Graphs

Definition ((c_e, c_v)-graph): We define a (c_e, c_v)**-graph** to be a simple undirected graph where each vertex and edge is assigned a colour from $[c_e], [c_v]$ respectively.

Definition (Isomorphism): A function $f: V(G) \to V(H)$ for G, H (c_e, c_v)-graphs is an isomorphism if it is a classic graph isomorphism and preserves colours. We write $G \cong H$.

Definition (Graph Family): If $\mathbb{G}^{c_e,c_v}$ is the class of all (c_e, c_v)-graphs then a **graph family** $\mathcal{G} \subseteq \mathbb{G}^{c_e,c_v}$ is just some subclass.

Flags

The choice of $\mathcal{G}$ informs all the following definitions.

Modified from (Silva, Filho, Sato - 2016: [Flag Algebras: A First Glance]).

Original definitions: Section 2.1 of (Razborov - 2007: [Flag Algebras]).

Definition (Type): A **type of size** $k \in \mathbb{N}_0$ is some $\sigma \in \mathcal{G}$ with $V(\sigma) = [k]$. We write $|\sigma|$ for the size of the type.

Definition (Embedding): An **embedding** of a type σ of size k into $F \in \mathcal{G}$ is an injective function $\theta: [k] \to V(F)$ such that θ is an isomorphism between σ and $F[\operatorname{im}\theta]$.

Definition (Flag): A pair (F, θ) where $F \in \mathcal{G}$ and θ is an embedding of σ into F is a σ**-flag.**

Notation: We denote the collection of all σ-flags with up to isomorphism $\mathcal{G}^\sigma$. Use a subscript $\mathcal{G}^\sigma_n$ to denote only flags of size n. If $\sigma = \emptyset$ we can drop the σ superscript.

Notation: We will often implicitly treat a type σ as a σ-flag. Specifically one with the $\mathrm{id}: [|\sigma|] \to [|\sigma|]$ embedding.

Definition (Labelled): We refer to a vertex in (F, θ) as labelled if it is in the image of θ, and often call $\operatorname{im}\theta$ the *labelled part of F.*

Definition (Flag Isomorphism): $f: V(F) \to V(F')$ is a σ-flag isomorphism from $(F, \theta) \to (F', \theta')$ if it is an isomorphism $F \to F'$ which preserves labels, meaning $f(\theta(i)) = \theta'(i)\ \forall\, i \in [|\sigma|]$. We can write $(F, \theta) \cong (F', \theta')$ if such an f exists.

Empty-flags: If $\sigma = \emptyset$, the empty graph then flags are just graphs with trivial embeddings. We usually just write F rather than (F, θ) and treat graphs as implicit $\emptyset$-flags.

Induced Count

Definition (Induced Count): Fix two σ-flags $(F, \theta), (G, \eta)$. We define the induced count of (F, θ) in (G, η) written $c((F, \theta); (G, \eta))$ (abbreviated as $c(F; G)$ if θ, η are implicit) as the number of induced copies of F in G s.t. the labelled part of F maps to the labelled part of G.

Specifically this is the number of subsets $\mathrm{im}(\eta) \subseteq U \subseteq V(G)$ such that $(F,\theta) \cong (G[U],\eta)$. Clearly $c(F,G) = 0$ if $|G| < |F|$.

Extension to multiple flags: Take a finite number of σ-flags $(F_i,\theta_i), i \in [t]$ and another σ-flag (G,η). We now define $c(F_1,\dots,F_t;G)$ as the number of t-tuples of induced copies of $F_1,\dots,F_t$ in G which are disjoint except precisely at the labelled part of G.

Precisely this is the number of subsets $U_1,\dots,U_t \subseteq V(G)$ such that $\mathrm{im}\,\eta = U_i \cap U_j \,\forall\, i,j \in [t]$ where $i \neq j$ and $(F_i,\theta_i) \cong (G[U_i],\eta)$ for all $i \in [t]$.

Definition (Fit): Note that if $c(F_1,\dots,F_t;G) > 0$ then $|G| - |\sigma| \geq \sum_{i=1}^{t} |F_i| - |\sigma|$. If this holds we say $F_1,\dots,F_t$ **fit in** G.

Classic Density

This is the classic density used in Razborov's flag algebras: [(Razborov - 2007)].

Definition (Induced Density): For σ-flags (F,θ) and (G,η) define $p((F,\theta),(G,\eta))$ (abbreviated as $p(F,G)$) as the proportion of subsets $\mathrm{im}\,\eta \subseteq U \subseteq V(G)$ such that $(F,\theta) \cong (G[U],\eta)$:

$$p(F;G) = \frac{c(F;G)}{\binom{|G|-|\sigma|}{|F|-|\sigma|}}.$$

Extension to finite set of flags: As with $c(F,G)$ we can extend to a finite collection of flags (F_i,θ_i), $i \in [t]$ where we normalise $c(F_1,\dots,F_t;G)$ by the total number of possible subsets $U_1,\dots,U_t \subseteq V(G)$ such that $U_i \cap U_j = \mathrm{im}\,\eta\,\forall i \neq j$.

$$p(F_1,\dots,F_t;G) = \frac{c(F_1,\dots,F_t;G)}{\binom{|G|-|\sigma|}{|F_1|-|\sigma|,\dots,|F_t|-|\sigma|,R}}$$

using multinomial coefficient notation where $R = (|G| - |\sigma|) - \sum_{i=1}^{t} |F_i| - |\sigma|$.

Probabilistic Interpretation: You can interpret $p(F_1,\dots,F_t;G)$ as the probability that a uniformly random tuple $U_1,\dots,U_t \subseteq V(G)$ such that $U_i \cap U_j = \mathrm{im}\,\eta \,\forall\, i \neq j$ has the property that $(F_i,\theta_i) \cong (G[U_i],\eta) \,\forall\, i \in [t]$.

Lemma (Chain Rule): *If $F_1,\dots,F_t$ are σ-flags which fit in G then for all $1 \leq s \leq t$ and every n such that $F_1,\dots,F_s$ fit into a σ flag of size n and a σ-flag and $F_{s+1},\dots,F_t$ fit in G we have:*

$$p(F_1,\dots,F_t;G) = \sum_{F \in \mathcal{G}_n^\sigma} p(F_1,\dots,F_s;F)p(F,F_{s+1},\dots,F_t;G).$$

Local Flags

> **Todo**
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> Needs an example of a density hard to describe with classic flags. Maybe something like paths of length 2 in a Δ-regular graph.

> **Todo**
>
> Highlight that Δ being the max degree is not actually required for these definitions to make sense. All that's required is that the Δ-parameter bounds

"degrees of freedom" in some way.

Consider the class of Δ-regular graphs. Usually we can WLOG assume a graph is Δ-regular where Δ is the max degree.

Assume $\Delta > 0$.

Definition (Local Density): Rather than normalising the induced count of σ-flags $c((F,\theta);(G,\eta))$ by $\binom{|G|-|\sigma|}{|F|-|\sigma|}$ we instead normalise by $\binom{\Delta}{|F|-|\sigma|}$.

$$\rho(F;G) = \frac{c(F;G)}{\binom{\Delta}{|F|-|\sigma|}}.$$

Note the $\rho \neq p$ notation.

Because of our choice of "normalisation" we are no longer guaranteed a $[0,1]$ codomain. The full range for ρ is $\mathbb{R}_{\geq 0}$.

Todo

Example of why ρ can be unbounded.

This unboundedness is undesirable. We will now define a restricted subset of the classic flags to the case where we do not have this divergent behaviour.

Once again these definitions are relative to some fixed family of graphs $\mathcal{G}$.

Todo

Needs an elucidatory example of a family of graphs $\mathcal{G}$ where we describe which structures have divergent density and which do not.

Definition (Local Type): A type σ is a **local type** if the function $\mathcal{G} \to \mathbb{R}_{\geq 0}$ given by $G \mapsto \rho(\sigma; G)$ is bounded ($\in O(1)$).

Definition (Local Flag): Let σ be a local type. Then (F,θ) is a local σ-flag if we have the following properties:

1. $(G,\eta) \to \rho((F,\theta),(G,\eta))$ is a bounded function.
2. If we label any of F's unlabelled vertices we get another local flag.

Todo

Is this second property always implied by the first?

What we're trying to capture here is that any "subflag" of F is also a local flag, meaning we can pin down F's vertices and continue to get bounded local behaviour.

Todo

Formalise this in terms of type extensions, inductive definition etc.

Notation: Write $\mathcal{L}^\sigma_n$ to be the set of all local σ-flags of size n up to isomorphism. Write $\mathcal{L}^\sigma$ for all local σ-flags. If $\sigma = \emptyset$ then we can drop the σ superscript.

Notation: As with classic flags if σ is a local type then we often implicitly treat it as a σ-flag with the identity embedding.

Note

The chain rule does not hold in general for local flags.