Problem Description

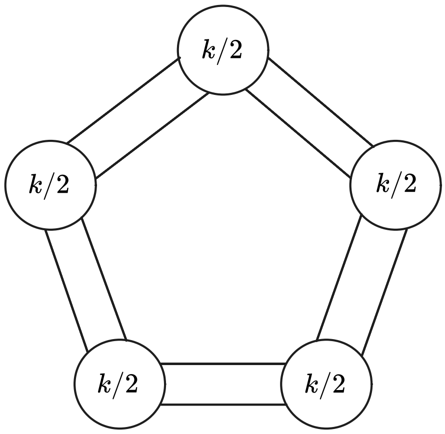
Theorem (Erdős' Pentagon Conjecture): The number of pentagons (5-cycles) in a simple triangle free graph G on n vertices is $\le \left(\frac{n}{5}\right)^5$.

We are interested in adding the requirement that G is regular, so we ask: Given a simple, triangle free, regular graph G what is the maximum number of pentagons in the graph.

Conjecture (Regular Erdős' Pentagon): The total number of pentagons in a regular triangle free graph is

$$\lesssim \frac{n}{5}\left(\frac{\Delta(G)}{2}\right)^4 = \frac{n}{5}\frac{\Delta(G)^4}{16}.$$

Tightness: For any $k \in \mathbb{N}$ even we can construct a blowup of the 5-cycle with $k/2$ vertices:



Each "supernode" consists of an independent set of $k/2$ vertices which are connected densely to the vertices in the adjacent supernodes.

Any vertex in this graph G has degree exactly k and G is triangle free, so G is a regular, triangle free graph on $n = \frac{5k}{2}$ vertices.

Consider any 5-cycle in this graph. If the 5-cycle contains only vertices from ≤ 4 supernodes then consider the subgraph induced by those supernodes. This subgraph is clearly bipartite so cannot contain an odd length cycle, which is a contradiction. Therefore this 5-cycle must contain exactly one vertex from each of the 5 supernodes. This choice of nodes uniquely identifies this cycle. Conversely, any choice of a vertex from each supernode forms a valid 5-cycle. Hence the set of 5-cycles is in 1-1 correspondence with all possible choices of 1 vertex from each supernode. Hence there are exactly $\left(\frac{k}{2}\right)^5$ 5-cycles.

The graph G has $\Delta(G) = k$ and $n = 5k/2$. We re-write

$$\left(\frac{k}{2}\right)^5 = \frac{k}{2}\left(\frac{k}{2}\right)^4 = \frac{n}{5}\left(\frac{\Delta(G)}{2}\right)^4$$

which shows G meets the bound of the conjecture.

We can construct an infinite increasing (in both n and $\Delta(G)$) sequence of such graphs. Hence if the conjecture is true it is tight.

Flag Argument

Locality

The factor of $\frac{n}{5}$ means we cannot apply local flags directly. Instead of bounding the total number of pentagons in the graph we instead try to bound the number of pentagons containing some fixed vertex v. Call this P_v.

If we show that $P_v \lesssim \frac{\Delta(G)^4}{k}$ then as each pentagon contains 5 such vertices $\sum_{v \in V(G)} P_v$ will count each pentagon 5-times, giving us a bound of $\lesssim \frac{n}{5}\frac{\Delta(G)^4}{k}$ on the total number of pentagons. If our tight example above is correct then we expect the extremal graphs to be vertex-transitive, meaning this approach should be effective.

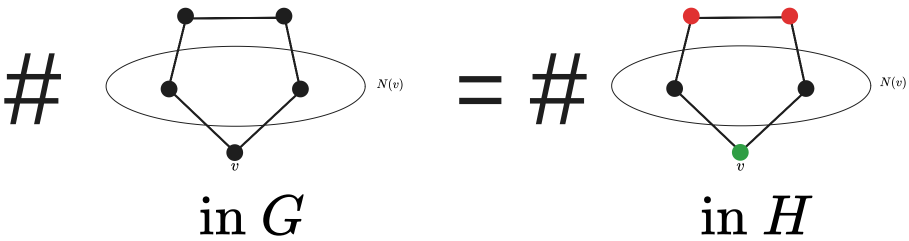
> **Note**
>
> If a graph is triangle free, then C_5 is a subgraph iff it is an induced subgraph, hence counting induced pentagons suffices.

Transformation

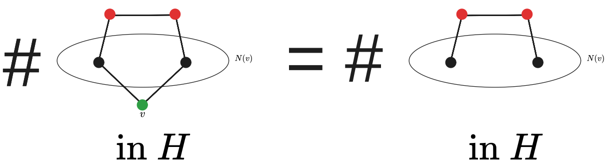
Let G be a simple, regular, triangle free graph and let $v \in V(G)$ be an arbitrary vertex. As G is triangle free $N(v)$ is an independent set. G is regular so we also have $|N(v)| = \Delta(G)$.

Consider a pentagon containing v: $\{v, u_1, u_2, u_3, u_4\}$. Two of u_i must be $\in N(v)$ and two must be $\notin N(v) \cup \{v\}$

If we construct a 3-coloured graph H which is a copy of G where v is coloured green, $N(v)$ is coloured black and the rest are coloured red then the number of pentagons in G containing v is the same as the number of pentagons in H containing 1 green node, 2 black nodes and 2 red nodes.



In fact, it suffices to count how many paths of length 4 there are in H starting and ending with black nodes and using only red nodes as middle nodes as the green node is densely connected to the black nodes.



If we remove v (the solitary green node) from the graph then this count is unchanged. This also reduces the degree of some small number of vertices by 1 but this is asymptotically irrelevant.

We have then taken counting the number of pentagons containing v in G and turned it into an equivalent problem of counting paths of length 4 starting and ending with black nodes in a black/red coloured graph H. This graph H has $\Delta(G)$ black nodes and is $\Delta(G)$ regular. The set of black nodes is independent.

Let $\mathcal{G}$ be the family of all red-black vertex coloured, regular graphs which are triangle free, have $\Delta(G)$ black nodes and the set of black nodes are independent. If we can bound the number of paths of length 4 starting and ending at black nodes in this family $\mathcal{G}$ then we have a bound on our number of pentagons containing G.

Local Flags

Given the above family of graphs $\mathcal{G}$, what are the local flags?

Lemma: *$F \in \mathcal{G}$ is a local $\emptyset$-flag iff each connected component of F contains a black node.*

> **Todo**
>
> Proof. (Only the $\Longleftarrow$ direction actually matters for the application).

In particular this tells us what our local types σ are.

Objective Function

We are interested in counting the number of paths of length 4, starting and ending at black nodes. This is precisely captured by the following $\emptyset$-flag:

This is as such a path can only appear as an induced subgraph as graphs in $\mathcal{G}$ are triangle free. We call this flag O. Importantly this is a local $\emptyset$-flag.

Constraints / Positive Elements

As $\mathcal{G}$ consists of graphs which are regular we get the constraints on regular graphs outlined in [Regularity Constraints].

There are exactly $\Delta(G)$ black nodes in any graph $G \in \mathcal{G}$. Therefore if we consider the flag $F_1 = \bullet$ we have $c(F_1; G) = \Delta(G)$ and hence $\rho(F_1; G) = 1$. In addition the set of black vertices is independent so $F_2 = \bullet\,\bullet$ counts all pairs of black vertices which is $\binom{\Delta(G)}{2}$ meaning $\rho(F_2; G) = 1$. This continues for any F_k consisting of k disconnected black vertices. In particular all such F_k are local flags.

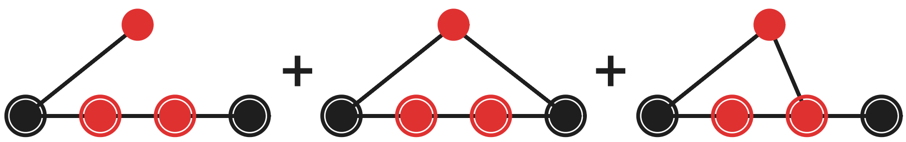
Semidefinite Program

We know from [Optimisation] how to use SDP to prove upper bounds on $\phi(f)$ for some $f \in \mathcal{L}^{\emptyset}$ and how there is a duality between real-valued constraints on limit functionals and elements of the semantic cone.

Take the basis $\mathcal{B}$ of $\emptyset$-flags of size 5.

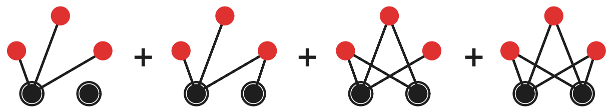
Our objective flag $O = \bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet$ is a flag of size 4. We need to write the same quantity in the basis of flags of size 5. As O is a local $\emptyset$-flag it is a local type and we can view O as a local O-flag in itself. This has the property that $[\![O]\!] = \frac{2}{4!}O = \frac{1}{12}O$.

Then as outlined in Regularity Constraints we have $\phi([\![O \cdot \mathrm{ext}_1^O]\!]) = \phi([\![O]\!]) = \frac{1}{12}\phi(O)$. $O \cdot \mathrm{ext}_1^O$ is a sum of flags of size 5:



We similarly need to translate our constraints $\phi(F_k) = 1$ into expressions over flags of size 5 by multiplying F_k by ext_i^σ vectors. We again view F_k as implicitly a local F_k-flag in itself. We can then compute $F_k \cdot (\mathrm{ext}_1^{F_k})^{5-k}$ to get sum of flags of size 5 where $\phi([\![F_k \cdot (\mathrm{ext}_1^{F_k})^{5-k}]\!]) = \phi([\![F_k]\!])$. Then $[\![F_k]\!] = F_k$ so $\phi([\![F_k \cdot (\mathrm{ext}_1^{F_k})^{5-k}]\!]) = \phi([\![F_k]\!]) = \phi(F_k)$.

For example, below is $F_2 \cdot (\mathrm{ext}_1^{F_2})^3$:



We can therefore add the constraints that $\phi([\![F_k \cdot (\mathrm{ext}_1^{F_k})^{5-k}]\!]) = 1$ to our list of constraints on the space of limit functionals.

As our $\mathcal{G}$ consists of regular graphs we also get all the $\phi([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) = 0$ constraints from Regularity Constraints for all local types σ of size 4.

We would like then to add the constraints that $\phi([\![f^2]\!]) \geq 0$ for all $f \in \mathcal{L}^\sigma$ where σ is a local type. In particular as we are working with flags of size 5 we want to choose our local type σ and basis $\mathcal{B}^\sigma$ such that f^2 results in flags of size 5 for $f \in \mathrm{span}\mathcal{B}^\sigma$. For example if $\sigma = \bullet$ (which is a local type) then any local σ-flag of size 3 has f^2 of size 5. Therefore letting $\mathcal{B}^\sigma$ be the local σ-flags of size 3 gives us a constraint $\phi([\![f^2]\!]) \geq 0$ which we can express in the basis $\mathcal{B}$ of local $\emptyset$-flags. The full list of local types σ and sizes k with which we can do this is:

- $\sigma = \bullet$ and $k = 3.$
- $\sigma = \bullet\ \bullet\ \bullet$ and $k = 4.$
- $\sigma = \bullet\!-\!\!\bullet\ \ \bullet$ and $k = 4.$
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This gives us 6 more constraints on limit functionals.

In summary our SDP is:

- Maximise $\phi([\![O \cdot \mathrm{ext}_1^O]\!])$
- Subject to:
  - $\phi(F) \geq 0 \;\forall\; F \in \mathcal{B}.$
  - $\phi([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) = 0$ for all local types σ where $|\sigma| = 4$ and all $i, j \in [|\sigma|].$
  - $\phi([\![F_k \cdot (\mathrm{ext}_1^{F_k})^{5-k}]\!]) = 1$ for all $k \in [5].$
  - $\phi([\![f^2]\!]) \geq 0$ for all $f \in \mathrm{span}\,\mathcal{B}^\sigma$ where $\mathcal{B}^\sigma$ is the set of all local σ-flags of size k for all σ, k from the above list.

This can be encoded as an SDP as described in Optimisation and software can be used to find a dual solution proving that $\phi(\llbracket O \cdot \mathrm{ext}_1^O \rrbracket) \leq \frac{1}{4}$ for all limit functionals $\phi \in \Phi^\emptyset$.

In particular this means $\frac{1}{12}\phi(O) \leq \frac{1}{4}$ hence for any Δ-increasing sequence of $\mathcal{G}$ graphs $(G_k)_{k\in\mathbb{N}}$ has the property that

$$\lim_{k\to\infty} \frac{c(O;G_k)}{\binom{\Delta(G_k)}{4}} \leq 3$$

meaning

$$\lim_{k\to\infty} \frac{c(O;G_k)}{\Delta(G_k)(\Delta(G_k)-1)(\Delta(G_k)-2)(\Delta(G_k)-3)} \leq \frac{3}{4!} = \frac{1}{8}.$$

$\Delta(G_k)(\Delta(G_k)-1)(\Delta(G_k)-2)(\Delta(G_k)-3) = \Delta(G_k)^4 + o(\Delta(G_k)^4)$ so

$$\begin{aligned}\lim_{k\to\infty} \frac{c(O;G_k)}{\Delta(G_k)^4} &= \lim_{k\to\infty} \frac{c(O;G_k)}{\Delta(G_k)^4} \frac{\Delta(G_k)^4 + o(\Delta(G_k)^4)}{\Delta(G_k)^4 + o(\Delta(G_k)^4)} \\ &= \lim_{k\to\infty} \frac{c(O;G_k)}{\Delta(G_k)^4 + o(\Delta(G_k)^4)}(1+o(1)) \\ &\leq (1+o(1))\frac{1}{8}.\end{aligned}$$

Therefore $c(O;G) \lesssim \frac{1}{8}\Delta(G)^4$.

$c(O;G)$ counted the number of paths of type in $G \in \mathcal{G}$. We saw above that this result implies the same bound for pentagons in some graph G passing through a fixed vertex v which prove the total number of pentagons in G is bounded by $\frac{n}{5}\frac{\Delta(G)^4}{8}$. This is a factor of $1/2$ away from the conjectured bound.