Note

Most of this document is recreating standard results from classic algebras. We only need to take care to make sure certain maps are well defined as maps to local flags. We also need to use the boundedness of local flags to make certain compactness arguments.

Graph Sequences

To formalise statements like "X is true as $\Delta(G) \to \infty$" we define graph sequences.

Note

These require that $\mathcal{G}$ is infinite, which we generally assume anyway.

Definition: (Δ-increasing sequence): A sequence of graphs $(G_k)_{k\in\mathbb{N}}$ is a **Δ-increasing sequence** if the sequence $(\Delta(G_k))_{k\in\mathbb{N}}$ is monotonically increasing and unbounded. This is easily extended to sequences of flags $((G_k, \theta_k))_{k\in\mathbb{N}}$.

Definition: (Local-Convergent Sequence): A sequence of σ-flags $(G_k)_{k\in\mathbb{N}}$ is **local-convergent** if $\lim_{k\to\infty} \rho(F; G)$ exists for all $F \in \mathcal{G}^\sigma_{\text{loc}}$.

Proposition: *Every sequence of σ-flags has a local-convergent subsequence.*

Proof:

Proof: Let $R_F \subseteq \mathbb{R}$ be the range of the function $\rho(F; \cdot)$ (as a function $\mathcal{G}^\sigma \to \mathbb{R}$) for each $F \in \mathcal{G}^\sigma_{\text{loc}}$. Then by definition of local flags this function is bounded so R_F is compact. Hence the space $\prod_{F\in\mathcal{G}^\sigma_{\text{loc}}} R_F$ is compact by Tychonoff and so every sequence in the space has a convergent subsequence.

($\blacksquare$)

The value of this proposition is that we can prove general results about the $\limsup$ or $\liminf$ of any sequence by just proving results about $\sup$ or $\inf$ of convergent sequences where all limits exist.

Definition (Limit Functional): Let $(G_k)_{k\in\mathbb{N}}$ be a local-convergent sequence of σ-flags. Define $\phi\colon \mathcal{G}^\sigma_{\text{loc}} \to \mathbb{R}$ as $\phi(F) := \lim_{k\to\infty} \rho(F; G_k)$. This is well defined as the sequence is local-convergent. ϕ is then easily extended linearly to the space $\mathbb{R}\mathcal{G}^\sigma_{\text{loc}}$:

$$\phi\left(\sum_{F\in\mathcal{G}^\sigma_{\text{loc}}} x_f \cdot F\right) := \sum_{F\in\mathcal{G}^\sigma_{\text{loc}}} x_f \cdot \phi(F).$$

We call ϕ a **limit functional.** Let Φ^σ denote the collection of all limit functionals $\mathbb{R}\mathcal{G}^\sigma_{\text{loc}} \to \mathbb{R}$.

Proposition *All limit functionals $\mathbb{R}\mathcal{G}^\sigma_{\text{loc}} \to \mathbb{R}$ are algebra homomorphisms $\mathcal{L}^\sigma \to \mathbb{R}$.*

✓ ***Proof*** ›

This follows immediately from the result which states
$\rho(f;G)\rho(f';G) = \rho(f\cdot f';G) + O(1/\Delta(G))$.

(■)

Todo

Is there a version of the completeness theorem? I suspect maybe not?

Positivity

Definition (Positive): We define a vector $f \in \mathcal{L}^\sigma$ to be **positive** if $\phi(f) \geq 0 \,\forall\, \phi \in \Phi^\sigma$.

Example

All single flags $F \in \mathcal{G}^\sigma_{\text{loc}}$ are positive as $\rho(F;G)$ is a non-negative function.

Write $f \geq 0$ to mean f is positive and extend this to $f \geq g \iff f - g \geq 0$.

Definition (Local Semantic Cone): The **(local) semantic cone** $\mathcal{C}^\sigma_{\text{sem}} \subseteq \mathcal{L}^\sigma$ is the cone[1] consisting of all positive elements of $\mathcal{L}^\sigma$.

Lemma: *If σ is a local type then the averaging operator $[\![\cdot]\!]\colon \mathcal{L}^\sigma \to \mathcal{L}^\emptyset$ preserves positivity. i.e. $[\![\mathcal{C}^\sigma_{\text{sem}}]\!] \subseteq \mathcal{C}^\emptyset_{\text{sem}}$.* (Local version of 3.1 in (Razborov - 2007)).

✓ ***Proof***

First note that as σ is a local type the image of $[\![\cdot]\!]$ is indeed $\subseteq \mathcal{L}^\emptyset$.

Razborov does this with measure theory, I want to avoid this.

Let $f \in \mathcal{C}^\sigma_{\text{sem}}$ and assume for contradiction that $[\![f]\!] \notin \mathcal{C}^\emptyset_{\text{sem}}$. There must then exist some $\phi \in \Phi^\emptyset$ such that $\phi(f) < 0$. Equivalently there is some Δ-convergent sequence of graphs $(G_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \rho(f;G_k) < 0$.

Then we know for fixed G_k we have

$$\rho([\![f]\!];G_k) = (1+o(1))\cdot\rho([\![\sigma]\!];G_k)\mathbb{E}_\theta[\rho(f,(G_k,\theta))]$$

(where $o(1)$ is relative to $\Delta(G_k)$). In particular this means we must have

$$\lim_{k\to\infty} \rho([\![\sigma]\!];G_k)\mathbb{E}_\theta[\rho(f,(G_k,\theta))] < 0.$$

We always have some constant C such that $C \geq \rho([\![\sigma]\!];G_k) \geq 0$ as σ is a local type so this is equivalent to

$$\lim_{k\to\infty} \mathbb{E}_\theta[\rho(f,(G_k,\theta))] < 0.$$

Hence there is some k_0 large enough such that $\mathbb{E}_\theta[\rho(f;(G_k,\theta))] < 0$ for all $k \geq k_0$. In particular there must exist some θ_k for each $k \geq k_0$ such that
$\rho(f,(G_k,\theta_k)) \leq \mathbb{E}_\theta[\rho(f;(G_k,\theta))]$.

This sequence $((G_k, \theta_k))_{k \geq k_0}$ is therefore a sequence of σ-flags and so must contain a Δ-convergent subsequence $(G'_k, \theta'_k)_{k \in \mathbb{N}}$. As each term $\rho(f; (G_k, \theta_k))$ is bounded above by $\mathbb{E}_\theta[\rho(f; (G_k, \theta))]$ the limit of this convergent subsequence must be $\leq \lim_{k \to \infty} \mathbb{E}_\theta[\rho(f; (G_k, \theta))] < 0$. Hence the corresponding limit functional $\phi' \in \Phi^\sigma$ has $\phi'(f) < 0$ which contradicts that $f \in \mathcal{C}^\sigma_{\text{sem}}$.

($\blacksquare$)

Example

For any $f \in \mathcal{L}^\sigma$ we have $f^2 \in \mathcal{C}^\sigma_{\text{sem}}$ so $[\![f^2]\!] \geq 0$ always holds.

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1. Closed under convex combination and scalar multiplication by non-negative reals.↩