_i \beta_i \right]^T$$

We are looking for vectors of the form $\lambda\emptyset - (f + r)$ for $r \in \mathcal{C}^{\emptyset}_{\text{sem}}$. We know each element of the basis is in the cone so it suffices to find $c_1, \ldots, c_k$ such that

$$\sum_{i=1}^{k} c_i \alpha_{i,j} \leq -f_j \,\forall\, j \in [\ell]$$

as the *slack* consists of some non-negative combination of basis elements which is in the semantic cone. Our $\lambda \in \mathbb{R}$ is then just $\sum_{i=1}^{k} c_i \beta_i$.

We can write the requirement on c in matrix form:

$$[c_1 \quad c_2 \quad \dots \quad c_k] \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,\ell} \\ \alpha_{2,1} & \alpha_{2,2} & \dots & \alpha_{2,\ell} \\ \vdots & \vdots & & \vdots \\ \alpha_{k,1} & \alpha_{k,2} & \dots & \alpha_{k,\ell} \end{bmatrix} \leq_{\mathbb{R}^\ell} -[f_1 \quad f_2 \quad \dots \quad f_\ell]$$

which we can write succinctly as $c^T A \leq -f^T$ by defining A appropriately.

⚠ Note

These inequalities are componentwise vector inequalities, not the inequalities derived from positivity via the semantic cone.

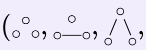
Then we are minimising the function $\sum_i c_i \beta_i = c^T \beta$ over all such $c \in \mathbb{R}^k$ such that $c^T A \leq -f^T$. Rewriting this as

$$\begin{aligned} \min_{c \in \mathbb{R}^k i} \quad & c^T \beta \\ \text{such that} \quad & c^T(-A) \geq f^T \\ & c \geq 0. \end{aligned}$$

puts our minimisation problem in the standard form for linear programming. Standard software can then be used to find an optimal (or at least good) solution.

Example (Mantel's as LP)

Using the basis

$\mathcal{B} = ($  $, \emptyset)$

allows us to write our three vectors $v_1, v_2, v_3 \in \mathcal{C}^{\emptyset}_{\text{sem}}$ as $v_1 = [-1, -1, -1, 1]^T$, $v_2 = [0, 1, 0, 0]^T$ and $v_3 = [1, -\frac{1}{3}, -\frac{1}{3}, 0]^T$. Then $f = [0, \frac{1}{3}, \frac{2}{3}, 0]^T$. Our matrix A and vector β are then

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \quad \beta = [1, 0, 0]^T.$$

We can easily check then that $c = [\frac{1}{2}, \frac{1}{3}, \frac{1}{2}]$ gives $c^T(-A) = f$ and $c^T \beta = \frac{1}{2}$.

Duality Interpretation

Linear programming problems have dual problems, in this case the dual would be: Maximise $f^T x$ over $x \in \mathbb{R}^\ell$ subject to $(-A)x \leq \beta$, $x \geq 0$. It is a theorem of linear programming that the optimal solution of this problem is at most the optimal solution of our original problem.

There is a useful interpretation of this dual version of our problem. We can think of it as an optimisation over a space of possible limit functionals ϕ. We can interpret $x \in \mathbb{R}^\ell_{\geq 0}$ as a vector corresponding to a possible assignment of values of some hypothetical limit functional ϕ_x to our basis flags: $\phi_x(F_i) = x_i$. Then the space of all such x is some relaxation of the true space of limit functionals. Each v_i that we know to be in the semantic cone places a restriction on this space as it requires that $\phi_x(v_i) \geq 0$.

In this interpretation

$$f^T x = \sum_{i=1}^{\ell} f_i x_i = \sum_{i=1}^{\ell} f_i \phi(F_i) = \phi\left(\sum_{i=1}^{\ell} f_i F_i\right) = \phi(f).$$

Hence $\max f^T x$ can be though of as $\max \phi(f)$. Then the constraint that $(-A)x \leq \beta$ can be similarly understood as requiring that for each $i \in [k]$ we have

$$-\sum_{j=1}^{\ell} \alpha_{i,j}\phi(F_j) \leq \beta_i \iff -\sum_{j=1}^{\ell} \alpha_{i,j}\phi(F_j) \leq \beta_i \phi(\emptyset) \iff 0 \leq \sum_{j=1}^{\ell} \alpha_{i,j}\phi(F_j) + \beta_i\phi(\emptyset)$$
$$\iff 0 \leq \phi\left(\sum_{j=1}^{\ell} \alpha_{i,j}F_j + \beta_i\emptyset\right) \iff 0 \leq \phi(v_i).$$

Therefore $(-A)x \leq \beta$ adds the constraint to our hypothetical ϕ function that each of our known $v_i \in \mathcal{C}^{\emptyset}_{\text{sem}}$ does indeed satisfy $\phi(v_i) \geq 0$ as required of a true limit functional.

Semidefinite Programming

Describing more positive elements

The drawback of the linear programming method is that it requires of us to preselect some fixed set of known elements of the semantic cone. What if instead we could encode an entire family of elements of the cone?

We know that $[\![\mathcal{C}^{\sigma}_{\text{sem}}]\!] \subseteq \mathcal{C}^{\emptyset}_{\text{sem}}$. In particular we know that $f^2 \geq 0 \, \forall f \in \mathcal{A}^{\sigma}$. Hence for every $f \in \mathcal{A}^{\sigma}$ we get a corresponding element in $\mathcal{C}^{\emptyset}_{\text{sem}}$ given by $[\![f^2]\!] \geq 0$.

Pick some type σ and basis $\mathcal{B}^{\sigma}$ of flags in $\mathcal{G}^{\sigma}$ to act as a basis for some subspace of $\mathcal{A}^{\sigma}$:

$$\mathcal{B}^{\sigma} = (F_1^{\sigma}, F_2^{\sigma}, \ldots, F_m^{\sigma}).$$

Let $\mathcal{B}$ then be some basis of flags in $\mathcal{G}^{\emptyset}$ such that $[\![f^2]\!] \in \operatorname{span}\mathcal{B}$ for all $f \in \operatorname{span}\mathcal{B}^{\sigma}$.

$$\mathcal{B} = (F_1, \ldots, F_{\ell}).$$

Given some $v \in \operatorname{span}\mathcal{B}^{\sigma}$ we want to know the expansion of $[\![v^2]\!]$ as a vector in $\operatorname{span}\mathcal{B}$. If $v = \sum_{i=1}^{m} a_i F_i^{\sigma} = [a_1, \ldots, a_m]^T$ then

$$[\![v^2]\!] = [\![\left(\sum_{i=1}^{m} a_i F_i^{\sigma}\right)\left(\sum_{j=1}^{m} a_j F_j^{\sigma}\right)]\!] = [\![\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j F_i^{\sigma} F_j^{\sigma}]\!] = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j [\![F_i^{\sigma} F_j^{\sigma}]\!].$$

Write $\operatorname{coef}_k(H)$ for the coefficient of F_k in H. Then

$$\operatorname{coef}_k([\![v^2]\!]) = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \operatorname{coef}_k([\![F_i^{\sigma} F_j^{\sigma}]\!]).$$

Then if we create a matrix

$$\operatorname{coef}_k(F^{\sigma}) := \begin{bmatrix} \operatorname{coef}_k([\![(F_1^{\sigma})^2]\!]) & \operatorname{coef}_k([\![F_1^{\sigma}F_2^{\sigma}]\!])) & \ldots & \operatorname{coef}_k([\![F_1^{\sigma}F_m^{\sigma}]\!]) \\ \operatorname{coef}_k([\![F_2^{\sigma}F_1^{\sigma}]\!]) & \operatorname{coef}_k([\![(F_2^{\sigma})^2]\!])) & \ldots & \operatorname{coef}_k([\![F_2^{\sigma}F_m^{\sigma}]\!]) \\ \vdots & \vdots & & \vdots \\ \operatorname{coef}_k([\![F_m^{\sigma}F_1^{\sigma}]\!]) & \operatorname{coef}_k([\![F_m^{\sigma}F_2^{\sigma}]\!])) & \ldots & \operatorname{coef}_k([\![(F_m^{\sigma})^2]\!]) \end{bmatrix}$$

we get

$$\mathrm{coef}_k([\![v^2]\!]) = v^T \cdot \mathrm{coef}_k(F^\sigma) \cdot v.$$

Note that $v^T \cdot \mathrm{coef}_k(F^\sigma) \cdot v = \mathrm{tr}(vv^T \mathrm{coef}_k(F^\sigma)) = \langle vv^T, \mathrm{coef}_k(F^\sigma)\rangle$ where $\langle A, B\rangle$ is a standard inner produce on real symmetric matrices.

Now given some fixed v we can compute using an inner product with $\mathrm{coef}_k(F^\sigma)$ the coefficients of $[\![v^2]\!]$. In theory then we could add v as a search variable and use $[\![v^2]\!]$ along with some fixed positive elements to build a positive element of the form $\lambda\emptyset - f$ as before. However we can do better than just searching over a single vector v.

Let P be a positive semidefinite matrix; It has a spectral decomposition $P = \sum_i \nu_i \nu_i^T$ for vectors ν_i. Then $[\![\sum_i \nu_i^2]\!]$ is a positive element. The coefficient of the expansion of this element is given by

$$\mathrm{coef}_k([\![\sum_i \nu_i^2]\!]) = \mathrm{coef}_k(\sum_i [\![\nu_i^2]\!]) = \sum_i \mathrm{coef}_k([\![\nu_i^2]\!]) = \sum_i \langle \nu_i \nu_i^T, \mathrm{coef}_k(F^\sigma)\rangle$$
$$= \langle \sum_i \nu_i \nu_i^T, \mathrm{coef}_k(F^\sigma)\rangle = \langle P, \mathrm{coef}_k(F^\sigma)\rangle.$$

Therefore any positive semidefinite matrix P gives us an element of the semantic cone whose coefficients we can compute with a matrix inner product with some fixed matrix.

Using these elements in search

Now we can return to the problem of searching for an element of the semantic cone: Assume we have $v_1, \ldots, v_k$ fixed elements of $\mathcal{C}^\emptyset_{\mathrm{sem}}$ and objective vector f as with the linear case. Fix some type σ and bases $\mathcal{B} = (F_1, \ldots, F_\ell, \emptyset)$ and $\mathcal{B}^\sigma = (F_1^\sigma, \ldots, F_m^\sigma)$ as above such that $v_1, \ldots, v_k, f \in \mathrm{span}\,\mathcal{B}$ and $[\![v^2]\!] \in \mathrm{span}\,\mathcal{B}$ for all $v \in \mathrm{span}\,\mathcal{B}^\sigma$. We want to find some minimum $\lambda \in \mathbb{R}$ such that $\lambda\emptyset - f \geq 0$.

We construct an element of the semantic cone from $v_1, \ldots, v_k$ and $\mathcal{B}^\sigma$ by choosing some $c_1, \ldots, c_k \geq 0$ and PSD $P = \sum_{i=1}^r \nu_i \nu_i^T$ over $\mathcal{B}^\sigma$. Then $V = \sum_{i=1}^k c_i v_i + \sum_{j=1}^r [\![\nu_j^2]\!] \in \mathcal{C}^\emptyset_{\mathrm{sem}}$.

Write $f = [f_1, \ldots, f_\ell]^T$, $v_i = [\alpha_{i,1}, \ldots, \alpha_{i,\ell}, \beta_i]^T$, $\beta = [\beta_1, \ldots, \beta_k]$ as before.

We want then that $\mathrm{coef}_i(V) = -f_i$ for all $i \in [\ell]$. As with the linear case the coefficients of $\emptyset$ form our objective function. We assume each $[\![\nu_i^2]\!]$ has no $\emptyset$ component. (This holds for certain in the local algebra case).

$$\mathrm{coef}_j(V) = \sum_{i=1}^k c_i \mathrm{coef}_j(v_i) + \sum_{i=1}^r \mathrm{coef}_j([\![\nu_i^2]\!]) = \sum_{i=1}^k c_i \alpha_{i,j} + \langle P, \mathrm{coef}_j(F^\sigma)\rangle.$$

Our optimisation problem then looks like:

$$\begin{aligned}
\min_{c \in \mathbb{R}^k} \quad & \sum_{i=1}^k c_i \beta_i \\
\text{such that} \quad & \sum_{i=1}^k c_i \alpha_{i,j} + \langle P, \mathrm{coef}_j(F^\sigma)\rangle = -f_j \;\forall\, j \in [\ell] \\
& c_i \geq 0 \;\forall\, i \in [k] \\
& P \succ 0.
\end{aligned}$$

This problem is in the form of a semidefinite programming problem. It is a simple exercise to convert this to the standard form by using diagonal matrices to adapt the linear equations and then block diagonals to combine several matrices into one.

Example (Mantel's via SDP)

Let $\sigma = \bullet$ and take the bases

$$\mathcal{B} = \left(\overset{\circ}{\circ\;\circ}, \overset{\circ}{\circ\!-\!\circ}, \overset{\circ}{\circ\diagup\diagdown\circ}, \emptyset\right) \qquad \mathcal{B}^\sigma = (\bullet\!-\!\circ, \bullet\;\circ).$$

Then $v_1 = -\overset{\circ}{\circ\;\circ} - \overset{\circ}{\circ\!-\!\circ} - \overset{\circ}{\circ\diagup\diagdown\circ} + \emptyset$, $v_2 = \overset{\circ}{\circ\!-\!\circ}$ are both $\in \mathcal{C}^\sigma_{\text{sem}}$ as before. Then we compute

$$\begin{aligned}
[\![F^\sigma]\!] &= \begin{bmatrix} [\![(\bullet\!-\!\circ)^2]\!] & [\![(\bullet\!-\!\circ)(\bullet\;\circ)]\!] \\ [\![(\bullet\!-\!\circ)(\bullet\;\circ)]\!] & [\![(\bullet\;\circ)^2]\!] \end{bmatrix} \\
&= \begin{bmatrix} [\![(\bullet\!-\!\circ)^2]\!] & [\![(\bullet\!-\!\circ)(\bullet\;\circ)]\!] \\ [\![(\bullet\!-\!\circ)(\bullet\;\circ)]\!] & [\![(\bullet\;\circ)^2]\!] \end{bmatrix} \\
&= \begin{bmatrix} [\![\overset{\bullet}{\circ\diagup\diagdown\circ}]\!] & [\![\frac{1}{2}\overset{\circ}{\bullet\!-\!\circ} + \frac{1}{2}\overset{\circ}{\bullet\diagup\diagdown\circ}]\!] \\ [\![\frac{1}{2}\overset{\circ}{\bullet\!-\!\circ} + \frac{1}{2}\overset{\circ}{\bullet\diagup\diagdown\circ}]\!] & [\![\overset{\bullet}{\circ\;\circ} + \overset{\bullet}{\circ\!-\!\circ}]\!] \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3}\overset{\circ}{\circ\diagup\diagdown\circ} & \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} + \frac{1}{3}\overset{\circ}{\circ\diagup\diagdown\circ} \\ \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} + \frac{1}{3}\overset{\circ}{\circ\diagup\diagdown\circ} & \overset{\circ}{\circ\;\circ} + \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} \end{bmatrix}.
\end{aligned}$$

Which means

$$\begin{aligned}
\operatorname{coef}_1([\![F^\sigma]\!]) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
\operatorname{coef}_2([\![F^\sigma]\!]) &= \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\
\operatorname{coef}_3([\![F^\sigma]\!]) &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}.
\end{aligned}$$

Our search then may return $c = [\frac{1}{2}, \frac{1}{3}]$ and $P = \frac{1}{2}\left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ which is PSD. Then

$$\begin{aligned}
\sum_{i=1}^k c_i\alpha_{i,1} + \langle P, \operatorname{coef}_1([\![F^\sigma]\!]) &= -\frac{1}{2} + 0 + \frac{1}{2} = 0 = -f_1 \\
\sum_{i=1}^k c_i\alpha_{i,2} + \langle P, \operatorname{coef}_2([\![F^\sigma]\!]) &= -\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = -\frac{1}{3} = -f_2 \\
\sum_{i=1}^k c_i\alpha_{i,3} + \langle P, \operatorname{coef}_3([\![F^\sigma]\!]) &= -\frac{1}{2} + 0 - \frac{1}{6} = -\frac{2}{3} = -f_3
\end{aligned}$$

as required. Then calculate $\lambda = \sum_i c_i\beta_i = \frac{1}{2}$ so $\frac{1}{2}\emptyset - f \in \mathcal{C}^\emptyset_{\text{sem}}$ proving Mantel's theorem.

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1. Again I'm ignoring the details of choosing a representative from the coset.↩